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Laplacian centrality: A new centrality measure for weighted networks

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ABSTRACT

The centrality of vertices has been a key issue in network analysis. For unweighted networks where edges are just present or absent and have no weight attached, many centrality measures have been presented, such as degree, betweenness, closeness, eigenvector and subgraph centrality. There has been a growing need to design centrality measures for weighted networks, because weighted networks where edges are attached weights would contain rich information. Some network measures have been proposed for weighted networks, including three common measures of vertex centrality: degree, closeness, and betweenness. In this paper, we propose a new centrality measure called the Laplacian centrality measure for weighted networks. The Laplacian energy is defined as $E_L(G) = \sum_i \lambda_i^2$, where λ_i's are eigenvalues of the Laplacian matrix of weighted network G. The importance (centrality) of a vertex v is reflected by the drop of the Laplacian energy of the network to respond to the deactivation (deletion) of the vertex from the network. We also prove an algebraic graph theory result that provides a structural description of the Laplacian centrality measure which is in terms of the number of all kinds of 2-walks. Laplacian centrality unveils more structural information about connectivity and density around v (further than its immediate neighborhood). That is, comparing with other standard centrality measures proposed for weighted networks (e.g. degree, closeness or betweenness centrality), Laplacian centrality is an intermediate measuring between global and local characterization of the importance (centrality) of a vertex. We further investigate the validness and robustness of this new centrality measure by illustrating this method to some classical weighted social network data sets and obtain reliable results, which provide strong evidences of the new measure's utility.

1. Introduction

Social network analysis (SNA) is the mapping and measuring of relationships and flows between people, groups, organizations, computers, Urls, and other connected information/knowledge entities. The vertices in the network represent people and groups while the edges show relationships between the vertices. SNA provides both a visual and a mathematical analysis of objects' relationships.

Note that whether the vertices represent individuals, organizations or even countries and the edges refer to communication, cooperation, friendship or trade, edges can be differentiated settings. These differences can be analyzed by defining a weighted network, in which edges are not just either present or absent, but have some form of weight attached to them. Many social network measures have been defined for binary situations in which a pair of vertices is either connected or

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not. If we use the binary version of a weighted network, much of the information contained in a weighted network topology cannot be described to the same extent or as richly. So, there has a growing need for network measures that directly account for weighted networks.

The centrality of vertices, or the identification of vertices which are more "central" than others, has been a key issue in network analysis. The findings of some important vertices with high centralities to characterize the properties on the networks have significant uses in many fields. These include the synchronization transition, epidemic spreading, and transmission of information. For example, in diffusive systems the vertices of large degree play a crucial role, which are decisive to resolve the traffic jam at a bottleneck [15]. Various centrality measurements have been proposed for unweighted networks: degree centrality, closeness centrality, betweenness centrality [6], eigenvector centrality and subgroup centrality [5].

Degree centrality is defined as the number of edges incident upon a vertex (i.e., the number of edges that a vertex is incident with). *Closeness* centrality was defined as the inverse sum of shortest distances to all other vertices from a focal vertex i, which is based on the mean geodesic distance (i.e., the shortest path) between i and all other vertices reachable from it, that is $\frac{n-1}{\sum_j d_{i,j}}$, where $d_{i,j}$ is the shortest distance between vertices i and j. *Betweenness* centrality assesses the degree to which a vertex lies on the shortest paths between pairs of other vertices. Betweenness centrality of the vertex i is defined by the number of shortest paths that pass through i. More specifically, let $L_{h,j}$ be the total number of shortest paths from a vertex h to another vertex j and $L_{h,j}(i)$ be the number of the shortest paths that pass through the vertex i. The betweenness centrality of the vertex i is defined as $\frac{2}{(n-1)\cdot(n-2)}\sum_{h\neq i}\sum_{j\neq i,j\neq h}\frac{L_{h,j}(i)}{L_{h,j}}$. *Eigenvector* centrality is another popular measure of the importance of a vertex in a network. It assigns relative scores to all vertices in the network based on the principle that connections to high-scoring vertices contribute more to the score of the vertex in question than equal connections to low-scoring vertex. The eigenvector centrality of the vertex i is defined as the ith component of the eigenvector corresponding to the greatest eigenvalue of the following characteristic equation $A\vec{X} = \lambda\vec{X}$, where A is the adjacency matrix of the network. Another centrality method named *subgraph centrality* also was presented by Estrada and Rodríguez-Velázquez [5], which characterizes the participation of each vertex in all subgraphs in a network, with more weight given smaller subgraphs than larger ones. The subgraph centrality of the vertex i is defined as $\sum_{k=0}^{\infty}\frac{u_k(i,i)}{k!}$, where $u_k(i,i)$ is the number of closed k-walks that vertex i participates in the network.

The above measures are all defined for unweighted networks. There have been several attempts to generalize degree, betweenness and closeness centrality measures to weighted networks. Degree centrality was extended to weighted networks by Barrat et al. [1] and defined as the sum of the weights attached to the edges connected to a vertex. The extensions of the closeness and betweenness centrality measures by Newman [11] and Brandes [2], respectively, rely on Dijkstra's [4] shortest path algorithm, which defines the shortest path between two vertices as the least costly path. Opsahl [12] provided a package written in R named "tnet" that currently can calculate degree, closeness and betweenness centrality for weighted social networks. Lately, Opsahl et al. [13] proposed a new generation of vertex centrality measures for weighted networks, which takes into consideration both the weight of edges and the number of edges associated with a vertex, where the relative importance of these two aspects are controlled by a tuning parameter α.

