Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

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Faculty of Mathematics, Physics and Informatics

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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	doc. RNDr. Robert Lukoťka, PhD.

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Názov: Edge colourings of signed cubic graphs
Hranové farbenia signovaných kubických grafov

Anotácia: Signované grafy sú grafy, ktorých hrany sú ohodnotené prvkami z {-1, 1}. Prepínanie signovaného grafu v jeho vrchole v je vynásobenie ohodnotenia incidentných hrán hodnotou -1. Grafy, ktoré možno získať sériou operácií prepínania sú ekvivalentné. Existuje veľa článkov, ktoré skúmajú rozšírenie štandardných grafových invariantov na signované grafy. Jednou zo skúmaných tém je farbenie signovaných grafov. Predmetom práce budú hranové farbenia signovaných kubických grafov. Hranové farbenia signovaných grafov začal skúmať Behr v článku [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. Cieľom práce je začať systematické štúdium hranovej 3-zafarbiteľnosti signovaných grafov.

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THESIS ASSIGNMENT

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Title: Edge colourings of signed cubic graphs

Annotation: Signed graphs are graphs, whose edges have assigned values from {-1, 1}. Switching at a vertex v of a graph is done by multiplying the values of all edges incident with v by -1. Graphs that can be obtained from each other by switching are called equivalent. There are plenty of papers studying generalization of standard graph invariants to signed graphs. One of these invariants is graph colouring. The thesis should focus on edge colourings of signed cubic graphs. The study of edge colourings of signed graphs was started by Behr [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. The aim of the thesis is to initiate the systematic study of 3-edge-colourability of signed cubic graphs.

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Abstrakt

Signované grafy definoval v roku 1953 Frank Harary ako model na štúdium sociálnych sietí. Problém farbenia signovaných grafov nebol preskúmaný do roku 1982, kedy Thomas Zaslavsky zverejnil prvé výsledky. Prepínanie vrcholov a izomorfizmus rozdeľuje signované grafy do tried ekvivalencie. V tejto práci prezentujeme algoritmus na generovanie prepínavo-izomorfne neekvivalentných signovaných grafov a algoritmus na konverziu problému hranového farbenia na 3SAT. Kombináciou týchto algoritmov vieme generovať malé 3-hranovo-nezafarniteľné signované kubické grafy a formulovať pozorovania o probléme hranového farbenia.

Kľúčové slová: signovaný graf, kubický graf, hranové farbenie, snark, prepínanie vrcholov, prepínavo-izomorfne-neekvivalentné grafy, generovanie grafov

Abstract

Signed graphs were defined by Frank Harary in year 1953 as a model for studying social networks. The problem of colouring, however, was not explored until 1982 when Thomas Zaslavsky published his first results. Vertex switching and isomorphism creates equivalence classes on signed graphs where each graph in a class can be switched and/or projected onto each other graph. In this thesis we present an algorithm for generating non-switching-isomorphic-equivalent signed graphs and an algorithm for edge colouring to 3SAT conversion. Combining these algorithms allows us to generate small non-3-edge-colourable cubic signed graphs and formulate some observations about the problem of edge colouring.

Keywords: signed graph, cubic graph, edge colouring, snark, vertex switching, non-switching-equivalent graphs, generating graphs

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Introduction

The problem of graph colouring has been known for a long time and is still relentlessly being studied today. Even in a problem this wide there are still areas to explore and improve. Edge colouring in combination with the concept of signed graphs remains more or less unexplored.

First discovered by the mathematician Frank Harary in 1953 for studying a question in social psychology, signed graphs remained idle until 1982 when Thomas Zaslavsky published multiple seminary papers on the topic. Many fundamental results in the study of nowhere-zero flows and the chromatic number of signed graphs have been established only recently and research the problem of signed edge colouring was started by Richard Behr in 2020. We expand on Behr's work to automate the process of finding signed edge colorings.

Signed graphs have been proven to be generalizations of simple graphs in many ways. Exponentially many signed graphs can be constructed given a simple underlying graph. We will show that this amount can be drastically reduced due to switching equivalence and isomorphism. Additionally, removing equivalence results in cleaner data for subsequent analysis.

The main result of our work is a database of signed snarks up to eighteen vertices obtained by processing a database of non-isomorphic cubic graphs. For each underlying graph we generate all non-switching-isomorphic signatures, which is an interesting problem in and of itself. This is achieved by transforming the signed graphs into unsigned graphs while preserving switching equivalence and using existing tools based on the automorphism group to filter them for isomorphisms. Then we transform each signature into a 3SAT instance solvable if and only if the signature is 3-edge-colorable. In addition to producing data for bulk analysis it is possible to process specific larger graphs.

