Lecture 12

Matrix and linear transformation

Matrix representation of linear transformation

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$$[I] = \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{4}\end{bmatrix}$$

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$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1+x_2 \\ x_1-x_2 \end{bmatrix}$$

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Prop: • Given V and W as above, with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ basis for V. Fix $\vec{l}_1, \ldots, \vec{l}_n$ be n vectors in W

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$$L(c_1B_1 + c_2B_2) = (c_1B_1 + c_2B_2) \cdot A$$

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$$L(c_1B_1 + c_2B_2) = (c_1B_1 + c_2B_2)\cdot A$$

$$= c_1B_1 \cdot A + c_2B_2 \cdot A$$

$$= c_1(B_1 \cdot A) + c_2(B_2 \cdot A)$$

Matrix representation of linear transformation

e.g.

- Let $M_{2\times 2}(\mathbb{R})$ be the space of 2×2 matrices :

  Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and we let $L : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

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$$= c_1 L(B_1) + c_2 L(B_2).$$

Matrix representation of linear transformation

Matrix representation of linear transformation

- We take the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for $M_{2\times 2}(\mathbb{R})$.

Matrix representation of linear transformation

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Matrix representation of linear transformation

- We take the basis $\underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}$ for $M_{2\times 2}(\mathbb{R})$.

$$\begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array}$$

$$L(v_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

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Matrix representation of linear transformation

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Matrix representation of linear transformation

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for $M_{2\times 2}(\mathbb{R})$.

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$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is the matrix representation.

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Therefore the matrix representation is $[L_P] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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$$\longrightarrow {}_{\vec{e}_i}[L_I]_{\vec{w}_i} \circ {}_{\vec{w}_i}[L_p]_{\vec{v}_i} = {}_{\vec{e}_i}[L_p]_{\vec{v}_i}$$

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$$[M] \cdot [L] = [M \circ L]$$

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- In wording, after the basis are fixed accordingly the composition of linear map correspond to matrix multiplication.

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$\updownarrow$

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$\vec{v}_1, \ldots, \vec{v}_n$ $\qquad$ $\vec{w}_1, \ldots, \vec{w}_n$

$\updownarrow$

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transition matrix

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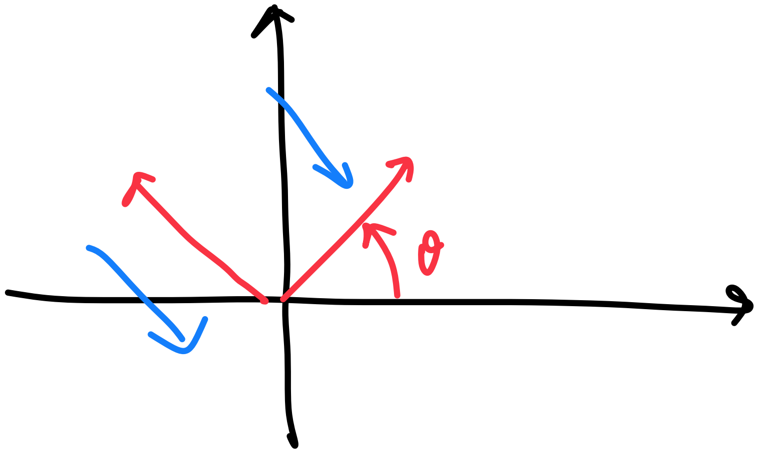

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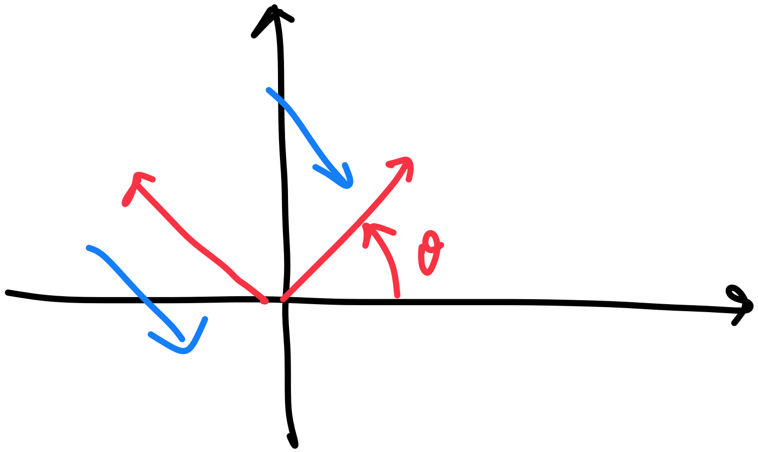

