Lecture 7

Solving Ax=b

Column space and null space

Column space and null space

- **e.g.:**

Consider $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

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- Find column space, null space of A.
- Find condition on b s.t. the equation is consistent
- Find general solutions for $Ax = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$

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pivot column.

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with two equations :

$$\begin{cases} 2x_3 + 2x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_4 \\ x_1 = -2x_2 - 2x_4 \end{cases}$$

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- Hence $N(A) = \left\{ \begin{pmatrix} -2x_2 - 2x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$

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$$\begin{cases} x_1 + 3x_3 = 0 \\ 2x_3 = 6 \end{cases} \implies x_3 = 3, \quad x_1 = -9.$$

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$$\begin{cases} x_1 + 3x_3 = 0 \\ 2x_3 = 6 \end{cases} \Longrightarrow \quad x_3 = 3, \quad x_1 = -9.$$

- therefore $x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$.

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Summary : A : $m \times n$ matrix

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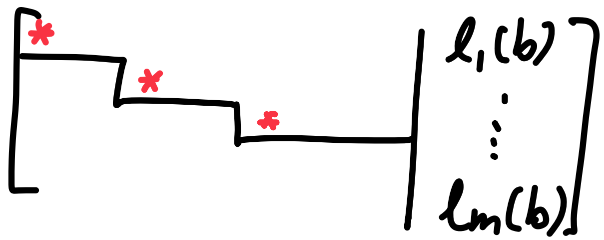
$$LP\,[A \mid b] =$$



Column space and null space

Summary: A : $m \times n$ matrix

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$$LP\,[A \mid b] = \left[\begin{array}{ccc|c} * & & & \ell_1(b) \\ & * & & \vdots \\ & & * & \\ & & & \ell_m(b) \end{array}\right]$$

- Column space:

  1. span by columns of A $\longleftrightarrow$ pivots columns.

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  2. given by $\ell_j(b) = 0$, for those j which corresponds to zero row in Row Echelon form of A

Column space and null space

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Summary :

Column space and null space

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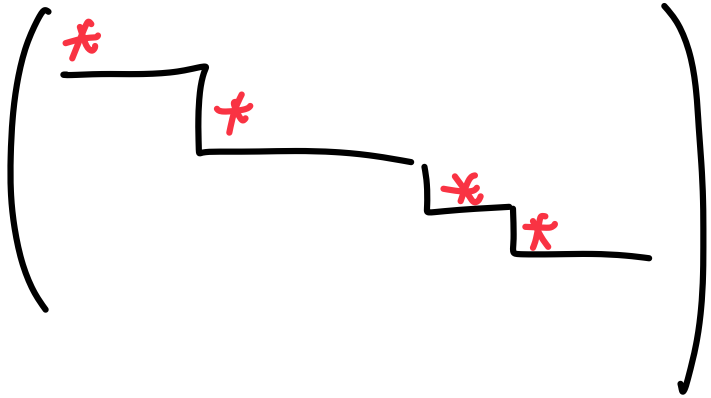
- Null space



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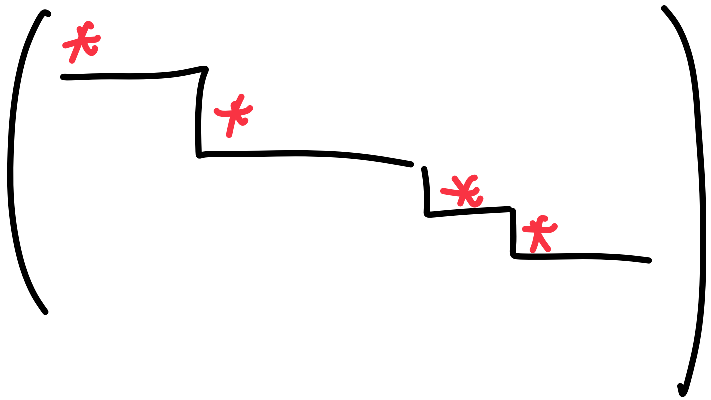
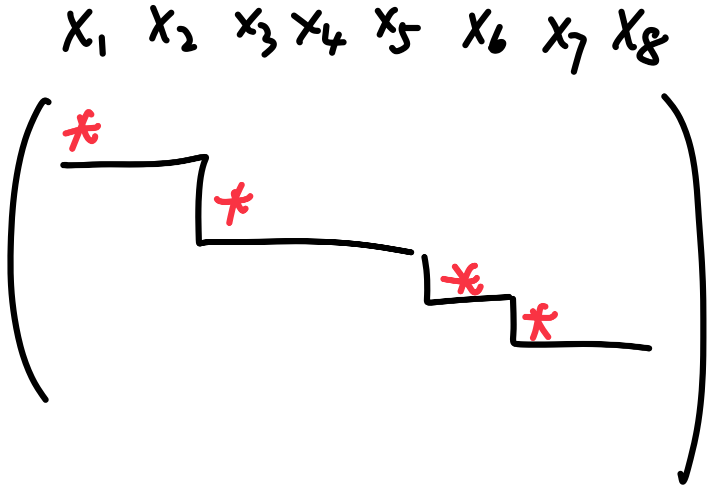