In the first chapter we define key concepts in the signed graph theory and describe the current state of research. We also mention the relationship to unsigned graphs and how it affects the colour set and its requirements. In the second chapter we describe the programs that generate non-switching-isomorphic signed graphs and signed snarks. In the third and final chapter we present some results achieved by using these tools and suggest options for future research that can be pursued.

Chapter 1

Preliminary Graph Theory

First, let's define some basic concepts of graph theory, starting with the graph itself.

1.1 Graphs

A graph is an algebraic structure most commonly used to describe relationships between objects. There are many definitions of a graph, the most abstract being simply a set V and a relation R on V denoting which elements of V are connected. Graphs in general are *directed*; if R is symmetric, the graph is *undirected*. For the purposes of this work we will be using a geometric definition and generally undirected graphs. An undirected graph is an ordered pair $G = (V, E)$, where V is a set of *vertices* and E is a set of *edges*, determining the incidence of vertices.

$$(\forall e \in E)\ e = uv = vu; u, v \in V$$

A *path* in a graph G from v to w; $v, w \in V$ is a sequence of vertices $(u_1, u_2, \ldots, u_n)$; $\{u_i \mid 1 \leq i \leq n\} \subseteq V$ such that $u_1 = v$, $u_n = w$ and $\{(u_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \subseteq E$. A graph is *connected* if there exists a path between every pair of vertices $v, w \in V$; $v \neq w$. A *degree* $\Delta(v)$ of a vertex v denotes how many edges are incident to this vertex. The highest degree of any vertex in G is denoted as $\Delta(G)$. A graph is *k-regular* if the degree of each vertex is exactly k. A *cubic graph* is a 3-regular graph.

As an example, the smallest cubic graph is a complete graph with 4 vertices K_4. (In a complete graph each vertex is incident with each other vertex.).

In general statements about graphs in later chapters we are referring to undirected cubic graphs.

We also need to define the set of *half-edges* or vertex-edge incidences of a graph (or signed graph)

$$\Sigma_G = \bigcup_{e=vw \in E_G} \{(e, v), (e, w)\}$$

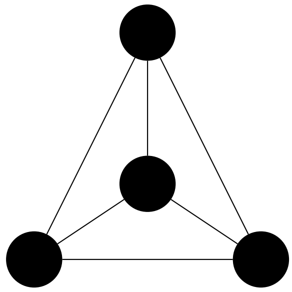

Figure 1.1: Smallest cubic graph.

because in case of signed edge colouring we will be colouring half-edges instead.

1.1.1 Colouring

When simple binary relationships between objects are not enough, weighted graphs and colouring offer a wider range of applications. Assigning colours to vertices or edges of graphs makes classifications of these objects more robust. A vertex colouring $\phi : V_G \to C$ of a graph G is a mapping from the vertex set of G to a set of colours C. An edge colouring $\chi : E_G \to C$ of a graph G is a mapping from the edge set of G to a set of colours C. A *proper vertex colouring* of G is a vertex colouring such that no two incident vertices share a colour. A *proper edge colouring* is an edge colouring such that no two incident edges have the same colour. A proper colouring using k colours is called a *k-colouring.*

As colouring in general is not very interesting, we will be considering only proper colourings henceforth. It is also important to define the set of "colours", especially when colouring signed graphs. It is most practical to use a subset of integers $C \subseteq \mathbb{Z}$ because it makes definitions and proofs clear. Additionally, it is important that a k-colouring uses a set of k colours.

The typical colouring problem is to find the minimum number of colours required for a proper colouring. This number is called the *chromatic number* for vertex colourings and *chromatic index* for edge colourings. Determining the chromatic number and index is useful in other areas of graph theory as well.

Theorem 1.1. *A graph is bipartite if and only if it has a proper vertex 2-colouring.*

For unsigned graphs these numbers are known.

Theorem 1.2 (Brooks[1])**.** *The chromatic number of a connected graph G is $\Delta(G)$ for all graphs except complete graphs and cycles of odd length, where the chromatic number is $\Delta(G) + 1$.*

Theorem 1.3 (Vizing)**.** *The chromatic index of a simple graph G is $\Delta(G)$ or $\Delta(G) + 1$.*

In other words, we can always colour the edges of a graph using at most $\Delta(G) + 1$ colours. The lower bound $\Delta(G)$ is trivial; we need exactly $\Delta(G)$ colours at the highest degree vertex in G to construct a proper colouring. The Vizing theorem proves the upper bound using Kempe chains.

