(1)

$$3 \quad 1 \quad 3 \quad 6 \quad = \quad 8$$

$$(3+1)/3\times 6 \quad = \quad (3+1)/(3/6) \quad = \quad 8$$

- Not easy to find, but easy to verify. (NP certificates)
- Could do exhaustive search (Arithmetic trees), but may be easier to "guess then check."
- Think outside the box: think about fractions, not only integers.

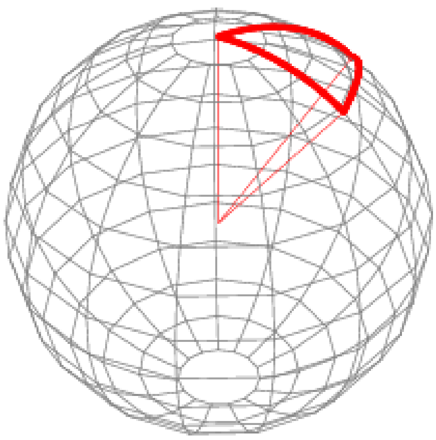
(2) Table form:

Action	A (3L Jug)	B (5L Jug)
	0	0
Fill A	3	0
Empty A into B	0	3
Fill A	3	3
Fill B from A (2L)	1	5
Empty B	1	0
Empty A into B	0	1
Fill A	3	1
Empty A into B	0	4

- Not easy to find, easy to verify. (NP certificates)
- Could do exhaustive search (Arithmetic trees), but may be easier to "guess then check."

(3) White (North pole), spherical triangle.

- Think outside the box.



(4) Python quick script:

```
lockers = [False for i in range(101)]

for run in range(1,101):
    for locker in range(run,101,run):
        lockers[ locker ] = not lockers[
            locker ]

for i in range(101):
    if lockers[i]:
        print(i)
```

- Quick code helps find the pattern. Use maths to justify.

(5) No.

Minimum is:
$$0+1+2+\cdots+9 = \frac{10\times(0+9)}{2} = 45 > 44.$$

- Sometimes we have to prove a solution does not exist.

(6) The answer is: $10+9=19$.

Use "baby-step giant-step": use first egg at floors $10, 20, \ldots, 100$. Once it breaks at floor $10k$ we then know it will break at $10k-9, \ldots, 10k-1$. Use second egg 9 times at most

In general, for n floor: use giant steps of size x: optimal when $(n/x+x-1)'=0$ i.e. $x=\sqrt{n}$. So number of steps is $2\sqrt{n}-1$. .

For 3 eggs: $(n/x+(2\sqrt{x}-1)-1)'=0$ i.e. $x=n^{2/3}$. So number of steps is: $3n^{1/3}-2$.

For 4 eggs, etc.

When we have "enough" eggs ($\ell \geq \log_2 n$) we can perform binary search: $\log_2 n$ steps.

- Special cases, sub-problems, generalizations.

(7) No.

There are more black squares than white squares. (Each domino covers exactly one black and one white).

- Idea of invariant.

(1) Give the truth table for the following propositions

Expression	**Meaning**
$a \wedge b$	a and b
$a \vee b$	a or b
$a \oplus b$	a xor b
$\neg a$ (or $\bar{a}$)	not a
$a \implies b$	a implies b, or: if a then b
$a \iff b$	a and b are equivalent, or: "a if and only if b"

It is usual to apply these "bit-wise" to the bits of integers, e.g. $\texttt{0011} \oplus \texttt{0101} = \texttt{0110}$.

Solution

$\neg a$	a	b	$a \wedge b$	$a \vee b$	$a \oplus b$	$a \implies b$	$a \iff b$
1	0	0	0	0	0	1	1
1	0	1	0	1	1	1	0
0	1	0	0	1	1	0	0
0	1	1	1	1	0	1	1

(2) Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of **natural numbers**, and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of **integers**.

Consider the following set definitions

- $A = \{a \in \{1, 2, 3, 4\} \mid (a < 2) \vee (a > 3)\}$
- $B = \{a \in \mathbb{N} \mid a < 9\}$
- $C = \{a \in \mathbb{N} \mid a > 2 \wedge a < 7\}$
- $D = \{i \in \mathbb{Z} \mid i^2 \leq 9\}$

a) Give an explicit enumeration for each set, i.e. write down the elements in the form $\{x_1, x_2, \ldots\}$.

b) What is the cardinality of each set?

c) Which of these sets are subsets of at least one other set?

