Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

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Study Programme: Computer Science
Field of Study: Computer Science
Department: Department of Computer Science
Supervisor: doc. RNDr. Robert Lukoťka, PhD.

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ZADANIE ZÁVEREČNEJ PRÁCE

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Názov: Edge colourings of signed cubic graphs
Hranové farbenia signovaných kubických grafov

Anotácia: Signované grafy sú grafy, ktorých hrany sú ohodnotené prvkami z {-1, 1}. Prepínanie signovaného grafu v jeho vrchole v je vynásobenie ohodnotenia incidentných hrán hodnotou -1. Grafy, ktoré možno získať sériou operácií prepínania sú ekvivalentné. Existuje veľa článkov, ktoré skúmajú rozšírenie štandardných grafových invariantov na signované grafy. Jednou zo skúmaných tém je farbenie signovaných grafov. Predmetom práce budú hranové farbenia signovaných kubických grafov. Hranové farbenia signovaných grafov začal skúmať Behr v článku [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. Cieľom práce je začať systematické štúdium hranovej 3-zafarbiteľnosti signovaných grafov.

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Spôsob sprístupnenia elektronickej verzie práce:
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THESIS ASSIGNMENT

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Study programme: Computer Science (Single degree study, master II. deg., full time form)
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Title: Edge colourings of signed cubic graphs

Annotation: Signed graphs are graphs, whose edges have assigned values from {-1, 1}. Switching at a vertex v of a graph is done by multiplying the values of all edges incident with v by -1. Graphs that can be obtained from each other by switching are called equivalent. There are plenty of papers studying generalization of standard graph invariants to signed graphs. One of these invariants is graph colouring. The thesis should focus on edge colourings of signed cubic graphs. The study of edge colourings of signed graphs was started by Behr [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. The aim of the thesis is to initiate the systematic study of 3-edge-colourability of signed cubic graphs.

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Acknowledgments: You can thank anyone who helped you with the thesis here (e.g. your supervisor).

Abstrakt

Slovenský abstrakt v rozsahu 100–500 slov, jeden odstavec. Abstrakt stručne sumarizuje výsledky práce. Mal by byť pochopiteľný pre bežného informatika. Nemal by teda využívať skratky, termíny alebo označenie zavedené v práci, okrem tých, ktoré sú všeobecne známe.

Kľúčové slová: Slovak, keywords, here

Abstract

Abstract in the English language (translation of the abstract in the Slovak language).

Keywords: English, keywords, here

Contents

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Introduction

TODO Introduction, for the purposes of Diplomovy seminar (1) I will put some introduction in other chapters.

Chapter 1

Terminology

Here we define the basic terminology used in this thesis.

1.1 Graphs

Definition 1. We write G for a graph and $V(G)$ and $E(G)$ for its vertes set and edge set respectively. We assume no graph constraints unless otherwise specified, so loops and duplicit edges are allowed in general.

Definition 2. We write $e = vw \in E(G)$ to indicate that the edge e of G has endpoints v and w.

Definition 3. A *k-regular* graph is a graph where each vertex has degree k.

Definition 4. A *circle* or a *circuit* is a connected 2-regular subgraph. A circle is *positive* if the product of its edge signs is positive and *negative* otherwise.

Definition 5. A *cubic* graph is a 3-regular graph.

Definition 6. A *chromatic number* of a graph G is the number of colors required for a proper vertex coloring of said graph.

1.2 Signed graphs

Signed graphs were introduced by Harary[1] in 1953 as a model for social networks. A signed graph has a value of $+1$ or -1 assigned to all edges, so each edge is positive or negative. They have proved to be a natural generalization of unsigned graphs in many ways and interesting observations may arise by applying ordinary graph theory to signed graphs.

Definition 7. A *signed graph* is a pair (G, Σ); $\Sigma \subseteq E(G)$, where Σ is a subset of the edge set of G and contains the negative edges.

Definition 8. Function $\sigma : E(G) \to \{+1; -1\}$ denotes the sign of an edge e.

A signed graph can also be defined as a pair (G, σ) using the sign function directly, but I found this definition more natural.

Definition 9. Given a signed graph (G, Σ), *switching* at a vertex v inverts the sign of each edge incident with v.

Using the previously mentioned definition of a signed graph, the resulting graph after a switching is the symmetric difference of Σ and the set of edges incident with v.

Definition 10. Two graphs are *equivalent* if one can be obtained from the other through a series of vertex switchings. Switching equivalence is an equivalence relation and we write $[(G, \Sigma)]$ for an equivalence class of (G, Σ) under this relation.

