Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

2025
Bc. Bohdan Jóža

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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	doc. RNDr. Robert Lukoťka, PhD.

Bratislava, 2025

Bc. Bohdan Jóža

71793490

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Bc. Bohdan Jóža
Študijný program: informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
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Sekundárny jazyk: slovenský

Názov: Edge colourings of signed cubic graphs
Hranové farbenia signovaných kubických grafov

Anotácia: Signované grafy sú grafy, ktorých hrany sú ohodnotené prvkami z {-1, 1}. Prepínanie signovaného grafu v jeho vrchole v je vynásobenie ohodnotenia incidentných hrán hodnotou -1. Grafy, ktoré možno získať sériou operácií prepínania sú ekvivalentné. Existuje veľa článkov, ktoré skúmajú rozšírenie štandardných grafových invariantov na signované grafy. Jednou zo skúmaných tém je farbenie signovaných grafov. Predmetom práce budú hranové farbenia signovaných kubických grafov. Hranové farbenia signovaných grafov začal skúmať Behr v článku [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. Cieľom práce je začať systematické štúdium hranovej 3-zafarbiteľnosti signovaných grafov.

Vedúci: doc. RNDr. Robert Lukoťka, PhD.
Katedra: FMFI.KI - Katedra informatiky
Vedúci katedry: prof. RNDr. Martin Škoviera, PhD.

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prof. RNDr. Rastislav Kráľovič, PhD.
garant študijného programu

....

študent vedúci práce

71793490

Comenius University Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname:	Bc. Bohdan Jóža
Study programme:	Computer Science (Single degree study, master II. deg., full time form)
Field of Study:	Computer Science
Type of Thesis:	Diploma Thesis
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Title: Edge colourings of signed cubic graphs

Annotation: Signed graphs are graphs, whose edges have assigned values from {-1, 1}. Switching at a vertex v of a graph is done by multiplying the values of all edges incident with v by -1. Graphs that can be obtained from each other by switching are called equivalent. There are plenty of papers studying generalization of standard graph invariants to signed graphs. One of these invariants is graph colouring. The thesis should focus on edge colourings of signed cubic graphs. The study of edge colourings of signed graphs was started by Behr [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. The aim of the thesis is to initiate the systematic study of 3-edge-colourability of signed cubic graphs.

Supervisor:	doc. RNDr. Robert Lukoťka, PhD.
Department:	FMFI.KI - Department of Computer Science
Head of department:	prof. RNDr. Martin Škoviera, PhD.
Assigned:	16.11.2022
Approved:	28.04.2023

prof. RNDr. Rastislav Kráľovič, PhD.
Guarantor of Study Programme

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Student Supervisor

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Abstrakt

Signované grafy definoval v roku 1953 Frank Harary ako model na štúdium sociálnych sietí. Problém farbenia signovaných grafov nebol preskúmaný do roku 1982, kedy Thomas Zaslavsky zverejnil prvé výsledky. Prepínanie vrcholov rozdeľuje signované grafy do tried ekvivalencie. V tejto práci prezentujeme algoritmus na generovanie neekvivalentných signovaných grafov v kontexte prepínania a izomorfizmu a algoritmus na určenie chromatického čísla signovaného grafu. Kombináciou týchto algoritmov generujeme malé 3-hranovo-nezafarniteľné signované kubické grafy a uľahčujeme budúci výskum tejto problematiky.

Kľúčové slová: signovaný graf, kubický graf, signované hranové farbenie, snark, prepínanie vrcholov, prepínací izomorfizmus, generovanie grafov, neekvivalentné grafy

Abstract

Signed graphs were defined by Frank Harary in year 1953 as a model for studying social networks. The problem of signed colouring, however, was not explored until 1982 when Thomas Zaslavsky published his first results. Vertex switching creates equivalence classes on signed graphs. In this thesis we present an algorithm for generating non-equivalent signed graphs under switching isomorphism and an algorithm to determine the chromatic index of a signed graph. Combining these algorithms allows us to generate small non-3-edge-colourable cubic signed graphs and make the systematic research in this topic easier.

Keywords: signed graph, cubic graph, signed edge coloring, snark, vertex switching, switching isomorphism, generating graphs, non-equivalent graphs

Contents

List of Figures

Introduction

The problem of graph colouring has been known for a long time and is still relentlessly being studied today. Even in a problem this wide there are still areas to explore and improve. Edge colouring in combination with the concept of signed graphs remains more or less unexplored.

First discovered by the mathematician Frank Harary in 1953 for studying a question in social psychology, signed graphs remained idle until 1982 when Thomas Zaslavsky published multiple seminary papers on the topic. Many fundamental results in the study of nowhere-zero flows and the chromatic number of signed graphs have been established only recently and research the problem of signed edge colouring was started by Richard Behr in 2020. We expand on Behr's work to automate the process of finding signed edge colorings.

Signed graphs have been proven to be generalizations of simple graphs in many ways. Exponentially many signed graphs can be constructed given a simple underlying graph. We will show that this amount can be drastically reduced due to switching equivalence and isomorphism. Additionally, removing equivalence results in cleaner data for subsequent analysis.

The main result of our work is a database of signed snarks up to eighteen vertices obtained by processing a database of non-isomorphic cubic graphs. For each underlying graph we generate all non-switching-isomorphic signatures, which is an interesting problem in and of itself. This is achieved by transforming the signed graphs into unsigned graphs while preserving switching equivalence and using existing tools based on the automorphism group to filter them for isomorphisms. Then we transform each signature into a 3SAT instance solvable if and only if the signature is 3-edge-colorable. In addition to producing data for bulk analysis it is possible to process specific larger graphs.

