Valuing American Options by Simulatioin: A Simple Least-Squares Approach

Bohdan Tkachuk

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1 Introduction

One of the most important problems in option pricing theory is the valuation and optimal exercise of derivatives with American-style exercise features. These types of derivatives are found in all major financial markets. In this seminar we consider method of valuing American option using Monte-Carlo simulation

2 Valuing of American option

2.1 Snell Envelope

X_t exercise process (money you get if you exercise at time t)
The value of an American option can be represented by Snell envelope:

$$\hat{V}_t = \max\left\{E_Q(\hat{X}_\tau) \mid \mathcal{F}_t : \tau \text{ stopping time in [t, T]}\right\} \tag{1}$$

Where $\hat{V}_t$ and $\hat{X}_t$ discounted option value and discounted exercise process at tome t.
Q is some EMM, maybe unique. Q depends on our model

2.2 Approximation of Snell Envelope

We focus the discussion on the case where the American option can only be exercised at the K discrete times $0 < t_1 < t_2 < ... < t_K = T$. Therefore we can approximate value of American Value

$$\hat{V}_t \approx \max\left\{E_Q(\hat{X}_\tau) \mid \mathcal{F}_\tau : \tau \text{ stopping time in } t_m = t, t_{m+1} ... t_K\right\} \tag{2}$$

$$\hat{V}_t \approx E_Q(\hat{X}_\tau \mid \mathcal{F}_{\tau^\star})\ \tau^\star \text{ is optimal stopping time} \tag{3}$$

Let's consider $E_Q(\hat{V}_{t_m} \mid \mathcal{F}_{t_{m-1}})$:

$$E_Q(\hat{V}_{t_m} \mid \mathcal{F}_{t_{m-1}}) = E_Q(E_Q(\hat{X}_{\tau^\star} \mid \mathcal{F}_{t_m}) \mid \mathcal{F}_{t_{m-1}}) = E_Q(\hat{X}_{\tau^\star} \mid \mathcal{F}_{t_{m-1}}) \tag{4}$$

2.3 Recursion of Snell Envelope

Snell envelope is defined recursively:

$$\hat{V}_T = \hat{X}_T \tag{5}$$

$$\hat{V}_{t_{m-1}} = \max\left\{\hat{X}_{t_{m-1}}, E_Q(\hat{V}_{t_m} \mid \mathcal{F}_{t_{m-1}})\right\} \tag{6}$$

and using (4)

$$\hat{V}_{t_{m-1}} = \max\left\{\hat{X}_{t_{m-1}}, E_Q(\hat{X}_{\tau^\star} \mid \mathcal{F}_{t_{m-1}})\right\} \tag{7}$$

Hence, first part of maximum means stopping value and second part means continuation value.

2.4 Approximation of conditional expectation

Let's consider $E_Q(\hat{X}_{\tau^\star} \mid \mathcal{F}_{t_{m-1}})$.We assume that conditional expectation is an element of the L_2 space of square-integrable function. There is some deterministic function $R \to R$ such that

$$E_Q(\hat{X}_{\tau^\star} \mid \mathcal{F}_{t_{m-1}}) = f_t(S_{t_{m-1}}) \tag{8}$$

where $S_{t_{m-1}}$ is stock price and Markov process.
Since L_2 is a Hilbert space, it has a countable orthonormal basis and conditional expectation can be represented as a linear combination of this basis (for some fixed t)

$$f_t(x) = \sum_{j=0}^{\infty} a_j l_j(x) \tag{9}$$

$l_0(x), l_1(x)...$ are given basis functions (e.g Laguerre polynomials)
$a_0, a_1... \in R$ are coefficient for f_t
From Probability Theory we know $E(X \mid l)$ minimize $Y \to E((X-Y)^2)$ where X is given r.v and l is σ-algebra. In our case f_t minimize (10) i.e $(a_j)_{j=0}^{\infty}$ minimize (11)

$$f_t \to E((\hat{X_{\tau^\star}} - f_t)^2) \tag{10}$$

$$(a_j)_{j=0}^{\infty} \to E((\hat{X_{\tau^\star}} - \sum_{j=0}^{\infty} a_j l_j(x))^2) \tag{11}$$

To approximate conditional expectation, we use first $M < \infty$ basis function

$$f_t(x) \approx \sum_{j=0}^{M} \tilde{a}_j l_j(x) \tag{12}$$

for some $\tilde{a}_0, \tilde{a}_1... \in R$
The next step to find corresponded coefficients $\tilde{a}_0, \tilde{a}_1... \in R$.

$$(a_j)_{j=0}^{M} \to E((\hat{X}_{\tau^\star} - \sum_{j=0}^{M} a_j l_j(x))^2) \tag{13}$$

2.5 Algorithm and Monte-Carlo simulation

Using Monte-Carlo simulation we re[lace expectation by arithmetic sum and using our recursion we find unknown optimal stopping time.

$$E((\hat{X}_{\tau^\star} - \sum_{j=0}^{M} a_j l_j(x))^2) \approx \frac{1}{N}(\hat{X}_{\tau^\star} - \sum_{j=0}^{M} a_j l_j(S_{t_{m-1}}))^2 \tag{14}$$

where N number of Monte-Carlo's paths, $\hat{X}_{\tau^\star}$ and $S_{t_{m-1}}$ simulated. Optimal time $\tau^\star$ we get from recursion comparing payoff and continuation value in t_m,

... t_K because we are going backward.
In general, regression minimize following problem:

$$(a_j)_{j=0}^{M} \to \sum_{i=0}^{N}((y_i - \sum_{j=0}^{M} a_j X_i)^2) \tag{15}$$

and from Econometric

$$\vec{\tilde{a}} = (X^T X)^{-1} X^T Y \tag{16}$$

Applying (14) we get $\tilde{a}_0$, ... $\tilde{a}_M$. And the last step to find optimal stopping time going backward. Let's consider that we know some optimal stopping time $\tau^\star$ in $\{t_{m-1}, ... t_K\}$. If $\hat{X_{t_m}} > X_{\tau^\star}$ then $\tau^\star = t_m$ if $\hat{X_{t_m}} < X_{\tau^\star}$ then $\tau^\star$ from before. Repeat algorithm for t_{m-1} ... t_0