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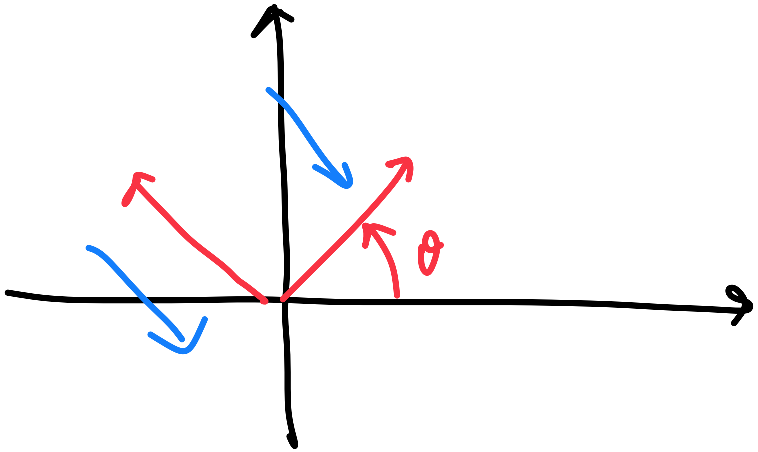

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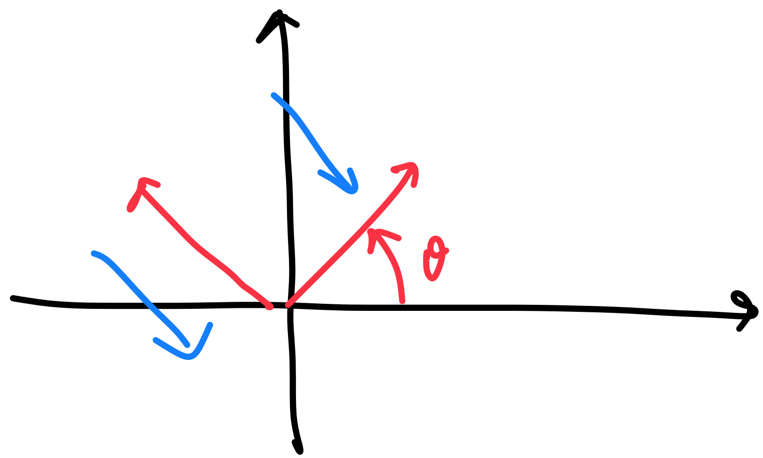

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$$_{\vec{w}_i}[P]_{\vec{v}_i} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Longleftrightarrow \begin{array}{l} P\vec{v}_1 = \vec{w}_1 \\ P\vec{v}_0 = 0. \end{array}$$

Matrix representation of linear transformation

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$$\underset{\vec{e}_1,\vec{e}_2}{\mathbb{R}^2} \xrightarrow{id} \underset{\vec{v}_1,\vec{v}_2}{\mathbb{R}^2} \xrightarrow{P} \underset{\vec{v}_1,\vec{v}_2}{\mathbb{R}^2} \xrightarrow{id.} \underset{\vec{e}_1,\vec{e}_2}{\mathbb{R}^2}.$$

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- Their matrix representation satisfies.

$$_{\vec{e}_i}[id]_{\vec{v}_i} \cdot {}_{\vec{v}_i}[P]_{\vec{v}_i} \cdot {}_{\vec{v}_i}[id]_{\vec{e}_i} = {}_{\vec{e}_i}[P]_{\vec{e}_i}$$

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- $${}_{\vec{e}_i}[P]_{\vec{e}_i} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

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this is a diagonalization of [P], later chapter.

