Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

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Bc. Bohdan Jóža

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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	doc. RNDr. Robert Lukoťka, PhD.

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Bc. Bohdan Jóža

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Bc. Bohdan Jóža
Študijný program:	informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor:	informatika
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Názov: Edge colourings of signed cubic graphs
Hranové farbenia signovaných kubických grafov

Anotácia: Signované grafy sú grafy, ktorých hrany sú ohodnotené prvkami z {-1, 1}. Prepínanie signovaného grafu v jeho vrchole v je vynásobenie ohodnotenia incidentných hrán hodnotou -1. Grafy, ktoré možno získať sériou operácií prepínania sú ekvivalentné. Existuje veľa článkov, ktoré skúmajú rozšírenie štandardných grafových invariantov na signované grafy. Jednou zo skúmaných tém je farbenie signovaných grafov. Predmetom práce budú hranové farbenia signovaných kubických grafov. Hranové farbenia signovaných grafov začal skúmať Behr v článku [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. Cieľom práce je začať systematické štúdium hranovej 3-zafarbiteľnosti signovaných grafov.

Vedúci:	doc. RNDr. Robert Lukoťka, PhD.
Katedra:	FMFI.KI - Katedra informatiky
Vedúci katedry:	prof. RNDr. Martin Škoviera, PhD.

Spôsob sprístupnenia elektronickej verzie práce:
bez obmedzenia

Dátum zadania: 16.11.2022

Dátum schválenia: prof. RNDr. Rastislav Kráľovič, PhD.
garant študijného programu

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študent vedúci práce

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Comenius University Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname: Bc. Bohdan Jóža
Study programme: Computer Science (Single degree study, master II. deg., full time form)
Field of Study: Computer Science
Type of Thesis: Diploma Thesis
Language of Thesis: English
Secondary language: Slovak

Title: Edge colourings of signed cubic graphs

Annotation: Signed graphs are graphs, whose edges have assigned values from {-1, 1}. Switching at a vertex v of a graph is done by multiplying the values of all edges incident with v by -1. Graphs that can be obtained from each other by switching are called equivalent. There are plenty of papers studying generalization of standard graph invariants to signed graphs. One of these invariants is graph colouring. The thesis should focus on edge colourings of signed cubic graphs. The study of edge colourings of signed graphs was started by Behr [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. The aim of the thesis is to initiate the systematic study of 3-edge-colourability of signed cubic graphs.

Supervisor: doc. RNDr. Robert Lukoťka, PhD.
Department: FMFI.KI - Department of Computer Science
Head of department: prof. RNDr. Martin Škoviera, PhD.

Assigned: 16.11.2022

Approved: prof. RNDr. Rastislav Kráľovič, PhD.
Guarantor of Study Programme

.... Student

.... Supervisor

Acknowledgments: You can thank anyone who helped you with the thesis here (e.g. your supervisor).

Abstrakt

Slovenský abstrakt v rozsahu 100–500 slov, jeden odstavec. Abstrakt stručne sumarizuje výsledky práce. Mal by byť pochopiteľný pre bežného informatika. Nemal by teda využívať skratky, termíny alebo označenie zavedené v práci, okrem tých, ktoré sú všeobecne známe.

Kľúčové slová: Slovak, keywords, here

Abstract

Abstract in the English language (translation of the abstract in the Slovak language).

Keywords: English, keywords, here

Contents

List of Figures

Introduction

TODO

Chapter 1

Preliminary Graph Theory

First, let's define some basic concepts of graph theory, starting with the graph itself.

1.1 Graphs

A graph is an algebraic structure most commonly used to describe relationships between objects. There are many definitions of a graph. The most abstract definition of a graph is simply a set V and a relation R on V denoting which elements of V are connected. Graphs in general are *directed*, if R is symmetric, the graph is *undirected*. For the purposes of this work we will be using a geometric definition and generally undirected graphs.

Definition 1. An undirected graph is an ordered pair $G = (V, E)$, where V is a set of *vertices* and E is a set of edges, i. e. a set of unordered pairs of vertices $\forall e \in E: \; e = (u, v); u, v \in V$.

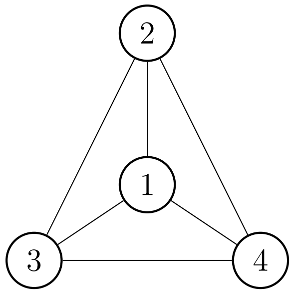
Definition 2. A *path* in a graph G from v to w; $v, w \in V$ is a sequence of vertices $(u_1, u_2, \ldots, u_n)$; $\{u_i \mid 1 \leq i \leq n\} \subseteq V$ such that $u_1 = v$, $u_n = w$ and $\{(u_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \subseteq E$. A graph is *connected* if there exists a path between every pair of vertices $v, w \in V$; $v \neq w$.

