Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

2024
Bc. Bohdan Jóža

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	doc. RNDr. Robert Lukoťka, PhD.

Bratislava, 2024
Bc. Bohdan Jóža

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Bc. Bohdan Jóža
Študijný program: informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor: informatika
Typ záverečnej práce: diplomová
Jazyk záverečnej práce: anglický
Sekundárny jazyk: slovenský

Názov: Edge colourings of signed cubic graphs
Hranové farbenia signovaných kubických grafov

Anotácia: Signované grafy sú grafy, ktorých hrany sú ohodnotené prvkami z {-1, 1}. Prepínanie signovaného grafu v jeho vrchole v je vynásobenie ohodnotenia incidentných hrán hodnotou -1. Grafy, ktoré možno získať sériou operácií prepínania sú ekvivalentné. Existuje veľa článkov, ktoré skúmajú rozšírenie štandardných grafových invariantov na signované grafy. Jednou zo skúmaných tém je farbenie signovaných grafov. Predmetom práce budú hranové farbenia signovaných kubických grafov. Hranové farbenia signovaných grafov začal skúmať Behr v článku [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. Cieľom práce je začať systematické štúdium hranovej 3-zafarbiteľnosti signovaných grafov.

Vedúci: doc. RNDr. Robert Lukoťka, PhD.
Katedra: FMFI.KI - Katedra informatiky
Vedúci katedry: prof. RNDr. Martin Škoviera, PhD.

Spôsob sprístupnenia elektronickej verzie práce:
bez obmedzenia

Dátum zadania: 16.11.2022

Dátum schválenia: prof. RNDr. Rastislav Kráľovič, PhD.
garant študijného programu

....

študent vedúci práce

Comenius University Bratislava
Faculty of Mathematics, Physics and Informatics

71793490

THESIS ASSIGNMENT

Name and Surname: Bc. Bohdan Jóža
Study programme: Computer Science (Single degree study, master II. deg., full time form)
Field of Study: Computer Science
Type of Thesis: Diploma Thesis
Language of Thesis: English
Secondary language: Slovak

Title: Edge colourings of signed cubic graphs

Annotation: Signed graphs are graphs, whose edges have assigned values from {-1, 1}. Switching at a vertex v of a graph is done by multiplying the values of all edges incident with v by -1. Graphs that can be obtained from each other by switching are called equivalent. There are plenty of papers studying generalization of standard graph invariants to signed graphs. One of these invariants is graph colouring. The thesis should focus on edge colourings of signed cubic graphs. The study of edge colourings of signed graphs was started by Behr [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. The aim of the thesis is to initiate the systematic study of 3-edge-colourability of signed cubic graphs.

Supervisor: doc. RNDr. Robert Lukoťka, PhD.
Department: FMFI.KI - Department of Computer Science
Head of department: prof. RNDr. Martin Škoviera, PhD.

Assigned: 16.11.2022

Approved: prof. RNDr. Rastislav Kráľovič, PhD.
Guarantor of Study Programme

.... Student

.... Supervisor

Acknowledgments: You can thank anyone who helped you with the thesis here (e.g. your supervisor).

Abstrakt

Slovenský abstrakt v rozsahu 100–500 slov, jeden odstavec. Abstrakt stručne sumarizuje výsledky práce. Mal by byť pochopiteľný pre bežného informatika. Nemal by teda využívať skratky, termíny alebo označenie zavedené v práci, okrem tých, ktoré sú všeobecne známe.

Kľúčové slová: Slovak, keywords, here

Abstract

Abstract in the English language (translation of the abstract in the Slovak language).

Keywords: English, keywords, here

Contents

Introduction **1**

1 Preliminary Graph Theory **3**

1.1 Graphs 3

1.1.1 Coloring 4

1.2 Signed graphs 5

1.2.1 Coloring 6

1.3 Motivation 7

2 Terminology **9**

2.1 Graphs 9

2.2 Signed graphs 9

2.3 Vertex coloring 10

2.4 Edge coloring 11

List of Figures

1.1 Example of a signed graph 5
1.2 Example of a switching 6

Introduction

TODO

Chapter 1

Preliminary Graph Theory

First, let's define some basic concepts of graph theory, starting with the graph itself.