1.2 Signed graphs

A *signed graph* $\Gamma = (G, \sigma)$ consists of an *underlying graph* G and a *sign function* $\sigma : E(G) \to \{+, -\}$ that assigns a sign ($+$ or $-$) to each edge of G. $\Gamma^+ = (V_\Gamma, E_{\Gamma^+})$ and $\Gamma^- = (V_\Gamma, E_{\Gamma^-})$ denote subgraphs of Γ with positive and negative edges respectively, $E_{\Gamma^+} = \sigma^{-1}(+1)$ and $E_{\Gamma^-} = \sigma^{-1}(-1)$. $\mathcal{S}(G)$ denotes the set of all signed graphs with the underlying graph G.

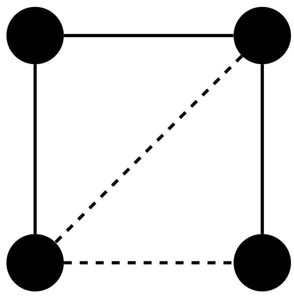

Figure 1.2: Example of a signed graph. Dashed edges are negative, solid edges are positive.

A fundamental concept in the theory of signed graphs is *vertex switching*. Switching vertex v of a signed graph reverses the sign of each edge incident with v. More generally, switching a signed subgraph reverses the sign of each edge between a vertex subset and its complement. Let's define switching by a *switching function*

$$\theta : V_\Gamma \to \{+1, -1\}$$

where vertices mapped to -1 are being switched. The new graph will have an altered sign function,

$$\Gamma^\theta = (G, \sigma^\theta); \quad \sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v)$$

If a signed graph can be obtained from another signed graph by switching, they are considered *switching equivalent*. Switching equivalence is an equivalence relation and thus forms equivalence classes on $\mathcal{S}(G)$. It makes sense to study properties of signed graphs that behave consistently under switching. An example of such a property is the signs of cycles. Switching doesn't change the sign of cycles because if a switched vertex is a part of a cycle, it will reverse the sign of two edges on that cycle leaving the sign product the same. In fact, the sign (or balance, defined below) of cycles is an alternative definition of a switching equivalence class, each combination of balance among a set of cycles that form a cycle space basis yields the same equivalence classes as the method used in this thesis and defined later. It is also important to point out that switching a set of vertices is equivalent to a sequence of one-vertex switches.

Directly related to vertex switching is the notion of *balance*. A cycle is *balanced* when the product of signs of its edges is positive and *unbalanced* otherwise. A signed graph Γ is balanced when each cycle in Γ is balanced. It is *antibalanced* if each cycle is unbalanced.

Theorem 1.4 (Harary). *A signed graph is balanced if and only if*

1. *for every pair of vertices, all paths between these vertices have the same sign*

2. *the vertices can be divided into two subsets (possibly empty) such that each edge with both ends in the same subset is positive and each edge with ends in different subsets is negative*

This is a generalization of the earlier mentioned bipartite graph theorem (Theorem 1.1).

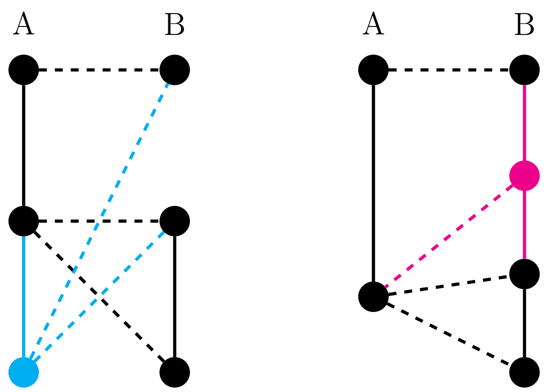

Figure 1.3: Switching a balanced graph.

An all-positive graph Γ is trivially balanced, all paths are positive and the division of vertices will be subsets V_Γ and $\emptyset$. Only graphs switching equivalent to an all-positive graph are balanced. Consider how the conditions in theorem 1.4 behave under switching.

Let's say we switch vertex $v \in V_\Gamma$. The sign of each path that ends at v from any other vertex will flip, because exactly one edge on that path changed signs. So if all paths between v and any other vertex, say u, had the same sign before switching, now they will have the opposite but still the same sign.

