Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

2024

Bc. Bohdan Jóža

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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	doc. RNDr. Robert Lukoťka, PhD.

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ZADANIE ZÁVEREČNEJ PRÁCE

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Študijný program: informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
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Typ záverečnej práce: diplomová
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Názov: Edge colourings of signed cubic graphs
Hranové farbenia signovaných kubických grafov

Anotácia: Signované grafy sú grafy, ktorých hrany sú ohodnotené prvkami z {-1, 1}. Prepínanie signovaného grafu v jeho vrchole v je vynásobenie ohodnotenia incidentných hrán hodnotou -1. Grafy, ktoré možno získať sériou operácií prepínania sú ekvivalentné. Existuje veľa článkov, ktoré skúmajú rozšírenie štandardných grafových invariantov na signované grafy. Jednou zo skúmaných tém je farbenie signovaných grafov. Predmetom práce budú hranové farbenia signovaných kubických grafov. Hranové farbenia signovaných grafov začal skúmať Behr v článku [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. Cieľom práce je začať systematické štúdium hranovej 3-zafarbiteľnosti signovaných grafov.

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Spôsob sprístupnenia elektronickej verzie práce:
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Dátum schválenia: prof. RNDr. Rastislav Kráľovič, PhD.
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študent vedúci práce

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THESIS ASSIGNMENT

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Study programme:	Computer Science (Single degree study, master II. deg., full time form)
Field of Study:	Computer Science
Type of Thesis:	Diploma Thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Edge colourings of signed cubic graphs

Annotation: Signed graphs are graphs, whose edges have assigned values from {-1, 1}. Switching at a vertex v of a graph is done by multiplying the values of all edges incident with v by -1. Graphs that can be obtained from each other by switching are called equivalent. There are plenty of papers studying generalization of standard graph invariants to signed graphs. One of these invariants is graph colouring. The thesis should focus on edge colourings of signed cubic graphs. The study of edge colourings of signed graphs was started by Behr [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. The aim of the thesis is to initiate the systematic study of 3-edge-colourability of signed cubic graphs.

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Student Supervisor

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Abstrakt

Slovenský abstrakt v rozsahu 100–500 slov, jeden odstavec. Abstrakt stručne sumarizuje výsledky práce. Mal by byť pochopiteľný pre bežného informatika. Nemal by teda využívať skratky, termíny alebo označenie zavedené v práci, okrem tých, ktoré sú všeobecne známe.

Kľúčové slová: Slovak, keywords, here

Abstract

Abstract in the English language (translation of the abstract in the Slovak language).

Keywords: English, keywords, here

Contents

List of Figures

Introduction

TODO Introduction, for the purposes of Diplomovy seminar (1) I will put some introduction in the Preliminaries chapter

Chapter 1

Terminology

1.1 Graphs

Here we define the basic terminology used in this thesis.

Definition 1. We write G for a graph and $V(G)$ and $E(G)$ for its vertes set and edge set respectively. We assume no graph constraints unless otherwise specified, so loops and duplicit edges are allowed in general.

Definition 2. We write $e = vw \in E(G)$ to indicate that the edge e of G has endpoints v and w.

Definition 3. A *k-regular* graph is a graph where each vertex has degree k.

Definition 4. A *circle* or a *circuit* is a connected 2-regular subgraph. A circle is *positive* if the product of its edge signs is positive and *negative* otherwise.

Definition 5. A *cubic* graph is a 3-regular graph.

Definition 6. A *chromatic number* of a graph G is the number of colors required for a proper vertex coloring of said graph.

1.2 Signed graphs

Signed graphs were introduced by Harary[1] in 1953 as a model for social networks. A signed graph has a value of $+1$ or -1 assigned to all edges, so each edge is positive or negative. They have proved to be a natural generalization of unsigned graphs in many ways and interesting observations may arise by applying ordinary graph theory to signed graphs.

Definition 7. A *signed graph* is a pair (G, Σ); $\Sigma \subseteq E(G)$, where Σ is a subset of the edge set of G and contains the negative edges.

Definition 8. Function $\sigma : E(G) \to \{+1; -1\}$ denotes the sign of an edge e.

A signed graph can also be defined as a pair (G, σ) using the sign function directly, but I found this definition more natural.

Definition 9. Given a signed graph (G, Σ), *switching* at a vertex v inverts the sign of each edge incident with v.

Using the previously mentioned definition of a signed graph, the resulting graph after a switching is the symmetric difference of Σ and the set of edges incident with v.

Definition 10. Two graphs are *equivalent* if one can be obtained from the other through a series of vertex switchings. Switching equivalence is an equivalence relation and we write $[(G, \Sigma)]$ for an equivalence class of (G, Σ) under this relation.

Additionally, switching doesn't change the signs of circuits in a graph, so two signed graphs are equivalent if their underlying graphs and the signs of all circuits are the same. Consequently, all properties depending only on the signs of the circuits are invariant for all graphs in $[(G, \Sigma)]$.

Definition 11. A circuit is *balanced* if the product of the signs of its edges is positive.

Definition 12. A signed graph is *balanced* if all of its circuits are balanced.

Balance is an important concept in the sign graph theory, becauase balanced signed graphs (G, Σ) are equivalent to $(G, \{\})$ (an all-positive graph with the same underlying graph).

Definition 13. A signed graph (G, Σ) is *antibalanced* if it is equivalent to $(G, V(G))$ (the same graph with all-negative signature).