But note that these standard methods generalized for weighted networks also inherit the weakness when they were used for unweighted case. For example, as have been stated, "The simplicity of degree method is an advantage and also is an disadvantage: only the local structure around a vertex is calculated and it does not take into consideration the global structure of the network; for example, although a vertex might be connected to many others, it might not be in a position to reach others quickly to access resources, such as information or knowledge [13]"; "Betweenness method considers the global network structure and also can be applied to networks with disconnected components, but it is not without limitations; for example, vertices in a network that generally do not lie on a shortest path between any two other vertices will receive the same score of zero [13]". Besides, these existing measures describe either the local environment around a vertex (e.g., degree centrality) or the more global position of a vertex in the network (e.g., closeness, betweenness and subgraph centrality). For example, from its definition, "subgraph centrality" tends to find the center (s) of whole network. But at the most time what we are really interested is to find the center for each community in the network. If the network is consist of more than two communities and with dramatically different sizes, the nodes in smaller community would get lower "subgraph centrality" ranks than the ones in larger community, so that the leader in smaller community will not come up with high rank. Thus, an intermediate (between local and global) characterization of the vertex centrality has been claimed as a necessity for the study of food web in [8,9] if species to community relations are to be understood. An intermediate centrality approach is also suggested to be the most appropriate if the relative importance of vertex is to be quantified in social networks.

In this paper, we propose a new centrality strategy for weighted networks, which permits one to consider more "intermediate" environmental information around a vertex. The centrality of some vertex v is characterized as a function in terms of the numbers of 2-walks that the vertex v takes part in the network, which implies the estimation of the centrality of a vertex involves not only the direct connections with its immediate neighborhood but also the importance of its neighbors. This strategy is called "Laplacian centrality method" because it is from the use of a matrix valued function that describes the so-called "*Laplacian energy*" of the network. The basic idea is that the importance (centrality) of a vertex is related to the ability of the network to respond to the deactivation of the vertex from the network. In particular, the relative drop of Laplacian energy in the network caused by the deactivation of this vertex from the network will be used as the indicator to show its importance in the network. We further investigate the validity and robustness of this new measure by applying this method to some classical data sets of social networks. Successful applications on those bench mark data sets are evidences of the utility of this proposed centrality measurement.

This paper is organized as follows. In Section 2, we give some useful graph theory notations and terminology. In Section 3, we present the definition of Laplacian centrality; In Section 4, we give a theorem to show a structural description of the Laplacian centrality. Analytical and numerical results based on various centrality measures applying on classic social network data sets are shown in Section 5; Time complexity of Laplacian method is discussed at Section 6; Conclusions are made in Section 7.

2. Graph theory notation and terminology

A social network usually is represented as a graph. The vertices are the individuals, and the edges represent the social links. In this paper, we consider the symmetric case where social networks are represented by undirected graphs. *Multiple edges* are two or more edges connecting the same two vertices. Graphs with multiple edges are called *multigraph*. A degenerate edge of a graph which joins a vertex to itself, is called a *loop*. The number of edges that are incident to a vertex is called the *degree* of the vertex. The *neighborhood* of a vertex v is the set of all vertices adjacent to v.

Graph entropy measures are always used for determining the structural information content of graphs, which has been proved to play an important role in a variety of problem areas, including biology, chemistry, and sociology [4]. Laplacian energy, which could be thought as one kind of graph entropy, representing a certain coherent measuring of a network, is used here to measure the importance (centrality) of a vertex by the relative drop of Laplacian energy in the network caused by the deactivation of this vertex from the network. In Section 3, we will introduce the definition of Laplacian energy of a network and Laplacian centrality of a vertex.

3. Laplacian centrality

3.1. Laplacian energy for a network

Let $G = (V,E,W)$ be a weighted network (or weighted graph) with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, edge set E, where each edge $e = (v_i, v_j)$ is attached with a weight $w_{i,j}$. If there is no edge between v_i and v_j, $w_{i,j} = 0$. Since we only consider undirected network without loops, then $w_{i,j} = w_{j,i}$ and $w_{i,i} = 0$.

We define

$$W(G) = \begin{pmatrix} 0 & w_{1,2} & \ldots & w_{1,n} \\ w_{2,1} & 0 & \ldots & w_{2,n} \\ . & . & . & . \\ w_{n,1} & w_{n,2} & \ldots & 0 \end{pmatrix}$$

and

$$X(G) = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ . & . & . & . \\ 0 & 0 & \ldots & x_n \end{pmatrix}$$

with $x_i = \sum_{j=1}^{n} w_{i,j} = \sum_{u \in N(v_i)} w_{v_i,u}$, and we call x_i the *sum-weight* of a vertex v_i, where $N(v_i)$ is the neighborhood of v_i.

Definition 1. The matrix $L(G) = X(G) - W(G)$ is called the **Laplacian matrix** of the weighed network G.

Definition 2. Let $G = (V,E,W)$ be a weighted network on n vertices, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of its Laplacian matrix. The **Laplacian energy** of G is defined as the following invariant:

$$E_L(G) = \sum_{i=1}^{n} \lambda_i^2.$$

At first, we will give some properties of Laplacian energy of G.

Theorem 1. *For any network $G = (V,E,W)$ on n vertices whose vertex sum-weights are $x_1, x_2, \ldots, x_n$ respectively, we have*

$$E_L(G) = \sum_{i=1}^{n} x_i^2 + 2\sum_{i<j} w_{i,j}^2.$$

Proof. Let

$$L(G) = \begin{pmatrix} x_1 & -w_{1,2} & \dots & -w_{1,n} \\ -w_{2,1} & x_2 & \dots & -w_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ -w_{n,1} & -w_{n,2} & \dots & x_n \end{pmatrix}.$$

Denote the characteristic polynomial of $L(G)$ as

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$$

whose roots are λ_i, $i = 1, 2, \ldots, n$.

We all know that

$$a_0 = (-1)^n det(L(G)),$$
$$a_1 = -tr(L(G)),$$

where $det(L(G))$ is the determinant of square matrix $L(G)$, and $tr(L(G))$ is the trace of square matrix $L(G)$.