Additionally, switching doesn't change the signs of circuits in a graph, so two signed graphs are equivalent if their underlying graphs and the signs of all circuits are the same. Consequently, all properties depending only on the signs of the circuits are invariant for all graphs in $[(G, \Sigma)]$.

Definition 11. A signed graph is *balanced* if all of its circuits are positive.

Balance is an important concept in the sign graph theory, becauase balanced signed graphs (G, Σ) are equivalent to $(G, \{\})$ (an all-positive graph with the same underlying graph).

Definition 12. A signed graph (G, Σ) is *antibalanced* if it is equivalent to $(G, V(G))$ (the same graph with all-negative signature).

Equivalent signed graphs have the same sets of positive circuits and same sets of negative circuits. Additionally, if (G, Σ) is balanced, then $(G, V(G) - \Sigma)$ is antibalanced. Given a partition (A, B) of $V(G)$, let $[A, B]$ denote the set of all edges with one end in A and the other in B.

Theorem 1 (Harary [1])**.** *A signed graph (G, Σ) is balanced if and only if there is a set $X \subseteq V(G)$ such that $\Sigma = [X, V(G) - X]$.*

1.3 Vertex coloring

Vertex and edge coloring is a deeply explored topic of graph theory, even in the field of signed graphs. The research was initiated by Zaslavsky[2] in the early 1980s and published in multiple seminary papers[3, 4, 5]. Máčajová, Raspaud and Škoviera expand on this topic in The chromatic number of a signed graph[6], focusing on the behaviour of colorings instead of the polynomial invariants, which Zaslavsky's research concentrates on.

Definition 13 (Zaslavsky)**.** A *proper vertex coloring* of a signed graph (G, Σ) is $\phi : V(E) \to \mathbb{Z}$ where for each edge $e = vw \in E(G)$: $\phi(v) \neq \sigma(e)\phi(w)$.

Vertices connected by a positive edge must not have the same color and vertices connected by a negative edge must not have opposite colors. This definition is natural mainly because of the consistency under vertex switching, but also other reasons discussed by Zaslavsky. Zaslavsky originally defined the coloring of a signed graph in k colors or $2k+1$ signed colors as a mapping $V(G) \to \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, (k-1), k\}$. A coloring is zero-free if no vertex is colored 0. He then defined the *chromatic polynomial* $\chi_G(\lambda)$ to be the function whose values for negative arguments $\lambda = 2k+1$ are the numbers of signed colorings in k colors. The *balanced chromatic polynomial* $\chi_G^b(\lambda)$ defined for positive arguments $\lambda = 2k$ are the numbers of zero-free signed colorings in k colors. Finally, the *chromatic number* $\gamma(G)$ of G is the smallest non-negative integer k such that $\chi(2k+1) > 0$ and the *strict chromatic number* $\gamma * (G)$ is the same for the balanced chromatic polynomial $\chi_G^b(2k) > 0$.

The Zaslavsky's definitions are sound, but they are not direct extensions of the chromatic polynomials and chromatic number for unsigned graphs. That is because they basically count the absolute values of colors. It makes sense to require a signed version of any graph invariant to agree with its underlying graph for balanced signed graphs. Máčajová et. al.[6] instead propose different definitions. They first define sets $M_n \subseteq \mathbb{Z}$ for each $n \geq 1$ as $M_n = \{\pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k$; $k \in \mathbb{N}$ and $M_n = \{0, \pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k+1$ respectively. We can then define a *proper n-coloring* that uses colors from M_n. The smallest n such that an *n-coloring* exists. In comparison to Zaslavsky, this way an n-coloring uses exactly n colors.

1.4 Edge coloring

In Edge coloring of signed graphs[7], Behr adopts the signed color sets defined by Máčajová et. al.

Definition 14 (Behr)**.** An *n-edge coloring* γ of (G, Γ) is an assignment of colors from M_n to each vertex-edge incidence of G such that $\gamma(v, e) = -\sigma(e)\gamma(w, e)$ for each edge $e = vw$. If an edge e exists such that $\gamma(v, e) = a$, then the color a is present at v.

The same condition for a *proper n-edge coloring* applies to the signed version, no color can be present more than once at any vertex.

Coloring each vertex-edge incidence makes signed edge coloring particularly interesting. This definition also behaves naturally under switching; if we switch a vertex and all colors present at said vertex, the coloring remains consistent. But again, we have to be mindful of the color 0 as in the case of vertex coloring.

