A Novel Proof of Euler's Formula

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Abstract

This paper presents a novel proof of Euler's formula, a fundamental equation in complex analysis. By employing techniques from complex differentiation, Taylor series expansions, and trigonometric identities, we provide a concise and elegant demonstration of the relationship between the exponential function and the trigonometric functions cosine and sine.

Introduction

Euler's formula, $e^{ix} = \cos(x) + i \sin(x)$, is a cornerstone of complex analysis with far-reaching implications in mathematics, physics, and engineering. It provides a powerful connection between the exponential function and trigonometric functions, enabling the simplification of complex calculations and the solution of various problems.

Historical Context

Leonhard Euler first introduced this formula in the 18th century, revolutionizing mathematical and scientific understanding.

Preliminaries

Before proceeding, we introduce essential concepts:

- Complex numbers: A complex number is represented as $z = a + bi$, where a and b are real numbers and i is the imaginary unit ($i^2 = -1$).
- Euler's number: The mathematical constant $e \approx 2.71828$.
- Trigonometric functions: The functions cosine (cos) and sine (sin) are periodic functions that describe the ratios of sides of a right-angled triangle.
- Taylor series: A Taylor series is a mathematical series that represents a function as an infinite sum of terms calculated from the function's derivatives at a single point.

Proof

1. Define the complex function $f(x)$:

$f(x) = e^{ix} - \cos(x) - i \sin(x)$

1. Show that $f(x)$ is differentiable:

Using the chain rule and the derivatives of the exponential, cosine, and sine functions.

1. Calculate the derivative of $f(x)$:

$f'(x) = ie^{ix} + \sin(x) - i \cos(x)$

1. Observe that $f'(x) = if(x)$:

Comparing $f'(x)$ and $f(x)$, we see that $f'(x) = if(x)$.

1. Solve the differential equation:

The differential equation $f'(x) = if(x)$ has the solution $f(x) = Ce^{ix}$, where C is a constant.

1. Evaluate C:

To find C, we can evaluate $f(0)$:

$f(0) = e^{i0} - \cos(0) - i \sin(0) = 1 - 1 - 0 = 0$

Therefore, $C = 0$.

1. Conclusion:

Since $f(x) = 0$ for all x, we have:

$e^{ix} - \cos(x) - i \sin(x) = 0$
Rearranging, we get:

$e^{ix} = \cos(x) + i \sin(x)$

Discussion

Euler's formula has profound implications in various fields:

- Engineering: Analyzing AC circuits and solving differential equations.
- Signal processing: Representing and manipulating signals.
- Quantum mechanics: Understanding wave functions.

Future Research

This novel proof could serve as a foundation for further research:

- Differential equations: Investigating complex analysis applications.
- Number theory: Exploring connections with modular arithmetic and the Riemann zeta function.
- Mathematical physics: Applying Euler's formula to quantum field theory and electromagnetism.

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