# Turing Machines (TMs)

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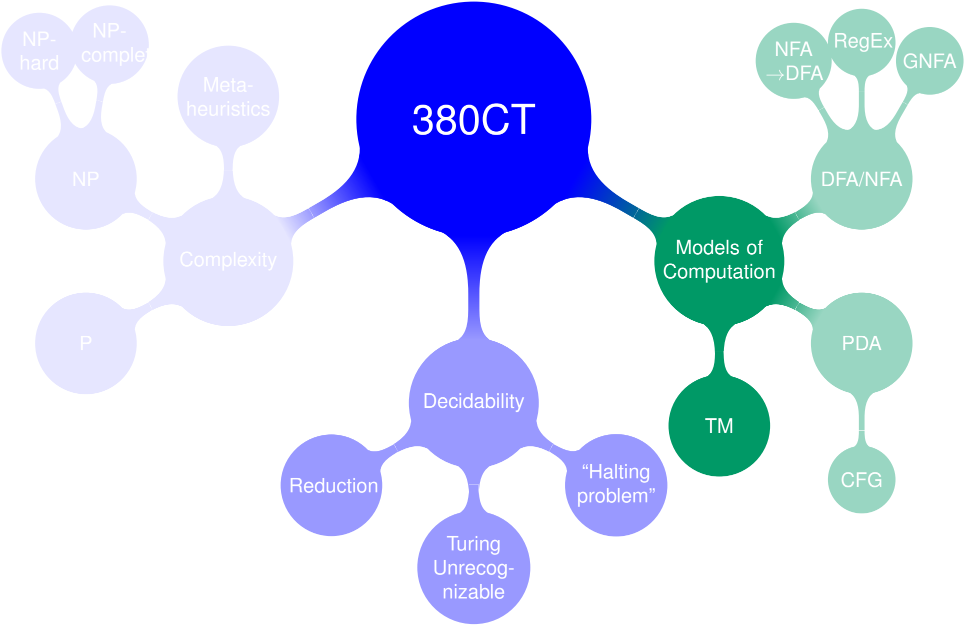


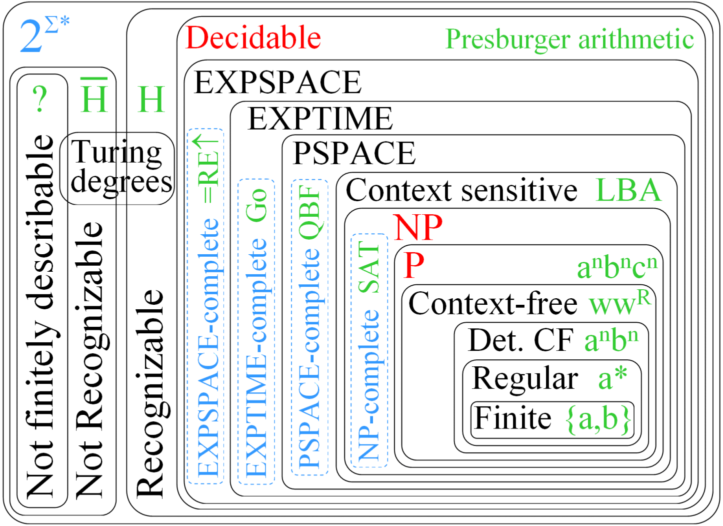
380CT
Complexity
NP
NP-hard
NP-complete
P
Meta-heuristics
Models of Computation
DFA/NFA
NFA →DFA
RegEx
GNFA
PDA
CFG
TM
Decidability
Reduction
Turing Unrecog-nizable
"Halting problem"

# Last week... Grammars/Chomsky Hierarchy

| Grammar | Languages | Automaton | Production rules |
|---|---|---|---|
| Type-0 | Recursively Enumerable | Turing Machine (TM) | $\alpha \to \beta$ (no restrictions) |
| Type-1 | Context Sensitive | Linear-bounded TM | $\alpha A \beta \to \alpha \gamma \beta$ |
| Type-2 | Context Free | PDA | $A \to \gamma$ |
| Type-3 | Regular | NFA/DFA | $A \to aB \mid a$ |