Definition 3. A *degree* $\Delta(v)$ of a vertex v denotes how many edges are incident to this vertex. $\Delta(G)$ is the highest degree of any vertex in G.

Definition 4. A graph is *k-regular* if the degree of each vertex is exactly k. A *cubic graph* is a 3-regular graph.

As an example, the K_4 graph is cubic.

In general statements about graphs in later chapters, we are referring to unordered cubic graphs.



1.1.1 Coloring

When simple binary relationships between objects are not enough, weighted graphs and coloring offer a wider range of applications. Assigning colors to vertices or edges of graphs makes classifications of these objects possible.

Definition 5. A vertex coloring of a graph G is a mapping from the vertex set of G to a set of colors C. An edge coloring of a graph G is a mapping from the edge set of G to a set of colors C.

Definition 6. A *proper vertex coloring* of G is a vertex coloring such that no two neighboring vertices share a color. A *proper edge coloring* is an edge coloring such that no two edges that share an endpoint have the same color. A proper coloring using k colors is called a *k-coloring*.

As coloring in general is not very interesting, we will be considering only proper colorings henceforth. It is also important to define the set of "colors", especially when coloring signed graphs. Although actual colors tend to be a nice visualization of a coloring, it is more practical to use a subset of integers $C \subseteq \mathbb{Z}$.

The canonical coloring problem is to find the minimum number of colors required for a proper coloring. This number is called the *chromatic number* for vertex colorings and *chromatic index* for edge colorings. Determining the chromatic number and index is useful in other areas of graph theory as well.

Theorem 1. *A graph is bipartite if and only if it has a proper vertex 2-coloring.*

For regular unsigned graphs these numbers are known.

Theorem 2 (Brooks)**.** *The chromatic number of a graph G is $\Delta(G)$ for all graphs except complete graphs and cycles of odd length, where the chromatic number is $\Delta(G) + 1$.[1]*

Theorem 3 (Vizing)**.** *The chromatic index of a simple graph G is $\Delta(G)$ or $\Delta(G) + 1$.TODO cite*

In other words, we can always color the edges of a graph using at most $\Delta(G) + 1$ colors where $\Delta(G)$ is the highest degree of any vertex in G. The lower bound $\Delta(G)$

is trivial; we need exactly $\Delta(G)$ colors at the highest degree vertex in G to construct a proper coloring. The Vizing theoremTODO CITE VIZING THEOREM proves the upper bound using Kempe chains.

1.2 Signed graphs

A signed graph is a graph in which each edge has either a positive or a negative sign. There are multiple definitions of a signed graph but for our purposes a sign function is most practical.

Definition 7. A *signed graph* $\Gamma = (G, \sigma)$ consists of a *underlying graph* G and a *sign function* $\sigma : E(G) \to \{+, -\}$ that assigns a sign to each edge of G.

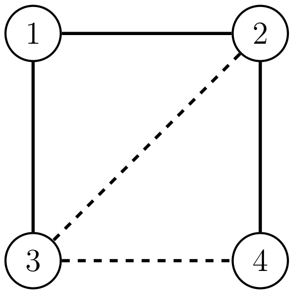


Figure 1.1: Example of a signed graph. Dashed lines indicate negative edges, solid lines positive edges.

A fundamental concept in the signed graphs theory is *balance*. The sign of a path is the product of the signs of its edges. A path is positive if and only if there is an even number of negative edges on it. A cycle is balanced if it is positive and a signed graph is balanced if each cycle in it is balanced[2].

Theorem 4 (Harary)**.** *A signed graph is balanced if and only if*

1. *for every pair of vertices, all paths between these vertices have the same sign*

2. *the vertices can be divided into two subsets (possibly empty) such that each edge with both ends in the same subset is positive and each edge with ends in different subsets is negative*

This is the generalization of the earlier mentioned bipartite graph theorem.TODO reference?

The proof uses the method of *switching*. Switching a vertex of a signed graph reverses the sign of each edge incident to it. More generally, switching a signed graph reverses the sign of each edge between a vertex subset and its complement.

We can prove by induction that a signed graph can be switched to an all-positive graph if and only if it is balanced. Both conditions in Harary's theorem apply to all all-positive graphs and graphs that can be switched from an all-positive graph. Consequently, all balanced graphs are equivalent to an all-positive graph, which is an alternative definition of a positive graph. Similarly, we call a graph *antibalanced* if it is equivalent to an all-negative graph, (all cycles of even length in and antibalanced graph are positive and cycles of odd length are negative).