Row Echelon form of A

Column space and null space

Summary:

- Null space



Row Echelon form of A

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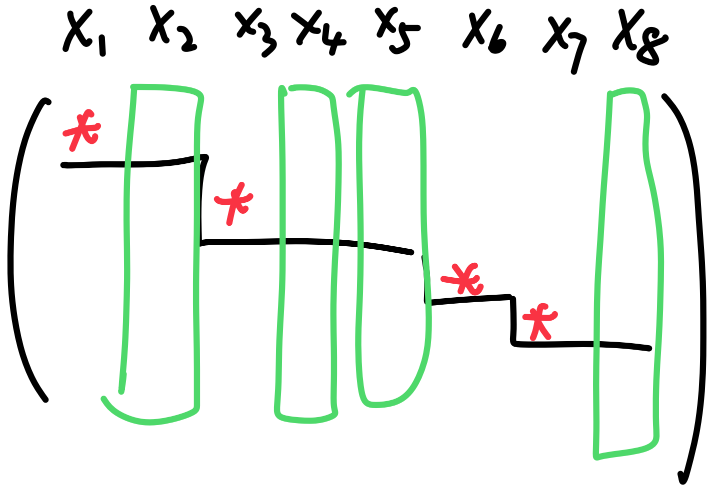

Row Echelon form of A

- variables corresponding to non-pivots columns $\leftrightarrow$ free

Column space and null space

Summary:

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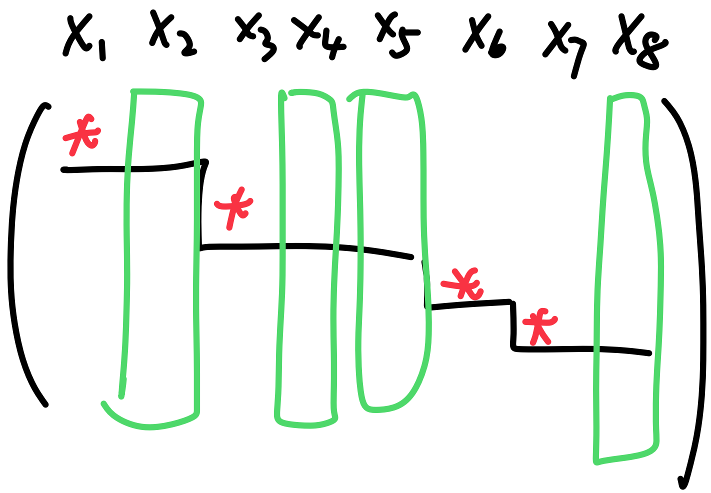

Row Echelon form of A

- variables corresponding to non-pivots columns $\longleftrightarrow$ free

  variables corresponding to pivots columns $\longleftrightarrow$ not free

Column space and null space

Summary:

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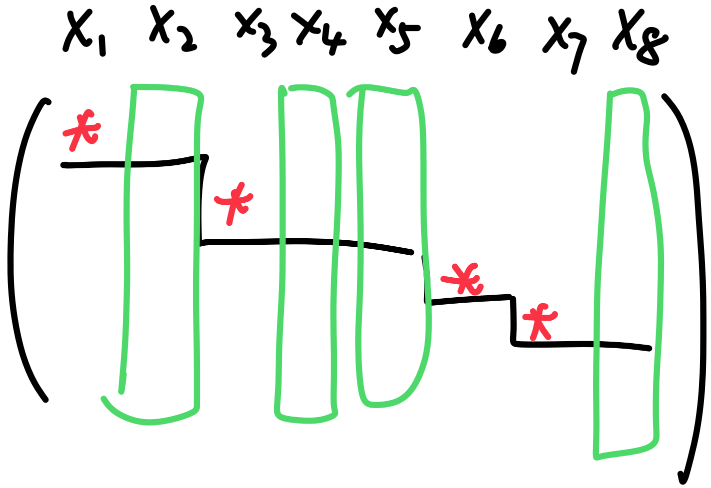

