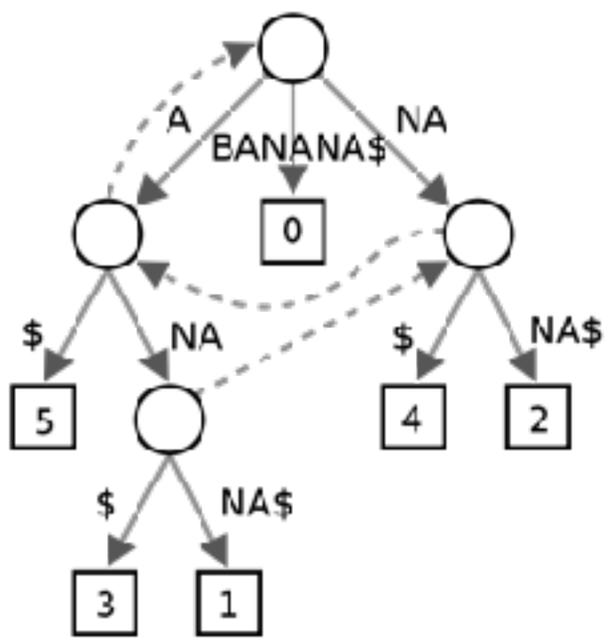
Burrows-Wheeler Transform & FM Index

Indexing with suffixes



Suffix Tree

6	$
5	A$
3	ANA$
1	ANANA$
0	BANANA$
4	NA$
2	NANA$

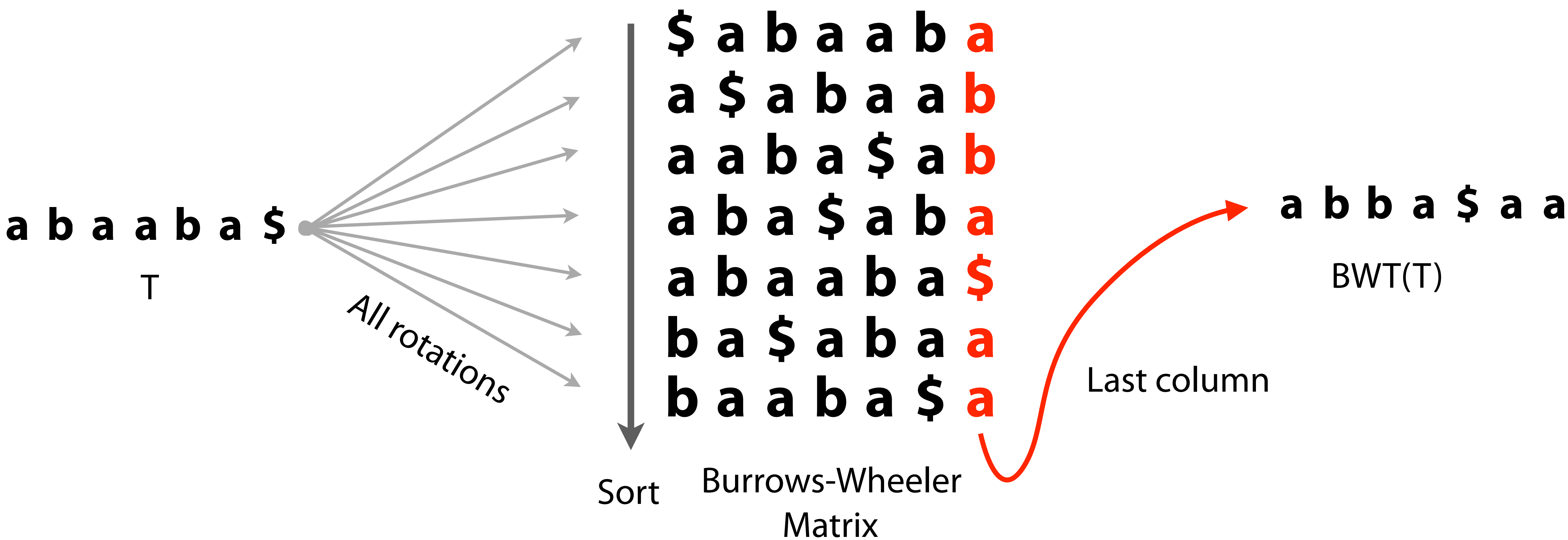
Suffix Array

$ BANAN **A**
A $BANA **N**
A NA$BA **N**
A NANA$ **B**
B ANANA **$**
N A$BAN **A**
N ANA$B **A**

FM Index

Burrows-Wheeler Transform

Reversible permutation of the characters of a string, used originally for compression



Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

All rotations

a b a a b a $

b a a b a $ a

a a b a $ a b

a b a $ a b a

b a $ a b a a

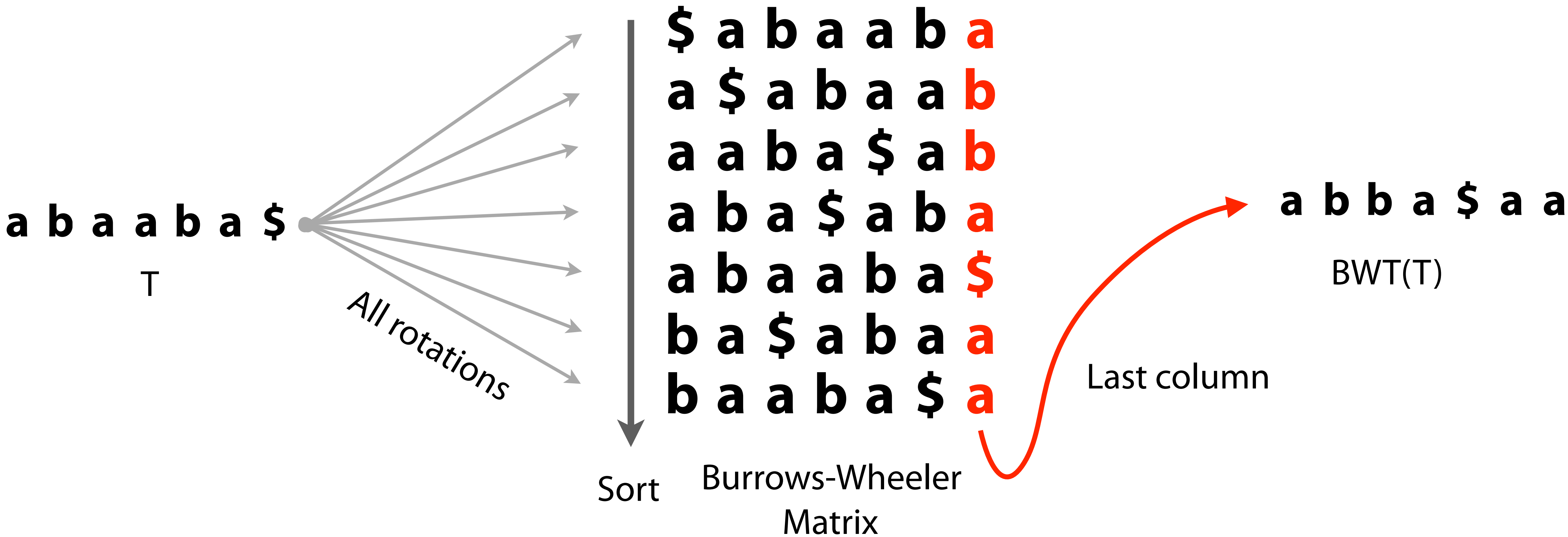
a $ a b a a b

$ a b a a b a

(then they repeat)

Burrows-Wheeler Transform

Reversible permutation of the characters of a string, used originally for compression



How is it useful for compression? How is it reversible? How is it an index?