Definition 8. If a signed graph can be obtained from another signed graph by switching, they are considered *equivalent*. For a single underlying graph, switching forms *equivalence classes* of signed graphs. Within a single equivalence class all graphs can be switched to each other.

It makes sense to study properties of signed graphs that behave consistently under switching. An example of such property is the sign of cycles. Switching a single vertex doesn't change the sign of cycles (cycles containing the vertex reverse signs for two edges resulting in the same product) and switching a set of vertices is equivalent to a sequence of one-vertex-switches (each edge within the set and within the complement gets reversed twice).

1.2.1 Coloring

The research in signed graph coloring was initiated by Zaslavsky[3] in the early 1980s and published in multiple seminal papers[4, 5, 6].

Definition 9. A *signed vertex coloring* $\phi(\Gamma)$ of a graph Γ is a mapping from the vertex set of Γ to a set of signed colors C. A *signed edge coloring* $\gamma(\Gamma)$ of a graph Γ is a mapping from the set of half-edges (vertex-edge incidences) of Γ to a set of colors C. Additionally, the half-edges must have the same color on positive edges and opposite colors on negative edges.

$$(\forall e = (u, v) \in E(\Gamma)) \;\; \gamma(e, u) = \sigma(e)\gamma(e, v)$$

Definition 10. A *proper vertex signed coloring* is a coloring $\phi(\Gamma)$ such that for each pair of neighboring vertices (u, v) $\phi(u) \neq \sigma(uv)\phi(v)$. In case of *proper edge signed coloring* the definition remains the same, because the coloring condition is already a part of the general coloring definition. Each color must be present at each vertex at most once (or adjacent half-edges have different colors). We are, again, assuming only proper colorings from now on.

Here it is even more important to define the color set. Zaslavsky[5] defined a k-coloring based on a signed color set $C_k = \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, (k-1), k\}$

and called colorings zero-free if the color 0 was not used. He then studied the properties of *chromatic polynomials* related to signed colorings, the number of colorings for a signed graph. (Balanced chromatic polynomials in case of zero-free colorings.)

This definition is consistent under switching. Assuming a graph Γ and a proper vertex coloring ϕ, if we obtain Γ' by switching vertex u, then $\phi' = \phi; \phi'(u) = -\phi(u)$ is a proper vertex coloring of Γ'. Similarly for edge coloring in which we reverse the sign of each half-edge incident to u.

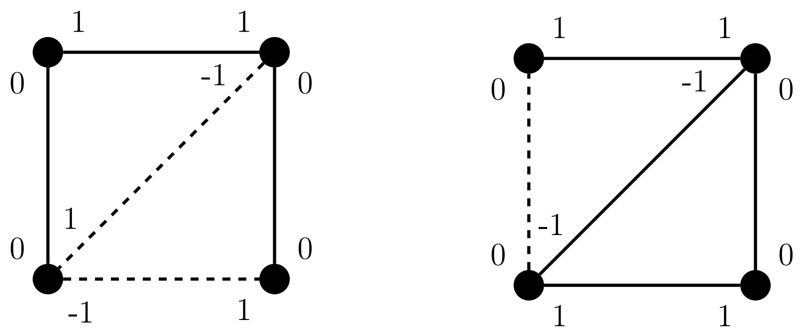


Figure 1.2: Example of a proper signed edge coloring on the left. We obtain the graph on the right by switching the bottom left vertex and the coloring remains correct and proper.

However, this definition is not a natural extension of the original color set of integers, because a k-coloring essentially uses $2k$ or $2k + 1$ signed colors. It is a desirable property for the color set because signed graphs themselves are an extension of unsigned graphs. A balanced signed graph is essentially equivalent to the unsigned underlying graph, so its chromatic number and index for instance should also match. In *The chromatic number of a signed graph*, Máčajová et al. define the color set a little bit differently: An n-coloring uses the color set $C_n = \{\pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k$ and $C_n = \{0, \pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k + 1$. We adopt this color set in this thesis.

We adopt the signed versions of coloring definitions from *The chromatic number of a signed graph*[7] and *Edge coloring signed graphs*[8]. In the latter, however, Behr colors the half-edges the same color if their sign is negative, not positive. Since there was no obvious advantage stated in the article and the definitions are somewhat equivalentTODO dokaz? bijekcia?, we find this version to be more natural, as the underlying graph can be interpreted as an "absolute value" of the signed graph, which is positive.