Matrix representation of linear transformation

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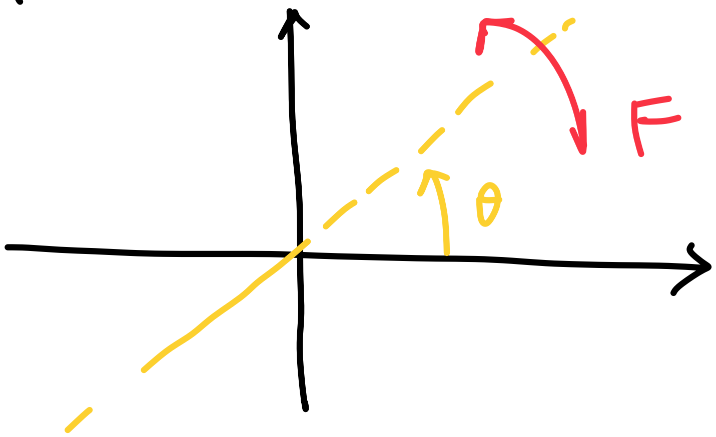
- Suppose we want to write down reflection along the line



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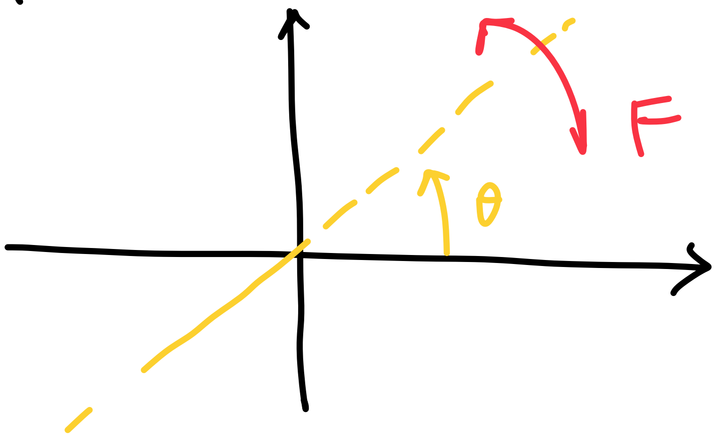

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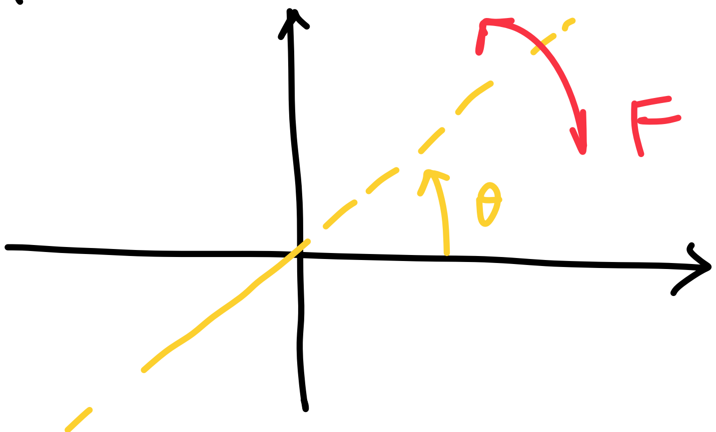

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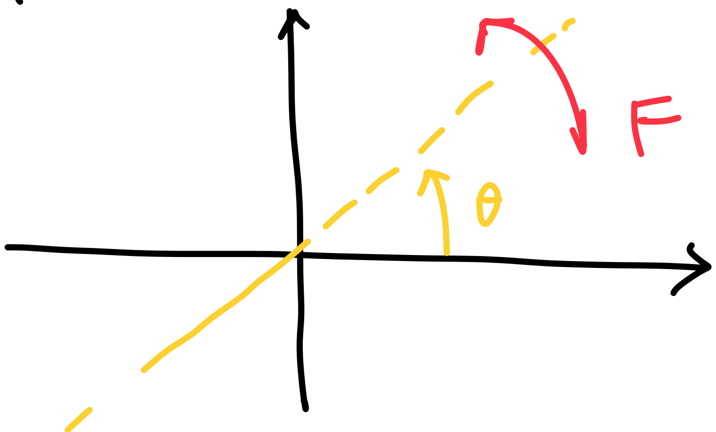

