HOME ASSIGNMENT

“POISSON PROCESS AND THE ELEMENTS OF ACTUARIAL MATHEMATICS”

КАЛІНІЧЕНКО НАЗАР КА-83

Cramér–Lundberg process $U(t)$ is defined as follows:

$$U(t) \;=\; u_0 + ct - \sum_{i=1}^{N(t)} X_i \;,\; t \geq 0$$

Claim distribution, according to personal assignment:

$$X_i \sim 0.3U(0,1) + 0.7U(2,4)$$

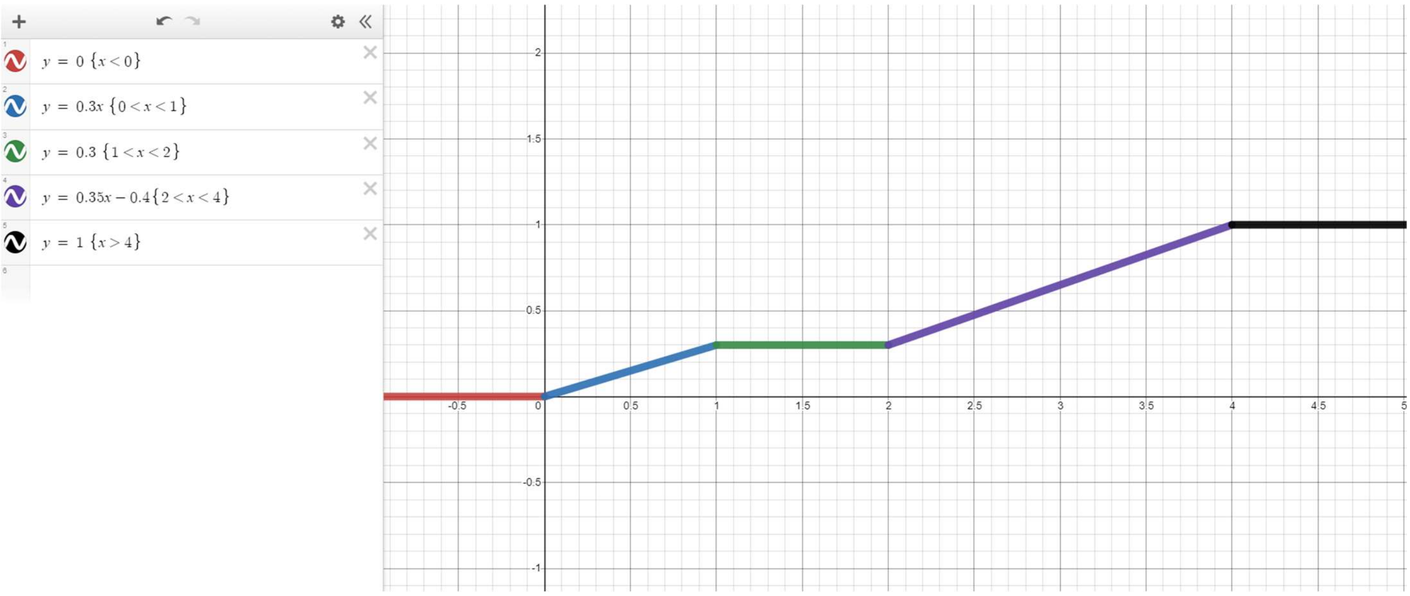
PRELIMINARY CALCULATIONS

Mean of *X*:

$$\mathbb{E}X = \mathbb{E}(0.3U(0,1) \,+\, 0.7U(2,4)) = 0.3\mathbb{E}U(0,1) \,+\, 0.7\mathbb{E}U(2,4) =$$
$$= 0.3 \cdot 0.5 \,+\, 0.7 \cdot 3 = 0.15 + 2.1 = 2.25$$

Distribution function of *X*:

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ 0.3x, & 0 < x \leq 1 \\ 0.3, & 1 < x \leq 2 \\ 0.35x - 0.4, & 2 < x \leq 4 \\ 1, & x > 4 \end{cases}$$