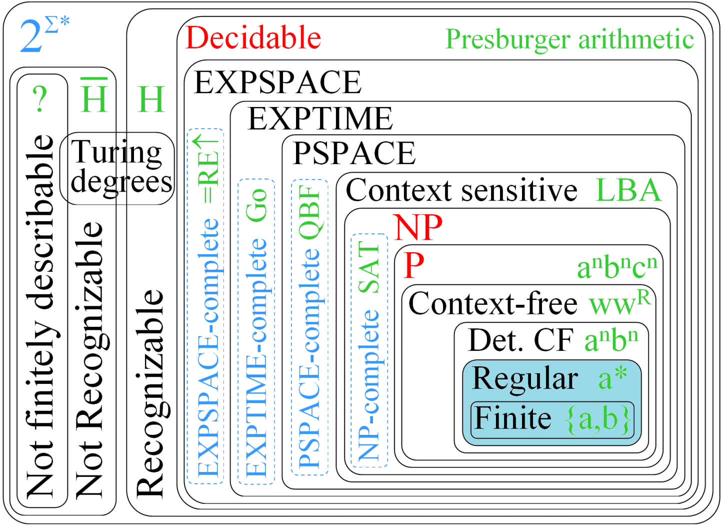
$a, b, \ldots$ *Terminals – constitute the strings of the language*
$A, B, \ldots$ *Non-terminals – should be replaced*
$\alpha, \beta, \ldots$ *Combinations of the above*

# The Extended Chomsky Hierarchy



# Chomsky Hierarchy

## The Extended Chomsky Hierarchy



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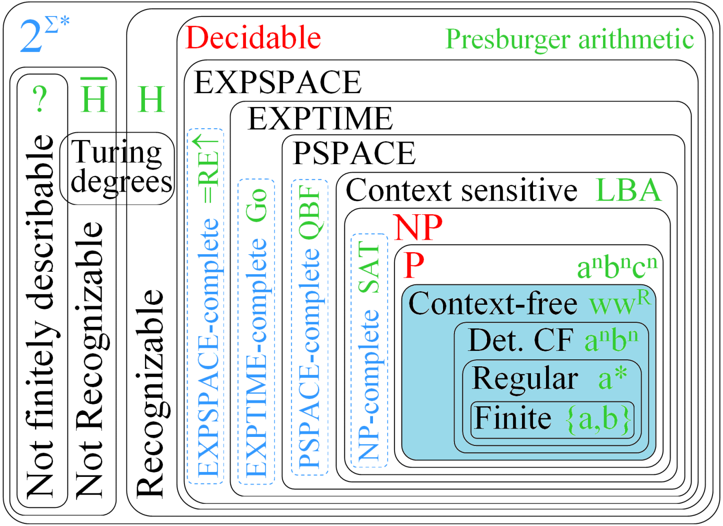
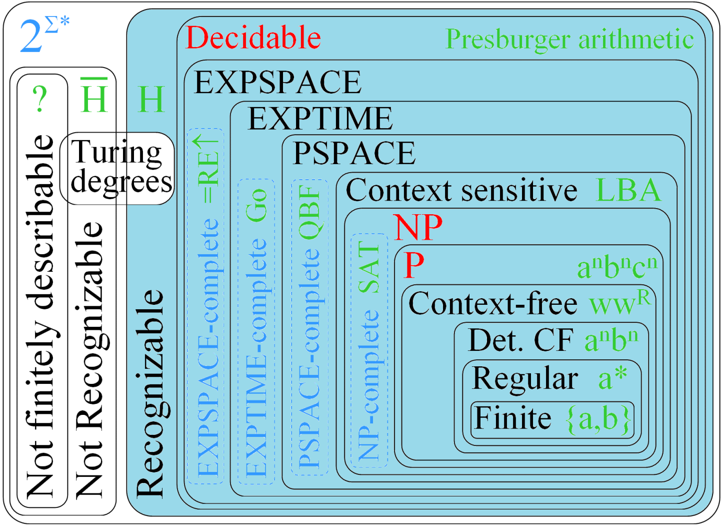


# Chomsky Hierarchy

## The Extended Chomsky Hierarchy



# TMs vs NFAs/PDAs

- TMs are similar to NFAs/PDAs, but have access to unlimited memory.
- No known model of computation is more powerful than the TM model.

The main differences are:

1. TMs may store the entire input string and refer to it **as often as needed.**
2. Dedicated states for **accepting** and **rejecting** which take **immediate effect**. (No need to reach the end of the input string.)

   $\rightarrow$ The TM has the potential to go on for ever, without reaching either an accept or reject state $\rightarrow$ **"Halting Problem"**.

## Example (TM to recognize $\{w\#w \mid w = \{0,1\}^*\}$)

- Scan the input to check it contains only a single $\#$ symbol. If not then **reject**.
- Zig-zag across the tape to corresponding symbols on either side of the $\#$ symbol, crossing off each matching pair. If they do not match then **reject**.
- When all symbols to the left of the $\#$ are crossed off, check for any remaining symbols to the right. If there are then **reject**, otherwise **accept**.