1.1 Graphs

A graph is an algebraic structure most commonly used to describe relationships between objects. There are many definitions of a graph. The most abstract definition of a graph is simply a set V and a relation R on V denoting which elements of V are connected. Graphs in general are *directed*, if R is symmetric, the graph is *undirected*. For the purposes of this work we will be using a geometric definition and generally undirected graphs.

Definition 1. An undirected graph is an ordered pair $G = (V, E)$, where V is a set of *vertices* and E is a set of edges, i. e. a set of unordered pairs of vertices $\forall e \in E: \ e = (u, v); u, v \in V$.

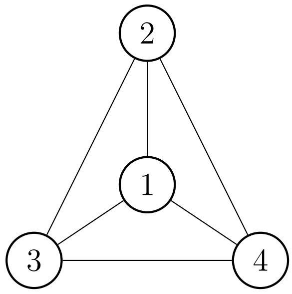
Definition 2. A *path* in a graph G from v to w; $v, w \in V$ is a sequence of vertices $(u_1, u_2, \dots, u_n)$; $\{u_i \mid 1 \leq i \leq n\} \subseteq V$ such that $u_1 = v$, $u_n = w$ and $\{(u_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \subseteq E$. A graph is *connected* if there exists a path between every pair of vertices $v, w \in V$; $v \neq w$.

Definition 3. A *degree* $\Delta(v)$ of a vertex v denotes how many edges are incident to this vertex. $\Delta(G)$ is the highest degree of any vertex in G.

Definition 4. A graph is *k-regular* if the degree of each vertex is exactly k. A *cubic graph* is a 3-regular graph.

As an example, the K_4 graph is cubic.

In general statements about graphs in later chapters, we are referring to unordered cubic graphs.



1.1.1 Coloring

When simple binary relationships between objects are not enough, weighted graphs and coloring offer a wider range of applications. Assigning colors to vertices or edges of graphs makes classifications of these objects possible.

Definition 5. A vertex coloring $\phi(G)$ of a graph G is a mapping from the vertex set of G to a set of colors C. An edge coloring $\gamma(G)$ of a graph G is a mapping from the edge set of G to a set of colors C.

Definition 6. A *proper vertex coloring* of G is a vertex coloring such that no two neighboring vertices share a color. A *proper edge coloring* is an edge coloring such that no two edges that share an endpoint have the same color. A proper coloring using k colors is called a *k-coloring*.

As coloring in general is not very interesting, we will be considering only proper colorings henceforth. It is also important to define the set of "colors", especially when coloring signed graphs. Although actual colors tend to be a nice visualization of a coloring, it is more practical to use a subset of integers $C \subseteq \mathbb{Z}$.

The canonical coloring problem is to find the minimum number of colors required for a proper coloring. This number is called the *chromatic number* for vertex colorings and *chromatic index* for edge colorings. Determining the chromatic number and index is useful in other areas of graph theory as well.

Theorem 1. *A graph is bipartite if and only if it has a proper vertex 2-coloring.*

For regular unsigned graphs these numbers are known.

Theorem 2 (Brooks)**.** *The chromatic number of a graph G is $\Delta(G)$ for all graphs except complete graphs and cycles of odd length, where the chromatic number is $\Delta(G) + 1$.[1]*

Theorem 3 (Vizing)**.** *The chromatic index of a graph G is $\Delta(G)$ or $\Delta(G) + 1$.TODO cite*

In other words, we can always color the edges of a graph using at most $\Delta(G) + 1$ colors where $\Delta(G)$ is the highest degree of any vertex in G. The lower bound $\Delta(G)$

is trivial; we need exactly $\Delta(G)$ colors at the highest degree vertex in G to construct a proper coloring. The Vizing theoremTODO CITE VIZING THEOREM proves the upper bound using Kempe chains.

1.2 Signed graphs

A signed graph is a graph in which each edge has either a positive or a negative sign. There are multiple definitions of a signed graph but for our purposes a sign function is most practical.

Definition 7. A *signed graph* $\Gamma = (G, \sigma)$ consists of a *base graph* G and a *sign function* $\sigma : E(G) \to \{+, -\}$ that assigns a sign to each edge of G.

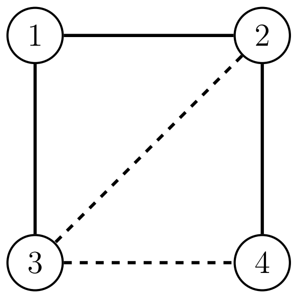


Figure 1.1: Example of a signed graph. Dashed lines indicate negative edges, solid lines positive edges.