Viète rules says

$$\sum_{i=1}^{n} \lambda_i = -a_{n-1}$$

and

$$\sum_{i<j} \lambda_i \lambda_j = a_{n-2}.$$

Note that $w_{i,j} = w_{j,i}$, by computing the coefficient a_{n-2} of λ^{n-2}, we would have,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} x_i, \tag{1}$$
$$\sum_{i<j} \lambda_i \lambda_j = \sum_{i<j} x_i x_j - \sum_{i<j} w_{i,j}^2. \tag{2}$$

Therefore,

$$E_L(G) = \sum_{i=1}^{n} \lambda_i^2 = \left(\sum_{i=1}^{n} \lambda_i\right)^2 - \sum_{i \neq j} \lambda_i \lambda_j = \left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i \neq j} x_i x_j + \sum_{i \neq j} w_{i,j}^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i \neq j} w_{i,j}^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i<j} w_{i,j}^2. \qquad \square$$

Corollary 1. *If H is an arbitrary subgraph of a network G, then $E_L(H) \leqslant E_L(G)$. And equality holds if and only if $V(G) - V(H)$ is a set of isolated vertices.*

3.2. Laplacian centrality for a vertex

Definition 3. If $G = (V,E,W)$ is a network of n vertices $\{v_1, v_2, \ldots, v_n\}$, let G_i be the network obtained by deleting v_i from G. The Laplacian centrality $C_L(v_i,G)$ of vertex v_i is defined as

$$C_L(v_i, G) = \frac{(\Delta E)_i}{E_L(G)} = \frac{E_L(G) - E_L(G_i)}{E_L(G)}$$

Obviously, by Corollary 1, $E_L(G) - E_L(G_i)$ must be non-negative. Since the denominators $E_L(G)$ are the same for all vertices, we just need to focus on the numerator $(\Delta E)_i$. In the next section, we will prove a graph theory result that provides a structural description of $(\Delta E)_i$.

4. Graph theoretical descriptions of Laplacian centrality

4.1. 2-walks

Let $G = (V,E,W)$ be a weighted network of n vertices $\{v_1, v_2, \ldots, v_n\}$. A walk of length k (we denote it by *k-walk*) is a sequence of (not necessary different) vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}}, v_{i_k}$ such that for each $i = 1, 2, \ldots, k$ there is an edge from $v_{i_{j-1}}$ to v_{i_j}. A walk $v_{i_0} \cdots v_{i_k}$ is *closed* if $v_{i_0} = v_{i_k}$.

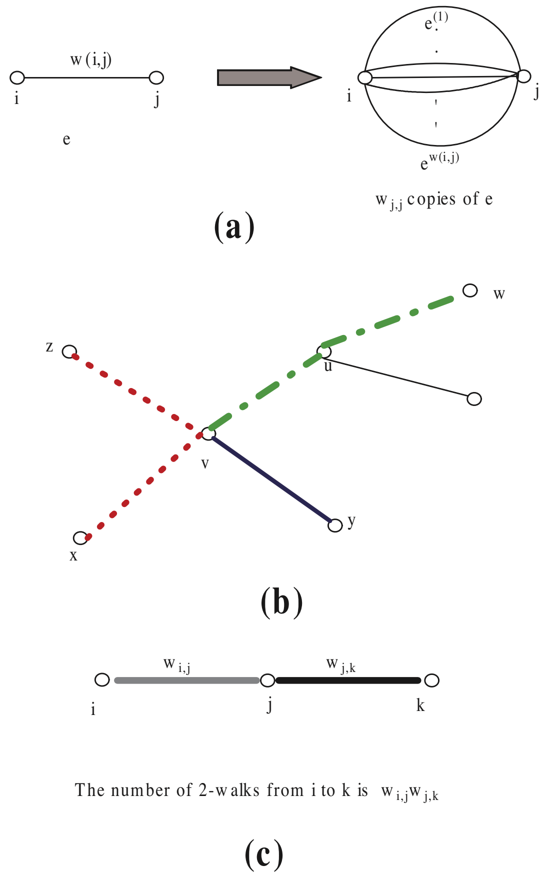

Fig. 1. 2-Walks in weighted network.

Clearly, every k-walk is related to a given subgraph in a network, thus the importance of some vertex in G could be measured by the number of k-walks it participates in the network. The more k-walks it participates in, the more important it is. For example, the degree centrality is actually using the number of 1-walks that vertex v participates in the network as an indicator of its importance. It is easy to see when k becomes bigger, the involved environment around the focal vertex grows larger. In this paper, we will focus on the case when $k = 2$.

For the sake of easy understanding, we always assume the weight $w_{i,j}$ for each edge $e = (v_i, v_j)$ is a non-negative integer.[1] By replacing e with $w_{i,j}$ copies of unweighted multiedges (see Fig. 1a), an weighted network $G = (V,E,W)$ is transferred as an unweighted multigraph. A 2-walk $v_0v_1v_2$ in the weighted network version will correspond to $(w_{v_0,v_1} \cdot w_{v_1,v_2})$ 2-walks in the unweighted multigraph version. For the number of 2-walks that a given vertex v takes part in a network $G = (V,E,W)$, we have the following observation.

Lemma 1. *Let $G = (V,E,W)$ be a weighted network and v be an arbitrary vertex of G. Then there are three types of 2-walks containing v with the following observations.*

Type 1. *Closed 2-walks containing the vertex v: the number of such 2-walks is*

$$NW_2^C(v) = \sum_{y_i \in N(v)} w_{v,y_i}^2.$$

(Blue[2] edges in Fig. 1b).

Type 2. *Non-closed 2-walks containing the vertex v as one of the end-vertices: the number of such 2-walks is*

$$NW_2^E(v) = \sum_{y_i \in N(v)} \left(\sum_{z_j \in \{N(y_i) - v\}} w_{v,y_i} w_{y_i,z_j} \right).$$

---

[1] We will see from the subsequent Corollary 2, it does not affect our result.

[2] For interpretation of color in Figs. 1, 4 and 5, the reader is referred to the web version of this article.

(Green edges in Fig. 1b).

Type 3. *Non-closed 2-walks containing the vertex v as the middle point, the number of such 2-walks is*

$$NW_2^M(v) = \sum_{y_i, y_j \in N(v), y_i \neq y_j} w_{y_i, v} w_{v, y_j}.$$

(Red edges in Fig. 1b).

4.2. Graph theoretical descriptions

From a graph theoretic perspective we have the following results.

Theorem 2. *$G = (V, E, W)$ is a weighted network of n vertices $\{v_1, v_2, \ldots, v_n\}$. Let H be the network obtained by deleting vertex v from G, then the drop of Laplacian energy with respect to v_i is*

$$(\Delta E)_i = E_L(G) - E_L(H) = 4 \cdot NW_2^C(v_i) + 2 \cdot NW_2^E(v_i) + 2 \cdot NW_2^M(v_i)$$

Proof. Note that $E_L(G) = \sum_{i=1}^n x^2(v_i) + 2\sum_{i<j} w_{i,j}^2$.