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

```
def rotations(t):
    """ Return list of rotations of input string t """
    tt = t * 2
    return [ tt[i:i+len(t)] for i in range(0, len(t)) ]

def bwm(t):
    """ Return lexicographically sorted list of t's rotations """
    return sorted(rotations(t))

def bwtViaBwm(t):
    """ Given T, returns BWT(T) by way of the BWM """
    return ''.join(map(lambda x: x[-1], bwm(t)))
```

Make list of all rotations

Sort them

Take last column

```
>>> bwtViaBwm("Tomorrow_and_tomorrow_and_tomorrow$")
'w$wwdd__nnoooaattTmmmrrrrrrooo__ooo'

>>> bwtViaBwm("It_was_the_best_of_times_it_was_the_worst_of_times$")
's$esttssfftteww_hhmmbootttt_ii__woeeaaressIi_______'

>>> bwtViaBwm('in_the_jingle_jangle_morning_Ill_come_following_you$')
'u_gleeeengj_mlhl_nnnnt$nwj__lggIolo_iiiiarfcmylo_oo_'
```

Burrows-Wheeler Transform

a b a a b a $

T

$ a b a a b **a**	**a** $ a b a a b
a $ a b a a **b**	**b** a $ a b a a
a a b a $ a **b**	**b** a a b a $ a
a b a $ a b **a**	**a** a b a $ a b
a b a a b a **$**	**$** a b a a b a
b a $ a b a **a**	**a** b a $ a b a
b a a b a $ **a**	**a** b a a b a $

BWT

Right contexts

BWT(T) orders T's characters according to alphabetical order of their right contexts in T

Right context

The right context of a position in *T* consists of everything that comes after it with "wrap around"

T: $\boxed{\textbf{a}}$ **b a a b a $**

Right context: **b a a b a $**

2

3 1

T: **a b a** $\boxed{\textbf{a}}$ **b a $**

Right context: **b a $ a b a**

Burrows-Wheeler Transform

Right context:
a b a $ a b

$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a

Right context:
a b a $ a b

Burrows-Wheeler Transform

Sorted by *right-context*

Gives “structure” to BWT(T), making it more compressible

final char (L)	sorted rotations
a	n to decompress. It achieves compression
o	n to perform only comparisons to a depth
o	n transformation} This section describes
o	n transformation} We use the example and
o	n treats the right-hand side as the most
a	n tree for each 16 kbyte input block, enc
a	n tree in the output stream, then encodes
i	n turn, set $L[i]$ to be the
i	n turn, set $R[i]$ to the
o	n unusual data. Like the algorithm of Man
a	n use a single set of probabilities table
e	n using the positions of the suffixes in
i	n value at a given point in the vector $R
e	n we present modifications that improve t
e	n when the block size is quite large. Ho
i	n which codes that have not been seen in
i	n with ch appear in the {\em same order
i	n with ch. In our exam
o	n with Huffman or arithmetic coding. Bri
o	n with figures given by Bell~\cite{bell}.

Figure 1: Example of sorted rotations. Twenty consecutive rotations from the sorted list of rotations of a version of this paper are shown, together with the final character of each rotation.

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

BWM is related to the suffix array

$	a	b	a	a	b	a
a	$	a	b	a	a	b
a	a	b	a	$	a	b
a	b	a	$	a	b	a
a	b	a	a	b	a	$
b	a	$	a	b	a	a
b	a	a	b	a	$	a

BWM(T)

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

SA(T)

Same order whether rows are rotations or suffixes

Burrows-Wheeler Transform

In fact, this gives us a new definition / way to construct BWT(T):

$$BWT[i] = \begin{cases} T[SA[i] - 1] & \text{if } SA[i] > 0 \\ \$ & \text{if } SA[i] = 0 \end{cases}$$

"BWT = characters just to the left of the suffixes in the suffix array"

$	a	b	a	a	b	a
a	$	a	b	a	a	b
a	a	b	a	$	a	b
a	b	a	$	a	b	a
a	b	a	a	b	a	$
b	a	$	a	b	a	a
b	a	a	b	a	$	a

BWM(T)

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

SA(T)

Burrows-Wheeler Transform

```
def suffixArray(s):
    """ Given T return suffix array SA(T).  We use Python's sorted
        function here for simplicity, but we can do better. """
    satups = sorted([(s[i:], i) for i in xrange(0, len(s))])
    # Extract and return just the offsets
    return map(lambda x: x[1], satups)
```

Make suffix array

```
def bwtViaSa(t):
    """ Given T, returns BWT(T) by way of the suffix array. """
    bw = []
    for si in suffixArray(t):
        if si == 0: bw.append('$')
        else: bw.append(t[si-1])
    return ''.join(bw) # return string-ized version of list bw
```

Take characters just to the left of the sorted suffixes

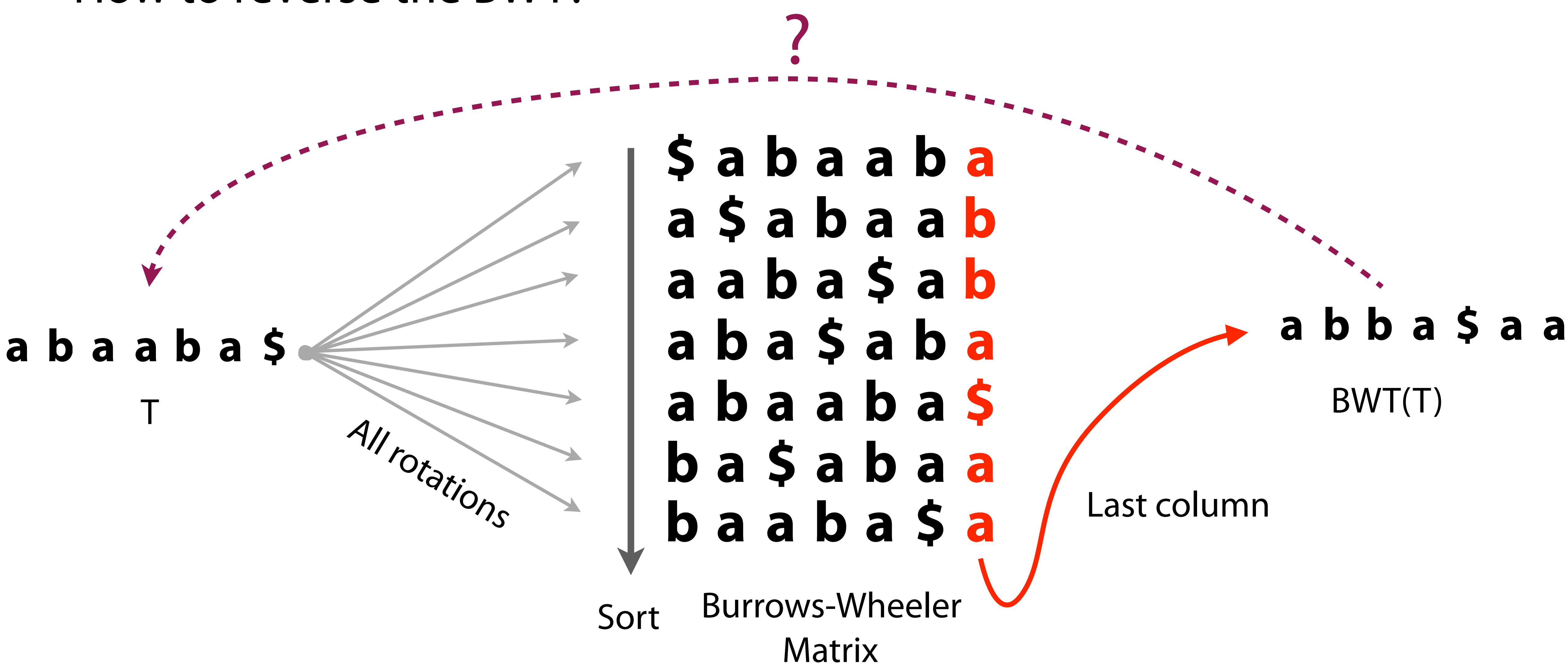
```
>>> bwtViaSa("Tomorrow_and_tomorrow_and_tomorrow$")
'w$wwdd__nnoooaattTmmmrrrrrrooo__ooo'

>>> bwtViaSa("It_was_the_best_of_times_it_was_the_worst_of_times$")
's$esttssfftteww_hhmmboottttt_ii__woeeaaressIi_______'

>>> bwtViaSa('in_the_jingle_jangle_morning_Ill_come_following_you$')
'u_gleeeengj_mlhl_nnnnt$nwj__lggIolo_iiiiarfcmylo_oo_'
```

Burrows-Wheeler Transform

How to reverse the BWT?



BWM has a key property called the *LF Mapping*...