A fundamental concept in the signed graphs theory is *balance*. The sign of a path is the product of the signs of its edges. A path is positive if and only if there is an even number of negative edges on it. A cycle is balanced if it is positive and a signed graph is balanced if each cycle in it is balanced[2].

Theorem 4 (Harary)**.** *A signed graph is balanced if and only if*

1. *for every pair of vertices, all paths between these vertices have the same sign*
2. *the vertices can be divided into two subsets (possibly empty) such that each edge with both ends in the same subset is positive and each edge with ends in different subsets is negative*

This is the generalization of the earlier mentioned bipartite graph theorem.TODO reference?

The proof uses the method of *switching*. Switching a vertex of a signed graph reverses the sign of each edge incident to it. More generally, switching a signed graph reverses the sign of each edge between a vertex subset and its complement.

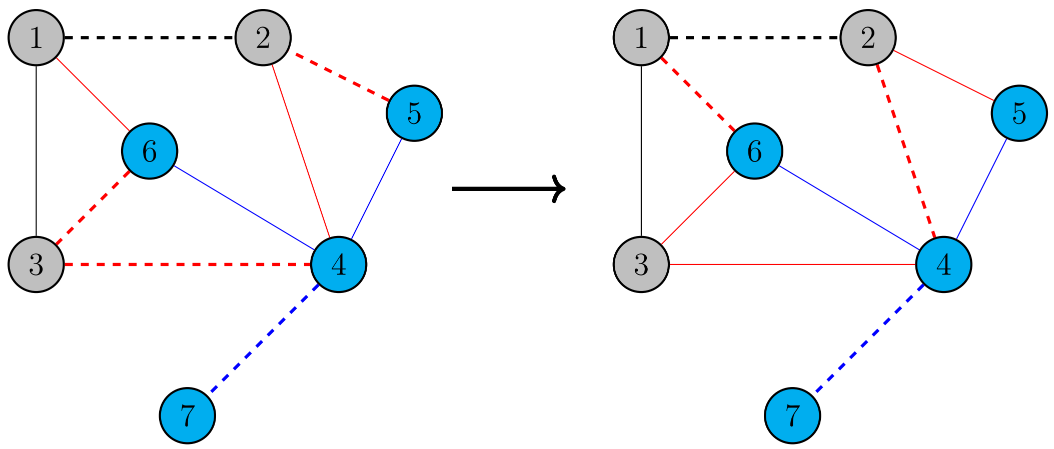


Figure 1.2: Example of a switching. Note that switching blue vertices and switching grey vertices results in the same transformation.

We can prove by induction that a signed graph can be switched to an all-positive graph if and only if it is balanced. Both conditions in Harary's theorem apply to all all-positive graphs and graphs that can be switched from an all-positive graph. Consequently, all balanced graphs are equivalent to an all-positive graph, which is an alternative definition of a positive graph. Similarly, we call a graph *antibalanced* if it is equivalent to an all-negative graph, (all cycles of even length in and antibalanced graph are positive and cycles of odd length are negative).

Definition 8. If a signed graph can be obtained from another signed graph by switching, they are considered *equivalent*. For a single base graph, switching forms *equivalence classes* of signed graphs. Within a single equivalence class all graphs can be switched to each other.

It makes sense to study properties of signed graphs that behave consistently under switching. An example of such property is the sign of cycles. Switching a single vertex doesn't change the sign of cycles (cycles containing the vertex reverse signs for two edges resulting in the same product) and switching a set of vertices is equivalent to a sequence of one-vertex-switches (each edge within the set and within the complement gets reversed twice).

1.2.1 Coloring

The research in signed graph coloring was initiated by Zaslavsky[3] in the early 1980s and published in multiple seminary papers[4, 5, 6].

1.3 Motivation

"In the study of various important and difficult problems in graph theory (such as the cycle double cover conjecture and the 5-flow conjecture), one encounters an interesting but somewhat mysterious variety of graphs called snarks. In spite of their simple definition [...] and over a century long investigation, their properties and structure are largely unknown." — Chladný, Škoviera [7]

TODO

Chapter 2

Terminology

Here we define the basic terminology used in this thesis.