Suppose we are able to divide V_Γ into two subsets A and B as per the second condition in theorem 1.4 and without loss of generality let $v \in A$. So all edges vx; $x \in A$ are positive and all edges $vy;\ y \in B$ are negative. After switching v we will construct a new division of V_Γ, subsets $A' = A \setminus \{v\}$ and $B' = B \cup \{v\}$. A' and B' is obviously a correct division of V_Γ and the second condition will still hold, since all edges incident with v flipped signs and at the same time changes whether they end in the same sunbset as v or not.

Connected to vertex switching is the notion of *balance*. The sign of a path is the product of the signs of its edges. A path is positive if and only if there is an even number of negative edges on it. A cycle is balanced if it is positive and a signed graph is balanced if each cycle in it is balanced[2].

1.2.1 Colouring

The research of signed graph colouring was initiated by Zaslavsky[3] in the early 1980s and published in multiple seminal papers [4, 5, 6]. Before defining signed vertex and edge colouring it is necessary to define the set of colours.

In the context of signed graphs and vertex switching we are looking for a set of signed integers with the idea of switching a color reversing its sign, same operation as with the signs of edges. Proper colourings of signed graphs will then be consistent under vertex switching because "reversing the sign" is a bijection on $\mathbb{Z}$. Zaslavsky [5] defined a k-colouring based on a signed colour set $C_k = \{-k, -(k-1), \dots, -1, 0, 1, \dots, (k-1), k\}$ and called colourings zero-free if the colour 0 was not used. He then studied the properties of *chromatic polynomials* related to signed colourings, the number of colourings for a signed graph. (Balanced chromatic polynomials in case of zero-free colourings.)

However, this definition is not a natural extension of the original colour set of integers, because a k-colouring essentially uses $2k$ or $2k+1$ signed colours. This is a desirable property for the colour set, since signed graphs themselves are an extension of unsigned graphs the signed color set should behave in a similar way. A balanced signed graph is essentially equivalent to the unsigned underlying graph, so its chromatic number and index for instance should also match. Máčajová et al. offer another definition: a k-colouring uses the colour set $C_k = \{\pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k$ and $C_k = \{0, \pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k+1$. Behr [7] also adopts this definition.

A vertex colouring $\phi : V_\Gamma \to C_k$ of a signed graph Γ is, similarly to unsigned graphs, a mapping from the vertex set of Γ to a set of signed colours C_k.

Edge colouring, however, needs to be defined differently to incorporate the additional information given by the edge signs. The definition of a k-coloring $\gamma : \Sigma_\Gamma \to C_k$ of a signed graph Γ inspired by Behr [7] is a mapping from the set of half-edges of Γ to a set of signed colours C_k such that

$$(\forall e = uv \in E_\Gamma)\ \ \gamma(e,u) = \sigma(e)\gamma(e,v)$$

In other words, half-edges that form a positive edge must have the same color and half-edges that form a negative edge must have opposite colors. The only difference to Behr's version is the usage of $\sigma(e)$ instead of $-\sigma(e)$, which is really only a matter of taste. In our version positive edges behave like unsigned instead of negative which seemed more natural. Between these definitions different signatures are colorable, but they are equivalent in the sense that there is a bijection between graphs colorable under our definition and Behr's definition. If signed graph $\Gamma = (G, \sigma)$ is edge colorable under our definition, then $\Gamma' = (G, -\sigma)$ with the sign of each edge reversed is colorable under Behr's.

An edge coloring of a signed graph is *proper* if, just as with unsigned graphs, each color is present at each vertex at most once. In case of a vertex coloring all neighbors of each vertex must have different colors. We will, again, consider only proper colorings from now on.

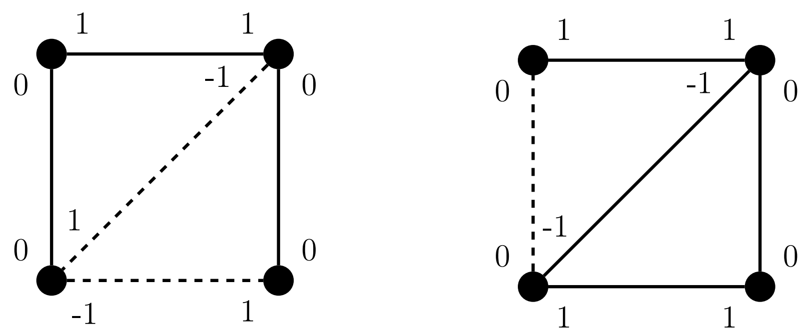

Figure 1.4: Example of a proper signed edge coloring on the left. We obtain the graph on the right by switching the bottom left vertex and the coloring remains correct and proper.

