Тема 8 „Понятие об интеграле"

3. Вычислить определённый интеграл:

$$\int_0^{\pi} 3x^2 \sin(2x)\, dx$$

$$\int_a^b f(x)\, dx = F(x) \Big|_a^b = F(b) - F(a)$$

$$F(x) = \int 3x^2 \sin(2x)\, dx = 3\int x^2 \sin(2x)\, dx =$$

$$U = x^2 \quad \Rightarrow \quad dU = 2x\, dx$$

$$dV = \sin 2x\, dx \quad \Rightarrow \quad V = -\frac{1}{2}\cos 2x$$

$$(2x)' = 2$$

$$dx = \frac{1}{2} d2x$$

$$= 3\left(-\frac{1}{2}x^2 \cdot \cos 2x + \frac{1}{\cancel{2}} \cdot \cancel{2} \int x \cdot \cos 2x\, dx\right) =$$

$$U = x \quad \Rightarrow \quad dU = dx$$

$$dV = \cos 2x\, dx \Rightarrow V = \frac{1}{2}\sin 2x$$

$$= 3\left(-\frac{1}{2}x^2 \cdot \cos 2x + \frac{1}{2}\left(x \cdot \sin 2x - \int \sin 2x\, dx\right)\right) =$$

$$= 3\left(-\frac{1}{2}x^2 \cdot \cos 2x + \frac{1}{2}\left(x \cdot \sin 2x - \frac{1}{2}\int \sin 2x\, d2x\right)\right) =$$

$$= 3\left(-\frac{1}{2}x^2 \cdot \cos 2x + \frac{1}{2}\left(x \cdot \sin 2x + \frac{1}{2}\cos 2x\right) + C\right) =$$

$$= -\frac{3}{2}x^2 \cdot \cos 2x + \frac{3}{2}x \cdot \sin 2x + \frac{3}{4}\cos 2x + C$$

$$F(a) = F(0) = -\frac{3}{2}\cdot 0 + \frac{3}{2}\cdot 0 + \frac{3}{4}\cos 0 + C =$$

$$= \frac{3}{4} + C.$$

$$F(b) = F(\pi) = -\frac{3}{2}\pi^2 \cos\underset{1}{2\pi} + \frac{3}{2}\pi\cdot \sin\underset{0}{2\pi} +$$

$$+ \frac{3}{4}\cos\underset{1}{2\pi} + C = -\frac{3}{2}\pi^2 + \frac{3}{4} + C$$

$$\int_0^{\pi} 3x^2 \sin(2x)\,dx = F(b) - F(a) =$$

$$= -\frac{3}{2}\pi^2 + \frac{3}{4} + C - \frac{3}{4} - C = -\frac{3}{2}\pi^2.$$

4. Найти неопределённый интеграл:

$$\int \frac{1}{\sqrt{x+1}}\,dx = 2\int \frac{\sqrt{x+1}}{\sqrt{x+1}}\,d\sqrt{x+1} = 2\sqrt{x+1} + C.$$

$$\left(\sqrt{x+1}\right)' = \frac{1}{2\sqrt{x+1}}\cdot 1$$

$$dx = 2\sqrt{x+1}\,d\sqrt{x+1}$$

Однородные ДУ

$$y' = 1 + \frac{2y}{x}$$

$$y = tx \quad , \quad y' = t'x + t$$
$$t = \frac{y}{x}$$

$$t'x + t = 1 + 2t$$

$$t'x - t = 1$$

$$t' = \frac{t+1}{x}$$

$$\frac{dt}{dx} = \frac{t+1}{x}$$

$$\frac{dt}{t+1} = \frac{dx}{x}$$

$$\int \frac{dt}{t+1} - \int \frac{dx}{x} = 0$$

$$\ln(t+1) - \ln x = C$$

$$\ln\left(\frac{y}{x} + 1\right) - \ln x = C$$

Тема 7. „Ряды"

6. Дана функция $f(x) = x^2$

а. Разложить функцию в ряд Фурье по косинусам на отрезке $[-\pi; \pi]$.

в. Построить график функции и её разложения.

а) По косинусам раскладывается чётная функция в ряд Фурье:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n x}{l}.$$

В данной задаче период разложения $T = 2\pi$, полупериод $l = \pi$.

Коэффициенты Фурье на $[-\pi; \pi]$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left(\frac{x^3}{3}\Big|_{\pi} - \frac{x^3}{3}\Big|_0\right) =$$

$$\int x^2 dx = \frac{x^3}{3}$$

$$= \frac{2\pi^3}{3\pi} = \frac{2}{3}\pi^2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = \frac{2}{\pi}\left(x^2 \cdot \frac{1}{n}\sin nx\Big|_0^{\pi} - \right.$$

$$U = x^2 \Rightarrow dU = 2x\,dx$$

$$dV = \cos nx\, dx \Rightarrow V = \frac{1}{n}\sin nx$$

$$UV\Big|_a^b - \int_a^b V\,dU$$

$$\left. - \frac{2}{n}\int_0^{\pi} x \cdot \sin nx\, dx\right) = \frac{2}{\pi}\left(x^2 \cdot \frac{1}{n}\sin nx\Big|_0^{\pi} - \right.$$