Tail distribution function of *X*:

$$\overline{F}_X(x) = \begin{cases} 1, & x \le 0 \\ 1 - 0.3x, & 0 < x \le 1 \\ 0.7, & 1 < x \le 2 \\ 1.4 - 0.35x, & 2 < x \le 4 \\ 0, & x > 4 \end{cases}$$

PROGRESS

CASE 1

1. $NPC: c > \lambda\mu = \lambda\mathbb{E}X = 1 \cdot 2.25 = 2.25$
2. $c = 2\lambda\mu = 2 \cdot 2.25 = 4.5$

3. $\varphi(u) = \varphi(0) + \frac{\lambda}{c}\int_0^u \varphi(u-y)\overline{F}_X(y)dy, \ \forall u \ge 0$

$$\varphi(u) \overset{NPC}{=} 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^u \varphi(u-y)\overline{F}_X(y)dy = 1 - \frac{1 \cdot 2.25}{4.5} + \frac{1}{4.5}\int_0^u \varphi(u-y)\overline{F}_X(y)dy =$$

$$= \frac{1}{2} + \frac{2}{9}\int_0^u \varphi(u-y)\overline{F}_X(y)dy$$

4. $\Phi(p) = \mathcal{L}\{\varphi(u)\} = \frac{1-\frac{\lambda\mu}{c}}{p[1-\frac{\lambda}{c}\hat{\overline{F}}_X(p)]}$, where $\hat{\overline{F}}_X(p) = \mathcal{L}\{\overline{F}_X(x)\}$

$$\hat{\overline{F}}_X(p) = \int_0^{+\infty} e^{-pt}\overline{F}_X(t)dt = \int_0^1 e^{-pt}(1-0.3t)dt + \int_1^2 e^{-pt}0.7dt + \int_2^4 e^{-pt}(1.4-0.35t)dt + \int_4^{+\infty} e^{-pt}0dt =$$

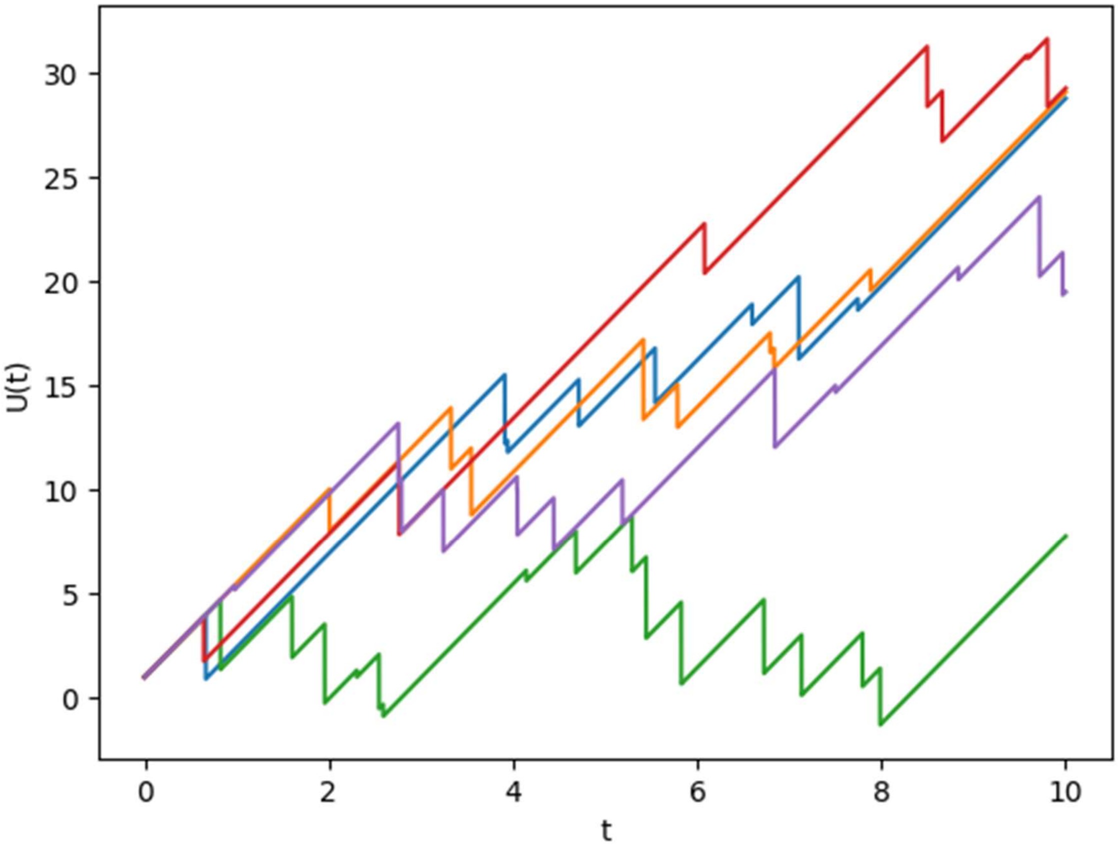
$$= \frac{e^{-p}(0.3 + e^p(-0.3+p) - 0.7p)}{p^2} + \frac{0.7e^{-2p}(-1+e^p)}{p} + \frac{e^{-4p}(0.35 + e^{2p}(-0.35+0.7p))}{p^2} + 0 =$$

