Lecture 5

Vector spaces

Vector spaces and subspaces

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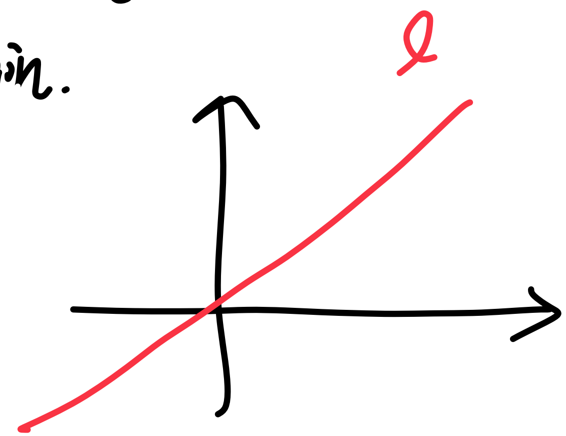

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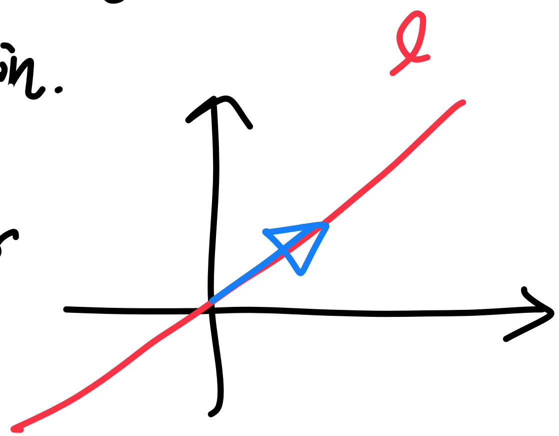
It is determined by one vector



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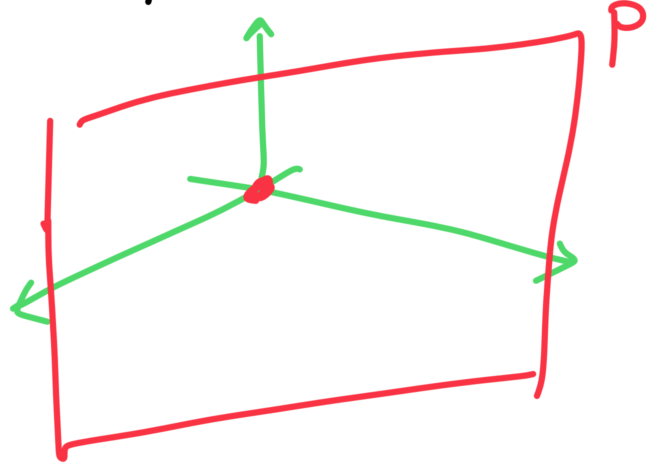

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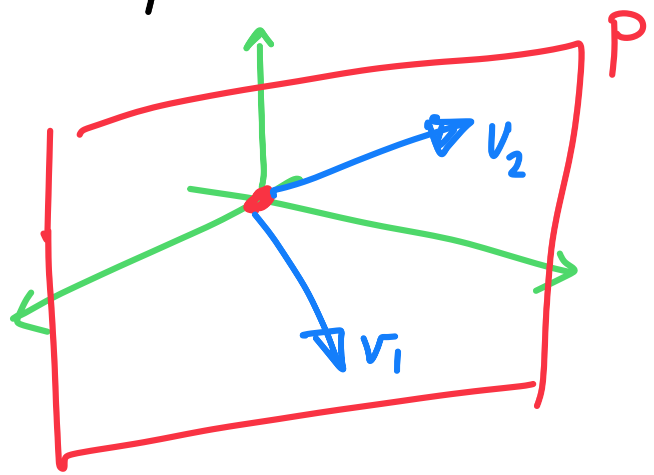

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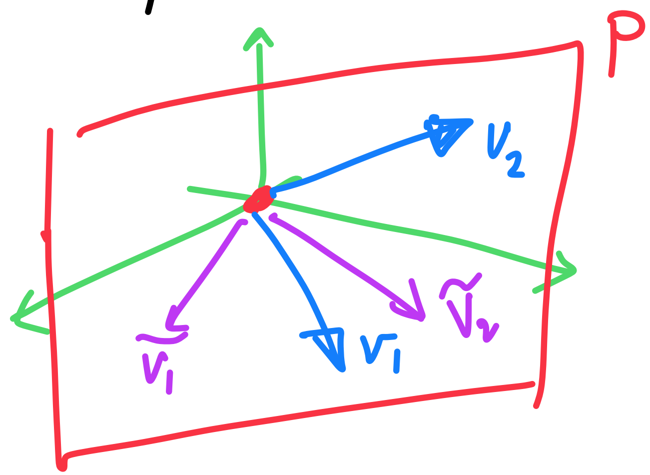

which can be specified by two vectors v_1, v_2.