Solution

a)
- $A = \{1, 4\}$
- $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $C = \{3, 4, 5, 6\}$
- $D = \{-3, -2, -1, 0, 1, 2, 3\}$

b)
- $\#A = 2$ (#A is also denoted by $|A|$)
- $\#B = 8$
- $\#C = 4$
- $\#D = 7$

c) $A \subset B$ and $C \subset B$.

(3) If the set A is $\{1,3,4\}$ and the set B is $\{3,5\}$, write down:

Expression	Meaning
$A \cup B$	union of A and B
$A \cap B$	intersection of A and B
$A - B$	A minus B
$A \times B$	Cartesian product of A and B: set of all possible pairs (a,b) where $a \in A$ and $b \in B$
2^B (or $\mathcal{P}(B)$)	power set of B: set of all subsets of B

Solution

- $A \cup B = \{1,3,4,5\}$
- $A \cap B = \{3\}$
- $A - B = \{1,4\}$
- $A \times B = \{(1,3),(1,5),(3,3),(3,5),(4,3),(4,5)\}$
- $2^B = \{\emptyset, \{3\}, \{5\}, \{3,5\}\}$

(4) Draw the (undirected) graph $G = (V, E)$, where

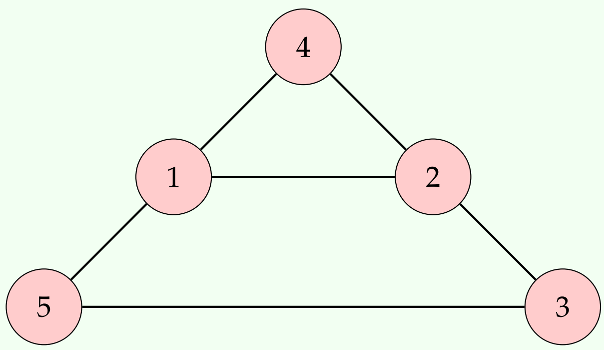
$$\begin{aligned} V &= \{1,2,3,4,5\} \\ E &= \{(1,2),(1,4),(2,3),(2,4),(3,5),(1,5)\} \end{aligned}$$

a) Is the graph connected?

b) What about the graph $G' = (V', E')$, where $V' = \{1,2,3,4\}$ and $E' = \{(1,3),(2,4)\}$?

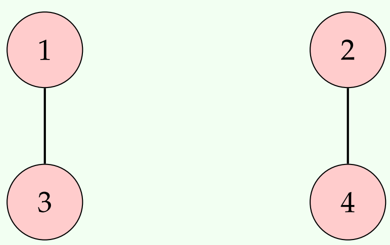
Solution

G:



G is connected.

G' (pronounced "G prime"):



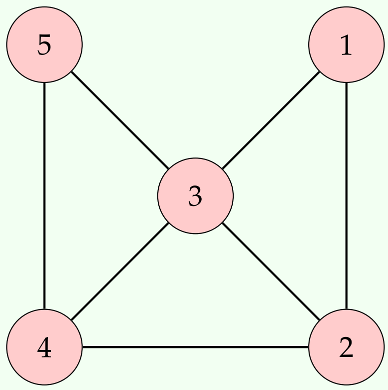
G' is not connected.

(5) Draw the graph $G = (V, E)$, where $V = \{1, \ldots, 5\}$ and

$$E = \{(a, b) \mid a, b \in V \wedge (a < b < a + 3)\}.$$

Solution

a	b
1	$2, 3$
2	$3, 4$
3	$4, 5$
4	5
5	



(6) Express the following expressions using O-notation

- $x + 5$
- 2016
- $543x + x^3 + 13$
- $2784x + 132 \times 1074$
- $x^{578} + 4685 + 2^x$
- $x^2 + x(\log x)^2 + 35$
- $x + x \log^2 x + 35$
- $2016^x + x^x + x!$
- $x^{86754} + x!$

Solution

$$\begin{aligned}
x + 5 &= O(x) \\
2016 &= O(1) \\
543x + x^3 + 13 &= O(x^3) \\
2784x + 132 \times 1074 &= O(x) \\
x^{578} + 4685 + 2^x &= O(2^x) \\
x^2 + x(\log x)^2 + 35 &= O(x^2) \\
x + x \log^2 x + 35 &= O(x \log^2 x) \\
2016^x + x^x + x! &= O(x^x) \\
x^{86754} + x! &= O(x!)
\end{aligned}$$