In the first chapter we define key concepts in the signed graph theory and describe the current state of research. We also mention the relationship to unsigned graphs and how it affects the colour set and its requirements. In the second chapter we describe the programs that generate non-switching-isomorphic signed graphs and signed snarks. In the third and final chapter we present some results achieved by using these tools and suggest options for future research that can be pursued.

Chapter 1

Preliminary Graph Theory

First, let's define some basic concepts of graph theory, starting with the graph itself.

1.1 Graphs

A graph is an algebraic structure most commonly used to describe relationships between objects. There are many definitions of a graph, the most abstract being simply a set V and a relation R on V denoting which elements of V are connected. Graphs in general are *directed*; if R is symmetric, the graph is *undirected*. For the purposes of this work we will be using a geometric definition and generally undirected graphs. An undirected graph is an ordered pair $G = (V, E)$, where V is a set of *vertices* and E is a set of *edges*, determining the incidence of vertices.

$$(\forall e \in E)\ e = uv = vu; u, v \in V$$

A *path* in a graph G from v to w; $v, w \in V$ is a sequence of vertices $(u_1, u_2, \ldots, u_n)$; $\{u_i \mid 1 \leq i \leq n\} \subseteq V$ such that $u_1 = v$, $u_n = w$ and $\{(u_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \subseteq E$. A graph is *connected* if there exists a path between every pair of vertices $v, w \in V;\ v \neq w$. A *degree* $\Delta(v)$ of a vertex v denotes how many edges are incident to this vertex. The highest degree of any vertex in G is denoted as $\Delta(G)$. A graph is *k-regular* if the degree of each vertex is exactly k. A *cubic graph* is a 3-regular graph.

As an example, the smallest cubic graph is a complete graph with 4 vertices K_4. (In a complete graph each vertex is incident with each other vertex.).

In general statements about graphs in later chapters we are referring to undirected cubic graphs.

We also need to define the set of *half-edges* or vertex-edge incidences of a graph (or signed graph)

$$\Sigma_G = \bigcup_{e=vw \in E_G} \{(e, v), (e, w)\}$$

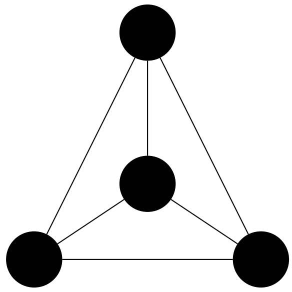

Figure 1.1: Smallest cubic graph.

because in case of signed edge colouring we will be colouring half-edges instead.

1.1.1 Colouring

When simple binary relationships between objects are not enough, weighted graphs and colouring offer a wider range of applications. Assigning colours to vertices or edges of graphs makes classifications of these objects more robust. A vertex colouring $\phi : V_G \to C$ of a graph G is a mapping from the vertex set of G to a set of colours C. An edge colouring $\chi : E_G \to C$ of a graph G is a mapping from the edge set of G to a set of colours C. A *proper vertex colouring* of G is a vertex colouring such that no two incident vertices share a colour. A *proper edge colouring* is an edge colouring such that no two incident edges have the same colour. A proper colouring using k colours is called a *k-colouring*.

As colouring in general is not very interesting, we will be considering only proper colourings henceforth. It is also important to define the set of "colours", especially when colouring signed graphs. It is most practical to use a subset of integers $C \subseteq \mathbb{Z}$ because it makes definitions and proofs clear. Additionally, it is important that a k-colouring uses a set of k colours.

The typical colouring problem is to find the minimum number of colours required for a proper colouring. This number is called the *chromatic number* for vertex colourings and *chromatic index* for edge colourings. Determining the chromatic number and index is useful in other areas of graph theory as well.

Theorem 1.1. *A graph is bipartite if and only if it has a proper vertex 2-colouring.*

For unsigned graphs these numbers are known.

Theorem 1.2 (Brooks [1])**.** *The chromatic number of a connected graph G is $\Delta(G)$ for all graphs except complete graphs and cycles of odd length, where the chromatic number is $\Delta(G) + 1$.*

Theorem 1.3 (Vizing)**.** *The chromatic index of a simple graph G is $\Delta(G)$ or $\Delta(G) + 1$.*

In other words, we can always colour the edges of a graph using at most $\Delta(G) + 1$ colours. The lower bound $\Delta(G)$ is trivial; we need exactly $\Delta(G)$ colours at the highest degree vertex in G to construct a proper colouring. The Vizing theorem proves the upper bound using Kempe chains.

1.2 Signed graphs

A *signed graph* $\Gamma = (G, \sigma)$ consists of an *underlying graph* G and a *sign function* $\sigma : E(G) \to \{+, -\}$ that assigns a sign ($+$ or $-$) to each edge of G. $\Gamma^+ = (V_\Gamma, E_{\Gamma^+})$ and $\Gamma^- = (V_\Gamma, E_{\Gamma^-})$ denote subgraphs of Γ with positive and negative edges respectively, $E_{\Gamma^+} = \sigma^{-1}(+1)$ and $E_{\Gamma^-} = \sigma^{-1}(-1)$. $\mathcal{S}(G)$ denotes the set of all signed graphs with the underlying graph G.

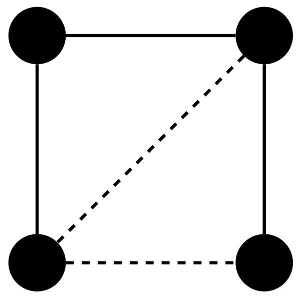

Figure 1.2: Example of a signed graph. Dashed edges are negative, solid edges are positive.