Without loss of generality, assume $H = G - v_1$. Let $N(v_1)$ be the neighborhood of vertex v_1 in G and $x'(v_i)$ be the corresponding sum-weight of vertex v_i in H. We will have:

$$x'(v_i) = \begin{cases} 0, & \text{if } i = 1; \\ x(v_i) - w_{v_1, v_i}, & \text{if } v_i \in N(v_1); \\ x(v_i), & \text{otherwise.} \end{cases} \tag{3}$$

So, by Theorem 1 and Eq. (3),

$$E_L(H) = \sum_{v_i \in N(v_1)} (x(v_i) - w_{v_1, v_i})^2 + \sum_{v_i \notin N(v_1)} x^2(v_i) + 2 \sum_{1<i<j} w_{i,j}^2$$

Thus, the drop of Laplacian energy with respect to v_1 is

$$\begin{aligned} (\Delta E)_{v_1} &= E_L(G) - E_L(H) \\ &= x^2(v_1) + \sum_{v_i \in N(v_1)} [x^2(v_i) - (x(v_i) - w_{v_1, v_i})^2] + 2\sum_{j=2}^n w_{1,j}^2 \\ &= x^2(v_1) + \sum_{v_i \in N(v_1)} [w_{v_1, v_i} \cdot (2x(v_i) - w_{v_1, v_i})] + 2\sum_{j=2}^n w_{1,j}^2 \\ &= x^2(v_1) + 2\sum_{v_i \in N(v_1)} w_{v_1, v_i} \cdot x(v_i) - \sum_{v_i \in N(v_1)} w_{v_1, v_i}^2 + 2\sum_{j=2}^n w_{1,j}^2 \\ &= \left(\sum_{v_i \in N(v_1)} w_{v_1, v_i}\right)^2 - \sum_{v_i \in N(v_1)} w_{v_1, v_i}^2 + 2\sum_{v_i \in N(v_1)} w_{v_1, v_i} \cdot x(v_i) + 2\sum_{j=2}^n w_{1,j}^2 \\ &= 2\sum_{v_i, v_j \in N(v_1), v_i \neq v_j} w_{v_1, v_i} \cdot w_{v_j, v_1} + 2\sum_{v_i \in N(v_1)} w_{v_1, v_i} \cdot \sum_{u \in N(v_i)} w_{v_i, u} + 2\sum_{j=2}^n w_{1,j}^2 \\ &= 2\sum_{v_i, v_j \in N(v_1), v_i \neq v_j} w_{v_1, v_i} \cdot w_{v_j, v_1} + 2\sum_{v_i \in N(v_1)} \sum_{u \in N(v_i)} w_{v_1, v_i} \cdot w_{v_i, u} + 2\sum_{y \in N(v_1)} w_{v_1, y}^2 \\ &= 2\sum_{v_i, v_j \in N(v_1), v_i \neq v_j} w_{v_1, v_i} \cdot w_{v_j, v_1} + 2\sum_{v_i \in N(v_1)} \left(\sum_{u \in \{N(v_i) - v_1\}} w_{v_1, v_i} \cdot w_{v_i, u} + w_{v_1, v_i}^2\right) + 2\sum_{y \in N(v_1)} w_{v_1, y}^2 \\ &= 2\sum_{v_i, v_j \in N(v_1), v_i \neq v_j} w_{v_1, v_i} \cdot w_{v_j, v_1} + 2\sum_{v_i \in N(v_1)} \sum_{u \in \{N(v_i) - v_1\}} w_{v_1, v_i} \cdot w_{v_i, u} + 4\sum_{y \in N(v_1)} w_{v_1, y}^2 \\ &= 2 \cdot NW_2^E(v_1) + 2 \cdot NW_2^M(v_1) + 4 \cdot NW_2^C(v_1) \qquad \square \end{aligned}$$

By Theorems 1 and 2, and Definition 3, the following corollary is obvious:

Corollary 2. *Let $G = (V, E, W)$ be a weighted network. If we redefine the weight $w_{i,j}$ of each edge $e = (v_i, v_j)$ as $\widetilde{w_{i,j}} = c \cdot w_{i,j}$, where c is any common constant. Then, each vertex has the same Laplacian centrality in both network $G = (V, E, W)$ and $\widetilde{G} = (V, E, \widetilde{W})$.*

From Theorem 2, we also notice the following facts.

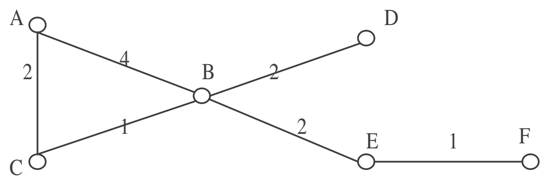

Fig. 2. A network with 6 vertices and 6 weighted edges, which has the same topology as Fig. 2 in [13] but with different weights.

Table 1
Scores based on various centrality methods.

	Degree	Betweenness	Closeness	Laplacian
A	6	4	0.125	0.70
B	9	8	0.143	0.90
C	**3**	**0**	**0.083**	**0.28**
D	2	0	0.091	0.22
E	**3**	**4**	**0.111**	**0.26**
F	1	0	0.059	0.04

First, the Laplacian centrality agrees with the standard measures on assignment of extremes. For example, if all edges in a network have the same weights, it will give the maximum value to the central vertex of a star, and equal value to the vertices of a cycle or a complete graph. Second, as we know, the degree centrality of v is actually considering the number of vertices which could be reachable from v directly, while the Laplacian centrality of a vertex involves the information of vertices that could be reachable from v within two steps. The Laplacian centrality of a vertex not only takes into account the local environment around it but also a bigger environment around its neighbors. Third, three types of 2-walks weight differently in the calculation of Laplacian centrality: the closed walk is assigned the biggest weight 4 and the other two types are assigned with weight 2. It is reasonable because the closed 2-walks reflect the most local environment of v, which should have the most influence on the centrality of v.

4.3. Two simple examples

In this section, we will give simple examples to show the differences between the known standard centrality measures (degree, closeness and betweenness) and Laplacian centrality for weighted network.