**Task:** Trace the TM on the following inputs:

`01#01`  `011#01`  `01#011`  `01##01`

## Turing Machine (TM)

- TM has an **infinite tape** (memory), divided into cells.
- It has a tape **head**, which may **read** and **write** symbols and **move** around.
- Initially: tape contains only the *input string*; *blank everywhere else*.
- If the machine needs to store information, it can write it on the tape.
- It has designated **accept** and **reject** states.
  Can only terminate on reaching one or the other; otherwise, it will just keep going. . . !
- **Transition function** $\delta$: given a $(\textit{state}, \textit{symbol})$ pair, TM **changes state, writes a symbol** and **moves left or right** by one cell.

# Turing Machine Computation

- Input is placed on tape; rest of the tape is blank.
- Head starts on the leftmost cell of the input.
- Computation proceeds according to the rules of $\delta$.
- Computation continues until it enters either an **accept** or **reject** state.
- Snapshot of tape and head at a given time $\rightarrow$ **configuration**

## Configuration

Notation $uqv$:

- $u$: string to left of head.
- $v$: string to right of head including the current head location.
- $q$: current state.

e.g. tape contains 10010, TM is in state $q_6$, and head is over the second zero $\rightarrow$ write: $10q_6010$

# TM languages classification

## Turing **decidable** languages

A language is **decidable** if some TM **decides** it.
Namely, given a string $w$:

- if $w$ is **in** the language then the TM will **accept** it.
- it $w$ is **not in** the language then the TM will **reject** it.

Such TMs are called **deciders.**

## Turing **recognizable** languages

A language is **recognizable** if some TM **recognizes** it.
Namely, given a string $w$:

- if $w$ is **in** the language: the TM will **accept** it.
- it $w$ is **not in** the language then the TM may **reject** it or **never halt**.

# Description of algorithms and TMs

Three possible levels of detail:

1. Formal description (Transition diagrams, etc.).
2. Implementation description.
   (Describe how TM manages tape and moves head).
3. High-level description (Pseudocode or higher).

Also, we usually specify how to **encode** objects (if not obvious/standard), and the exact **input** and **output**.

## Example ($\{0^{2^n} \mid n \geq 0\}$ – High level)

This language consists of all strings of 0's whose length is a power of 2.

$$\{0, 00, 0000, 00000000, 0^{16}, 0^{32}, \ldots\}$$

**Input:** String $s \in \{0\}^+$.
**Output:** *true* if $|s|$ is a power of $2$; *false* otherwise.

**while** $|s|$ *is even* **do**
    $s \leftarrow$ *half of s*
**end while**
**if** $|s| = 1$ **then**
    **return** *true*
**else**
    **return** *false*
**end if**

# Example ($\{0^{2^n} \mid n \geq 0\}$ – Implementation level)

1. Scan left to right across the tape, crossing off every other 0.
2. If only a single 0 remains then **accept**.
3. If an odd number of 0's remain then **reject**.
4. Return to the left hand end of the tape.
5. Go to step 1.

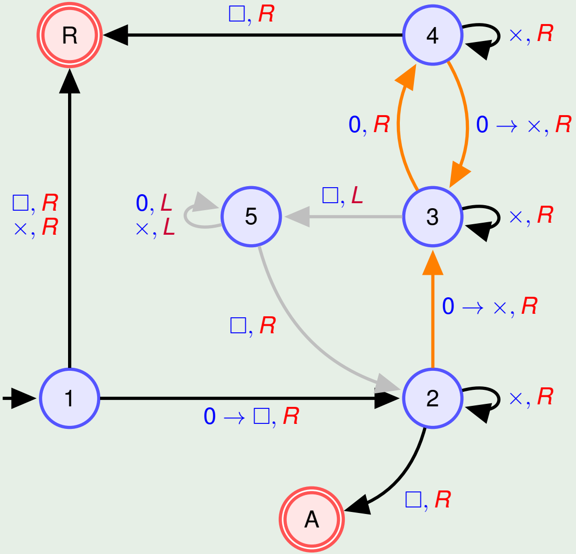
**Task:** Trace the following inputs:

$$0, 0^2, 0^3, 0^4, 0^7$$

# Example ($\{0^{2^n} \mid n \geq 0\}$ – Formal description)

Formal description:

- $Q = \{1, 2, 3, 4, 5, \mathrm{A}, \mathrm{R}\}$
- $\Sigma = \{0\}$
- $\Gamma = \{0, x, \square\}$
- The start, accept and reject states are 1, A and R, respectively.
- $\delta$ is given by the state diagram:




**Notation:**
$a \to b, R$: on reading a on the tape: replace it with b,then move to the right.
$a, R$: shorthand for $a \to a, R$

## Formal Definition of a TM

A Turing Machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_{\text{start}}, q_{\text{accept}}, q_{\text{reject}})$ where

- $Q$ is the finite set of states
- $\Sigma$ is the input alphabet, not containing the special *blank symbol*: $\square$
- $\Gamma$ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subset \Gamma$
- $\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{R, L\}$ is the transition function
- $q_{\text{start}}$ is the start state
- $q_{\text{accept}}$ is the accept state
- $q_{\text{reject}}$ is the reject state, where $q_{\text{accept}} \neq q_{\text{reject}}$

# Multi-tape TMs

- A **multi-tape TM**, is a TM with *more than one tape*.
- More transitions need to be defined, but it **simplifies computations**

**Example ($\{w\#w \mid w = \{0,1\}^*\}$)**

One-tape: zig-zag around $\#$ crossing off matching symbols. Requires two loops.

Multi-tape: write the second half in the second tape, then use a single loop to check it matches the first half.

**Equivalence**

Every multi-tape TM has an equivalent single-tape TM.

# Nondeterministic Turing Machines (NTMs)

- A configuration can have **zero or more** subsequent configurations.
  $\rightarrow$ TM may be in many configurations at the same time.
  Imagine the TM self-replicating as it goes along.
- Subtlety: If a non-deterministic TM is a decider then **all** branches need to reject for it to **reject** a string.
- Deterministic and nondeterministic TMs recognize the same languages!

**Equivalence**

Every NTM has an equivalent deterministic TM.

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**Equivalence**

Every NTM has an equivalent deterministic TM.

The closest we have to an NTM is **DNA computation**: the processed units are artificially manufactured chromosomes (capable of self-replication). This still is not really nondeterministic as there is a finite limit to the number of DNA strands which may exist during computation.

Quantum computers are also equivalent to Turing Machines.

# History: nature of computing

Questions about this first arose in the context of pure Mathematics:

- Gottlob Frege (1848–1925)
- David Hilbert (1862–1943)
- George Cantor (1845–1918)
- Kurt Gödel (1906–1978)
- 1936:
  - Gödel and Stephen Kleene (1909-1994): **Partial Recursive Functions**
  - Gödel, Kleene and Jacques Herbrand (1908–1931)
  - Alonzo Church (1903–1995): **Lambda Calculus**
  - Alan Turing (1912–1954): **Turing Machine**
- 1943: Emil Post (1897–1954): **Post Systems**
- 1954: A.A. Markov: Theory of Algorithms – **Grammars**
- 1963: Shepherdson and Sturgis: **Universal Register Machines**

**Equivalence of all of these models $\rightarrow$ "Church-Turing Thesis"**

All these models define exactly the same class of computable functions!
$\rightarrow$ Anything that is computable can be computed by some Turing machine.

# The Church-Turing Thesis

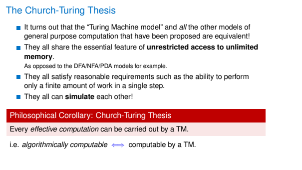
- It turns out that the "Turing Machine model" and *all* the other models of general purpose computation that have been proposed are equivalent!
- They all share the essential feature of **unrestricted access to unlimited memory.**

  As opposed to the DFA/NFA/PDA models for example.
- They all satisfy reasonable requirements such as the ability to perform only a finite amount of work in a single step.
- They all can **simulate** each other!

**Philosophical Corollary: Church-Turing Thesis**

Every *effective computation* can be carried out by a TM.

i.e. *algorithmically computable* $\iff$ computable by a TM.




See http://plato.stanford.edu/entries/church-turing/ and http://en.wikipedia.org/wiki/Church-Turing_thesis for discussion.

# Algorithms

In a sense, the Church-Turing thesis implies that the underlying class of "algorithms" described by all these models of computation is the same, and corresponds to the natural intuitive concept of *algorithms*.

$$\text{Intuitive concept of algorithms} \quad = \quad \text{Turing machine algorithms}$$

# Next week...

## Limits of computation...

Even a TM **cannot** solve certain problems!

Such problems are beyond the theoretical limits of computation!