2.1 Graphs

Using the standard notation we write G for a graph and $V(G)$ and $E(G)$ for its vertes set and edge set respectively. We assume no graph constraints unless otherwise specified, e. g. loops and duplicit edges are generally allowed.

We write $e = vw \in E(G)$ to indicate that the edge e of G has endpoints v and w. In the context of edge and vertex coloring it makes sense to define the next terms. In *regular graphs* all vertices have the same degree, specifically a *k-regular* graph is a graph where each vertex has degree exactly k (there is no vertex with more than k edges). A *cubic* graph is a 3-regular graph. A *circuit* is a connected 2-regular subgraph. A *factor* of a graph G is a *spanning subgraph* (a subgraph covering all vertices of G). A *k-factor* is a k-regular spanning subgraph and *k-factorization* partitions all edges of G into disjoint k-factors. A circuit is *positive* if the product of its edge signs is positive and *negative* otherwise. In the context of flows in signed graphs, we will be talking about *signed circuits* as the signed equivalent to circuits on unsigned graphs. A *chromatic number* of a graph G is the number of colors required for a proper vertex coloring of said graph.

2.2 Signed graphs

Signed graphs were introduced by Harary[2] in 1953 as a model for social networks. A signed graph has a value of $+1$ or -1 assigned to all edges, so each edge is positive or negative. They have been proven to be a natural generalization of unsigned graphs in many ways and interesting observations may arise by applying ordinary graph theory to signed graphs.

A *signed graph* is a pair (G, Σ); $\Sigma \subseteq E(G)$, where Σ is a subset of the edge set of G and contains the negative edges.

Function $\sigma : E(G) \to \{+1; -1\}$ denotes the sign of an edge e.

A signed graph can also be defined as a pair (G, σ) using the sign function directly, but I found this definition more natural.

Given a signed graph (G, Σ), *switching* at a vertex v inverts the sign of each edge incident with v.

Using the previously mentioned definition of a signed graph, the resulting graph after a switching is the symmetric difference of Σ and the set of edges incident with v.

Two graphs are *equivalent* if one can be obtained from the other through a series of vertex switchings. Switching equivalence is an equivalence relation and we write $[(G, \Sigma)]$ for an equivalence class of (G, Σ) under this relation.

Additionally, switching doesn't change the signs of circuits in a graph, so two signed graphs are equivalent if their underlying graphs and the signs of all circuits are the same. Consequently, all properties depending only on the signs of the circuits are invariant for all graphs in $[(G, \Sigma)]$.

A signed graph is *balanced* if all of its circuits are positive and unblanaced otherwise.

Balance is an important concept in the sign graph theory, becauase balanced signed graphs (G, Σ) are equivalent to $(G, \{\})$ (an all-positive graph with the same underlying graph).

A signed graph (G, Σ) is *antibalanced* if it is equivalent to $(G, V(G))$ (the same graph with all-negative signature).

Equivalent signed graphs have the same sets of positive circuits and same sets of negative circuits. Additionally, if (G, Σ) is balanced, then $(G, V(G) - \Sigma)$ is antibalanced. (Performing the switchings necessary to transofrm (G, Σ) to an all-positive graph after flipping all signs leads to an all-negative graph.) Given a partition (A, B) of $V(G)$, let $[A, B]$ denote the set of all edges with one end in A and the other in B. Harary[2] characterized balanced graphs:

Theorem 5. *A signed graph (G, Σ) is balanced if and only if there is a set $X \subseteq V(G)$ such that $\Sigma = [X, V(G) - X]$.*

2.3 Vertex coloring

Vertex and edge coloring is a deeply explored topic of graph theory, even in the field of signed graphs. The research was initiated by Zaslavsky[3] in the early 1980s and his results were published in multiple seminary papers[4, 5, 6]. Máčajová, Raspaud and Škoviera expand on this topic in The chromatic number of a signed graph[8], focusing on the behaviour of colorings instead of the polynomial invariants, which Zaslavsky concentrated on in his research.

A *proper vertex coloring* of a signed graph (G, Σ) is $\phi : V(E) \to \mathbb{Z}$ where for each edge $e = vw \in E(G)$: $\phi(v) \neq \sigma(e)\phi(w)$.

Vertices connected by a positive edge must not have the same color and vertices connected by a negative edge must not have opposite colors.