A fundamental concept in the theory of signed graphs is *vertex switching*. Switching vertex v of a signed graph reverses the sign of each edge incident with v. More generally, switching a signed subgraph reverses the sign of each edge between a vertex subset and its complement. Let's define switching by a *switching function*

$$\theta : V_\Gamma \to \{+1, -1\}$$

where vertices mapped to -1 are being switched. The new graph will have an altered sign function,

$$\Gamma^\theta = (G, \sigma^\theta); \quad \sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v)$$

If a signed graph can be obtained from another signed graph by switching, they are considered *switching equivalent*. Switching equivalence is an equivalence relation and thus forms equivalence classes on $\mathcal{S}(G)$. It makes sense to study properties of signed graphs that behave consistently under switching. An example of such a property is the signs of cycles. Switching doesn't change the sign of cycles because if a switched vertex is a part of a cycle, it will reverse the sign of two edges on that cycle leaving the sign product the same. In fact, the sign (or balance, defined below) of cycles is an alternative definition of a switching equivalence class, each combination of balance among a set of cycles that form a cycle space basis yields the same equivalence classes as the method used in this thesis and defined later. It is also important to point out that switching a set of vertices is equivalent to a sequence of one-vertex switches.

Directly related to vertex switching is the notion of *balance*. A cycle is *balanced* when the product of signs of its edges is positive and *unbalanced* otherwise. A signed graph Γ is balanced when each cycle in Γ is balanced. It is *antibalanced* if each cycle is unbalanced. Note that balanced doesn't imply *all-positive*, there are balanced graphs with negative edges (see section 1.2).

Theorem 1.4 (Harary [2])**.** *A signed graph is balanced if and only if*

1. *for every pair of vertices, all paths between these vertices have the same sign*
2. *the vertices can be divided into two subsets (possibly empty) such that each edge with both ends in the same subset is positive and each edge with ends in different subsets is negative*

This is a generalization of the earlier mentioned bipartite graph theorem (Theorem 1.1).

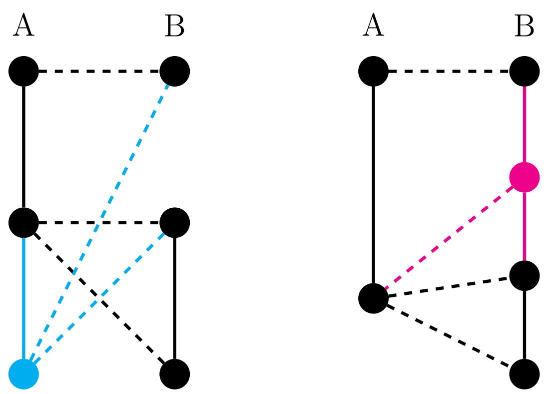

Figure 1.3: Switching a balanced graph.

An all-positive graph Γ is trivially balanced, all paths are positive and the division of vertices will be subsets V_Γ and $\emptyset$. Only graphs switching equivalent to an all-positive graph are balanced. Consider how the conditions in theorem 1.4 behave under switching.

Let's say we switch vertex $v \in V_\Gamma$. The sign of each path that ends at v from any other vertex will flip, because exactly one edge on that path changed signs. So if all paths between v and any other vertex, say u, had the same sign before switching, now they will have the opposite but still the same sign.

Suppose we are able to divide V_Γ into two subsets A and B as per the second condition in theorem 1.4 and without loss of generality let $v \in A$. So all edges $vx;\ x \in A$ are positive and all edges $vy;\ y \in B$ are negative. After switching v we will construct a new division of V_Γ, subsets $A' = A \setminus \{v\}$ and $B' = B \cup \{v\}$. A' and B' is obviously a correct division of V_Γ and the second condition will still hold, since all edges incident with v flipped signs and at the same time changes whether they end in the same sunbset as v or not.

Connected to vertex switching is the notion of *balance*. The sign of a path is the product of the signs of its edges. A path is positive if and only if there is an even number of negative edges on it. A cycle is balanced if it is positive and a signed graph is balanced if each cycle in it is balanced[2].

1.2.1 Colouring

The research of signed graph colouring was initiated by Zaslavsky[3] in the early 1980s and published in multiple seminal papers [4, 5, 6]. Before defining signed vertex and edge colouring it is necessary to define the set of colours.

In the context of signed graphs and vertex switching we are looking for a set of signed integers with the idea of switching a color reversing its sign, same operation as with the signs of edges. Proper colourings of signed graphs will then be consistent under vertex switching because "reversing the sign" is a bijection on $\mathbb{Z}$. Zaslavsky [5] defined a k-colouring based on a signed colour set $C_k = \{-k, -(k-1), \dots, -1, 0, 1, \dots, (k-1), k\}$ and called colourings zero-free if the colour 0 was not used. He then studied the properties of *chromatic polynomials* related to signed colourings, the number of colourings for a signed graph. (Balanced chromatic polynomials in case of zero-free colourings.)

However, this definition is not a natural extension of the original colour set of integers, because a k-colouring essentially uses $2k$ or $2k+1$ signed colours. This is a desirable property for the colour set, since signed graphs themselves are an extension of unsigned graphs the signed color set should behave in a similar way. A balanced signed graph is essentially equivalent to the unsigned underlying graph, so its chromatic number and index for instance should also match. Máčajová et al. offer another definition: a k-colouring uses the colour set $C_k = \{\pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k$ and $C_k = \{0, \pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k+1$. Behr [7] also adopts this definition.