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Column space and null space

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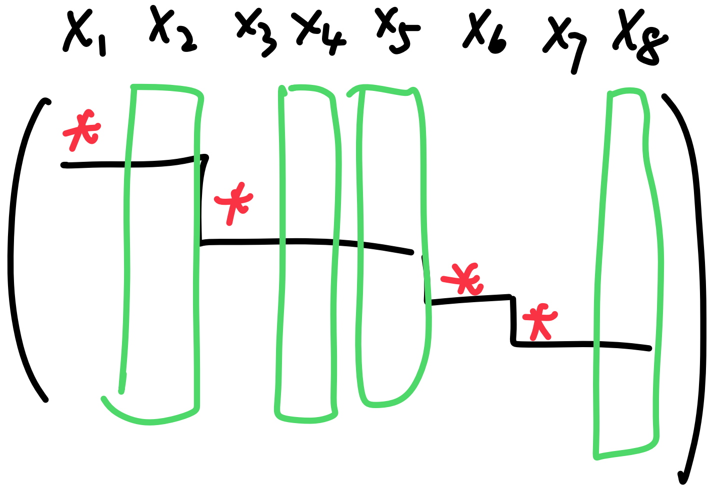

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  null space

  particular solution

Column space and null space

Summary:

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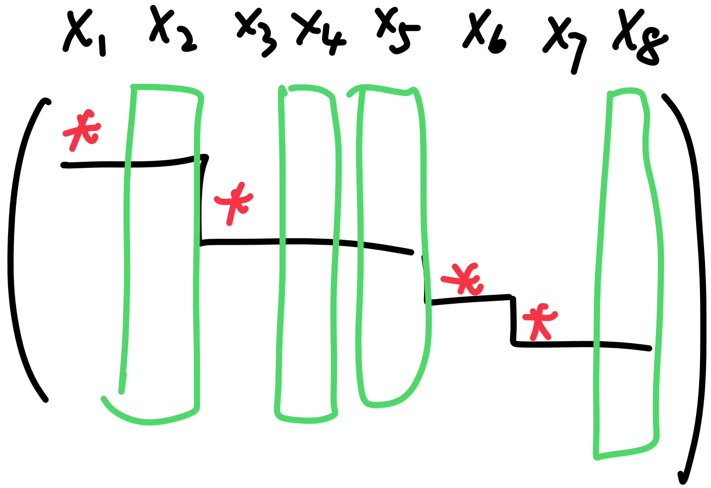

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- variables corresponding to non-pivots columns $\longleftrightarrow$ free

  variables corresponding to pivots columns $\longleftrightarrow$ not free

- Solving $Ax=b$, solution is $x_n + x_p$

  null space (for x_n); particular solution (for x_p)

- Find x_p : set free variables be any constant.

Column space and null space

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<u>Remarks:</u>

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1. Permuting rows may be needed in the Elimination, since our focus is not LU factorization, we don't need to fix the permutation P at the beginning.

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3. We have the equality

$$\#\ \text{pivot columns} + \#\ \text{free variables of } N(A) = n$$

Rank of matrix

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Def: • The rank of a $m \times n$ matrix A is the no. of pivots in its row echelon form.

Rank of matrix

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- For $n \times n$ matrix A: A non-singular
  $\iff A$ invertible
  $\iff \text{rk}(A) = n$.

- $\text{rk}(AB) \leq \min(\text{rk}(A), \text{rk}(B))$.

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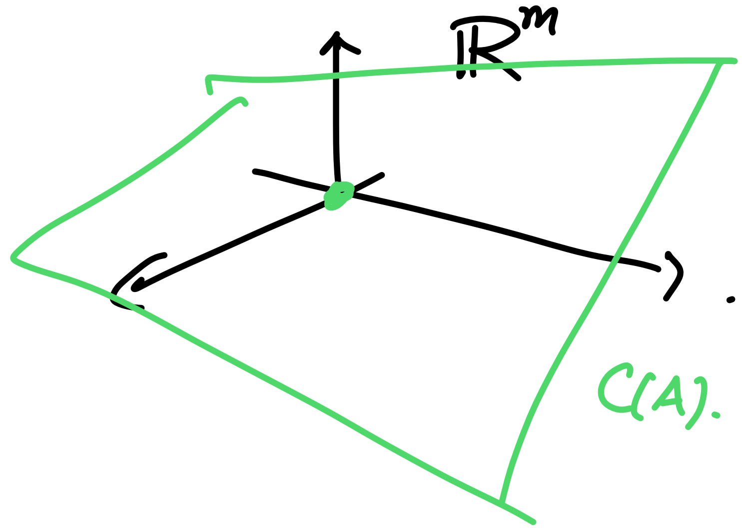