Equivalent signed graphs have the same sets of balanced circuits and same sets of unbalanced circuits. Additionally, if (G, Σ) is balanced, then $(G, V(G) - \Sigma)$ is antibalanced. Given a partition (A, B) of $V(G)$, let $[A, B]$ denote the set of all edges with one end in A and the other in B.

Theorem 1 (Harary [1])**.** *A signed graph (G, Σ) is balanced if and only if there is a set $X \subseteq V(G)$ such that $\Sigma = [X, V(G) - X]$.*

1.3 Vertex coloring

Vertex and edge coloring is a deeply explored topic of graph theory, even in the field of signed graphs. The research was initiated by Zaslavsky[2] in the early 1980s and published in multiple seminary papers[3, 4, 5]. Máčajová, Raspaud and Škoviera expand on this topic in The chromatic number of a signed graph[6], focusing on the behaviour of colorings instead of the polynomial invariants, which Zaslavsky's research concentrates on.

Definition 14 (Zaslavsky)**.** A *proper vertex coloring* of a signed graph (G, Σ) is $\phi : V(E) \to \mathbb{Z}$ where for each edge $e = vw \in E(G)$: $\phi(v) \neq \sigma(e)\phi(w)$.

Vertices connected by a positive edge must not have the same color and vertices connected by a negative edge must not have opposite colors. This definition is natural mainly because of the consistency under vertex switching, but also other reasons discussed by Zaslavsky. Zaslavsky originally defined the coloring of a signed graph in k colors or $2k+1$ signed colors as a mapping $V(G) \to \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, (k-1), k\}$. A coloring is zero-free if no vertex is colored 0. He then defined the *chromatic polynomial* $\chi_G(\lambda)$ to be the function whose values for negative arguments $\lambda = 2k+1$ are the numbers of signed colorings in k colors. The *balanced chromatic polynomial* $\chi_G^b(\lambda)$ defined for positive arguments $\lambda = 2k$ are the numbers of zero-free signed colorings in k colors. Finally, the *chromatic number* $\gamma(G)$ of G is the smallest non-negative integer k such that $\chi(2k+1) > 0$ and the *strict chromatic number* $\gamma*(G)$ is the same for the balanced chromatic polynomial $\chi_G^b(2k) > 0$.

The Zaslavsky's definitions are sound, but they are not direct extensions of the chromatic polynomials and chromatic number for unsigned graphs. That is because they basically count the absolute values of colors. It makes sense to require a signed version of any graph invariant to agree with its underlying graph for balanced signed graphs. Máčajová et. al.[6] instead propose different definitions. They first define sets $M_n \subseteq \mathbb{Z}$ for each $n \geq 1$ as $M_n = \{\pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k$; $k \in \mathbb{N}$ and $M_n = \{0, \pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k+1$ respectively. We can then define a *proper n-coloring* that uses colors from M_n. The smallest n such that an *n-coloring* exists. In comparison to Zaslavsky, this way an n-coloring uses exactly n colors.

1.4 Chromatic number

Now we will explore some properties of the chromatic number of signed graphs as defined by Máčajová et. al. The proofs can be found in [6]. If (G, Σ) has a positive loop, a proper coloring is not possible. So for the rest of this section we assume only graphs without positive loops. It is also good to keep in mind that the color 0 behaves differently from the other colors, because $0 = -0$. (For example if there is a negative loop at a vertex v, then $\phi(v) \neq 0$.) First, let's compare the chromatic number of a signed graph to the chromatic number of its underlying graph.

Theorem 2 (Máčajová et. al.)**.** *For every loopless signed graph* (G, Σ) *we have*

$$\chi((G,\Sigma)) \leq 2\chi(G) - 1$$

Furthermore, this boundary is sharp.

The idea for the first part is that each coloring of G using colors in $\{0, 1, \ldots, n-1\}$ is also a signed coloring of (G,Σ) using colors from $\{0, \pm 1, \pm 2, \ldots, \pm(n-1)\} = M_{2n-1}$.

A signed graph is antibalanced if the sign product on every even circuit is positive and on every odd circuit negative. (Each such graph is equivalent to an all-negative signature). Based on theorem 1, we can partition the vertex set of an antibalanced signed graph into two sets such that each edge with one end in the first set and the other end in the second set is positive and edges inside the sets are negative. A balanced antibalanced signed graph has to be bipartite, so antibalanced signed graphs are a natural generalization of bipartite graphs.

Proposition 1 (Máčajová et. al.). *A signed graph is 2-colorable ($\chi((G,\Sigma)) \leq 2$) if and only if it is antibalanced.*

If the graph is antibalanced, we can switch some vertices to make it all-negative and color all vertices 1. If the graph is 2-colorable, we can partition the vertices into positive (V_1) and negative (V_{-1}). The edges within the sets have to be negative and edges between the sets have to be positive, which fulfils the condition for an antibalanced graph.

Proposition 2. *If (G,Σ) is a signed complete graph on n vertices, then $\chi((G,\Sigma)) \leq n$. Furthermore, $\chi((G,\Sigma)) = n$ if and only if (G,Σ) is balanced.*

[6] proves the Brooks theorem[7] for signed graphs and concludes by proving the 5 color theorem for planar signed graphs.

Theorem 3. *Let (G,Σ) be a simple connected signed graph. If (G,Σ) is not a balanced copmlete graph, a balanced odd circuit or an unbalanced even circuit, then $\chi((G,\Sigma)) \leq \Delta(G)$.*

($\Delta(G)$ is the maximum degree of G)

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