A vertex colouring $\phi : V_\Gamma \to C_k$ of a signed graph Γ is, similarly to unsigned graphs, a mapping from the vertex set of Γ to a set of signed colours C_k.

Edge colouring, however, needs to be defined differently to incorporate the additional information given by the edge signs. The definition of a k-coloring $\gamma : \Sigma_\Gamma \to C_k$ of a signed graph Γ inspired by Behr [7] is a mapping from the set of half-edges of Γ to a set of signed colours C_k such that

$$(\forall e = uv \in E_\Gamma) \ \ \gamma(e,u) = \sigma(e)\gamma(e,v)$$

In other words, half-edges that form a positive edge must have the same color and half-edges that form a negative edge must have opposite colors. The only difference to Behr's version is the usage of $\sigma(e)$ instead of $-\sigma(e)$, which is really only a matter of taste. In our version positive edges behave like unsigned instead of negative which seemed more natural. Between these definitions different signatures are colorable, but they are equivalent in the sense that there is a bijection between graphs colorable under our definition and Behr's definition. If signed graph $\Gamma = (G, \sigma)$ is edge colorable under our definition, then $\Gamma' = (G, -\sigma)$ with the sign of each edge reversed is colorable under Behr's.

An edge coloring of a signed graph is *proper* if, just as with unsigned graphs, each color is present at each vertex at most once. In case of a vertex coloring all neighbors of each vertex must have different colors. We will, again, consider only proper colorings from now on.

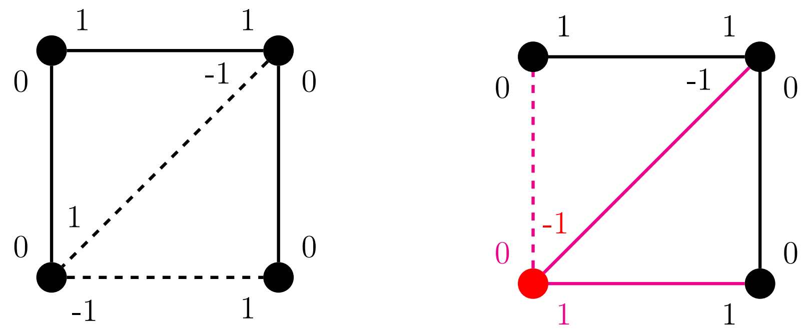

Figure 1.4: Example of a proper signed edge coloring on the left. We obtain the graph on the right by switching the bottom left vertex and the coloring remains correct and proper.

1.3 Motivation

Behr proved a signed version of the Vizing's theorem.

Theorem 1.5 (Signed Vizing's theorem [7])**.** *The chromatic index of a simple signed graph* Γ *is* $\Delta(\Gamma)$ *or* $\Delta(\Gamma) + 1$.

This opens the door for research into edge 3-coloring and cubic signed snarks.

"In the study of various important and difficult problems in graph theory (such as the cycle double cover conjecture and the 5-flow conjecture), one encounters an interesting but somewhat mysterious variety of graphs called snarks. In spite of their simple definition [...] and over a century long investigation, their properties and structure are largely unknown." — Chladný, Škoviera [8]

The exact definition of a snark may vary from paper to paper but a snark is essentially a cubic graph with chromatic index four (its edges can't be coloured with three colours). Every cubic graph with a loop or a bridge is trivially a snark, triangles (cycles of length three) can be contracted into a single vertex and cycles of length four can also be simplified. Therefore many definitions forbid these properties by considering true snarks only graphs with girth (length of the shortest cycle) at least five. Even stronger, sometimes only cyclically 4-edge-connected graphs are considered (there is no subset of three or fewer edges such that their removal will disconnect the graph into two subgraphs each containing a cycle). One of the alternative formulations of the four colour theorem is that each snark is non-planar.

Signed snarks, however, are not subject to the same trivial cases or reductions. For instance, signed graphs with loops or bridges might have colorable signatures and only balanced triangles can be contracted into one vertex. That is why we will be considering any connected cubic signed graph with chromatic index 4 a *signed snark*.

1.4 Previous research

Máčajová et al. expanded upon Zaslavsky's research by studying the properties of the chromatic number of signed graphs, ultimately proving a signed version of the famous Brooks' [1] theorem.

Theorem 1.6 (Signed Brooks' Theorem)**.** *Let* Γ *be a simple connected signed graph. If* Γ *is not a balanced complete graph, a balanced odd circuit or an unbalanced even circuit, then* $\chi(\Gamma) \leq \Delta(\Gamma)$*.*

In addition to the Signed Vizing's theorem Behr [7] proved that there is a bijection between proper edge colorings of Γ and proper vertex colorings of the negative of the line graph of Γ.

Chapter 2

Generating signed snarks

Since the structure of snarks is generally unknown, brute force is still the best approach. Considering one underlying graph, the number of signed graphs is simply too big for an efficient analysis. Filtering them up to switching-isomorphism reduces these numbers to manageable amounts and ensures clean and usable data. Bagheri, Moghaddamfar, Ramezani [9] establish a method of determining the non-switching-isomorphic signed graphs based on the action of its automorphism group. Our aim is to automate this process using a different idea and making analysis of small cubic signed graphs possible.

2.1 Switching equivalence

Recall that two signed graphs $\Gamma \sim \Sigma$ are *switching-equivalent* if a switching function θ exists such that $\Gamma^\theta = \Sigma$. Given an underlying graph G there are $2^{|E_G|}$ possible signed graphs constructed from G. However, provided that G is connected, only $2^{|E_G|-|V_G|+1}$ of them are mutually non-switching-equivalent.