Behr [7] proved the signed version of the Vizing's theorem.

1.3 Motivation

"In the study of various important and difficult problems in graph theory (such as the cycle double cover conjecture and the 5-flow conjecture), one encounters an interesting but somewhat mysterious variety of graphs called snarks. In spite of their simple definition [. . .] and over a century long investigation, their properties and structure are largely unknown." — Chladný, Škoviera [8]

By Vizing's theorem, cubic graphs are edge-colourable either with three ("class one" graphs) or four colours ("class two" graphs). The exact definition of a snark may vary from paper to paper but a snark is essentially a cubic graph with chromatic index four (its edges can't be coloured with three colours). Every cubic graph with a loop or a bridge is a "snark", triangles (cycles of length three) can be contracted into a single vertex and cycles of length four can also be simplified. Therefore many definitions forbid these properties by considering true snarks only graphs with girth (length of the shortest cycle) at least five. Even more strongly, only cyclically 4-edge-connected graphs are considered (there is no subset of three or fewer edges such that their removal will disconnect the graph into two subgraphs each containing a cycle). One of the alternative formulations of the four colour theorem is that each snark is non-planar. Snarks are important in a multitude of graph theory areas and thus it makes sense to investigate the reach of signed snarks too.

1.4 Previous research

In *The chromatic number of a signed graph*[9] Máčajová et al. continue Zaslavsky's research by studying the properties of the chromatic number of signed graphs, ultimately proving a signed version of the famous Brooks'[1] theorem.

Theorem 1.5 (Signed Brooks' Theorem)**.** *Let* Γ *be a simple connected signed graph. If* Γ *is not a balanced complete graph, a balanced odd circuit or an unbalanced even circuit, then* $\chi(\Gamma) \leq \Delta(\Gamma)$*.*

Edge colouring signed graphs defines a version of the signed edge colouring and proves a signed version of the equally fundamental Vizing's theorem.

Theorem 1.6 (Signed Vizing's Theorem)**.** *Let* Γ *be a simple signed graph. The chromatic index of* Γ *is* $\Delta(\Gamma)$ *or* $\Delta(\Gamma) + 1$*.*

Chapter 2

Non-switching-isomorphic graphs

Considering one underlying graph, the number of signed graphs is simply too big for an efficient analysis. Filtering them for switching-isomorphism reduces them to manageable amounts and ensures clean and usable data. Bagheri, Moghaddamfar, Ramezani[10] establish a method of determining the non-switching-isomorphic signed graphs based on the action of its automorphism group. Our aim is to automate this process using a different idea and making analysis of small cubic signed graphs possible.

2.1 Switching equivalence

Two signed graphs are *switching-equivalent* if one can be obtained from the other with a series Given an underlying graph G there are $2^{|E_G|}$ possible signed graphs constructed from G. However, provided that G is connected, only $2^{|E_G|-|V_G|+1}$ or them are mutually non-switching equivalent.

Theorem 2.1. *Let G be a simple unsigned connected graph with n vertices and m edges. There are 2^{m-n+1} mutually non-switching equivalent graphs on G.*

Proof. Bagheri, Moghaddamfar, Ramezani[10] prove this theorem but we present a simpler version. The idea is to use a spanning tree $S \subseteq G$ and show that each switching equivalence class of G has exactly one element that is all-positive on S. Since S contains $n-1$ edges, there are $2^{m-(n-1)}$ different graphs all-positive on S. Suppose we have a signed graph Γ all-positive on S and we switch some vertices. If we switch no vertices or all vertices, the graph stays the same, so we will have a non-empty set of switched vertices A and a non-empty set of unswitched vertices B. At least one edge of S must have one end in A and the other end in B, otherwise G would not be connected or S would not be a spanning tree. After this switching all edges with both ends in either A or B will retain the same sign (not reversed or reversed twice) and edges with one end in A and on end in B will have its sign reversed. Therefore every possible switching from Γ will result in a graph that is not all-positive on S. $\square$

We use this approach to reduce the number of signed graphs that need to be filtered for isomorphisms.

2.2 Isomorphism

A *canonical form* of a graph G is a graph isomorphic to G such that each other graph isomorphic to G has the same canonical form. Since there are known ways of converting unsigned graphs to canonical forms[11], we aim to define a conversion from signed graphs to unsigned graphs such that two signed graphs are switching-isomorphic if and only if they are isomorphic after the conversion.