We can observe that negative edges behave in the same way as unsigned edges. So each proper n-edge coloring of a all-negative signed graph corresponds to a proper unsigned edge coloring of its underlying graph. This is one of the reasons for the importance of natural definitions: the signed graphs themselves are in a way a generalization of unsigned graphs, so in the field of signed graphs, we are looking for natural generalizations of concepts defined on unsigned graphs.

Chapter 2

Present research

In this chapter we offer an overview of the recent research in the field of signed graphs.

2.1 Chromatic number

Now we will explore some properties of the chromatic number of signed graphs as defined by Máčajová et. al. The proofs can be found in [6]. If (G, Σ) has a positive loop, a proper coloring is not possible. So for the rest of this section we assume only graphs without positive loops. It is also good to keep in mind that the color 0 behaves differently from the other colors, because $0 = -0$. (For example if there is a negative loop at a vertex v, then $\phi(v) \neq 0$.) First, let's compare the chromatic number of a signed graph to the chromatic number of its underlying graph.

Theorem 2 (Máčajová et. al.). *For every loopless signed graph* (G, Σ) *we have*

$$\chi((G, \Sigma)) \leq 2\chi(G) - 1$$

Furthermore, this boundary is sharp.

The idea for the first part is that each coloring of G using colors in $\{0, 1, \ldots, n-1\}$ is also a signed coloring of (G, Σ) using colors from $\{0, \pm 1, \pm 2, \ldots, \pm(n-1)\} = M_{2n-1}$.

A signed graph is antibalanced if the sign product on every even circuit is positive and on every odd circuit negative. (Each such graph is equivalent to an all-negative signature). Based on theorem 1, we can partition the vertex set of an antibalanced signed graph into two sets such that each edge with one end in the first set and the other end in the second set is positive and edges inside the sets are negative. A balanced antibalanced signed graph has to be bipartite, so antibalanced signed graphs are a natural generalization of bipartite graphs.

Proposition 1 (Máčajová et. al.). *A signed graph is 2-colorable* ($\chi((G, \Sigma)) \leq 2$) *if and only if it is antibalanced.*

If the graph is antibalanced, we can switch some vertices to make it all-negative and color all vertices 1. If the graph is 2-colorable, we can partition the vertices into positive (V_1) and negative (V_{-1}). The edges within the sets have to be negative and edges between the sets have to be positive, which fulfils the condition for an antibalanced graph.

Proposition 2 (Máčajová et. al.). *If (G, Σ) is a signed complete graph on n vertices, then $\chi((G, \Sigma)) \leq n$. Furthermore, $\chi((G, \Sigma)) = n$ if and only if (G, Σ) is balanced.*

[6] proves the Brooks' theorem[8] for signed graphs and concludes by proving the five color theorem for planar signed graphs. Let's observe the maximmum degree Δ of signed graphs with regard of the chromatic number. If we color the vertices greedily using any ordering of the vertices, for each vertex at most Δ colors are taken by previous neighbors. Hence $\chi((G, \Sigma)) \leq \Delta + 1$. The maximum chromatic number $\Delta + 1$ is reached in case of a balanced complete graph and a balanced old ciricuit, similar to the unsigned version. There is one more signed case, however: even unbalanced circuits.

Theorem 3 (Máčajová et. al.). *Let (G, Σ) be a simple connected signed graph. If (G, Σ) is not a balanced copmlete graph, a balanced odd circuit or an unbalanced even circuit, then $\chi((G, \Sigma)) \leq \Delta(G)$.*

($\Delta(G)$ is the maximum degree of G)