- We can choose different sets of vectors $\tilde{v}_1, \tilde{v}_2$ express the same plane.

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- We extract these properties into a notion of vector space.

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Def: • A vector space is a set V (elements of V are called vectors) equipped with two operations.

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• They satisfy the 8 compatibility conditions.

(the conditions in red in the following example)

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e.g. $\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \ x_1, x_2, x_3 \text{ real numbers} \right\}.$

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$\cdot$: $c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$, $+$: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

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Satisfying:

1. $$\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right)$$

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Abstractly

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1. $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

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2. $(c_1+c_2)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c_2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

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4. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

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2. $(c_1+c_2)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c_2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ 2. $(c_1+c_2)\vec{x} = c_1\vec{x}+c_2\vec{x}$

3. $c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ 3. $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$.

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Cont'd:

5. $c_1\left(c_2\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = (c_1c_2)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

Vector spaces and subspaces

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5. $$c_1\left(c_2\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=(c_1c_2)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

5. $$c_1(c_2\vec{x})=(c_1c_2)\vec{x}$$

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5. $c_1(c_2 \vec{x}) = (c_1 c_2)\vec{x}$

6. $1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

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5. $c_1(c_2\vec{x}) = (c_1c_2)\vec{x}$

6. $1.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

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6. $1\cdot\vec{x} = \vec{x}$.

7. There exists $\begin{bmatrix}0\\0\\0\end{bmatrix} \in \mathbb{R}^3$ s.t.

$$\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

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5. $c_1\left(c_2\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=(c_1c_2)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

5. $c_1(c_2\vec{x})=(c_1c_2)\vec{x}$

6. $1.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

6. $1\cdot\vec{x}=\vec{x}$.

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$$\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

7. $\exists\ \vec{0}$ s.t. $\vec{0}+\vec{x}=\vec{x}$ for all $\vec{x}$.

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7. $\exists\, \vec{0}$ s.t. $\vec{0}+\vec{x}=\vec{x}$ for all $\vec{x}$.

8. For any $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3$, there exists $\begin{bmatrix}-x_1\\-x_2\\-x_3\end{bmatrix}$ s.t. $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} + \begin{bmatrix}-x_1\\-x_2\\-x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

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5. $c_1\left(c_2\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=(c_1c_2)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

5. $c_1(c_2\vec{x})=(c_1c_2)\vec{x}$

6. $1\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

6. $1\cdot\vec{x}=\vec{x}$.

7. There exists $\begin{bmatrix}0\\0\\0\end{bmatrix}\in\mathbb{R}^3$ s.t.

$\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

7. $\exists\,\vec{0}$ s.t. $\vec{0}+\vec{x}=\vec{x}$ for all $\vec{x}$.

8. For any $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in\mathbb{R}^3$, there exists $\begin{bmatrix}-x_1\\-x_2\\-x_3\end{bmatrix}$ s.t. $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}-x_1\\-x_2\\-x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

8. for any $\vec{x}$, there exists $-\vec{x}$ s.t. $\vec{x}+(-\vec{x})=0$

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- it is "isomorphic" to $\mathbb{R}^4$.

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$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = -x_3,\ x_2 = -x_3 \Rightarrow W = \left\{ \begin{bmatrix} -c \\ -c \\ c \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$

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- $\left(x_1\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\right) + \left(y_1\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + y_2\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\right) = (x_1+y_1)\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + (x_2+y_2)\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$

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- If we let $W = \left\{ x_1\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \;\middle|\; x_1, x_4 \in \mathbb{R} \right\}$.

  It is a subspace of $\mathbb{R}^3$.