The first simple example is a weighted network with 6 vertices and 6 weighted edges, see Fig. 2. We will use Opsahl's package "tnet" to calculate centralities,[3] where the functions degree, closeness and betweenness are respectively with default value of parameter $\alpha = 1$, see Table 1.

Vertices C and E in this network will be paid attention here. They gain the same score 3 based on their sum-weights, and we cannot tell which one is more important based on degree method. Vertex E is a cut vertex in this network, thus it is regarded to be more important than vertex C based on betweenness centrality. And because the sum of distance from vertex C to all other vertices is bigger than that from vertex E, that explains why vertex E gets a higher score than vertex C based on closeness centrality.[4] In Laplacian centrality method, the number of 2-walks that vertex C involves in is more than that of vertex E, so vertex C gets higher rank than vertex E.

It is easy to see that betweenness and closeness centrality method are more global, while degree method is local and Laplacian method is intermediate. To further illustrate the differences between Laplacian and degree method, we present another example, see Fig. 3. For the sake of simplicity, each edge's weight is assumed to be 1. Based on degree centrality, u has higher ranking than v because the degree of u is 6 while the degree of v is 4. But based on Laplacian method, v would have higher ranking than u because

$$(\Delta E)_u = 4 \cdot NW_2^C(u) + 2 \cdot NW_2^E(u) + 2 \cdot NW_2^M(u) = 78.$$

and

$$(\Delta E)_v = 4 \cdot NW_2^C(v) + 2 \cdot NW_2^E(v) + 2 \cdot NW_2^M(v) = 84,$$

[3] *Tnet* is available for downloaded online: http://toreopsahl.com/tnet/.

[4] In closeness and betweenness, the distance between any two vertices is defined as the reciprocal of their edge weight.

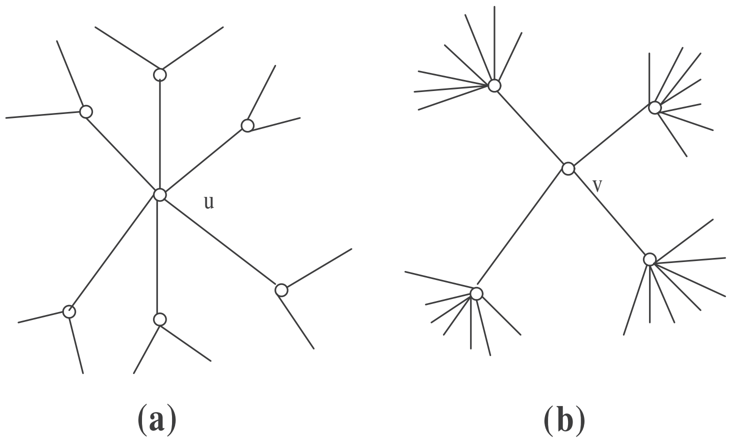

Fig. 3. An unweighted simple example to show the difference of degree and Laplacian method.

5. Applications and experimental results

In this section, we focus onto illustrate the utility of Laplacian centrality on real weighed networks. We apply this measure to two commonly used *weighted* network data sets: the Freeman's EIES data set [7], and Zachary's ''karate club'' network of 1977 [16]. We also test our method on one *unweighted* terrorist networks: the 9/11 hijacker network. As we will see from the following subsections, for the first data set (''the Freeman's EIES data set''), we detect the same most two popular scientists by Laplacian centrality measure as other methods; for the second data set (Zachary's ''karate club'' data set), only Laplacian method and degree method successfully find the two centers–the administrator and instructor; and for the third one (unweighted 9/11 hijacker network), only Laplacian method could find the four pilots as the top four important person. These results all give a strong evidence of the utility of the new method. We will discuss them detailedly in the following subsections.

5.1. Freeman's EIES network

Freeman's EIES dataset was collected in 1978 [7] and contains three different network relations among researchers working on social network analysis. While the first two networks are the inter-personal relationships among the researchers at the beginning and at the end of the study, the edges in the third network are defined as the number of messages sent among 32 of the researchers on an electronic communication tool. Here we test all centrality measure on the third weighted network.

The centrality scores of these 32 scientists based on three standard centrality methods for weighted networks (degree, closeness and betweenness) and the new Laplacian centrality method are given in Table 2, where the vertices are sorted by their Laplacian centrality scores. The top four are indicated in bold. As we see from the table, based on Laplacian centrality, Lin Freeman and Barry Wellman also get the two highest scores as other methods, which give an evidence of the utility of Laplacian method. Furthermore, since each centrality method are focusing on different aspects of the network, the orderings of these 32 scientists based on different methods are expected to be different. Here, to see the difference visually, we plot the detailed ranking information of these 32 vertices based on the four centrality measures in Fig. 5 (top), where the *X*-axis represents the 32 scientists which are sorted in the decreasing order of their Laplacian centrality. We will see from Fig. 5 (top) the curve based on betweenness ranking appears odd, that is because many actors get the same score (e.g. 0) and will get a same rank (5th) under betweenness centrality, which is exactly one shortcoming of betweenness centrality measure mentioned in [13]. The plotted figure also reflect the fact that the first two nodes (Lin Freeman and Barry Wellman) get the same rank (No. 1 and No. 2) based on all the methods.

5.2. Zachary's karate club network

The second social network we are testing here is the well known ''karate club'' of Zachary [16]. Zachary observed 34 members of a karate club over two years. During the course of observation, the club members split into two groups because of the disagreement between the administrator of the club and the club's instructor, and the members of one group left to start their own club. Zachary constructed a weighted network (whose adjacency matrix is given in Fig. 3 of [16]), where each member in the club is represented by a vertex, each edge is drawn if the two members are friends outside the club activities, and the weight assigned for edge is the number of contexts in which interaction took place between the two individuals involved. Fig. 4 shows the network, with the administrator and instructor represented by vertices 1 and 34 respectively. Red

Table 2
Scores of 32 social network scientist of Freeman's EIES network based on various centrality methods.