Theorem 2.1. *Let G be a simple unsigned connected graph with n vertices and m edges. There are 2^{m-n+1} mutually non-switching-equivalent graphs based on G.*

Proof. Bagheri, Moghaddamfar, Ramezani[9] prove this theorem in a different way but we present a version that is actually used in the implementation. The idea is to use a spanning tree S of graph G and show that each switching equivalence class of G has exactly one element that is all-positive on S. Since S contains $n-1$ edges, there are $2^{m-(n-1)}$ different signed graphs all-positive on S. Suppose we have a signed graph Γ all-positive on S and we switch some vertices. If we switch no vertices or all vertices, the graph stays the same, so we will have a non-empty set of switched vertices A and a non-empty set of unswitched vertices B. At least one edge of S must have one end in A and the other end in B, otherwise G would not be connected or S would not be a spanning tree. After this switching all edges with both ends in either A or B will retain the same sign (not reversed or reversed twice) and edges with one end in A and on end in B will have its sign reversed. Therefore every possible switching from Γ that would change the signature will result in a graph that is not all-positive on S. □

We use this approach to reduce the number of signed graphs that need to be filtered for isomorphisms.

2.2 Switching isomorphism

Two signed graphs Γ_1 and Γ_2 are *isomorphic* if a bijection $\phi : V_{\Gamma_1} \to V_{\Gamma_2}$ exists that preserves vertex adjacency and edge signs.

$$(\forall u, v \in V_\Gamma) \;\; uv \in E_{\Gamma_1} \iff \phi(u)\phi(v) \in E_{\Gamma_2}$$

$$(\forall e = uv \in E_\Gamma) \;\; \sigma_{\Gamma_1}(uv) = \sigma_{\Gamma_2}(\phi(u)\phi(v))$$

Two graphs are *switching isomorphic* if one graph is isomorphic to a switching equivalent of the other. This is the natural definition but in the generating algorithm and its proof of correctness we will be referring to an equivalent definition using the balance of cycles.

Lemma 2.1. *Two connected signed graphs $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ with the same underlying graph are switching equivalent if and only if all cycles in G have the same balance in Γ_1 and Γ_2.*

Proof. We already know that switching doesn't change the balance of cycles so switching equivalence trivially implies cycle balance consistency. Now suppose we have two graphs with the same underlying graph and the same balance of each cycle. Let S again be a spanning tree of G. Based on theorem 2.1 each equivalence class has exactly one signature that is all-positive on S so we will switch both graphs to their respective signatures that are all-positive on S. Each cycle has exactly one edge outside of S and since S is all-positive the balance of each cycle is determined by the sign of this one edge. In other words, there is a one-to-one correspondence between the balance of cycles and signature of edges outside of S. Each cycle has the same balance in both graphs, therefore each edge outside of S has the same sign in both graphs. Each edge in S also has the same signature in both graphs of course so the signatures are identical and Γ_1 and Γ_2 are switching equivalent. □

Theorem 2.2. *Let $\mathcal{C}(\Gamma)$ denote the set of all cycles in Γ. Two signed graphs $\Gamma_1 = (G_1, \sigma_1)$ and $\Gamma_2 = (G_2, \sigma_2)$ are switching isomorphic if a bijection $\phi : V_{\Gamma_1} \to V_{\Gamma_2}$ exists such that*

$$(\forall u, v \in V_{\Gamma_1}) \;\; uv \in E_{\Gamma_1} \iff \phi(u)\phi(v) \in E_{\Gamma_2}$$

$$(\forall c = v_0 v_1 v_2 \dots v_k \in \mathcal{C}(\Gamma_1); v_0 = v_k)$$

$$\sigma_1(v_0v_1)\sigma_1(v_1v_2)\dots\sigma_1(v_{k-1}v_k) = \sigma_2(\phi(v_0)\phi(v_1))\sigma_2(\phi(v_1)\phi(v_2))\dots\sigma_2(\phi(v_{k-1})\phi(v_k))$$

In other words, instead of preserving edge signs the bijection preserves cycle balance.

Proof. Follows mostly from lemma 2.1 with one exception. The definition by itself doesn't preserve the signs of edges that are not part of any cycle, i.e. *bridges* (edges whose removal disconnects the graph). However, we can always reverse the sign of a bridge "for free" by switching all vertices on one side of the bridge. The only edge that changes signs is the actual bridge. Thus in the context of switching isomorphism the sign of bridges is irrelevant. □

The *cycle space* $\mathcal{E}(\Gamma)$of a graph is the collection of its even-degree spanning subgraphs. It can be described as a vector space over the two-element Galois field: the elements are even-degree subgraphs as vectors over $\mathbb{Z}_2$. The additive operation is the symmetric difference from the subgraph perspective (given two even-degree subgraphs A and B, $A \oplus B$ contains edges that are in either A or B but not both) and simple sum from $\mathbb{Z}_2$ perspective. Scalar multiplication is trivially defined, multiplication by 0 yields a graph with no edges (zero in the cycle space) and multiplication by 1 is identity. Cycles are trivial elements of the cycle space because each even-degree subgraph is the sum of some cycles in the context of this vector space. Consequently, there must be a *cycle basis*, a set of linearly independent cycles that generates the entire cycle space.