- $\left(x_1\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\right) + \left(y_1\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + y_2\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\right) = (x_1+y_1)\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + (x_2+y_2)\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$

- $c\left(x_1\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\right) = (cx_1)\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + (cx_2)\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$

Subspaces

Subspaces

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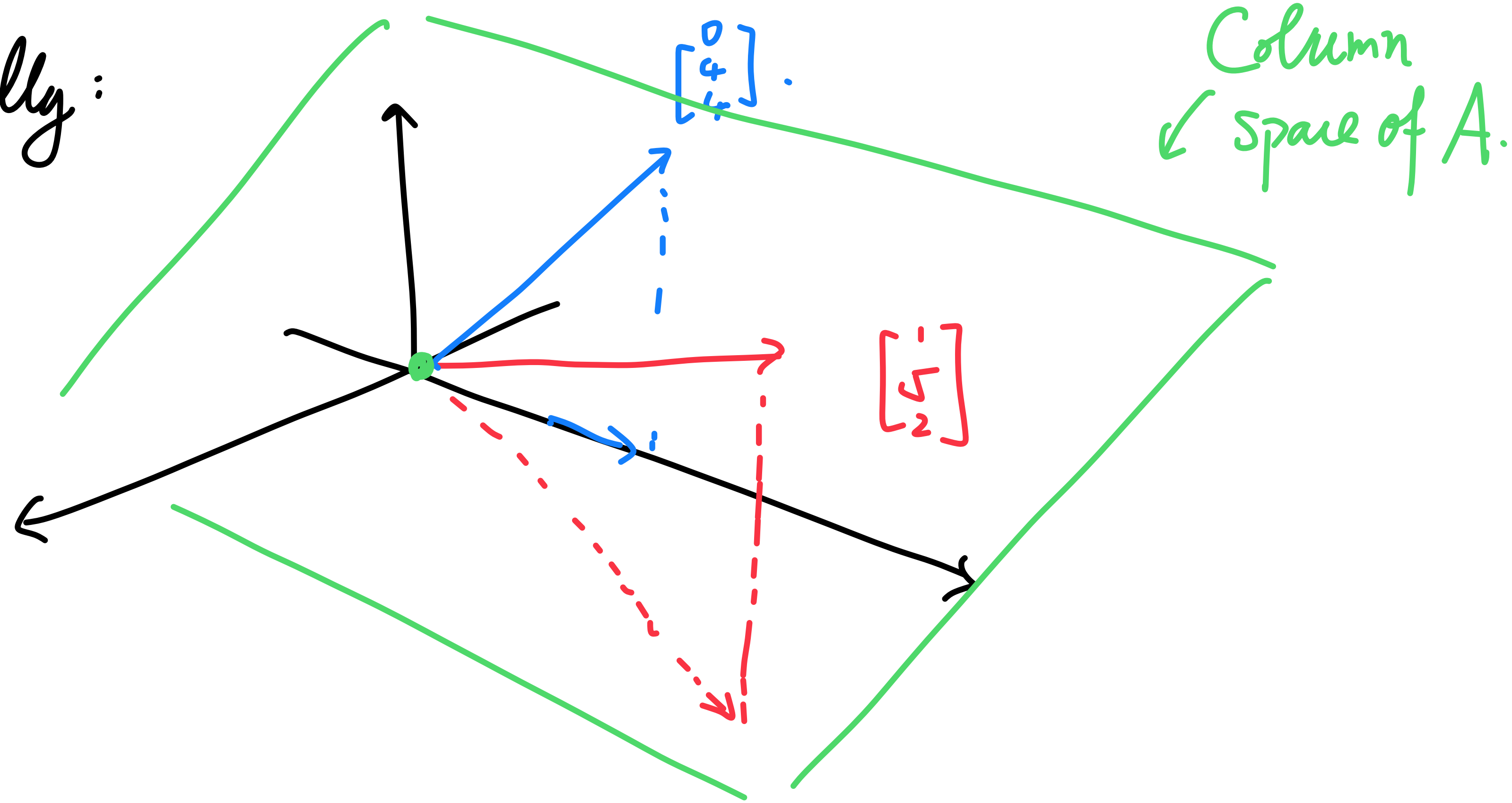