$$= -\frac{0.3}{p^2} + \frac{0.35e^{-4p}}{p^2} - \frac{0.35e^{-2p}}{p^2} + \frac{0.3e^{-p}}{p^2} + \frac{1}{p}$$

$$\Phi(p) = \frac{1 - \frac{1 \cdot 2.25}{4.5}}{p[1 - \frac{1}{4.5}(-\frac{0.3}{p^2} + \frac{0.35e^{-4p}}{p^2} - \frac{0.35e^{-2p}}{p^2} + \frac{0.3e^{-p}}{p^2} + \frac{1}{p})}$$

$$\Phi(p) = \frac{\frac{1}{2}}{p[1 - \frac{2}{9}(-\frac{0.3}{p^2} + \frac{0.35e^{-4p}}{p^2} - \frac{0.35e^{-2p}}{p^2} + \frac{0.3e^{-p}}{p^2} + \frac{1}{p})}$$

5.



6. Percent ruined $= 38.30\%$

7. $F_{\tilde{X}}(x) = \frac{1}{\mathbb{E}X}\int_0^x \overline{F}_X(t)dt = \frac{1}{2.25}\int_0^x \overline{F}_X(t)dt =$

$$= \begin{cases} 0, & x \leq 0 \\ \frac{1}{\mathbb{E}X}\int_0^x (1-0.3t)dt, & 0 < x \leq 1 \\ \frac{1}{\mathbb{E}X}\int_0^1 (1-0.3t)dt + \frac{1}{\mathbb{E}X}\int_1^x 0.7dt, & 1 < x \leq 2 \\ \frac{1}{\mathbb{E}X}\int_0^1 (1-0.3t)dt + \frac{1}{\mathbb{E}X}\int_1^2 0.7dt + \frac{1}{\mathbb{E}X}\int_2^x (1.4-0.35t)dt, & 2 < x \leq 4 \\ 1, & x > 4 \end{cases}$$

$$
= \begin{cases} 0, & x \leq 0 \\ \frac{4}{9}(x - 0.15x^2), & 0 < x \leq 1 \\ \frac{4}{9}(0.85) + \frac{4}{9}(0.7(x-1)), & 1 < x \leq 2 \\ \frac{4}{9}(0.85) + \frac{4}{9}(0.7) + \frac{4}{9}(-0.175x^2 + 1.4x - 2.1), & 2 < x \leq 4 \\ 1, & x > 4 \end{cases}
$$

Follow up with $\varphi(u) = \mathbb{P}\{\sum_{i=1}^{G} \tilde{X}_i \leq u\}$, where $G \sim Geom(1 - \frac{\lambda\mu}{c}) = Geom(\frac{1}{2})$

