

Problem set 6

Linear algebra 6

October 22, 2025

1. Let's consider the vector space P_2 consisting of polynomials with degree less than or equal to 2. Suppose we have a basis

$$w_1 = x^2 + x + 4, \quad w_2 = 4x^2 - 3x + 2, \quad w_3 = 2x^2 + 3$$

and another basis

$$v_1 = x^2 - x + 1, \quad v_2 = x + 1, \quad v_3 = x^2 + 1.$$

Find the transition matrix from the basis w_i 's to v_i 's.

2. Let P_2 be the vector space consisting of polynomials with degree less than or equal to 2, and $T : P_2 \rightarrow \mathbb{R}^3$ given by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} 13a_0 + a_1 + 4a_2 \\ a_0 + 13a_1 + 4a_2 \\ 4a_0 + 4a_1 + 10a_2 \end{bmatrix}.$$

Take a basis

$$v_1 = 1 + x - 2x^2, \quad v_2 = 1 - x, \quad v_3 = 1 + x + x^2$$

for P_2 and the standard basis e_1, e_2, e_3 for \mathbb{R}^3 .

- (a) Prove that T is a linear transformation.
- (b) Find the matrix representation of T with respect to the chosen basis.
3. Let $L : V \rightarrow W$ be an isomorphism from vector space V to W . Given a basis v_1, v_2, \dots, v_n of V , show that $L(v_1), L(v_2), \dots, L(v_n)$ is a basis for W .
4. Given two linear transformations $L_1, L_2 : V \rightarrow W$, and we let $T : V \rightarrow W$ defined by $T(v) = L_1(v) + L_2(v)$ for every $v \in V$.
 - (a) Show that T is a linear transformation.
 - (b) If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, with $L_1 = L_A$ and $L_2 = L_B$ for $m \times n$ matrices A, B , then show that $T = L_{A+B}$.

- (c) Show that $C(A+B) \subset C(A)+C(B)$, and hence $\text{rk}(A+B) \leq \text{rk}(A) + \text{rk}(B)$ for $m \times n$ matrices A, B .
5. A linear transformation $L : V \rightarrow W$ is said to be *injective* if two distinct elements v_1, v_2 are distinct after applying L , or in mathematical language $L(v_1) \neq L(v_2)$ whenever $v_1 \neq v_2$. It is said to be *surjective* if any element in W is $L(v)$ for some $v \in V$, in the other words $\text{Im}(L) = W$.
- (a) Prove that L is injective if and only if $\text{Ker}(L) = \{0\}$.
 - (b) Fixing a basis v_1, \dots, v_n of V , and a basis w_1, \dots, w_m for W , prove that its matrix representation $[L]$ with respect to these basis is invertible if and only if it is both injective and surjective.