2.2.1 Transformation

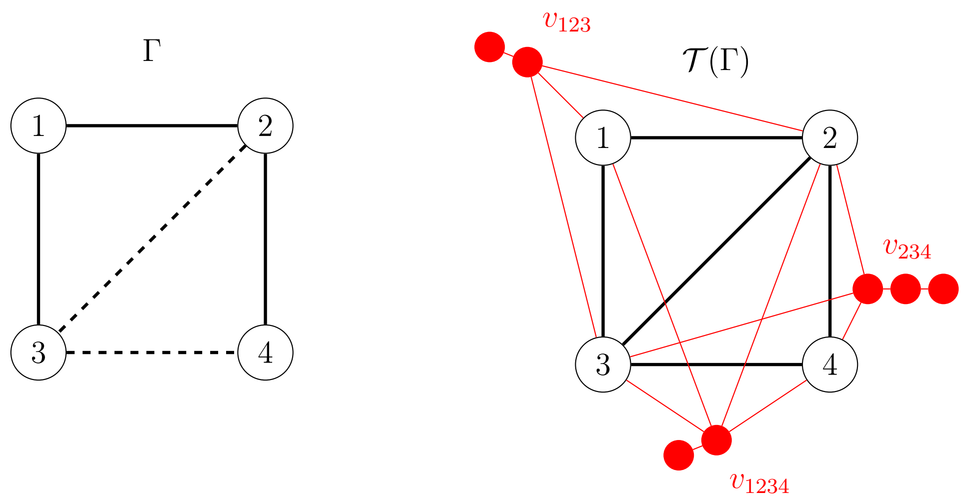

Figure 2.1: Transformation to an unsigned graph preserving cycle balance.

The following transformation of signed graphs to unsigned graphs preserves switching isomorphism and allows us to use existing tools to determine whether two signed graphs are switching isomorphic.

Theorem 2.3. *From a connected signed graph Γ we will construct an unsigned graph $\mathcal{T}(\Gamma)$ in the following way. Starting with the underlying graph, for each cycle c in $\mathcal{C}(\Gamma)$ we add one cycle vertex v_c and connect it to each vertex of the original cycle. We represent their balance with "tails", i.e. one or two new vertices connected to the cycle vertex, a_c and possibly b_c. Cycle vertices for balanced cycles will have a tail of length one and for unbalanced cycles a tail of length two.*

Signed graphs Γ_1 and Γ_2 with minimal degree three or more are switching isomorphic if and only if $\mathcal{T}(\Gamma_1)$ and $\mathcal{T}(\Gamma_2)$ are isomorphic.

Proof. The tails are either a vertex of degree one or an additional vertex of degree two. Since degrees of vertices are preserved under isomorphism and both original vertices and cycle vertices have degree at least three, balanced tails will be projected only onto balanced tails and unbalanced tails onto unbalanced tails. By extension cycle vertices will be projected only onto other cycle vertices, because each tail is connected to exactly one cycle vertex. Consequently the original vertices will also be projected only onto each other.

Suppose there is a switching isomorphism ϕ between Γ_1 and Γ_2. We will construct an isomorphism ψ between $\mathcal{T}(\Gamma_1)$ and $\mathcal{T}(\Gamma_2)$. For each $v \in V_{\Gamma_1}$ there is a vertex v in $\mathcal{T}(\Gamma_1)$ and we put $\psi(v) = \phi(v)$. For each cycle $c = v_0v_1v_2 \dots v_k \in \mathcal{C}(\Gamma_1);\ v_0 = v_k;\ v_i \in V_{\Gamma_1}$ there will again be a cycle $c^{\phi} = \phi(v_0)\phi(v_1)\phi(v_2)\dots\phi(v_k) \in \mathcal{C}(\Gamma_2);\ \phi(v_0) = \phi(v_k);\ \phi(v_i) \in V_{\Gamma_2}$ in Γ_2 with the same balance by definition of switching isomorphism. Consequently, there will again be cycle vertices $v_c \in \mathcal{T}(\Gamma_1)$ and $v_{c^{\phi}} \in \mathcal{T}(\Gamma_2)$ with tails of the same length because of the same balance of c and c^{ϕ}, so we can put $\psi(v_c) = v_{c^{\phi}}$, $\psi(a_c) = a_{c^{\phi}}$ and possibly $\psi(b_c) = b_{c^{\phi}}$ (recall that a_c and b_c are tail vertices of v_c). Based on the definition of $\mathcal{T}$ here are no other vertices in $\mathcal{T}(\Gamma_1)$ or $\mathcal{T}(\Gamma_2)$ and so ψ is an isomorphism between $\mathcal{T}(\Gamma_1)$ and $\mathcal{T}(\Gamma_2)$.

Now suppose that $\mathcal{T}(\Gamma_1)$ and $\mathcal{T}(\Gamma_2)$ are isomorphic and let ψ be their bijection. As shown above, ψ reduced to V_{Γ_1}, let's call it ϕ^*, will be an isomorphism between Γ_1 and Γ_2 that doesn't preserve edge signs. So for each cycle $c = v_0v_1v_2 \dots v_k \in \mathcal{C}(\Gamma_1);\ v_0 = v_k;\ v_i \in V_{\Gamma_1}$ there will be a cycle $c^{\phi^*} = \phi^*(v_0)\phi^*(v_1)\phi^*(v_2)\dots\phi^*(v_k) \in \mathcal{C}(\Gamma_2);\ \phi^*(v_0) = \phi^*(v_k);\ \phi^*(v_i) \in V_{\Gamma_2}$ (recall that original vertices can be projected only onto original vertices). Based on the definition of $\mathcal{T}$, $\mathcal{T}(\Gamma_1)$ will have a cycle vertex v_c connected to each vertex in c and $\mathcal{T}(\Gamma_2)$ will have a cycle vertex $v_{c^{\phi^*}}$ connected to each vertex in c^{ϕ^*}. It must be true that $\psi(v_c) = v_{c^{\phi^*}}$, otherwise there is another vertex $\psi(v_c)$ in $\mathcal{T}(\Gamma_2)$ that is connected to all vertices of c^{ϕ^*}, which would mean that there are two cycles with the same vertices in Γ_2, which is a contradiction. The projection of the tail of v_c

must be the tail of $v_{c^{\phi^*}}$ for the same reason, by definition of $\mathcal{T}()$ a cycle vertex only has one tail. Finally, due to the nature of the transformation c and c^{ϕ^*} must have the same balance. □