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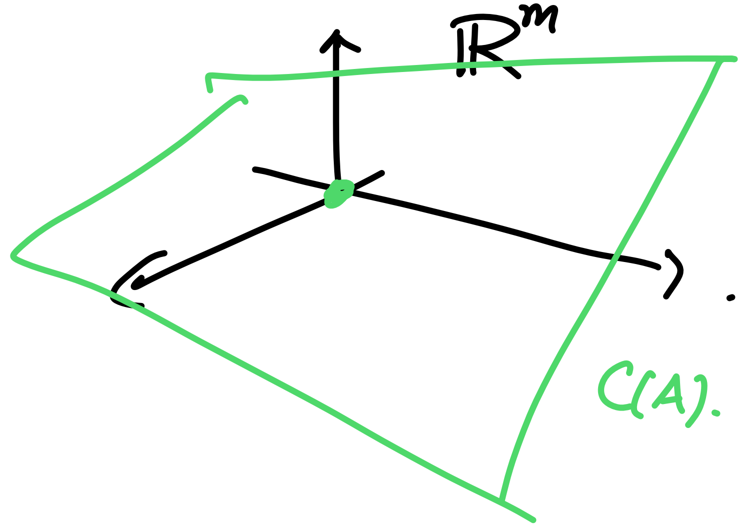

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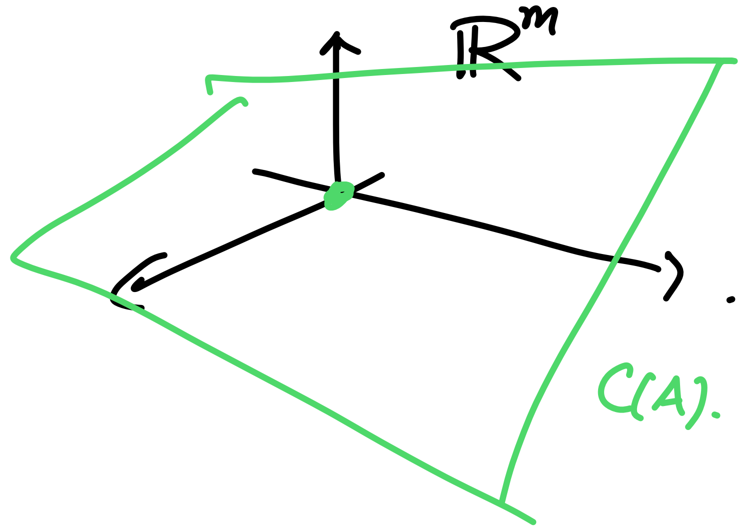

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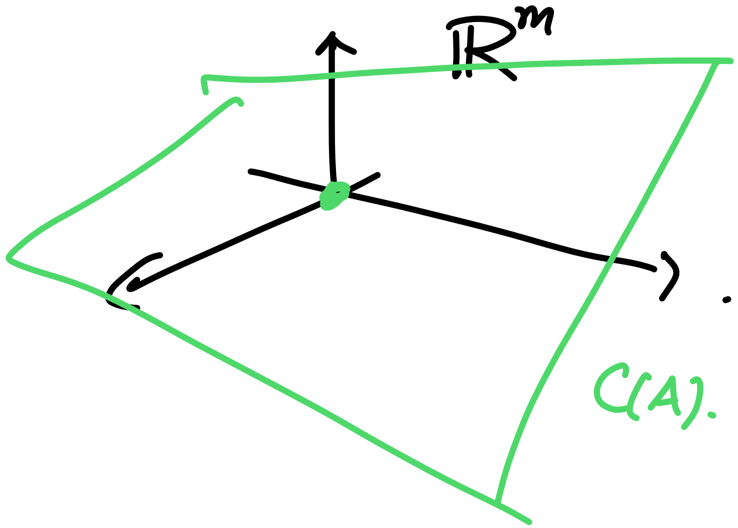

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  = minimal no. of columns needed for spanning $C(A)$.

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- $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ are redundant because they can be expressed in terms of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

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e.g.: • $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$, then if we look at

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- $\exists$ E invertible 3×3 matrix s.t.

$$A = E \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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fact: $\vec{a}_1, \dots, \vec{a}_n$ l.i $\iff$ $Q\vec{a}_1, \dots, Q\vec{a}_n$ l.i

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- Conversely, it is also true.