Percent ruined $= 39.67\%$

8. $\Phi^{-1}(p) = \mathcal{L}^{-1}\{\Phi(p)\} = \varphi(u) \quad (exact)$

$\mathbb{P}\{Ruin\} = 1 - \varphi(u) = 1 - \Phi^{-1}(p)$

Percent ruined $= 39.3\%$

CASE 2

1. $NPC: c > \lambda\mu = \lambda\mathbb{E}X = 5 \cdot 2.25 = 11.25$
2. $c = 1.05\lambda\mu = 1.05 \cdot 11.25 = 11.8125$

3. $\varphi(u) = \varphi(0) + \frac{\lambda}{c}\int_0^u \varphi(u-y)\overline{F}_X(y)dy, \ \forall u \geq 0$

$$\varphi(u) \overset{NPC}{=} 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^u \varphi(u-y)\overline{F}_X(y)dy = 1 - \frac{5 \cdot 2.25}{11.8125} + \frac{5}{11.8125}\int_0^u \varphi(u-y)\overline{F}_X(y)dy =$$

$$= \frac{1}{21} + \frac{80}{189}\int_0^u \varphi(u-y)\overline{F}_X(y)dy$$

4. $\Phi(p) = \mathcal{L}\{\varphi(u)\} = \frac{1-\frac{\lambda\mu}{c}}{p[1-\frac{\lambda}{c}\hat{\overline{F}}_X(p)]}$, where $\hat{\overline{F}}_X(p) = \mathcal{L}\{\overline{F}_X(x)\}$

$$\hat{\overline{F}}_X(p) = \int_0^{+\infty} e^{-pt}\overline{F}_X(t)dt = \int_0^1 e^{-pt}(1-0.3t)dt + \int_1^2 e^{-pt}0.7dt + \int_2^4 e^{-pt}(1.4-0.35t)dt + \int_4^{+\infty} e^{-pt}0dt =$$

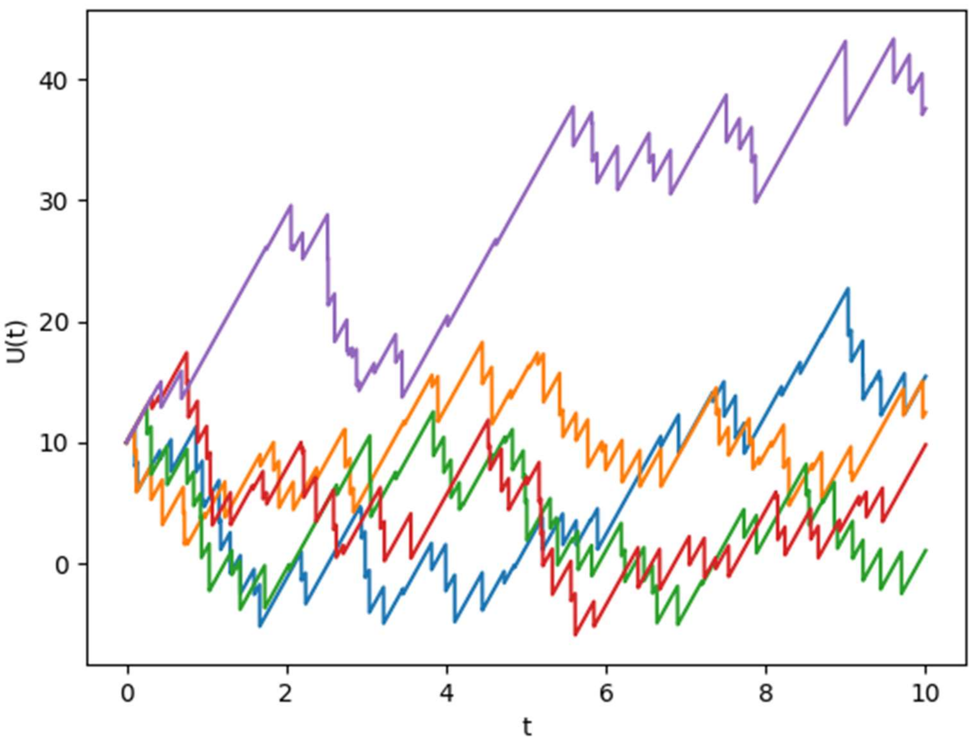
$$= \frac{e^{-p}(0.3 + e^p(-0.3+p) - 0.7p)}{p^2} + \frac{0.7e^{-2p}(-1+e^p)}{p} + \frac{e^{-4p}(0.35 + e^{2p}(-0.35+0.7p))}{p^2} + 0 =$$