Burrows-Wheeler Transform: T-ranking

Give each character in *T* a rank, equal to # times the character occurred previously in *T*. Call this the *T-ranking*.

$$\mathbf{a}_0\ \mathbf{b}_0\ \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}_1\ \mathbf{a}_3\ \$$$

Ranks aren't explicitly stored; they are just for illustration

Now let's re-write the BWM including ranks...

Burrows-Wheeler Transform

BWM with T-ranking:

$$
\begin{array}{ccccccc}
F & & & & & & L \\
\$ & a_0 & b_0 & a_1 & a_2 & b_1 & a_3 \\
a_3 & \$ & a_0 & b_0 & a_1 & a_2 & b_1 \\
a_1 & a_2 & b_1 & a_3 & \$ & a_0 & b_0 \\
a_2 & b_1 & a_3 & \$ & a_0 & b_0 & a_1 \\
a_0 & b_0 & a_1 & a_2 & b_1 & a_3 & \$ \\
b_1 & a_3 & \$ & a_0 & b_0 & a_1 & a_2 \\
b_0 & a_1 & a_2 & b_1 & a_3 & \$ & a_0
\end{array}
$$

Look at first and last columns, called *F* and *L*

And look at just the **a**s

as occur in the same order in *F* and *L*. As we look down columns, in both cases we see: a_3, a_1, a_2, a_0

Burrows-Wheeler Transform

BWM with T-ranking:

F						*L*
\$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	\$	a_0	b_0	a_1	a_2	$\mathbf{b}_1$
a_1	a_2	b_1	a_3	\$	a_0	$\mathbf{b}_0$
a_2	b_1	a_3	\$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	\$
$\mathbf{b}_1$	a_3	\$	a_0	b_0	a_1	a_2
$\mathbf{b}_0$	a_1	a_2	b_1	a_3	\$	a_0

Same with **b**s: $\mathbf{b}_1$, $\mathbf{b}_0$

Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

F						*L*
$	a_0	b_0	a_1	a_2	b_1	$\mathbf{a_3}$
$\mathbf{a_3}$	$	a_0	b_0	a_1	a_2	$\mathbf{b_1}$
$\mathbf{a_1}$	a_2	b_1	a_0	$	a_0	$\mathbf{b_0}$
$\mathbf{a_2}$	b_1	a_3	$	a_0	b_0	$\mathbf{a_1}$
$\mathbf{a_0}$	b_0	a_1	a_2	b_1	a_3	**$**
$\mathbf{b_1}$	a_3	$	a_0	b_0	a_1	$\mathbf{a_2}$
$\mathbf{b_0}$	a_1	a_2	b_1	a_3	$	$\mathbf{a_0}$

LF Mapping: The i^{th} occurrence of a character *c* in *L* and the i^{th} occurrence of *c* in *F* correspond to the *same* occurrence in *T* (i.e. have same rank)

However we rank occurrences of *c*, ranks appear in the same order in *F* & *L*

Burrows-Wheeler Transform: LF Mapping

Why does the LF Mapping hold?

Why are these **a**s in this order relative to each other?

$\$$	a	b	a	a	b	a_3
a_3	$\$$	a	b	a	a	b_1
a_1	a	b	a	$\$$	a	b_0
a_2	b	a	$\$$	a	b	a_1
a_0	b	a	a	b	a	$\$$
b_1	a	$\$$	a	b	a	a_2
b_0	a	a	b	a	$\$$	a_0

They're sorted by right-context

$\$$	a	b	a	a	b	a_3
a_3	$\$$	a	b	a	a	b_1
a_1	a	b	a	$\$$	a	b_0
a_2	b	a	$\$$	a	b	a_1
a_0	b	a	a	b	a	$\$$
b_1	a	$\$$	a	b	a	a_2
b_0	a	a	b	a	$\$$	a_0

Why are these **a**s in this order relative to each other?

They're sorted by right-context

Occurrences of *c* in *F* are sorted by right-context. Same for *L*!

Whatever ranking we give to characters in *T*, rank orders in *F* and *L* will match

Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

F						*L*
\$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	\$	a_0	b_0	a_1	a_2	b_1
a_1	a_2	b_1	a_3	\$	a_0	b_0
a_2	b_1	a_3	\$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	\$
b_1	a_3	\$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	\$	a_0

We’d like a different ranking so that for a given character, ranks are in ascending order as we look down the F / L columns...

Burrows-Wheeler Transform: LF Mapping

BWM with B-ranking:

F						*L*
\$	a_3	b_1	a_1	a_2	b_0	$\mathbf{a}_0$
$\mathbf{a}_0$	\$	a_3	b_1	a_1	a_2	$\mathbf{b}_0$
$\mathbf{a}_1$	a_2	b_0	a_0	\$	a_3	$\mathbf{b}_1$
$\mathbf{a}_2$	b_0	a_0	\$	a_3	b_1	$\mathbf{a}_1$
$\mathbf{a}_3$	b_1	a_1	a_2	b_0	a_0	**\$**
$\mathbf{b}_0$	a_0	\$	a_3	b_1	a_1	$\mathbf{a}_2$
$\mathbf{b}_1$	a_1	a_2	b_0	a_0	\$	$\mathbf{a}_3$

Ascending rank

F now has very simple structure: a **\$**, a block of **a**s *with ascending ranks*, a block of **b**s *with ascending ranks*

Burrows-Wheeler Transform

Say T has 300 **A**s, 400 **C**s, 250 **G**s and 700 **T**s and **\$** < **A** < **C** < **G** < **T**

Which BWM row (0-based) begins with $\mathbf{G}_{100}$? (Ranks are B-ranks.)

Skip row starting with **\$** (1 row)

Skip rows starting with **A** (300 rows)

Skip rows starting with **C** (400 rows)

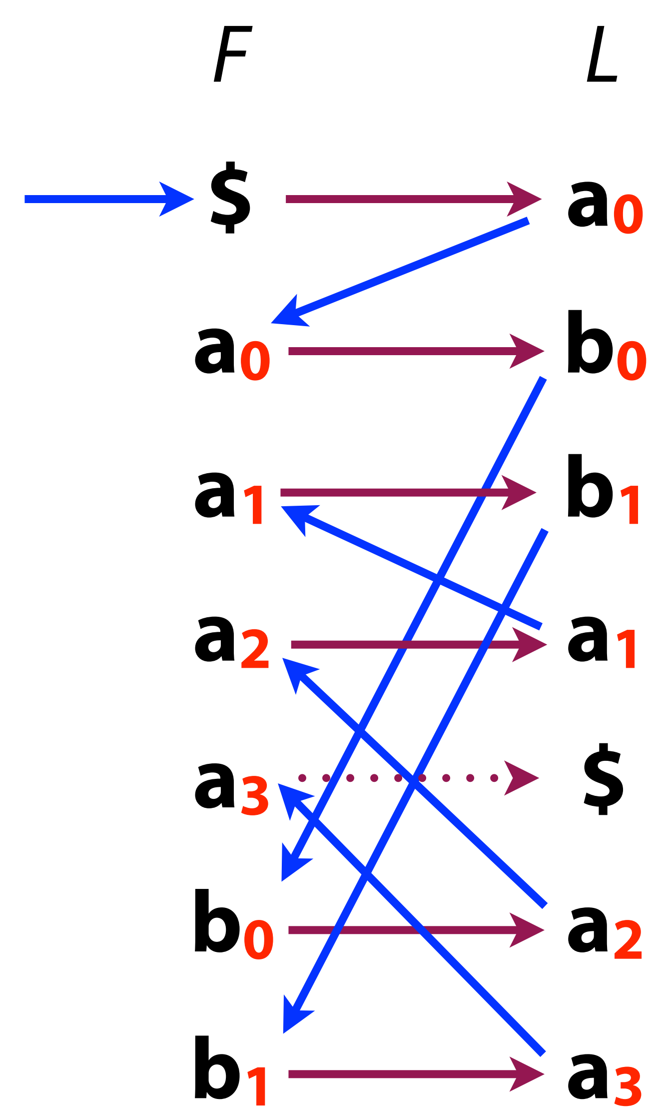
Skip first 100 rows starting with **G** (100 rows)

Answer: row 1 + 300 + 400 + 100 = **row 801**

Burrows-Wheeler Transform: reversing

Reverse BWT(T) starting at right-hand-side of *T* and moving left

T: $\mathbf{a_3\ b_1\ a_1\ a_2\ b_0\ a_0\ \$}$



Start in first row. *F* must have **$**.