	Degree	Betweenness	Closeness	Laplacian
LIN FREEMAN	**3449**	**422**	**0.0557**	**0.6544**
BARRY WELLMAN	**2221**	**188**	**0.0527**	**0.3772**
RUSS BERNARD	**1721**	**31**	**0.0487**	**0.2670**
DOUG WHITE	**1500**	0	**0.0498**	**0.2517**
LEE SAILER	1157	**2**	0.0457	0.1801
SUE FREEMAN	1324	0	0.0439	0.1725
PAT DOREIAN	869	0	0.0423	0.1183
NICK MULLINS	783	0	0.0420	0.1017
AL WOLFE	423	0	0.0393	0.0563
STEVE SEIDMAN	470	0	0.0348	0.0548
MAUREEN HALLINAN	276	0	0.0373	0.0462
RON BURT	446	0	0.0336	0.0457
JOHN BOYD	392	0	0.0328	0.0454
PAUL HOLLAND	345	0	0.0316	0.0443
RICHARD ALBA	377	0	0.0297	0.0386
JACK HUNTER	318	0	0.0316	0.0372
JOEL LEVINE	284	0	0.0348	0.0365
CAROL BARNER-BARRY	274	0	0.0306	0.0346
DAVOR JEDLICKA	246	0	0.0308	0.0306
NICK POUSHINSKY	171	0	0.0335	0.0282
MARK GRANOVETTER	214	0	0.0304	0.0274
CLAUDE FISCHER	206	0	0.0324	0.0271
CHARLES KADUSHIN	175	0	0.0203	0.0169
GARY COOMBS	124	0	0.0241	0.0167
BRIAN FOSTER	148	0	0.0212	0.0161
PHIPPS ARABIE	189	0	0.0170	0.0155
DON PLOCH	149	0	0.0218	0.0153
NAN LIN	125	0	0.0192	0.0151
SAM LEINHARDT	112	0	0.0230	0.0143
ED LAUMANN	101	0	0.0192	0.0130
EV ROGERS	101	0	0.0179	0.0111
JOHN SONQUIST	78	0	0.0166	0.0097

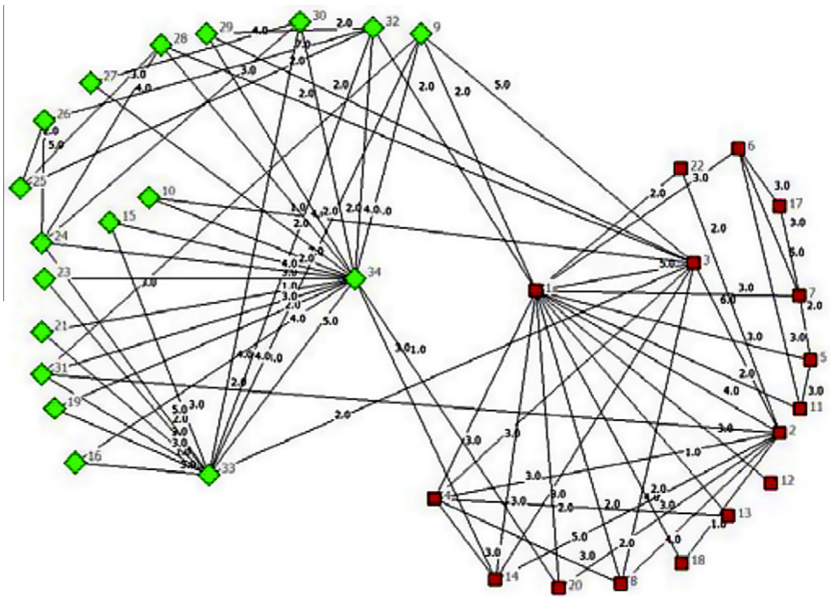

Fig. 4. Zachary's karate club network.

squares represent individuals associated with the administrator and green diamond represent those associated with the instructor.

Table 3
Scores of 34 members of Zachary's karate club network based on various centrality methods.

	Degree	Betweenness	Closeness	Laplacian
1	**42**	**209**	0.0157	**0.2544**
2	29	38	0.0153	0.1725
3	33	**178**	**0.0167**	0.2166
4	18	32	0.0129	0.0965
5	8	0	0.0108	0.0350
6	14	4	0.0111	0.0571
7	13	28	0.0111	0.0541
8	13	0	0.0125	0.0789
9	17	71	0.0163	0.1222
10	3	0	0.0093	0.0218
11	8	1	0.0095	0.0309
12	3	0	0.0105	0.0216
13	4	0	0.0090	0.0174
14	17	29	0.0156	0.1189
15	5	0	0.01020	0.0366
16	7	0	0.012	0.0549
17	6	0	0.0080	0.0173
18	3	0	0.0090	0.0192
19	3	0	0.0090	0.0226
20	5	0	0.0100	0.0331
21	4	0	0.0101	0.0280
22	4	0	0.0090	0.0246
23	5	0	0.0109	0.0382
24	21	6	0.0132	0.1294
25	7	0	0.0100	0.0227
26	14	9	0.0128	0.0645
27	6	0	0.0100	0.0282
28	13	18	0.0135	0.0752
29	6	0	0.0114	0.0365
30	13	7	0.0114	0.0707
31	11	0	0.0129	0.0709
32	21	47	0.0150	0.1310
33	38	84	0.0149	0.2371
34	**48**	172	**0.0165**	**0.3067**

The scores based on all centrality methods are listed in Table 3. The top two individuals are also indicated in bold. As we will see, based on Laplacian centrality, the administrator and instructor (#1 and #34) get the two highest scores as we expect. Note that *only Laplacian method and degree* method can give such a desire result. We also present detailed ranking information of these 34 vertices based on four centrality measures in Fig. 5 (Middle), where the X-axis represents these 34 actors sorted in the decreasing order of the Laplacian centrality score, and Y-axis corresponds to their ranking. We find that degree method and Laplacian method give the same top five centers, but afterwards they differ a lot. That is because the top 5 vertices all have big degrees, and the number of closed 2-walks dominates its corresponding Laplacian score. And we also find that vertex 3 get higher ranks based on both betweenness and closeness method which are No. 2 and No. 1 respectively, that is because of its relatively neutral position with two groups, it has many contacts with both the members of "administrator" group and the members of "instructor" group, thus it is misunderstood as one of the "centers" by either betweenness or closeness method.