$$= -\frac{0.3}{p^2} + \frac{0.35e^{-4p}}{p^2} - \frac{0.35e^{-2p}}{p^2} + \frac{0.3e^{-p}}{p^2} + \frac{1}{p}$$

$$\Phi(p) = \frac{1 - \frac{5 \cdot 2.25}{11.8125}}{p[1 - \frac{5}{11.8125}(-\frac{0.3}{p^2} + \frac{0.35e^{-4p}}{p^2} - \frac{0.35e^{-2p}}{p^2} + \frac{0.3e^{-p}}{p^2} + \frac{1}{p})}$$

$$\Phi(p) = \frac{\frac{1}{21}}{p[1 - \frac{80}{189}(-\frac{0.3}{p^2} + \frac{0.35e^{-4p}}{p^2} - \frac{0.35e^{-2p}}{p^2} + \frac{0.3e^{-p}}{p^2} + \frac{1}{p})}$$

5.



6. Percent ruined $= 69.5\%$

7. $F_{\tilde{X}}(x) = \dfrac{1}{\mathbb{E}X}\displaystyle\int_0^x \overline{F}_X(t)dt = \dfrac{1}{2.25}\int_0^x \overline{F}_X(t)dt =$

$$= \begin{cases} 0, & x \le 0 \\ \frac{1}{\mathbb{E}X}\int_0^x (1-0.3t)dt, & 0 < x \le 1 \\ \frac{1}{\mathbb{E}X}\int_0^1 (1-0.3t)dt + \frac{1}{\mathbb{E}X}\int_1^x 0.7dt, & 1 < x \le 2 \\ \frac{1}{\mathbb{E}X}\int_0^1 (1-0.3t)dt + \frac{1}{\mathbb{E}X}\int_1^2 0.7dt + \frac{1}{\mathbb{E}X}\int_2^x (1.4-0.35t)dt, & 2 < x \le 4 \\ 1, & x > 4 \end{cases}$$

$$= \begin{cases} 0, & x \le 0 \\ \frac{4}{9}(x - 0.15x^2), & 0 < x \le 1 \\ \frac{4}{9}(0.85) + \frac{4}{9}(0.7(x-1)), & 1 < x \le 2 \\ \frac{4}{9}(0.85) + \frac{4}{9}(0.7) + \frac{4}{9}(-0.175x^2 + 1.4x - 2.1), & 2 < x \le 4 \\ 1, & x > 4 \end{cases}$$

Follow up with $\varphi(u) = \mathbb{P}\{\sum_{i=1}^{G} \tilde{X}_i \le u\}$, where $G \sim Geom(1 - \frac{\lambda\mu}{c}) = Geom(\frac{1}{21})$

Percent ruined $= 69.55\%$

8. $\Phi^{-1}(p) = \mathcal{L}^{-1}\{\Phi(p)\} = \varphi(u) \quad (exact)$

$\mathbb{P}\{Ruin\} = 1 - \varphi(u) = 1 - \Phi^{-1}(p)$

Percent ruined $= 69.61\%$

CONCLUSION

We have been looking into methods of approximating the ruin probability in the Cramér–Lundberg model.

The first method of brute-force Monte Carlo has proved to be quite effective, though very demanding in terms of calculating time and resources. However, it managed to predict the probability less than 2.6% error from the true value.

The second method's prerequisites have been the modified claim distribution (integrated tail distribution), which required evaluating additional integrals and modelling another random variable. But the extra labor had been worth it, because this method succeeded at approximating, its error was less than 1%. It also ran incredibly quickly compared to the first method.

The numeric approximation of the inverse Laplace transform has been used to verify the results.

CODE

Programmed implementation (Python) of the model can be found at:
https://github.com/NazarNintendo/ruin-theory