L contains character just prior to **$**: $\mathbf{a_0}$

Jump to row *beginning* with $\mathbf{a_0}$.

L contains character just prior to $\mathbf{a_0}$: $\mathbf{b_0}$.

Repeat for $\mathbf{b_0}$, get $\mathbf{a_2}$

Repeat for $\mathbf{a_2}$, get $\mathbf{a_1}$

Repeat for $\mathbf{a_1}$, get $\mathbf{b_1}$

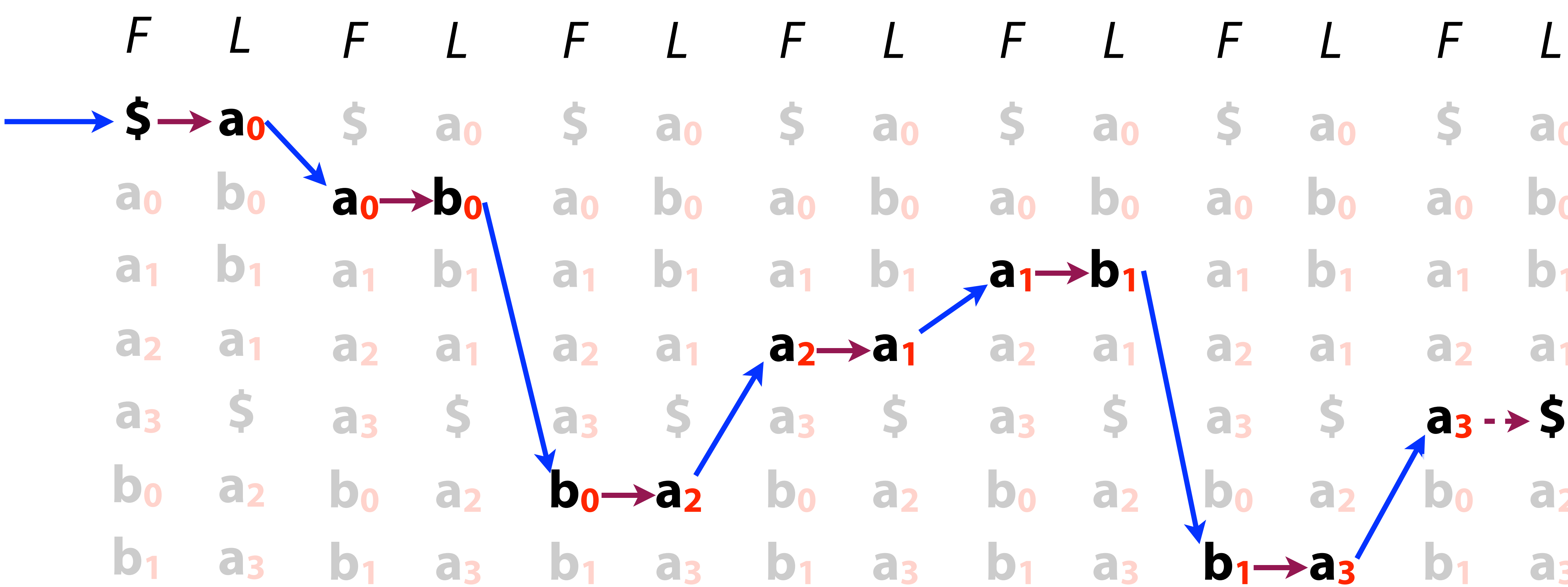
Repeat for $\mathbf{b_1}$, get $\mathbf{a_3}$

Repeat for $\mathbf{a_3}$, get **$** (done)

In reverse order, we saw = $\mathbf{a_3\ b_1\ a_1\ a_2\ b_0\ a_0\ \$}$ = *T*

Burrows-Wheeler Transform: reversing

Another way to visualize:



T: $a_3\, b_1\, a_1\, a_2\, b_0\, a_0\, \$$

Burrows-Wheeler Transform: reversing

```
def rankBwt(bw):
    ''' Given BWT string bw, return parallel list of B-ranks.  Also
        returns tots: map from character to # times it appears. '''
    tots = dict()
    ranks = []
    for c in bw:
        if c not in tots: tots[c] = 0
        ranks.append(tots[c])
        tots[c] += 1
    return ranks, tots
```

{ a: 4, b: 2, $: 1}

L

a_0
b_0
b_1
a_1
$\$$
a_2
a_3

Like when we did it by eye, the code depends on *knowing the ranks* of all the characters in L

But **ranks** is big! We'll fix this later

Burrows-Wheeler Transform

We’ve seen how BWT is useful for compression:

- Sorts characters by right-context, making a more compressible string

And how it’s reversible:

- Repeated applications of LF Mapping, recreating *T* from right to left

How is it used as an index?

FM Index

FM Index: an index combining the BWT *with a few small auxiliary data structures*

Core of index is ***F*** and ***L*** from BWM:

L is the same size as *T*

F can be represented as array of $|\Sigma|$ integers

L is compressible (but even uncompressed, it's small compared to suffix array)

We're discarding *T*

F						***L***
$	$	a	b	a	a	b	**a**
a	$	a	b	a	a	**b**
a	a	b	a	$	a	**b**
a	b	a	$	a	b	**a**
a	b	a	a	b	a	**$**
b	a	$	a	b	a	**a**
b	a	a	b	a	$	**a**

Not stored in index

Paolo Ferragina, and Giovanni Manzini. "Opportunistic data structures with applications." *Foundations of Computer Science, 2000. Proceedings. 41st Annual Symposium on*. IEEE, 2000.

FM Index: querying

How to query?

```
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
```

FM Index: querying

Can we query like the suffix array?

```
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
```

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

We don’t have these columns, and we don’t have T.
Binary search not possible.

FM Index: querying

Look for range of rows of BWM(T) with *P* as prefix

Start with shortest suffix, then match successively longer suffixes

P = **aba**

Easy to find all the rows beginning with **a**

F						*L*
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	**$**
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

FM Index: querying

We have rows beginning with **a**, now we want rows beginning with **ba**

$P =$ **aba**

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

Look at those rows in *L*.
b_0, b_1 are **b**s occuring just to left.

Use LF Mapping. Let new range delimit those **b**s

$P =$ **aba**

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

Now we have the rows with prefix **ba**

FM Index: querying

We have rows beginning with **ba**, now we seek rows beginning with **aba**

P = **aba**

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

← a_2, a_3 occur just to left.

P = **aba**

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

Use LF Mapping →

Now we have the rows with prefix **aba**

FM Index: querying

$P =$ **aba**

Got the same range, [3, 5), we would have got from suffix array

	F						L
	$	a	b	a	a	b	a_0
	a_0	$	a	b	a	a	b_0
	a_1	a	b	a	$	a	b_1
[3, 5)	a_2	b	a	$	a	b	a_1
	a_3	b	a	a	b	a	$
	b_0	a	$	a	b	a	a_2
	b_1	a	a	b	a	$	a_3

Where are these?

	6	$
	5	a $
	2	a a b a $
[3, 5)	3	a b a $
	0	a b a a b a $
	4	b a $
	1	b a a b a $

Unlike suffix array, we don't immediately know *where* the matches are in T...