Lemma 2.2. *Let $\mathcal{C}^*(\Gamma)$ be the set of all cycles in Γ with length up to some n such that this set generates the cycle space $\mathcal{E}(\Gamma)$. When using this set of cycles instead of all cycles in $\mathcal{T}(\Gamma)$, theorem 2.3 still holds.*

We are using a generating subset of $\mathcal{E}(\Gamma)$ (which is also the generating subset of $\mathcal{C}(\Gamma)$) to be more efficient. If a cycle c is the sum of some cycles in $\mathcal{C}^*(\Gamma)$, their balance directly determines the balance of c. So any isomorphism that preserves the balance of $\mathcal{C}^*(\Gamma)$ will preserve the balance of $\mathcal{C}(\Gamma)$ as well.

It is, however, necessary to use all cycles of length up to n for $\mathcal{T}$ because of the nature of isomorphism. In theory we would only need an isomorphism that preserves the balance of any cycle basis in Γ_1 because if Γ_1 and Γ_2 are isomorphic, $\mathcal{E}(\Gamma_1)$ and $\mathcal{E}(\Gamma_2)$ are also isomorphic and the projection of any basis in $\mathcal{E}(\Gamma_1)$ will be a basis in $\mathcal{E}(\Gamma_2)$. The issue with this approach is that it doesn't guarantee that cycles of the same length will be projected onto each other, which is required by $\mathcal{T}$.

2.3 Coloring

To determine the chromatic index of a cubic graph is an NP-complete problem. By extension, determining the chromatic index of a signed cubic graph is also NP-complete, because of the trivial reduction from the signed chromatic index problem to the unsigned chromatic index problem. Instead of designing an algorithm we decided to implement a conversion from the chromatic index problem to 3SAT and using a highly optimized SAT solver in the hope for better effectiveness.

2.3.1 Conversion to 3SAT

For any cubic signed graph Γ we will construct a 3SAT formula $F(\Gamma)$ that is satisfiable if and only if the graph is 3-colorable. There will be three literals for each half-edge $ev \in \Sigma_\Gamma$, one for each colour from $C_3 = \{-1, 0, 1\}$. Let's call them x_{ev}^{-1}, x_{ev}^{0} and x_{ev}^{1}. In any evaluation of these literals that satisfy F exactly one of them will be true denoting the colour of the half-edge. This will be guaranteed using three constituent formulas. Let $\Gamma = (G, \sigma)$

$$F_1(\Gamma) = \bigwedge_{e=vw\in E_\Gamma} (x_{ev}^{-1} \vee x_{ev}^{0} \vee x_{ev}^{1}) \wedge (x_{ew}^{-1} \vee x_{ew}^{0} \vee x_{ew}^{1})$$

The first formula ensures that each half-edge is coloured and is the only one containing clauses of length 3. The next formula will enforce the correctness of the colouring, restricting the colours of half edges that form one complete edge. Illegal signatures for each edge are negated using DeMorgan rules, resulting in a convenient CNF form. No edge can be coloured 0 on one side and 1 or -1 on the other

$$F_{abs}(e = vw) = \neg(x^0_{ev} \wedge x^1_{ew}) \wedge \neg(x^1_{ev} \wedge x^0_{ew}) \wedge \neg(x^0_{ev} \wedge x^{-1}_{ew}) \wedge \neg(x^{-1}_{ev} \wedge x^0_{ew})$$

$$F_{abs}(e = vw) = (\neg x^0_{ev} \vee \neg x^1_{ew}) \wedge (\neg x^1_{ev} \vee \neg x^0_{ew}) \wedge (\neg x^0_{ev} \vee \neg x^{-1}_{ew}) \wedge (\neg x^{-1}_{ev} \vee \neg x^0_{ew})$$

and the colours must be the same if the edge is positive and opposite if the edge is negative, which is equivalen to the following.

$$F_{sign}(e = vw) = (\neg x^{-1}_{ev} \vee \neg x^{\sigma(e,w)}_{ew}) \wedge (\neg x^1_{ev} \vee \neg x^{-\sigma(e,w)}_{ew})$$

$$F_2(\Gamma) = \bigwedge_{e=vw\in E_\Gamma} F_{abs}(e) \wedge F_{sign}(e)$$

Lastly we need to ensure that the colouring is proper. Let $N(v) = \{e \mid (e, v) \in \Sigma_\Gamma\}$ be the set of half-edges incident to v.

$$F_3(\Gamma) = \bigwedge_{\substack{v\in V_\Gamma \\ e_1,e_2\in N(v);\ e_1\neq e_2}} (\neg x^{-1}_{e_1v} \vee \neg x^{-1}_{e_2v}) \wedge (\neg x^0_{e_1v} \vee \neg x^0_{e_2v}) \wedge (\neg x^1_{e_1v} \vee \neg x^1_{e_2v})$$

Each pair of half-edges with a common vertex must have different colours.