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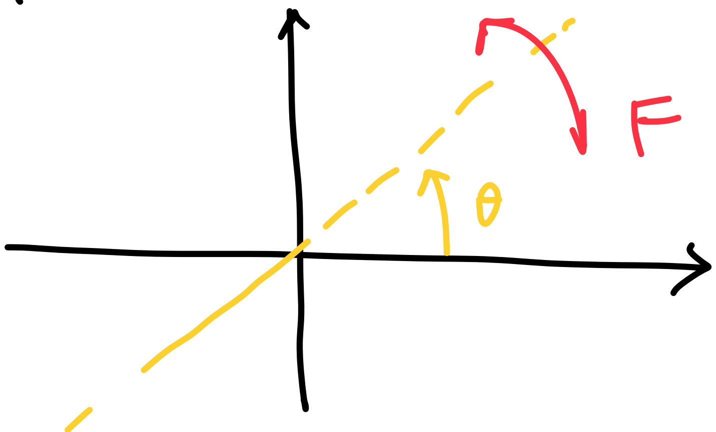

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- $${}_{\vec{e}_i}[F]_{\vec{e}_i} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

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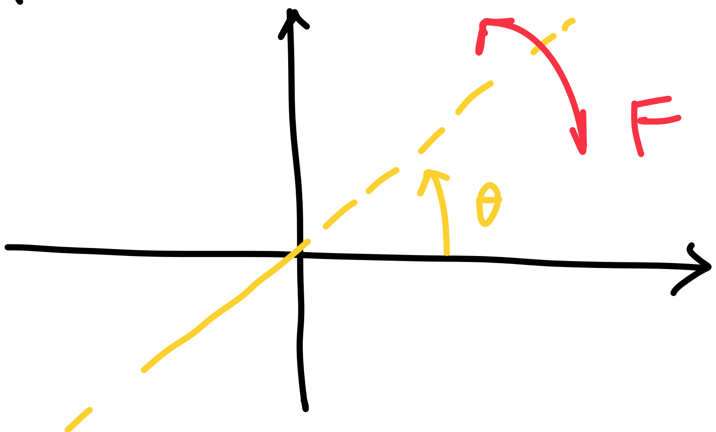

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Image and Kernel of linear transformation

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Image and Kernel of linear transformation

Image and Kernel of linear transformation

- Now with $L: V \longrightarrow W$ equipped with basis $\vec{v}_1, \ldots, \vec{v}_n$ for V, $\vec{w}_1, \ldots, \vec{w}_m$ for W.

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- Matrix representation: (say $m=2$, $n=3$).

  write $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ then

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$$L(\vec{v}) = [\vec{w}_1, \vec{w}_2] [L] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

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- $N([L]) \longleftrightarrow \mathrm{Ker}(L)$ given by the 1-1 correspondence.

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \longrightarrow [\vec{v}_1, \vec{v}_2, \vec{v}_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Image and Kernel of linear transformation

Image and Kernel of linear transformation

- $C([L]) \longleftrightarrow \mathrm{Im}(L)$ is a 1-1 correspondence.

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- $\mathrm{Ker}(L) \subseteq V$ is a subspace

  $\mathrm{Im}(L) \subseteq W$ is a subspace.

Image and Kernel of linear transformation

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  $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \longrightarrow [w_1, w_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

- $\mathrm{Ker}(L) \subseteq V$ is a subspace

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- $\dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = \dim(V)$.

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Claim: f, g are linear and $f \circ g = id_{\mathbb{R}_+}$

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- Assume L is an isomorphism, then we have $T : W \longrightarrow V$ s.t. $T \circ L = id_V$, $L \circ T = id_W$.
- then we take their matrix representation:

$$_{\vec{v}_i}[T]_{\vec{w}_i} \cdot {}_{\vec{w}_i}[L]_{\vec{v}_i} = {}_{\vec{v}_i}[id]_{\vec{v}_i} = I_{n \times n}.$$

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$\Longrightarrow$ $m = n$ and $[L]^{-1} = [T]$

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$m = rk(I_{m\times m}) \leq rk([L]) \leq n$

$n = rk(I_{n\times n}) \leq rk([L]) \leq m$.

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$$T(\vec{w}) := [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]\,[L]^{-1}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$ i.e. $[T] = [L]^{-1}$

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- Then ${}_{\vec{v}_i}[T \circ L]_{\vec{v}_i} = I_{n\times n}$

  ${}_{\vec{w}_i}[L \circ T]_{\vec{w}_i} = I_{n\times n}$. $\Rightarrow$ T is the inverse of L.

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linear transformation $L: V \to W$
(with a choice of basis).

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- Composition
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- $\mathrm{Ker}(L)$
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