FM Index: querying

When *P* does not occur in *T*, we eventually fail to find next character in *L*:

$P =$ **bba**

	F					L	
	$\$$	a	b	a	a	b	a_0
	a_0	$\$$	a	b	a	a	b_0
	a_1	a	b	a	$\$$	a	b_1
	a_2	b	a	$\$$	a	b	a_1
	a_3	b	a	a	b	a	$\$$
Rows with **ba** prefix	b_0	a	$\$$	a	b	a	a_2 ← No **b**s!
	b_1	a	a	b	a	$\$$	a_3

FM Index: querying

If we *scan* characters in the last column, that can be slow, $O(m)$

$P =$ **ab**a

F						L
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

Scan, looking for **b**s

FM Index: lingering issues

(1) Scanning for preceding character is slow

$\$\ a\ b\ a\ a\ b\ a_0$
$a_0\ \$\ a\ b\ a\ a\ b_0$
$a_1\ a\ b\ a\ \$\ a\ b_1$
$a_2\ b\ a\ \$\ a\ b\ a_1$
$a_3\ b\ a\ a\ b\ a\ \$$
$b_0\ a\ \$\ a\ b\ a\ a_2$
$b_1\ a\ a\ b\ a\ \$\ a_3$

$O(m)$ scan

(2) Storing ranks takes too much space

m integers

```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

(3) Need way to find where matches occur in *T*:

$\$\ a\ b\ a\ a\ b\ a_0$
$a_0\ \$\ a\ b\ a\ a\ b_0$
$a_1\ a\ b\ a\ \$\ a\ b_1$
$a_2\ b\ a\ \$\ a\ b\ a_1$
$a_3\ b\ a\ a\ b\ a\ \$$
$b_0\ a\ \$\ a\ b\ a\ a_2$
$b_1\ a\ a\ b\ a\ \$\ a_3$

Where?

FM Index: fast rank calculations

Is there an fast way to determine which **b**s precede the **a**s in our range?

F						*L*
\$	a	b	a	a	b	$\mathbf{a}_0$
$\mathbf{a}_0$	\$	a	b	a	a	$\mathbf{b}_0$
$\mathbf{a}_1$	a	b	a	\$	a	$\mathbf{b}_1$
$\mathbf{a}_2$	b	a	\$	a	b	$\mathbf{a}_1$
$\mathbf{a}_3$	b	a	a	b	a	**\$**
$\mathbf{b}_0$	a	\$	a	b	a	$\mathbf{a}_2$
$\mathbf{b}_1$	a	a	b	a	\$	$\mathbf{a}_3$

FM Index: fast rank calculations

Idea: pre-calculate cumulative # **a**s, **b**s in *L* up to every row:

	Tally	
L	**a**	**b**
a	**1**	0
b	1	**1**
b		
a		
$		
a		
a		

FM Index: fast rank calculations

Idea: pre-calculate cumulative # **a**s, **b**s in *L* up to every row:

L	*Tally* **a**	**b**
a	**1**	0
b	1	**1**
b	1	**2**
a	**2**	2
$	2	2
a	**3**	2
a	**4**	2

FM Index: fast rank calculations

Tally

F	*L*	**a**	**b**
$	**a**	1	0
a	**b**	1	1
a	**b**	1	2
a	**a**	2	2
a	**$**	2	2
b	**a**	3	2
b	**a**	4	2

0 **b**s up to & including this row

2 **b**s up to & including this row

So $\mathbf{b}_0$ and $\mathbf{b}_1$ must be in there!

FM Index: fast rank calculations

F	*L*	*Tally* **a**	*Tally* **b**
$	**a**	1	0
a	**b**	1	1
a	**b**	1	2
a	**a**	2	2
a	**$**	2	2
b	**a**	3	2
b	**a**	4	2

2 **a**’s up to & including this row

4 **a**’s up to & including this row

So $\mathbf{a}_2$ and $\mathbf{a}_3$ must be in there!

O(1) time; 2 lookups
regardless of range size

FM Index: fast rank calculations

Tally

F	*L*	**a**	**b**
$	**a**	1	0
a	**b**	1	1
a	**b**	1	2
a	**a**	2	2
a	**$**	2	2
b	**a**	3	2
b	**a**	4	2

m

$\vdash |\Sigma| \dashv$

Tally is $m \times |\Sigma|$ integers

Too big!

FM Index: fast rank calculations

Next idea: pre-calculate # **a**s, **b**s in *L* up to *some* rows, e.g. every 5th row. Call pre-calculated rows *checkpoints*.

Tally

F	*L*	**a**	**b**	
$	**a**	1	0	Checkpoint 1
a	**b**			
a	**b**			
a	**a**			
a	**$**			
b	**a**	3	2	Checkpoint 2
b	**a**			

FM Index: fast rank calculations

Next idea: pre-calculate # **a**s, **b**s in *L* up to *some* rows, e.g. every 5th row. Call pre-calculated rows *checkpoints*.

F	*L*	*Tally* **a**	*Tally* **b**
$	**a**	1	0
a	**b**		
a	**b**		
a	**a**		
a	**$**		
b	**a**	3	2
b	**a**		

Lookup here succeeds as usual

Oops: not a checkpoint

But there's one nearby

To resolve a lookup for a non-checkpoint row, walk to nearest checkpoint. Use tally at that checkpoint, *adjusted for characters we saw along the way*.

FM Index: fast rank calculations

What goes here?

$482 + 2 = 484$

Checkpoint above

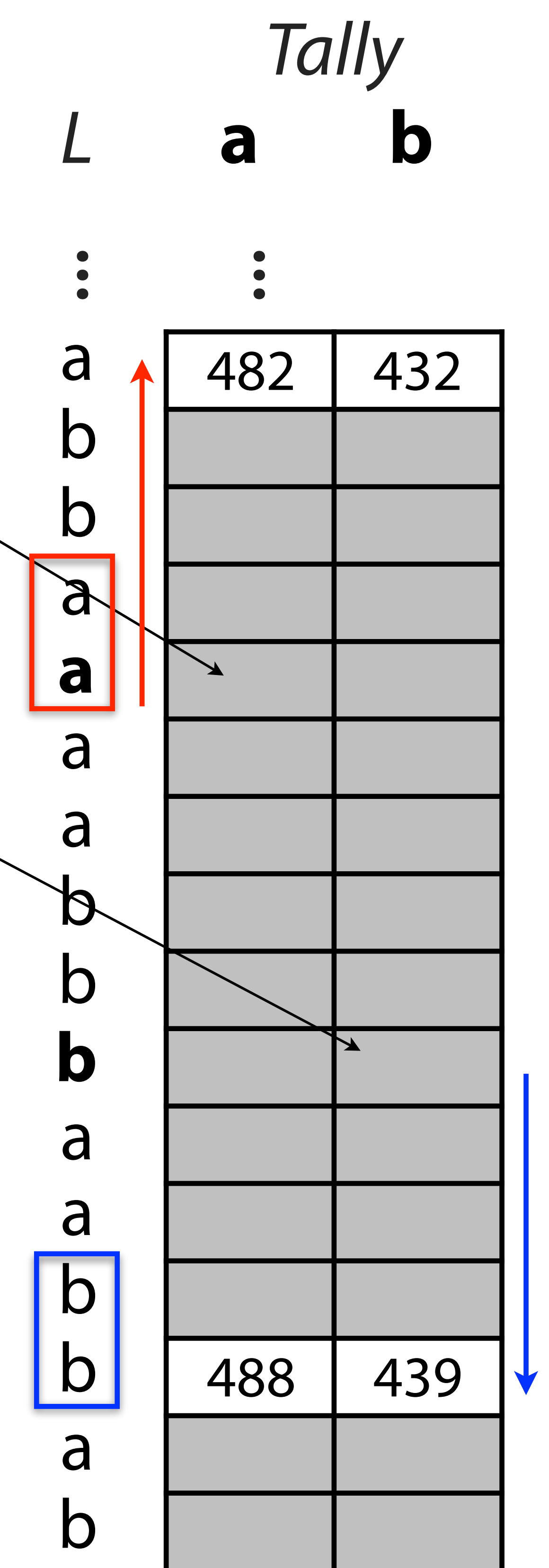
as along the way

What's goes here?

$439 - 2 = 437$

Checkpoint below

bs along the way



If checkpoints are $O(1)$ distance apart, lookups are $O(1)$

FM Index: a few problems

Solved! At the expense of adding checkpoints ($O(m)$ integers) to index.

(1)

F						*L*
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

This scan is $O(m)$ work

$O(1)$ with checkpoints

(2) Ranking takes too much space

m integers

```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

Still $O(m)$ space to store checkpoints, but *we control the constant*

FM Index: a few problems

Not yet solved:

(3) Where are these occurrences in *T*?