The *cycle space* of a graph is the collection of its Eulerian (even-degree) spanning subgraphs. It can be described as a vector space over the two-element Galois field: the elements are Eulerian subgraphs, the additive operation is symmetric difference (given two graphs the result contains edges that are in exactly one of them) and trivial scalar multiplication. Cycles are trivial elements of the cycle space because each Eulerian subgraph is the sum of some cycles in the context of this vector space. Consequently, there must be a *cycle basis*, a set of linearly independent cycles that generates this cycle space.

2.3 Transformation

Cycles behave well under both isomorphism and vertex switching. Two isomorphic graphs have the same number of cycles of the same length and the balance of cycles is consistent under vertex switching so two isomorphic *signed* graphs will have the same number of balanced and unbalanced cycles of the same length. The idea is to consider all cycles of progressively bigger length until we construct a set that generates the whole cycle space. Given this set of cycles we construct an unsigned graph and show that two signed graphs are switching-isomorphic if and only if they are isomorphic after processed in the following way.

Theorem 2.2. *Starting with the unsigned underlying graph, for each cycle we add one cycle vertex and connect it to each vertex of the original cycle. We represent their balance with tails, each balanced cycle vertex will have a tail of length one and unbalanced cycles will have a tail of length two. Signed graphs with minimal degree three or more are switching-isomorphic if and only if they are isomorphic after processed this way.*

Proof. The balance tails are either a vertex of degree one or an additional vertex of degree two. Since degrees of vertices are preserved under isomorphism and both original vertices vertices and cycle vertices have minimal degree three, balanced tails will be projected only onto balanced tails and unbalanced tails onto unbalanced tails.

By extension cycle vertices will be projected only onto cycle vertices, because each tail is connected to exactly one cycle vertex. Consequently the original vertices will also be projected only onto each other.

Any isomorphism between two signed graphs transformed this way consists of two parts. Based on the above, reduction to the set of original vertices results in a correct isomorphism between the underlying graphs. The part that is projecting cycle vertices and tails ensures that the signatures are switching-equivalent. □

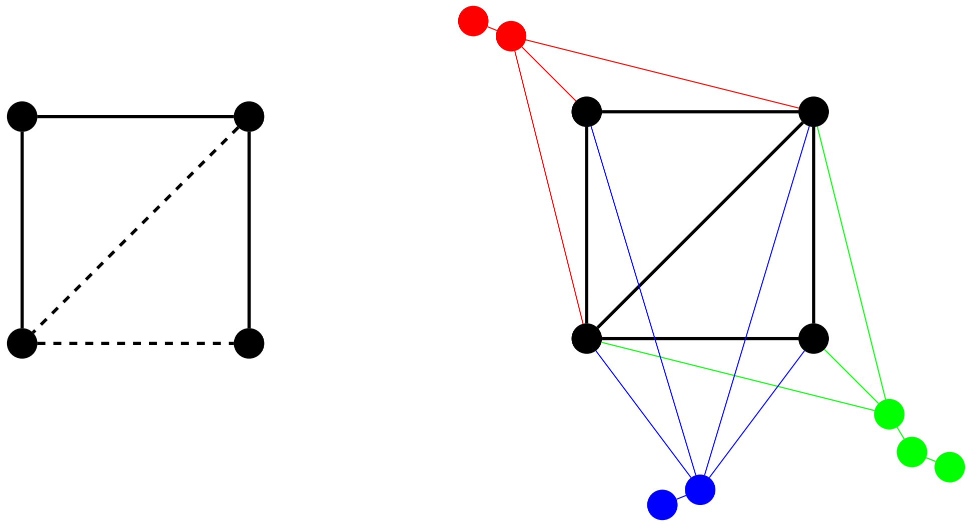

Figure 2.1: Example of a transformation to an unsigned graph.

Chapter 3

Coloring signed graphs

Since the structure of snarks is generally unknown, the most efficient way of systematically generating snarks is still a brute-force approach.

3.1 Chromatic index problem

To determine the chromatic index of a cubic graph is an NP-complete problem. By extension, determining the chromatic index of a signed cubic graph is also NP-complete, because of the trivial reduction from signed chromatic index problem to unsigned chromatic index problem. Instead of designing an algorithm we decided to implement a conversion from the chromatic index problem to 3SAT and using a highly optimized SAT solver in the hope for better effectiveness.