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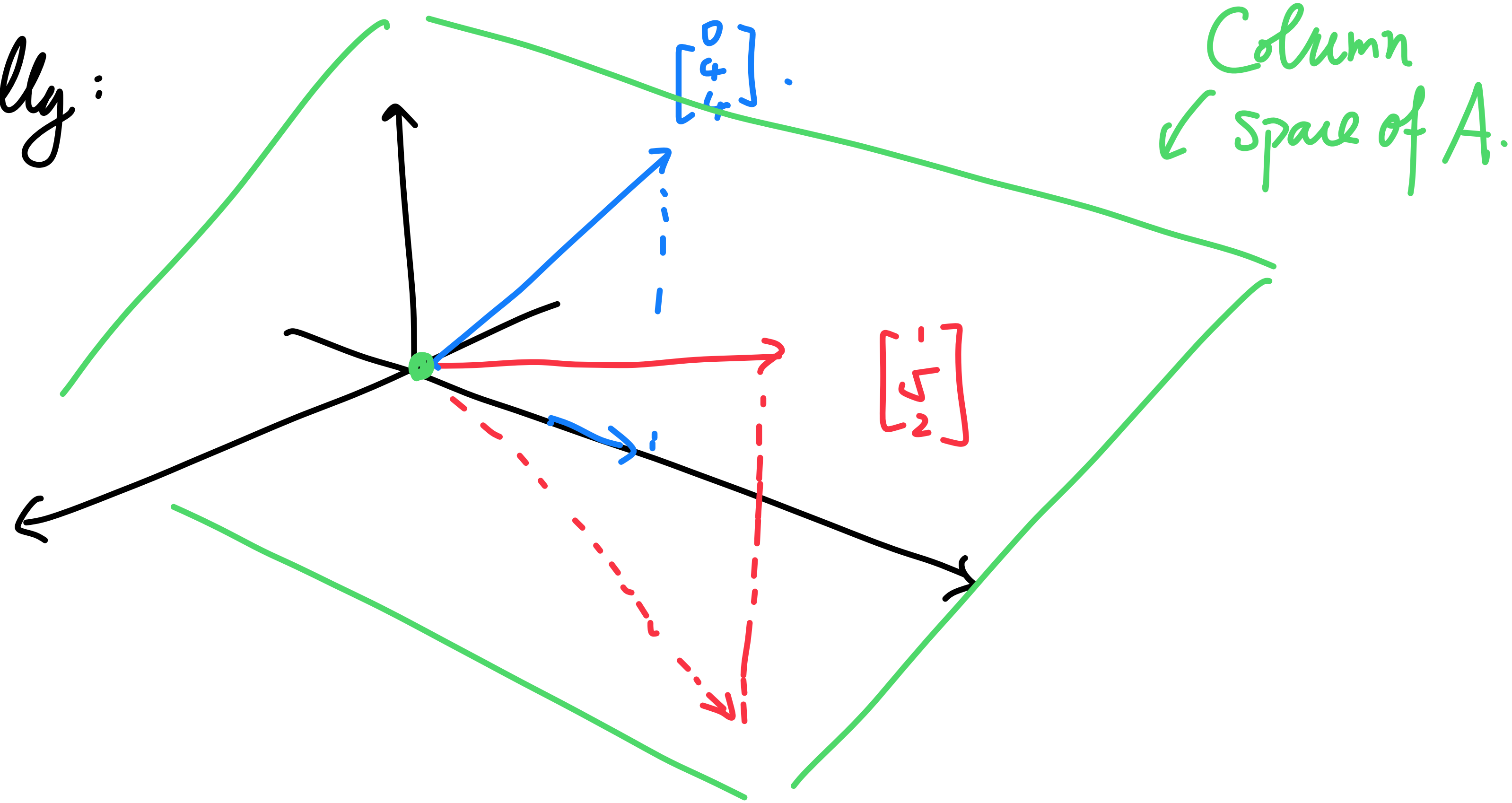

Def: Given a $m \times n$ matrix $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$, the column space of A is defined to be.

$$C(A) = \{ c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n \mid c_i \in \mathbb{R} \}$$

Vector spaces and subspaces

Vector spaces and subspaces

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Rk: Maybe you notice that $-\vec{v} = (-1)\cdot \vec{v}$ for general vector space.

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- $W_3 = \{ f \mid f(x) = ax + b \text{ for } x \in \mathbb{R} \}$
- $W_4 = \{ f \mid f$ is twice differentiable and satisfies

  $$\frac{d^2 f}{dx^2} + c_1 \frac{df}{dx} + c_2 = 0 \}.$$

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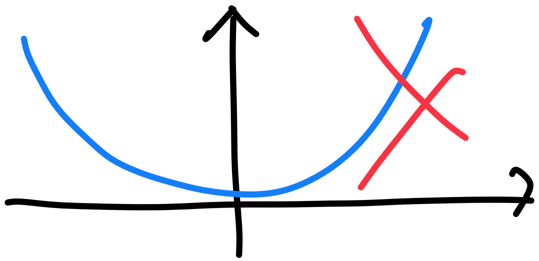