$	a	b	a	a	b	**a**$_0$
a$_0$	$	a	b	a	a	**b**$_0$
a$_1$	a	b	a	$	a	**b**$_1$
a$_2$	b	a	$	a	b	**a**$_1$
a$_3$	b	a	a	b	a	**$**
b$_0$	a	$	a	b	a	**a**$_2$
b$_1$	a	a	b	a	$	**a**$_3$

If we had suffix array, we could look up offsets...

F						*L*
$	a	b	a	a	b	**a**
a	$	a	b	a	a	**b**
a	a	b	a	$	a	**b**
a	b	a	$	a	b	**a**
a	b	a	a	b	a	**$**
b	a	$	a	b	a	**a**
b	a	a	b	a	$	**a**

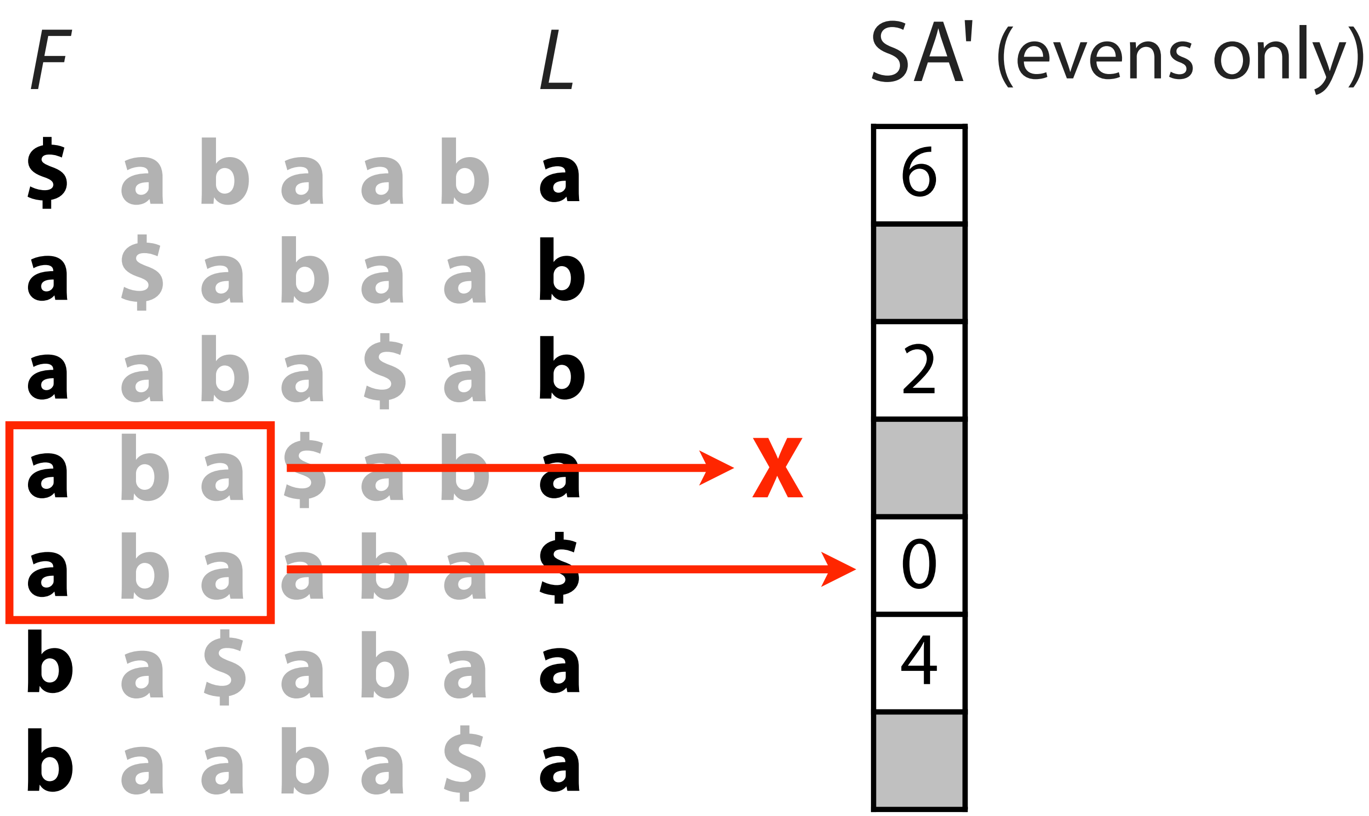
Offsets: 0, 3

SA	
6	**$**
5	**a $**
2	**a a b a $**
3	**a b a $**
0	**a b a a b a $**
4	**b a $**
1	**b a a b a $**

...but we don't; we are trying to avoid storing *m* integers

FM Index: resolving offsets

Idea: store some suffix array elements, but not all



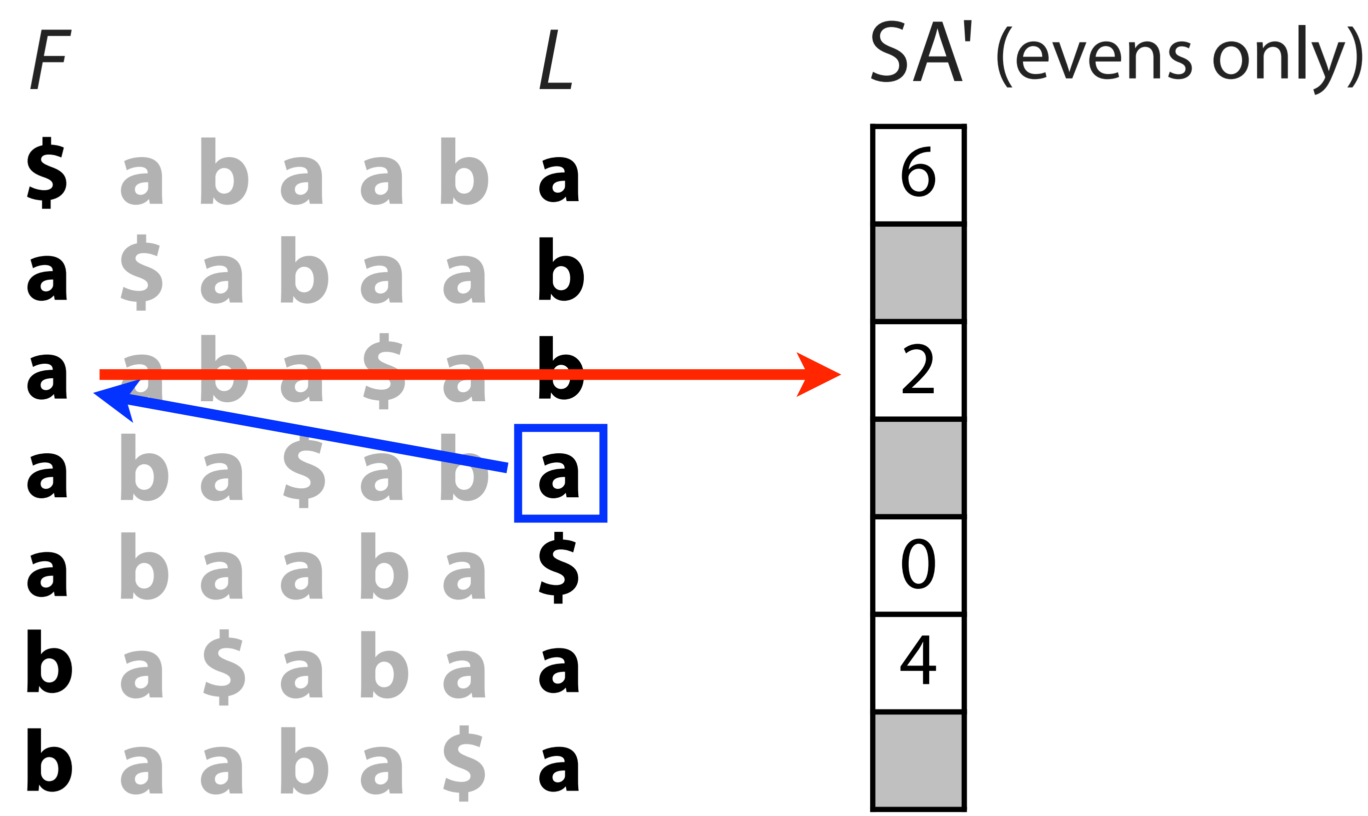
Lookup for row 4 succeeds

Lookup for row 3 fails - SA entry was discarded

FM Index: resolving offsets

LF Mapping tells us that "**a**" at the end of row 3 corresponds to...