5.3. Unweighted network–9/11 hijacker network

As we have mentioned, unweighted network could be regarded as a special case of weighted network where the weight of each edge is set as 1. To show the utility of Laplacian method further, we test the applications of Laplacian method on one unweighted terrorist networks: the 9/11 hijacker network. Through public data, Krebs [10] examined the network centered around the 19 hijackers of these events, which contains numerous additional individuals involved in the support network behind the 19 hijackers who actually conducted the suicide mission. These co-conspirators were conduits for money, communications routes, and provided needed skills and knowledge. Fig. 4 in [10] shows the hijackers and their network neighborhood - their direct and indirect associates.

There were four commercial airplanes hijacked in this terror plot. The following is the list of all hijackers on these airplanes.

American Airlines (AA 11): Mohamed Atta (pilot), Abdulaziz al-Omari, Satam al-Suqami, Wail al-Shehri, Waleed al-Shehri.
American Airlines (AA 77): Hani Hanjour (pilot), Nawaf al-Hazmi, Salem al-Hazmi, Khalid al-Mihdhar, Majed Moqed.

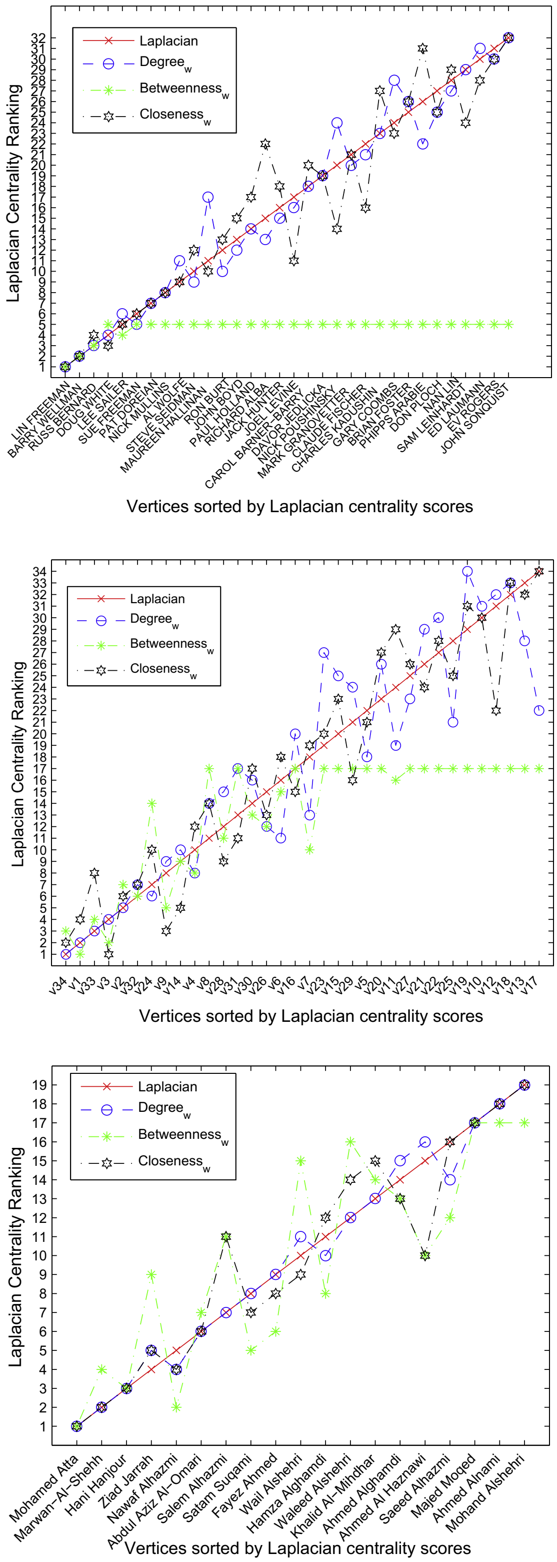

Fig. 5. Ranking information based on all centrality methods for these 3 data sets considered here. (Top) The 32 scientists in Freeman's EIES network; (Middle) 34 actors in Zachary's karate club; (Below) these 19 hijackers in the 9/11 hijacker network.

Table 4
Scores of 19 hijacks based on various centrality methods.

Names	Scores			
	Laplacian	Degree	Betweenness	Closeness
Mohamed Atta	**1.0000**	**1.0000**	**1.0000**	**1.0000**
Marwan al-Shehhi	**0.7628**	**0.8182**	**0.1495**	**0.7939**
Hani Hanjour	**0.4817**	**0.5909**	**0.2147**	**0.7591**
Ziad Jarrah	**0.3716**	0.4545	0.0289	0.7222
Nawaf al-Hazmi	0.3472	**0.5000**	**0.2617**	**0.7536**
Abdul Aziz al-Omari	0.3399	0.4091	0.0387	0.7222
Salem al-Hazmi	0.2738	0.3636	0.0217	0.6228
Satam Suqami	0.2738	0.3636	0.0857	0.6980
Fayez Ahmed	0.2714	0.3636	0.0438	0.6933
Wail al-Shehri	0.2249	0.2727	0.0042	0.6842
Hamza al-Ghamdi	0.1883	0.3182	0.0376	0.6154
Waleed al-Shehri	0.1809	0.2727	0.0013	0.5714
Khalid al-Mihdhar	0.1540	0.2727	0.0095	0.5652
Ahmed al-Ghamdi	0.1369	0.2273	0.0118	0.5778
Ahmed al-Haznawi	0.1345	0.1818	0.0260	0.6797
Saeed al-Hazmi	0.1296	0.2727	0.0198	0.5652
Majed Moqed	0.1174	0.1818	0.0000	0.5591
Ahmed al-Nami	0.0733	0.1364	0.0000	0.5503
Mohand al-Shehri	0.0440	0.0909	0.0000	0.5306

United Airlines (UA 175): Marwan al-Shehhi (pilot), Fayez Ahmed, Hamza al-Ghamdi, Ahmed al-Ghamdi, Mohand al-Shehri.
United Airlines (UA 93): Ziad Jarrah (pilot), Ahmed al-Haznawi, Saeed al-Ghamdi, Ahmed al-Nami.