2.2 Orientation of a signed graph

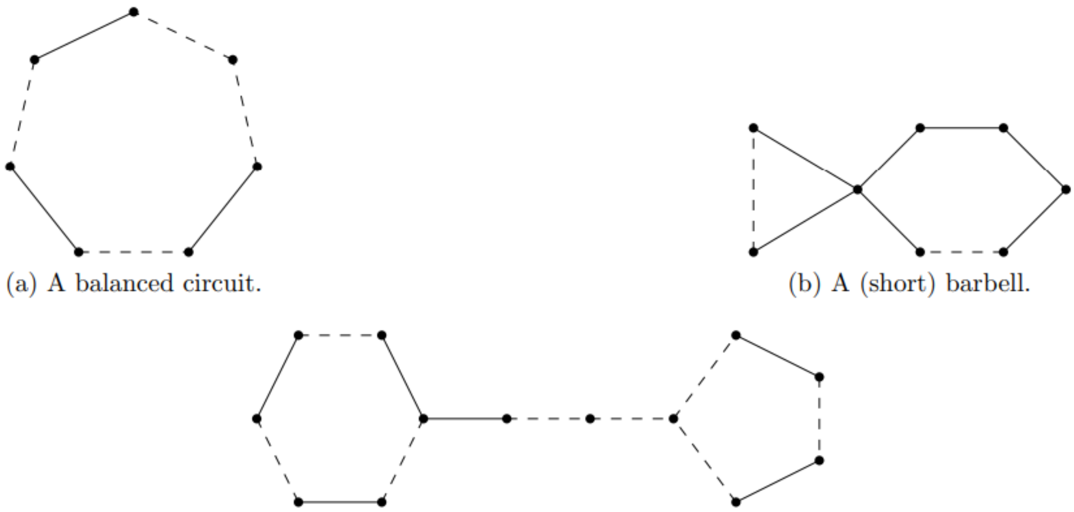
Nowhere-zero flows in signed graphs: A survey[9] captures the recent knowledge about nowhere-zero flows and circuit covers in signed graphs. Nowhere-zero flows is a dual problem to the already mentioned vertex coloring. It was originally introduced on signed graphs by Edmonds and Johnson[10] for expressing algorithms on matchings, but the first to systematically study this was Bouchet [11]. To understand the problem of nowhere-zero flows on signed graphs, we first define *signed circuits* (known to be the circuits of the associated signed graphic matroid). They are the equivalent of circuits on unsigned graphs for signed graphs. [9] recognizes two types of signed circuits (section 2.2):

- balanced circuits
- barbells; the union of two unbalanced vertices connected by a (possibly trivial) path P with endvertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$ such that $C_1 - v_1$ is disjoint from $P \cup C_2$ and $C_2 - v_2$ is disjoint from $P \cup C_1$

We refer to the original, unsigned circuits as *ordinary circuits*.

In order to assign signed edges an orientation, we perceive them as two half-edges. An *orientation* of an edge e consists of directions assigned to each half-edge of e section 2.2. An edge is *consistently oriented* if exactly one of the half-edges h, h' making up e points toward the corresponding endvertex. If both of them point to their respective endvertex e is *extroverted* and if none of them does, e is *introverted*. We say that an oriented edge is *incoming* at a vertex v if its half-edge incident with v points towards v and *outgoing* at v otherwise.

An *orientation* (often referred to as *bidirection*) of a signed graph (G, Σ) is the assignment of orientation to each edge of G in such a way that the positive edges are exactly the consistently oriented ones. An oriented signed graph is also called an *bidirected graph*.



(a) A balanced circuit.

(b) A (short) barbell.

(c) A (long) barbell.

Figure 2.1: Signed circuits (dashed lines indicate negative edges)

(a) A consistently oriented edge.

(b) An introverted edge.

(c) An extroverted edge.

Figure 2.2: Edge orientation

2.3 Nowhere-zero flows

Now to the problem itself. Let Γ be an Abelian group. A Γ-flow in (G, Σ) consists of an orientation of (G, Σ) and a function $\phi : E(G) \to \Gamma$ such that the usual conservation law is satisfied: for each vertex v the sum of $\phi(e)$ over the incoming edges e equals the sum of $\phi(e)$ over the outgoing edges e[9]. A Γ-flow is *nowhere-zero* if the value 0 is never used for any edge. A $\mathbb{Z}$-flow is said to be a k-flow ($k \geq 2$, k is an integer) if for each edge e: $|\phi(e)| \leq k$. If a signed graph (G, Σ) admits a nowhere-zero k-flow, its *flow-number* $\Phi(G, \Sigma)$ is defined as the smallest k such that (G, Σ) admits a nowhere-zero k-flow. Otherwise $\Phi(G, \Sigma)$ is defined as ∞.

A signed graph is said to be *flow-admissible* if it admits at least one nowhere-zero $\mathbb{Z}$-flow.

Theorem 4 (Bouchet [11])**.** *A signed graph (G, Σ) is flow-admissible if and only if each every edge of (G, Σ) belongs to a signed circuit.*

Consequently, nowhere-zero flows on signed graphs are a generalization of the same concept on unsigned graphs, because the definition of a flow on an all-positive signed graph corresponds to the definition of a flow on an unsigned graph.

Directly from the previous theorem follows

Corollary 1. *A signed graph with one negative edge is not flow-admissible.*

Tutte[12] proved that an unsigned graph G admits a nowhere-zero k-flow if and only if it admits a nowhere-zero $\mathbb{Z}_k$ flow. However, this is not true for signed graphs in general. For example an unbalanced circuit admits a $\mathbb{Z}_2$-flow, but no integer flow.

Buchet stated the following conjecture, mirroring its importance with the similar Tutte's 5-flow conjecture.