1.3 Motivation

"In the study of various important and difficult problems in graph theory (such as the cycle double cover conjecture and the 5-flow conjecture), one encounters an interesting but somewhat mysterious variety of graphs called snarks. In spite of their simple definition [. . .] and over a century long investigation, their properties and structure are largely unknown." — Chladný, Škoviera [9]

By Vizing's theorem, cubic graphs are colorable either with three ("class one" graphs) or four colors ("class two") graphs. The exact definition of a snark may vary from paper to paper but a snark is essentially a cubic graph with chromatic index four (its edges can't be colored with three colors). The definition varies across papers, trivial cases are generally excluded. Every cubic graph with a loop or a bridge is a "snark", triangles (cycles of length three) can be contracted into a single vertex and cycles of length four can also be simplified. Therefore many definitions forbid these properties by considering true snarks only graphs with girth (length of the shortest cycle) at least five. Even more strongly, only cyclically 4-edge-connected graphs are considered (there is no subset of three or fewer edges such that their removal will disconnect the graph into two subgraphs each containing a cycle). One of the alternative formulations of the four color theorem is that each snark is non-planar. Snarks are important in a multitude of graph theory areas and thus it makes sense to investigate the reach of signed snarks too.

1.4 Previous research

In *The chromatic number of a signed graph*[7] Máčajová et al. continue Zaslavsky's research by studying the properties of the chromatic number of signed graphs, ultimately proving a signed version of the famous Brooks'[1] theorem.

Theorem 5 (Signed Brooks' Theorem)**.** *Let* Γ *be a simple connected signed graph. If* Γ *is not a balanced complete graph, a balanced odd circuit or an unbalanced even circuit, then* $\chi(\Gamma) \leq \Delta(\Gamma)$*.*

Edge coloring signed graphs defines a version of the signed edge coloring and proves a signed version of the equally fundamental Vizing's theorem.

Theorem 6 (Signed Vizing's Theorem)**.** *Let* Γ *be a simple signed graph. The chromatic index of* Γ *is* $\Delta(\Gamma)$ *or* $\Delta(\Gamma) + 1$*.*

TODO

Chapter 2

Generating signed snarks

Since the structure of snarks is generally unknown, the most efficient way of systematically generating snarks is still a brute-force approach.

2.1 Chromatic index problem

To determine the chromatic index of a cubic graph is an NP-complete problem. By extension, determining the chromatic index of a signed cubic graph is also NP-complete, because of the trivial reduction from signed chromatic index problem to unsigned chromatic index problem. Instead of designing an algorithm we decided to implement a conversion from the chromatic index problem to 3SAT and using a highly optimized SAT solver anticipating better effectiveness.

2.1.1 Conversion to 3SAT

For any cubic signed graph Γ we will construct a 3SAT formula $F(\Gamma)$ that is satisfiable if and only if the graph is 3-colorable. There will be three literals for each half-edge ev of Γ, one for each color from $C_3 = \{-1, 0, 1\}$. Let these be x_{ev}^{-1}, x_{ev}^{0} and x_{ev}^{1}. In any evaluation of these literals that satisfy F exactly one of them will be true denoting the color of the half-edge. This will be guaranteed using three constituent formulas. Let $\Gamma = ((V, E), \sigma)$

$$F_1 = \bigwedge_{e=vw \in E} (x_{ev}^{-1} \vee x_{ev}^{0} \vee x_{ev}^{1}) \wedge (x_{ew}^{-1} \vee x_{ew}^{0} \vee x_{ew}^{1})$$

The first formula ensures that each half-edge is colored and is the only set containing clauses of length 3. The next formula will enforce the correctness of the coloring, restricting the colors of half edges that for one complete edge. Illegal signatures for each edge are negated using DeMorgan rules, resulting in a convenient CNF form. No edge can be colored 0 on one side and 1 or -1 on the other $(\neg(x_{ev}^{0} \wedge x_{ew}^{1}) = (\neg x_{ev}^{0} \vee \neg x_{ew}^{1}))$

and the colors must be the same if the edge is positive $((\neg x^1_{ev} \vee \neg x^{-1}_{ew}))$ and opposite if the edge is negative $((\neg x^1_{ev} \vee \neg x^1_{ew}))$.