$$U = x \Rightarrow dU = dx$$

$$dV = \sin nx\, dx \Rightarrow V = -\frac{1}{n}\cos nx$$

$$\left. - \frac{2}{n}\left(-\frac{1}{n}x \cdot \cos nx\Big|_0^{\pi} + \frac{1}{n}\int \cos nx\, dx\right)\right) =$$

$$= \frac{2}{\pi}\left(x^2 \cdot \frac{1}{n}\sin nx\Big|_0^{\pi} - \frac{2}{n}\left(-\frac{1}{n}x \cdot \cos nx\Big|_0^{\pi} + \right.\right.$$

$$\left.\left. + \frac{1}{n^2}\cdot \sin nx\Big|_0^{\pi}\right)\right) = \frac{2}{\pi}\left(\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \right.$$

$$-\frac{2}{n^3}\sin nx \Big|_0^{\pi} \Big) = \frac{2}{\pi}\Big(\frac{\pi^2}{n}\underset{0}{\sin n\pi} + \frac{2\pi}{n^2}\cos \pi n -$$

$$-\frac{2}{n^3}\underset{0}{\sin \pi n} - 0 - 0 + \frac{2}{n^3}\underset{0}{\sin 0}\Big) =$$

$$= \frac{2\pi}{n^2}\cos \pi n \cdot \frac{2}{\pi} = 4 \cdot \frac{\cos \pi n}{n^2} = 4 \cdot \frac{(-1)^n}{n^2}$$

Таким образом:

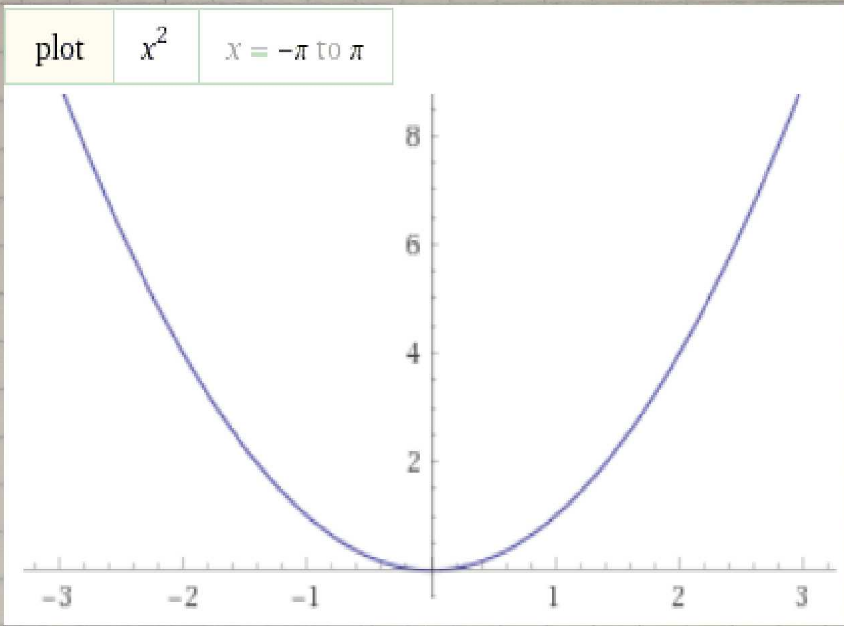
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx =$$

$$= \frac{2\pi^2}{3 \cdot 2} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2} \cdot \cos nx\right] =$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos nx\right].$$

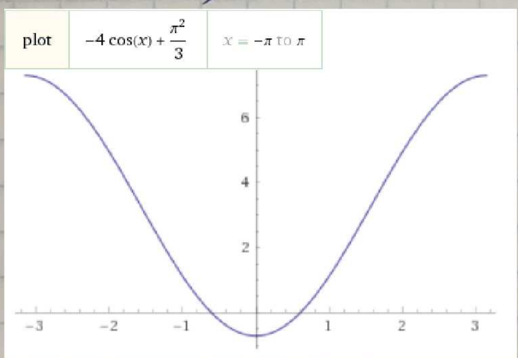
6. Построить график функции и её разложения:

График функции

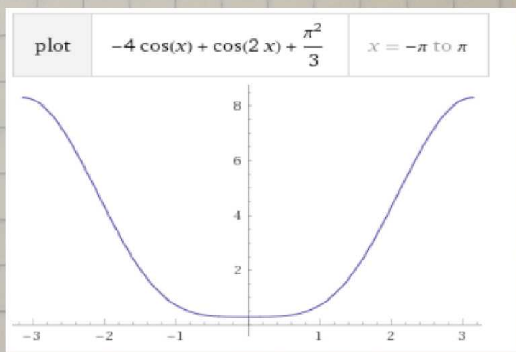


Графики разложения

S(0)



S(1)



S(2)