... "**a**" at the beginning of row 2



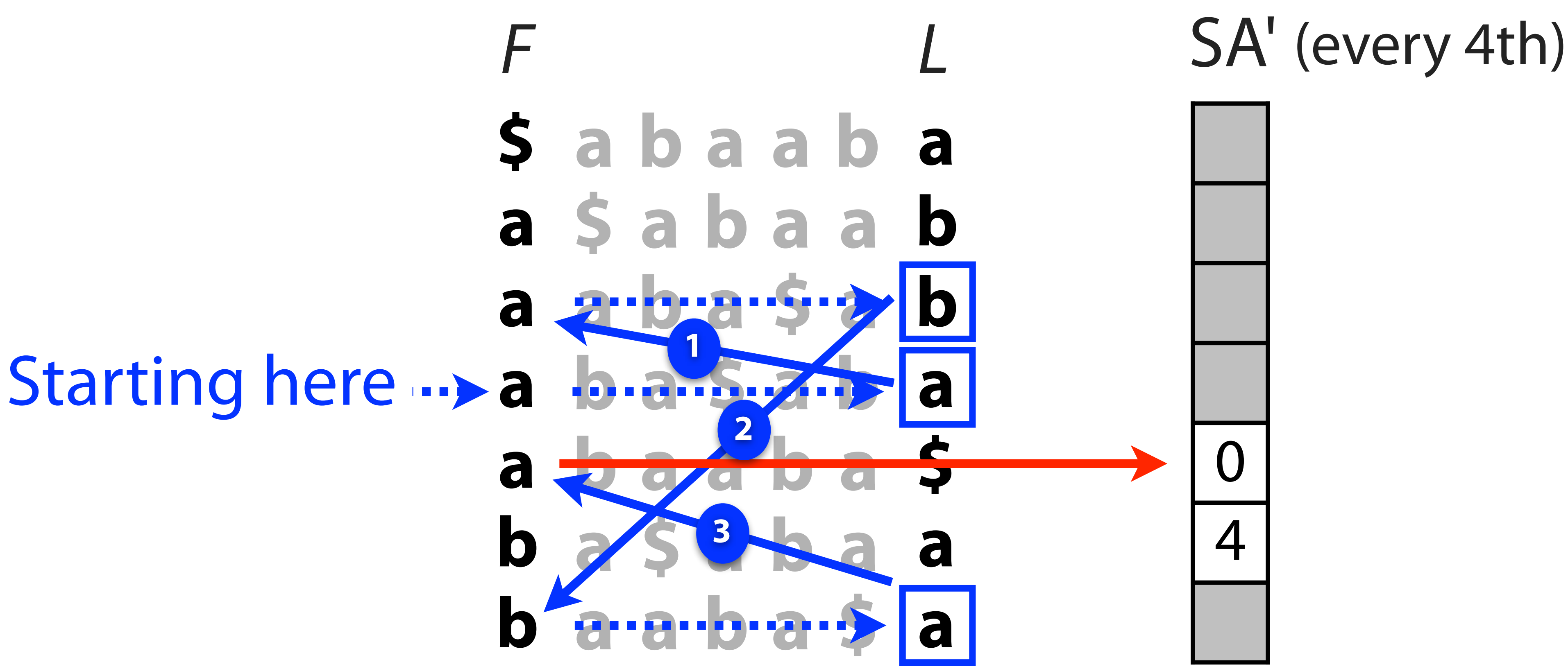
Row 2 of suffix array = 2

Missing value in row 3 = 2 (row 2's SA val) + 1 (# steps to row 2) = **3**

If saved SA values are O(1) positions apart in *T*, resolving offset is O(1) time

FM Index: resolving offsets

Many LF-mapping steps may be required to get to a sampled row:



Missing value = 0 (SA elt at destination) + 3 (# steps to destination) = **3**

FM Index: problems solved

Solved! At the expense of adding some SA values ($O(m)$ integers) to index

Call this the "SA sample"

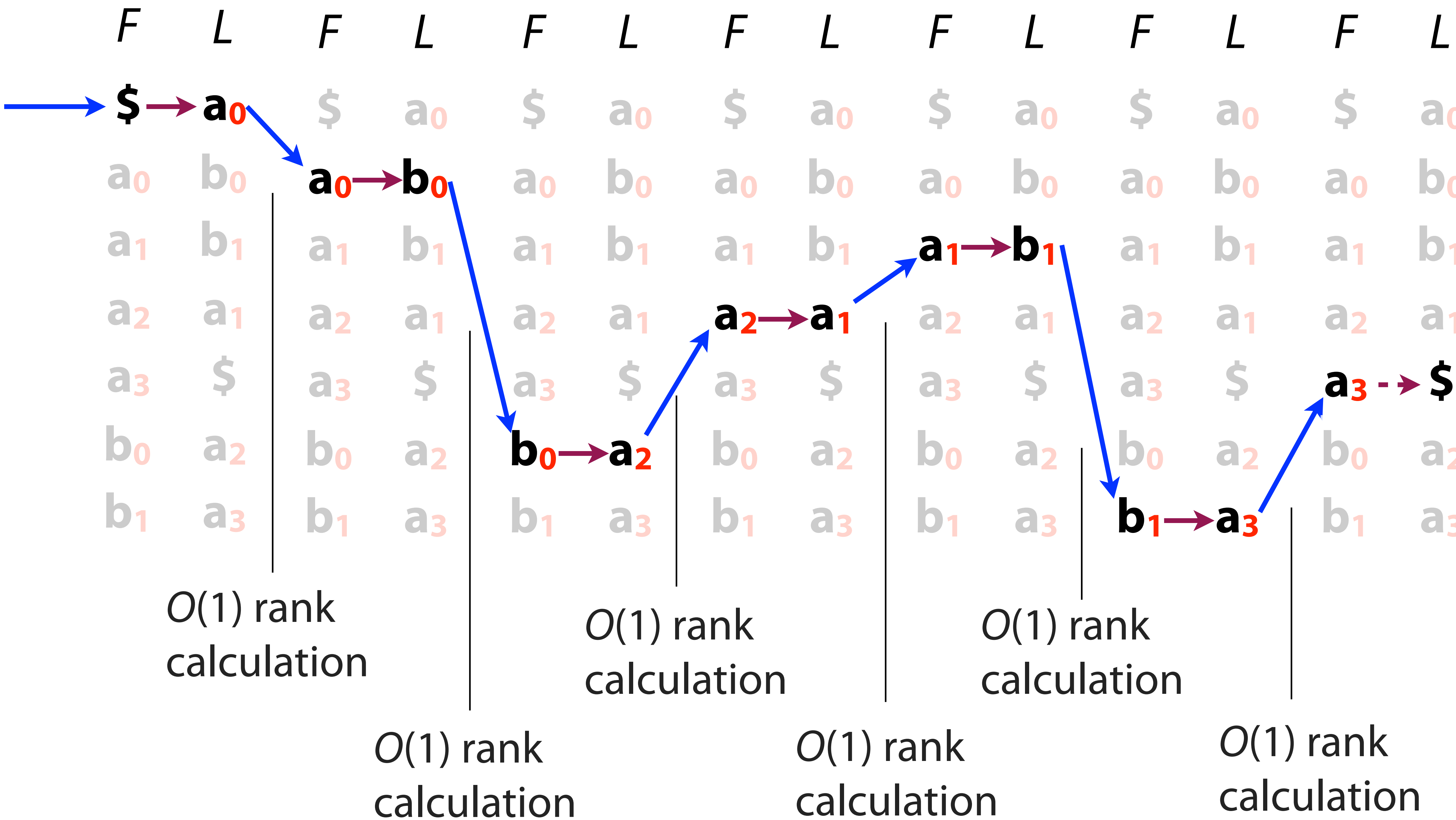
(3) Need a way to find where these occurrences are in *T*:

$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

With SA sample we can do this in $O(1)$ time per occurrence

FM Index

$|T| = m$



Reversing BWT(T) in FM Index is $O(m)$ time

FM Index

$|T| = m, |P| = n$

$P =$ **aba**

F						L
$\$$	a	b	a	a	b	a_0
a_0	$\$$	a	b	a	a	b_0
a_1	a	b	a	$\$$	a	b_1
a_2	b	a	$\$$	a	b	a_1
a_3	b	a	a	b	a	$\$$
b_0	a	$\$$	a	b	a	a_2
b_1	a	a	b	a	$\$$	a_3

$P =$ **aba**

F						L
$\$$	a	b	a	a	b	a_0
a_0	$\$$	a	b	a	a	b_0
a_1	a	b	a	$\$$	a	b_1
a_2	b	a	$\$$	a	b	a_1
a_3	b	a	a	b	a	$\$$
b_0	a	$\$$	a	b	a	a_2
b_1	a	a	b	a	$\$$	a_3

$P =$ **aba**

F						L
$\$$	a	b	a	a	b	a_0
a_0	$\$$	a	b	a	a	b_0
a_1	a	b	a	$\$$	a	b_1
a_2	b	a	$\$$	a	b	a_1
a_3	b	a	a	b	a	$\$$
b_0	a	$\$$	a	b	a	a_2
b_1	a	a	b	a	$\$$	a_3

2 $O(1)$ rank calculations

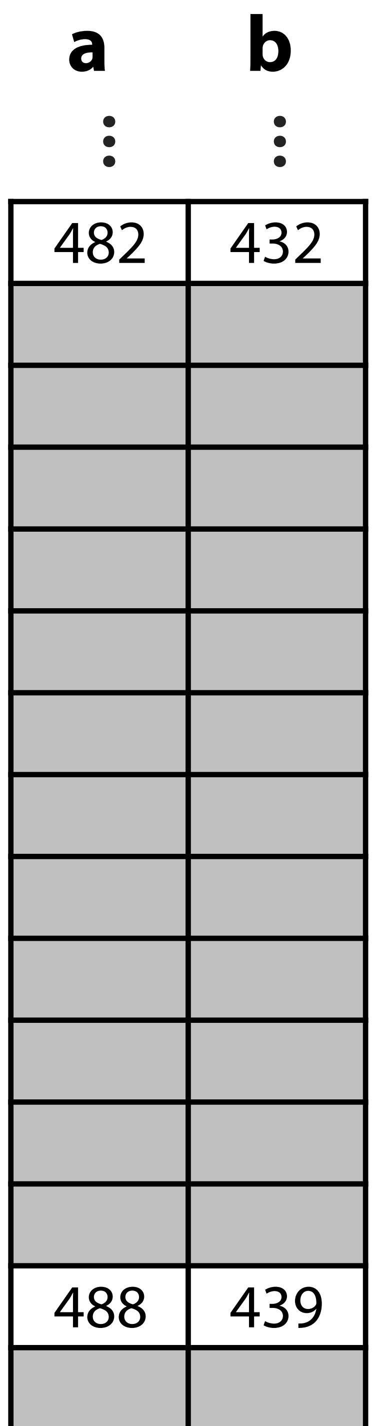
2 $O(1)$ rank calculations

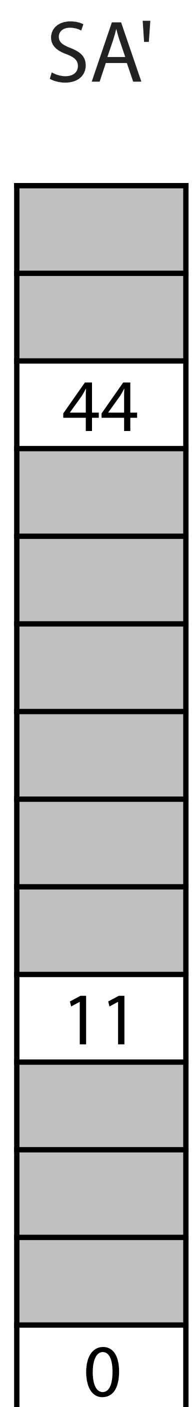
Determining of P occurs in T in FM Index is $O(n)$ time

FM Index

Let ***a*** = fraction of rows we keep

Let ***b*** = fraction of SA elements we keep





FM Index consists of these, plus *L* and *F* columns

Note: suffix tree/array didn't have parameters like ***a*** and ***b***

FM Index

Components of FM Index:

First column (F): $\sim |\Sigma|$ integers

Last column (L): m characters

SA sample: $m \cdot a$ integers, a is fraction of SA elements kept

Checkpoints: $m \cdot |\Sigma| \cdot b$ integers, b is fraction of tallies kept

For DNA alphabet (2 bits / nt), T = human genome, ***a*** = 1/32, ***b*** = 1/128 :

First column (F): 16 bytes

Last column (L): 2 bits * 3 billion chars = 750 MB

SA sample: 3 billion chars * 4 bytes / 32 = ~ 400 MB

Checkpoints: 3 billion * 4 alphabet chars * 4 bytes / *128* = ~ 400 MB

Total ≈ 1.5 GB

~0.5 bytes per input char

(blue indicates what we can adjust by changing *a* & *b*)

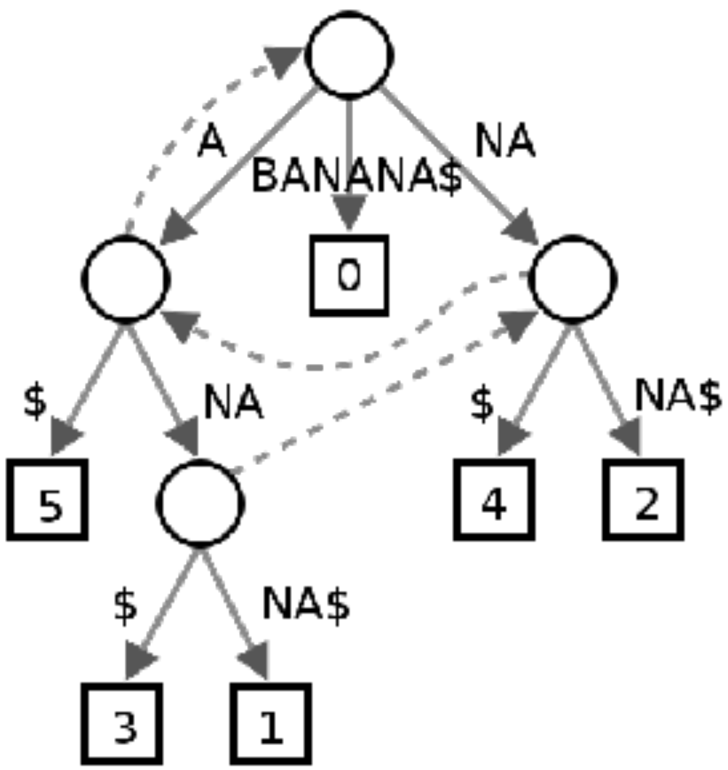
FM Index: small memory footprint

Paolo Ferragina, and Giovanni Manzini. "Opportunistic data structures with applications." *Foundations of Computer Science, 2000. Proceedings. 41st Annual Symposium on*. IEEE, 2000.

FM Index described here is simplified version of what's described in paper

Also discussed in paper: compressing BWT(T) for further savings (and selectively decompression portions of it at query time)

FM Index: small memory footprint



Suffix tree

≥ 45 GB

6	$
5	A$
3	ANA$
1	ANANA$
0	BANANA$
4	NA$
2	NANA$

Suffix array

≥ 12 GB

$ BANAN **A**
A $BANA **N**
A NA$BA **N**
A NANA$ **B**
B ANANA **$**
N A$BAN **A**
N ANA$B **A**

FM Index

~ 1.5 GB

Suffix index bounds

	Suffix tree	**Suffix array**	**FM Index**
Time: Does P occur?	$O(n)$	$O(n \log m)$	$O(n)$
Time: Count k occurrences of P	$O(n + k)$	$O(n \log m)$	$O(n)$
Time: Report k locations of P	$O(n + k)$	$O(n \log m + k)$	$O(n + k)$
Space	$O(m)$	$O(m)$	$O(m)$
Needs T?	*yes*	*yes*	*no*
Bytes per input character	>15	~4	~0.5

$m = |T|$, $n = |P|$, $k =$ # occurrences of P in T