We apply Laplacian method on the whole unweighted network which has 62 vertices, and only list the centrality scores for these 19 hijackers that we are interested. The values are normalized (dividing by the highest score of each method), see Table 4. We see that the four most important centers in 9/11 hijacker network based on *Laplacian centrality* is different from that based on all the other methods. When we trace back to the original references to check the centers' identities of Laplacian method, to our surprise, they comprise all four pilots of the different flights (AA 11, AA 77, UA 175 and UA 93), which is thought to be very reasonable by political analysts based on the fact that a pilot is much more important when a flight was hijacked because they need much more money and time to train, and also is consistent with our intuition of Laplacian method that "the importance (centrality) of a vertex is reflected by the drop of the Laplacian energy of the network to respond to the deactivation (deletion) of the vertex from the network", which gives us a strong evidence of the utility of Laplacian method. We also present detailed ranking information of these 19 hijacks based on four centrality measures in Fig. 5 (Below), where the X-axis represents these 19 actors sorted in the decreasing order of the Laplacian centrality score, and Y-axis corresponds to their ranking. From this figure, we could see that the pilot Ziad Jarrah gets higher ranks (i.e., 4th) in Laplacian method than in others (degree (5th), closeness (5th) and also betweenness (9th)), which is exactly the interesting facts that Laplacian method has showed.

6. Time complexity of Laplacian centrality

For the sake of comparison, we first present the complexity of all standard methods—degree, closeness and betweenness centralities. The data structure of the input graph is the adjacency list of G, which presents the adjacency relation of all edges and their weights of the input graph G. Assume there are m edges and n nodes.

Degree centralities. Though degree measuring is rather intuitively heuristic and its processing is pretty straightforward, the time complexity is $O(m)$.

Closeness and betweenness centralities. The most basic step in these two algorithms is the search for the shortest paths between every pair of vertices. Computing the shortest paths between any two vertices is the necessary step. Its fastest algorithm is Floyd-Warshall algorithm whose time complexity is $O(n^3)$ [14]. Hence, the total time complexity for either closeness method or betweenness method is at least $O(n^3)$.

Laplacian centrality. By glancing at the definition of Laplacian centrality (Definition 2), one might initially guess that the computational complexity would be relatively high since each step it involves the calculation of eigenvalues. However, by applying an algebraic graph theory result (Theorem 2), we are able to design a much faster algorithm provides a structural result that graphically describes the Laplacian centrality.

At first, by scanning the adjacency list, we could get all needed information (including the neighborhood, corresponding edge weights, the sum-weight and also degree) for each vertex. Clearly, this step could be finished in $O(m)$. Then, by Theorem 2 and Lemma 1, we compute the Laplacian centrality for each vertex v by calculating $NW_2^C(v), NW_2^E(v)$ and $NW_2^M(v)$

respectively. Denote the degree of vertex v by d_v, and denote Δ as the maximum degree, that is $\Delta = \max_{v\in V(G)} d_v$. Clearly, computing Type 1 walks $NW_2^C(v)$ needs d_v additions and d_v multiplications since $NW_2^C(v) = \sum_{j\in N(v)} w_{j,v}^2$; computing Type 2 walks $NW_2^E(v)$ needs d_v additions, d_v subtraction and d_v multiplications since $NW_2^E(v) = \sum_{j\in N(v)}[w_{v,j}\cdot(x_j - w_{v,j})]$; computing Type 3 walks $NW_2^M(v)$ needs $\binom{d_v}{2}$ additions and $\binom{d_v}{2}$ multiplications since $NW_2^M(v) = \sum_{i,j\in N(v), i\neq j} w_{i,v}w_{v,j}$. Thus, for a vertex v of degree d_v, it costs totally $5d_v + d_v\cdot(d_v - 1)$ operations. So for the whole graph G, the worst time complexity would be $\sum_{v\in V(G)}\left(5d_v + d_v^2\right) = O(m + n\Delta^2) = O(n\Delta^2)$, where n is the number of vertices. Thus, the total complexity for computing Laplacian centrality for network G with n vertices and maximum degree Δ would be no more than $O(n\Delta^2)$.

To be clear, the time complexities for each method are summarized in the following table. We could see that Laplacian method has higher time complexity than the most straightforward method—degree method, but would run faster than the other two methods.

	Degree	Closeness	Betweenness	Laplacian
Time complexity	$O(m)$	$O(n^3)$	$O(n^3)$	$O(n\Delta^2)$

7. Concluding remarks and future research

As we know from Corollary 2, we allow fractional weights of edges for practical applications. Furthermore, we also could control the relative importance of the weight of edges and the number of edges associated with a vertex by a tuning parameter α as Opsahl did in Ref. [13]. That is, we could redefine $w_{i,j} := w_{i,j}^\alpha$ for each edge (v_i, v_j) where $\alpha \geqslant 0$. Notice that when $\alpha = 0$, the networks are treated as unweighted, thus the impact of weights are zero; when $\alpha = 1$, the weights will act thoroughly, that is also what we are doing in this paper. To save space and also for the sake of easy understanding of the graph theoretical interpretations, here we present Laplacian centrality method with default value $\alpha = 1$, and also compare other methods with the same value of $\alpha = 1$.

We propose a new centrality measure–Laplacian centrality in this paper, which is applicable to weighted networks, but we have no intention to compete with other existing methods, since for any particular research project we will have to identify which centrality measure is most meaningful or useful. As we have illustrated in Theorem 2, the Laplacian centrality of some vertex is actually related to the number of 2-walks it participates in. That is, it not only takes into account the local environment immediately around it but also a bigger environment around its neighbors. It is an intermediate between global and local characterization of the position of a vertex in weighted networks. Thus it is supposed to have advantages when vertices to *community* (not whole network) relations are to be understood. The applications on two classical weighted data sets and one unweighted network have showed strong evidences of its utility. Unfortunately, Laplacian centrality also has a limitation that it cannot apply on directed networks directly. Laplacian matrix of directed network is not symmetric, thus the eigenvalues are complex numbers instead of real numbers. What is the relationship between Laplacian energy of directed graph and its graphical structure is still unknown. Characterizing the relationship and introducing a similar "intermediate" centrality method for directed graph is also one of our future research directions.

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