Theorem 2.4. *3SAT formula $F(\Gamma) = F_1(\Gamma) \wedge F_2(\Gamma) \wedge F_3(\Gamma)$ constructed in the way described above is satisfiable if and only if Γ is 3-colourable.*

Proof. Follows from the construction of F encapsulating all properties of a proper signed 3-colouring. □

Note that we don't need to explicitly ensure that for each half-edge exactly one literal is true, in case where multiple literals for different colors are true for the same edge we could choose any color from among them and the coloring would be proper.

2.4 Implementation

The programming language of choice was C++ due to its speed and convenience. Nauty [10] is a program for computing the automorphism group of graphs and most importantly the *canonical labeling*. Kissat [11], the winner of the SAT competition 2024, is a simple and fast SAT solver with easy integration.

2.4.1 Data structures

TODO

Chapter 3

Results

We found all signed snarks up to 18 vertices.

TODO

N	G	non-equivalent signatures per G	signed G	signed snarks
4	1	8	8	0
6	2	16	32	0
8	5	32	160	1
10	19	64	1216	48
12	85	128	10 880	227
14	509	256	130 304	2768
16	4060	512	2 078 720	31 869
18	41 301	1024	42 292 224	437 381

Figure 3.1: Basic signed graph data. Here signed snarks were not yet filtered for isomorphisms.

The smallest signed snark is smaller than the Petersen graph (smallest snark), it is the projection of a cube.

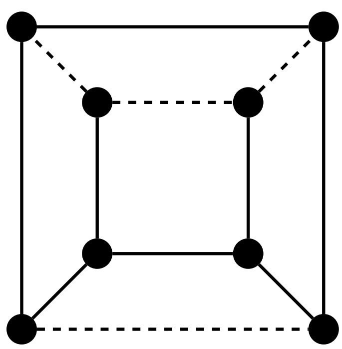

Figure 3.2: Smallest snark

Similarly to regular snarks, there are trivial properties of signed graphs that don't allow the possibility of a 3-edge-colouring. The following unsigned graph is the smallest graph that doesn't have a 3-edge-colourable signature.

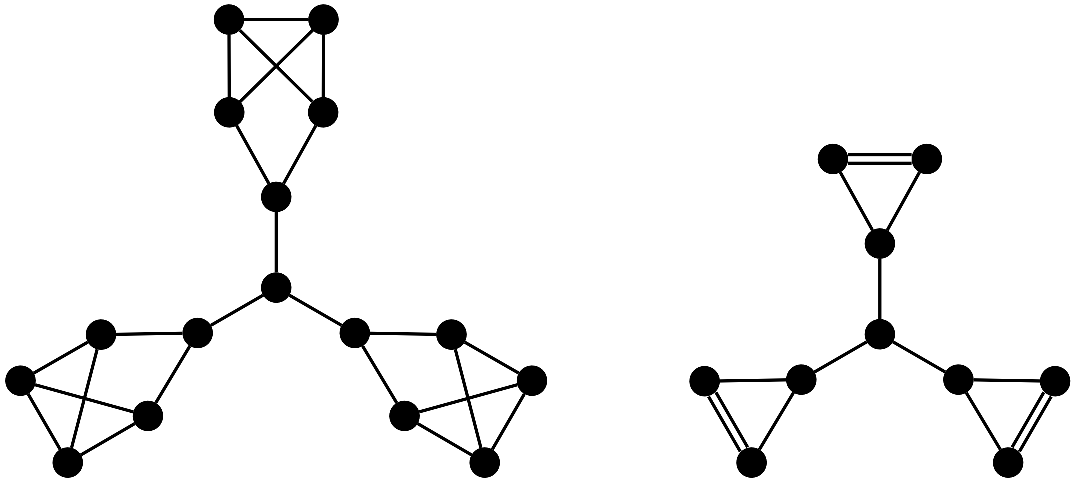

Figure 3.3: Smallest simple graph without a 3-edge-colourable signature and a simplified version allowing duplicate edges.

Theorem 3.1. *An unsigned graph G has a signature that admits a 3-edge-colouring if and only if it has a 1-factor (perfect matching).*

Proof. If there is a signature and a 3-edge-colouring on it, the edges coloured 0 form by the definition of a proper edge colouring a perfect matching. Now let $M \subseteq E(G)$ be a 1-factor. Let's assign the colour 0 to these edges again and remove them from G. After removing a 1-factor from a cubic graph we obtain a 2-factor, a set of disjunct cycles (if two cycles would have a common vertex, its degree in the original graph would have to be at least 4). According to Theorem 1.6 for the cycles to be colourable, we assign any balanced signature to even cycles and any unbalanced signature to odd cycles. All cycles from this 2-factor will now be 2-edge-colourable with colours 1 and -1 and combined with the 0-coloured 1-factor we obtain a 3-edge-colourable signed cubic graph. $\square$

The graph in Chapter 3 has no 1-factor. (The middle vertex has to be connected to one of the three triangles and the other two triangles will not have a matching.)

3.0.1 Future research

There are multiple directions we intend to take our research into this topic in the future. The analysis of small signed snarks can be taken further by inspecting different classes of graphs and searching for similarities. By optimizing the filtering algorithm, bigger graphs can be included.

Conclusion

TODO

In this thesis we outlined an algorithm to filter signed graphs that are not 3-edge-colourable. We analysed the first results and showed that for any 3-edge-colourable signature a cubic graph has to admit a perfect matching.

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Appendix A

Source code

The latest version of the source code can be found on `https://github.com/Bohdanator/signed-cubic-graphs`