Conjecture 1. *Every flow-admissible signed graph admits a nowhere-zero 6-flow.*

The value 6 would be best possible, since there exist graphs that admit no nowhere-zero 5-flows. Bouchet originally also proved the theorem for value 216. This number was improved multiple times, the lowest value is held by DeVos[13].

Theorem 5 (DeVos)**.** *Every flow-admissible signed graph admits a nowhere-zero 12-flow.*

Generally, assumptions about a graph's connectivity allow for better bounds on its flow number. This doesn't have to be true in case of signed graphs (See corollary 1). So we need to assume flow-admissibility as well. These are the most recently proven bounds for various connectivity assumptions.

Theorem 6 (Cheng, Lu, Luo, Zhang)**.** *Every flow-admissible 2-edge-connected signed graph has a nowhere-zero 11-flow.*

Theorem 7 (Wu, Ye, Zang, Zhu)**.** *Every flow-admissible 3-edge-connected signed graph admits a nowhere-zero 8-flow.*

And finally, using signed circular flows;

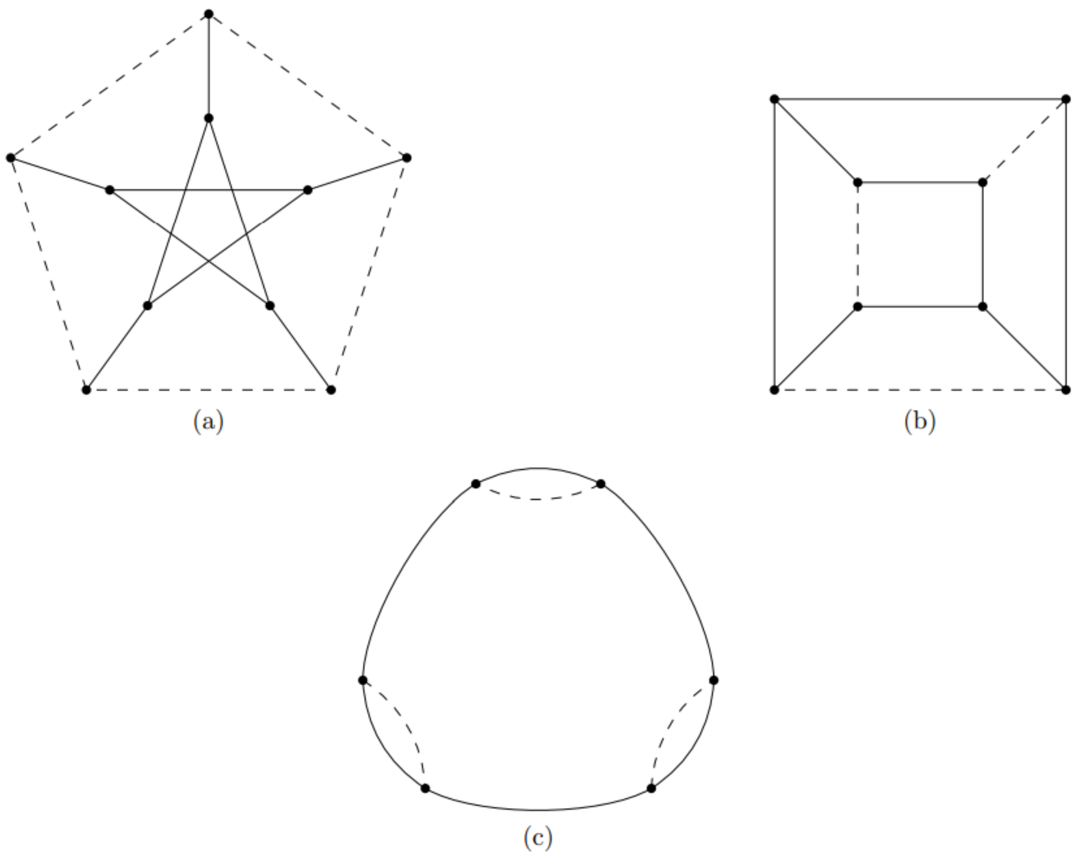


Figure 2.3: Signed graphs with no nowhere-zero 5-flows

Theorem 8 (Raspaud, Zhu)**.** *Every flow-admissible 4-edge-connected signed graph admits a nowhere-zero 4-flow.*

In the field of signed regular graphs, most of the research is focused on signed cubic graphs. Máčajová and Škoviera[14] characterized signed cubic graphs with flow number 3 or 4. I will later expand on this topic as it will probably be closely related with my work in this thesis.

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