3.1.1 Conversion to 3SAT

For any cubic signed graph Γ we will construct a 3SAT formula $F(\Gamma)$ that is satisfiable if and only if the graph is 3-colourable. There will be three literals for each half-edge ev of Γ, one for each colour from $C_3 = \{-1, 0, 1\}$. Let these be x_{ev}^{-1}, x_{ev}^{0} and x_{ev}^{1}. In any evaluation of these literals that satisfy F exactly one of them will be true denoting the colour of the half-edge. This will be guaranteed using three constituent formulas. Let $\Gamma = ((V, E), \sigma)$

$$F_1 = \bigwedge_{e=vw \in E} (x_{ev}^{-1} \vee x_{ev}^{0} \vee x_{ev}^{1}) \wedge (x_{ew}^{-1} \vee x_{ew}^{0} \vee x_{ew}^{1})$$

The first formula ensures that each half-edge is coloured and is the only set containing clauses of length 3. The next formula will enforce the correctness of the colouring, restricting the colours of half edges that for one complete edge. Illegal signatures for each edge are negated using DeMorgan rules, resulting in a convenient CNF form. No edge can be coloured 0 on one side and 1 or -1 on the other $(\neg(x_{ev}^{0} \wedge x_{ew}^{1}) = (\neg x_{ev}^{0} \vee \neg x_{ew}^{1}))$

and the colours must be the same if the edge is positive $((\neg x^1_{ev} \vee \neg x^{-1}_{ew}))$ and opposite if the edge is negative $((\neg x^1_{ev} \vee \neg x^1_{ew}))$.

$$F_2 = \bigwedge_{e=vw \in E} (\neg x^0_{ev} \vee \neg x^1_{ew}) \wedge (\neg x^0_{ev} \vee \neg x^{-1}_{ew}) \wedge (\neg x^{-1}_{ev} \vee \neg x^{\sigma(e,w)}_{ew}) \wedge (\neg x^1_{ev} \vee \neg x^{-\sigma(e,w)}_{ew}) \wedge (\dots v \rightleftarrows w \dots)$$

The first four clauses illustrate the condition from the "perspective" of v, they will be repeated for w as well by switching instances of v and w. Lastly we need to ensure the colouring is proper. Let $N(v) = \{(v,w) \mid (v,w) \in E;\ w \in V\}$ be the set of edges incident to v.

$$F_3 = \bigwedge_{\substack{v \in V \\ e_1, e_2 \in N(v);\ e_1 \neq e_2}} (\neg x^{-1}_{e_1 v} \vee \neg x^{-1}_{e_2 v}) \wedge (\neg x^0_{e_1 v} \vee \neg x^0_{e_2 v}) \wedge (\neg x^1_{e_1 v} \vee \neg x^1_{e_2 v})$$

Each pair of half-edges with a common vertex has to have different colours. Note that we don't need to explicitly ensure that for each half-edge exactly one literal is true, only that at least one is true, because it is a consequence of the properness of the colouring.

Theorem 3.1. *3SAT formula $F(\Gamma) = F_1 \wedge F_2 \wedge F_3$ constructed in the way described above is satisfiable if and only if Γ is 3-colourable.*

Proof. Follows from the construction of F encapsulating all properties of a proper signed 3-colouring. □

3.1.2 Generating algorithm

The algorithm first finds a spanning tree and assigns positive signs to all edges in it. Edges are enumerated and the spanning tree edges will be ignored. We can now imagine that positive sign means zero and negative sign means one. The remaining edges form a binary number in this way. To obtain the next representative we simply increment this number by one. This means flipping the lowest consecutive sequence of ones and the first instance of zero. We keep reversing the sign of edges from lowest to highest until we flip a positive edge for the first time or run out of edges. If we run out of edges, we basically went from the number $2^{\frac{n}{2}+1} - 1$ to 0. So starting with any signature that is all-positive on the spanning tree, we will have generated all equivalence classes after $2^{\frac{n}{2}+1}$ incrementations. The spanning tree, however, has to remain the same during the entire process.

3.2 Implementation

We achieved our results using the following implementation. The programming language of choice was C++ over Python due to its speed and a base of tools for graph computation. We implement a simple data structure to represent signed graphs as opposed to nauty and other optimized structures because there is little support for signed graphs "out of the box". Additionally, there is no need to optimize for the graph size. Cubic graphs and unsigned snarks are generated using snarkhunter. Our SAT solver of choice is the winner of the SAT Competition 2020[12], kissat. It is a “condensed and improved reimplementation of CaDiCaL in C”.

Chapter 4

Results

We found all signed snarks up to 18 vertices.