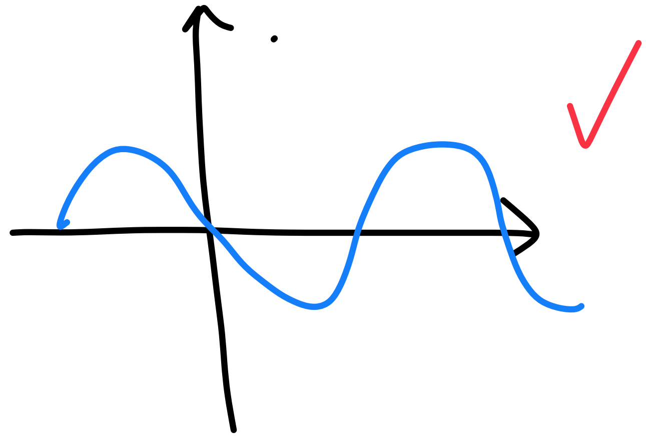

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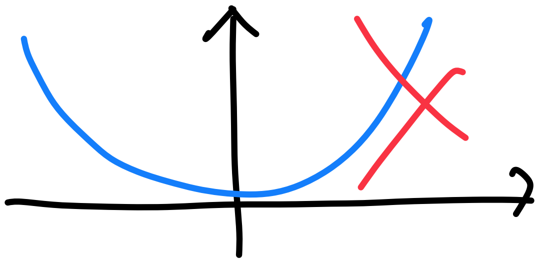

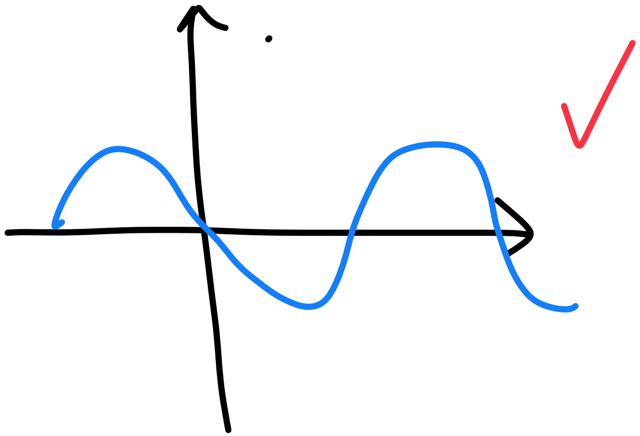

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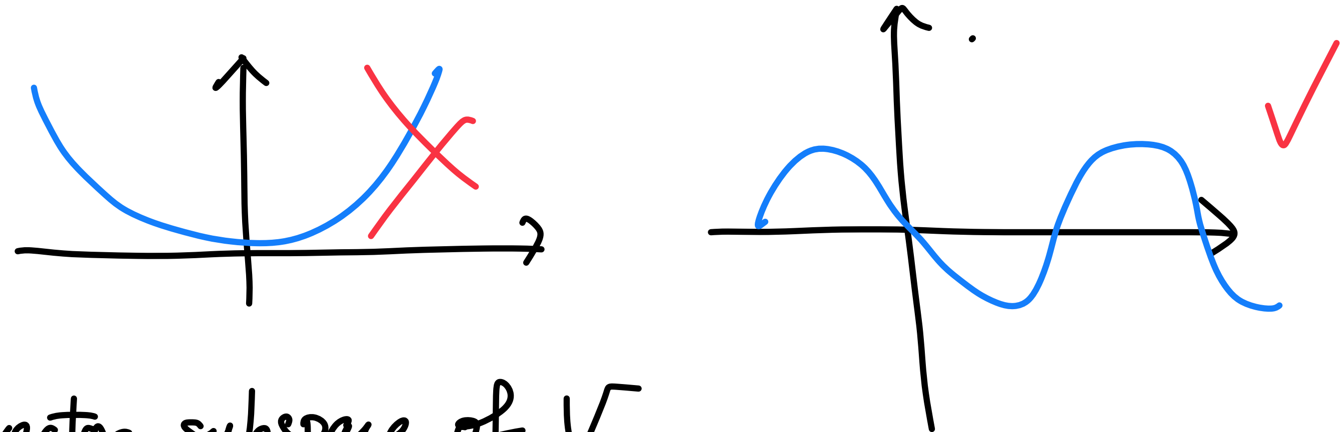

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