This definition is natural mainly because of the consistency under vertex switching, but also other reasons discussed by Zaslavsky. What is not as natural is the first attempt at a set of signed colors. Unlike colorings on unsigned graphs, here it is practical to assign signed colors from $\mathbb{Z}$ and so arises the problem of defining a signed color set of k colors. Zaslavsky originally defined the coloring of a signed graph in k colors or $2k+1$ signed colors as a mapping $V(G) \to \{-k, -(k-1), \dots, -1, 0, 1, \dots, (k-1), k\}$. A coloring is zero-free if no vertex is colored 0. He then defined the *chromatic polynomial* $\chi_G(\lambda)$ to be the function whose values for negative arguments $\lambda = 2k + 1$ are the numbers of signed colorings in k colors. The *balanced chromatic polynomial* $\chi_G^b(\lambda)$ defined for positive arguments $\lambda = 2k$ are the numbers of zero-free signed colorings in k colors. Finally, the *chromatic number* $\gamma(G)$ of G is the smallest non-negative integer k such that $\chi(2k + 1) > 0$ and the *strict chromatic number* $\gamma * (G)$ is the same for the balanced chromatic polynomial $\chi_G^b(2k) > 0$.

The Zaslavsky's definitions are sound, but they are not direct extensions of the chromatic polynomials and chromatic number for unsigned graphs. That is because they basically count the absolute values of colors. It makes sense to require a signed version of any graph invariant to agree with its underlying graph for balanced signed graphs. Máčajová et. al.[8] instead propose different definitions. They first define sets $M_n \subseteq \mathbb{Z}$ for each $n \geq 1$ as $M_n = \{\pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k$; $k \in \mathbb{N}$ and $M_n = \{0, \pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k + 1$ respectively. We can then define a *proper n-coloring* that uses colors from M_n. The smallest n such that an *n-coloring* exists. In comparison to Zaslavsky, this way an n-coloring uses exactly n colors.

2.4 Edge coloring

In Edge coloring of signed graphs[9], Behr adopts the signed color sets defined by Máčajová et. al. and using these signed colors defines a proper edge coloring on signed graphs.

An *n-edge coloring* γ of (G, Γ) is an assignment of colors from M_n to each vertex-edge incidence of G such that $\gamma(v, e) = -\sigma(e)\gamma(w, e)$ for each edge $e = vw$. If an edge e exists such that $\gamma(v, e) = a$, then the color a is present at v.

The same condition for a *proper n-edge coloring* applies to the signed version, no color can be present more than once at any vertex. The *chromatic index* of a signed graph (G, Γ) $\chi'((G, \Gamma))$ is the smallest n such that (G, Γ) is n-edge-colorable.

Coloring each vertex-edge incidence makes signed edge coloring particularly interesting. This definition also behaves naturally under switching; if we switch a vertex and all colors present at said vertex, the coloring remains consistent. But again, we have to be mindful of the color 0 as in the case of vertex coloring.

We can observe that negative edges behave in the same way as unsigned edges. So each proper n-edge coloring of an all-negative signed graph corresponds to a proper unsigned edge coloring of its underlying graph. This is one of the reasons for the importance of natural definitions: the signed graphs themselves are in a way a generalization of unsigned graphs, so in the field of signed graphs, we are looking for natural generalizations of concepts defined on unsigned graphs.

Bibliography

[1] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc. Math. Phys. Sci.*, 37:194–197, 1941.

[2] F. Harary. On the notion of balance of a signed graph. *Michigan Math J.*, 2(2):143–146, 1953.

[3] T. Zaslavsky. Signed graphs. *Discrete Applied Mathematics*, 4:47–74, 1982.

[4] T. Zaslavsky. Chromatic invariants of signed graphs. *Discrete Math.*, 42:287–312, 1982.

[5] T. Zaslavsky. Signed graph coloring. *Discrete Math.*, 39:215–228, 1982.

[6] T. Zaslavsky. Chromatic invariants of signed graphs. *Discrete Math.*, 52:279–284, 1982.

[7] M. Škoviera M. Chladný. Factorisation of snarks. *The Electronic Journal of Combinatorics*, 17(1), February 2010.

[8] M. Škoviera E. Máčajová, A. Raspaud. The chromatic number of a signed graph. *Electron J. Comb.*, pages 1–10, 2016.

[9] R. Behr. *Discrete Math.*, 2019.