N	G	non-equivalent signatures per G	signed G	signed snarks
4	1	8	8	0
6	2	16	32	0
8	5	32	160	1
10	19	64	1216	48
12	85	128	10 880	227
14	509	256	130 304	2768
16	4060	512	2 078 720	31 869
18	41 301	1024	42 292 224	437 381

Figure 4.1: Basic signed graph data. Here signed snarks were not yet filtered for isomorphisms.

The smallest signed snark is smaller than the Petersen graph (smallest snark), it is the projection of a cube.

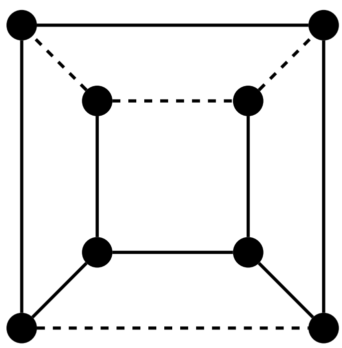

Figure 4.2: Smallest snark

Similarly to regular snarks, there are trivial properties of signed graphs that don't allow the possibility of a 3-edge-colouring. The following unsigned graph is the smallest graph that doesn't have a 3-edge-colourable signature.

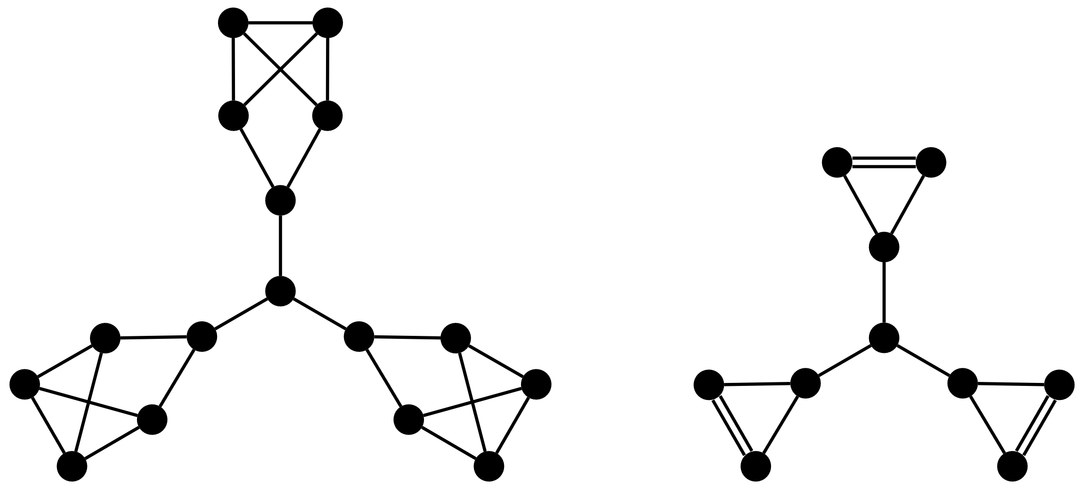

Figure 4.3: Smallest simple graph without a 3-edge-colourable signature and a simplified version allowing duplicate edges.

Theorem 4.1. *An unsigned graph G has a signature that admits a 3-edge-colouring if and only if it has a 1-factor (perfect matching).*

Proof. If there is a signature and a 3-edge-colouring on it, the edges coloured 0 form by the definition of a proper edge colouring a perfect matching. Now let $M \subseteq E(G)$ be a 1-factor. Let's assign the colour 0 to these edges again and remove them from G. After removing a 1-factor from a cubic graph we obtain a 2-factor, a set of disjunct cycles (if two cycles would have a common vertex, its degree in the original graph would have to be at least 4). According to Theorem 1.5 for the cycles to be colourable, we assign any balanced signature to even cycles and any unbalanced signature to odd cycles. All cycles from this 2-factor will now be 2-edge-colourable with colours 1 and -1 and combined with the 0-coloured 1-factor we obtain a 3-edge-colourable signed cubic graph. □

The graph in Chapter 4 has no 1-factor. (The middle vertex has to be connected to one of the three triangles and the other two triangles will not have a matching.)

4.0.1 Future research

There are multiple directions we intend to take our research into this topic in the future. The analysis of small signed snarks can be taken further by inspecting different classes of graphs and searching for similarities. By optimizing the filtering algorithm, bigger graphs can be included.

Conclusion

In this thesis we outlined an algorithm to filter signed graphs that are not 3-edge-colourable. We analysed the first results and showed that for any 3-edge-colourable signature a cubic graph has to admit a perfect matching.

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Appendix A

Source code

The latest version of the source code can be found on `https://github.com/Bohdanator/signed-cubic-graphs`