$$F_2 = \bigwedge_{e=vw\in E} (\neg x^0_{ev} \vee \neg x^1_{ew}) \wedge (\neg x^0_{ev} \vee \neg x^{-1}_{ew}) \wedge (\neg x^{-1}_{ev} \vee \neg x^{\sigma(e,w)}_{ew}) \wedge (\neg x^1_{ev} \vee \neg x^{-\sigma(e,w)}_{ew}) \wedge (\dots v \rightleftarrows w \dots)$$

The first four clauses illustrate the condition from the "perspective" of v, they will be repeated for w as well by switching instances of v and w. Lastly we need to ensure the coloring is proper. Let $N(v) = \{(v,w) \mid (v,w) \in E;\ w \in V\}$ be the set of edges incident to v.

$$F_3 = \bigwedge_{\substack{v\in V\\ e_1,e_2\in N(v);\ e_1\neq e_2}} (\neg x^{-1}_{e_1v} \vee \neg x^{-1}_{e_2v}) \wedge (\neg x^0_{e_1v} \vee \neg x^0_{e_2v}) \wedge (\neg x^1_{e_1v} \vee \neg x^1_{e_2v})$$

Each pair of half-edges with a common vertex has to have different colors. Note that we don't need to explicitly ensure that for each half-edge exactly one literal is true, only that at least one is true, because it is a consequence of the properness of the coloring.

Theorem 7. *3SAT formula $F(\Gamma) = F_1 \wedge F_2 \wedge F_3$ constructed in the way described above is satisfiable if and only if Γ is 3-colorable.*

Proof. Follows from the construction of F encapsulating all properties of a proper signed 3-coloring. □

2.2 Equivalence

Signed graphs can be equivalent in a combination two ways, switching-equivalent or isomorphic. Let's explore the switching equivalence first.

2.2.1 Signed equivalence classes

On any base graph G there are $2^{|E(G)|}$ possible signed graphs. Zaslavsky[3] enumerated the switching equivalence classes and described a representative for each class.

Theorem 8. *Let G be a simple unsigned base graph and $T \subseteq E(G)$ a spanning tree of G. Then all signed graphs that have an all-positive signature on T are not switching-equivalent and each equivalence class based on G has exactly one representative among them. There are $2^{|E(G)|-|V(G)|+1}$ switching classes on G.*

Proof. Take any signed graph constructed this way. Switching no vertices and all vertices results in the same graph. To obtain a different graph, at least one vertex will not be switched and at least one vertex will be switched. The set of switched vertices $A \neq \emptyset$ and the set of untouched vertices $B \neq \emptyset$ are a partition of $V(G)$. Since G is connected, there is at least one edge between A and B and at least one of them is in T. This edge will change its sign based on the definition of switching. So any graph we obtain by switching one of the graphs from theorem 8 will not be all-positive on T, making all these graphs belong to different equivalence classes. $\square$

According to Theorem 8, on a base cubic graph with n vertices there are $2^{\frac{n}{2}+1}$ equivalence classes, one for each signature of edges that are not in the spanning tree. Note that $\frac{n}{2}$ is always an integer since the number of vertices in a cubic graph is even. The following algorithm generates all non-equivalent representatives.

2.2.2 Generating algorithm

The algorithm first finds a spanning tree and assigns positive signs to all edges in it. Edges are enumerated and the spanning tree edges will be ignored. We can now imagine that positive sign means zero and negative sign means one. The remaining edges form a binary number in this way. To obtain the next representative we simply increment this number by one. This means flipping the lowest consecutive sequence of ones and the first instance of zero. We keep reversing the sign of edges from lowest to highest until we flip a positive edge for the first time or run out of edges. If we run out of edges, we basically went from the number $2^{\frac{n}{2}+1} - 1$ to 0. So starting with any signature that is all-positive on the spanning tree, we will have generated all equivalence classes after $2^{\frac{n}{2}+1}$ incrementations. The spanning tree, however, has to remain the same during the entire process.

2.2.3 Isomorphism

Signed graphs can be isomorphic if and only if their base graphs are isomorphic. We get the homomorphism by taking away signatures. This is a key observation because if we base our signed graphs on non-isomorphic graphs the only candidates for isomorphisms are based on the same graph. In Enumerating Switching Isomorphism Classes of Signed Graphs[10] non-equivalent graphs are enumerated using a one-to-one correspondence between switching isomorphism classes and signed double covers of Γ. The algorithm first generates all possible signed graphs from a base graph and then filters them for isomorphisms. This approach, although correct, is too inefficient for our purposes. The algorithm generates all signed graphs for each base graph, which for $n = 18$ is already too much, 41301 cubic graphs with 18 vertices results in 10.8 billion signed graphs.

A second approach is based on the cycle space of Γ and Eulerian graphs. TODO

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