Lecture 4

Inverse and transpose matrices

Partitioned matrices

Partitioned matrices

- <u>Recall</u> : A : a $m \times k$ matrix
  B : a $k \times n$ matrix

  We treat $B = [\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n]$

Partitioned matrices

- <u>Recall</u>: A : a $m\times k$ matrix

  B : a $k\times n$ matrix

  We treat $B = [\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n]$

- The multiplication is given by

$$AB = [A\vec{b}_1, \dots, A\vec{b}_n]$$

<u>i.e.</u> column by column

Partitioned matrices

Partitioned matrices

- There is also a picture given by rows:

$$\begin{bmatrix} - a_1 - \\ \vdots \\ - a_i - \\ \vdots \\ - a_m - \end{bmatrix} B = \begin{bmatrix} (- a_1 -) B \\ \vdots \\ (- a_i -) B \\ \vdots \\ (- a_n -) B \end{bmatrix}$$

Partitioned matrices

- There is also a picture given by rows:

$$\begin{bmatrix} -a_1- \\ \vdots \\ -a_i- \\ \vdots \\ -a_m- \end{bmatrix} B = \begin{bmatrix} (-a_1-)B \\ \vdots \\ (-a_i-)B \\ \vdots \\ (-a_n-)B \end{bmatrix}$$

- These two are examples of more general block multiplication (分塊乘法)

Partitioned matrices

Partitioned matrices

- In general we may partition a matrix into smaller block:

Partitioned matrices

- In general we may partition a matrix into smaller block:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

Partitioned matrices

- In general we may partition a matrix into smaller block:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Partitioned matrices

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Partitioned matrices

- In general we may partition a matrix into smaller block:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Partitioned matrices

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Partitioned matrices

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Partitioned matrices

Partitioned matrices

- We can write the multiplication by a block partition:

  Case 1):

Partitioned matrices

- We can write the multiplication by a block partition:

Case 1):

$$A[B_1, B_2] = [AB_1, AB_2].$$

Partitioned matrices

- We can write the multiplication by a block partition:

Case 1):
$$\underset{m\times k}{A}\,[\underset{k\times n_1}{B_1}, \underset{k\times n_2}{B_2}] = [AB_1, AB_2].$$

Partitioned matrices

- We can write the multiplication by a block partition:

Case 1):

$$\underset{m\times k}{A}\,[\underset{k\times n_1}{B_1}, \underset{k\times n_2}{B_2}] = [AB_1, AB_2].$$

i.e.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \end{pmatrix} \right]$$

Case 2):

Partitioned matrices

- We can write the multiplication by a block partition:

<u>Case 1)</u>: $\underset{m\times k}{A}\,[\underset{k\times n_1}{B_1}, \underset{k\times n_2}{B_2}] = [AB_1, AB_2]$.

e.g.: B_1 $\quad$ B_2

<u>i.e.</u>

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \end{pmatrix} \right]$$

<u>Case 2)</u>:

Partitioned matrices

- We can write the multiplication by a block partition:

<u>Case 1)</u>: $\underset{m\times k}{A}\,[\underset{k\times n_1}{B_1}, \underset{k\times n_2}{B_2}] = [AB_1, AB_2]$.

e.g.: B_1 B_2

<u>i.e.</u>

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}_{B_1}, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underbrace{\begin{pmatrix} b_{13} \\ b_{23} \end{pmatrix}}_{B_2} \right]$$

<u>Case 2)</u>:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

Partitioned matrices

- We can write the multiplication by a block partition:

<u>Case 1)</u>:
$$\underset{m\times k}{A}\,[\underset{k\times n_1}{B_1}, \underset{k\times n_2}{B_2}] = [AB_1, AB_2].$$

e.g.: B_1 B_2

<u>i.e.</u>
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{13} \\ b_{23} \end{pmatrix}\right]$$

<u>Case 2)</u>:
$$\begin{bmatrix} \underset{m_1\times k}{A_1} \\ \underset{m_2\times k}{A_2} \end{bmatrix} \underset{k\times n}{B} = \begin{bmatrix} A_1B \\ A_2B \end{bmatrix}$$

Partitioned matrices

Partitioned matrices

- Case 3) :

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1 B_1 + A_2 B_2$$

Partitioned matrices

- Case 3) :

$$\begin{bmatrix} \underset{\uparrow \atop m\times k_1}{A_1} & \underset{\uparrow \atop m\times k_2}{A_2} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1B_1 + A_2B_2$$

$k_1 \times n$

$k_2 \times n$

Partitioned matrices

- Case 3) :

$$\begin{array}{cc} \begin{bmatrix} A_1 & A_2 \end{bmatrix} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ m\times k_1 \quad m\times k_2. & \end{array} = \overbrace{A_1B_1 + A_2B_2}^{m\times n.}$$

$k_1\times n$

$k_2\times n$

Partitioned matrices

- Case 3) :

$$\begin{bmatrix} \underset{m\times k_1}{A_1} & \underset{m\times k_2}{A_2} \end{bmatrix} \begin{bmatrix} \overset{k_1\times n}{B_1} \\ \underset{k_2\times n}{B_2} \end{bmatrix} = \overbrace{A_1B_1 + A_2B_2}^{m\times n}$$

e.g.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + b_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

Partitioned matrices

- Case 3) :

$$\underbrace{\begin{bmatrix} A_1 & A_2 \end{bmatrix}}_{\substack{A_1:\ m\times k_1 \\ A_2:\ m\times k_2}} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}_{\substack{B_1:\ k_1\times n \\ B_2:\ k_2\times n}} = \overbrace{A_1B_1 + A_2B_2}^{m\times n.}$$

e.g.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + b_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} b_3 \end{bmatrix}$$

Partitioned matrices

- Case 3) :

$$\underbrace{\begin{bmatrix} A_1 & A_2 \end{bmatrix}}_{m\times k_1 \quad m\times k_2.} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \overbrace{A_1B_1 + A_2B_2}^{m\times n.}$$

B_1: $k_1\times n$, B_2: $k_2\times n$

e.g.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + b_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} b_3 \end{bmatrix}$$

Partitioned matrices

Partitioned matrices

- Case 4:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Partitioned matrices

- Case 4:

$$\begin{array}{c c} & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Partitioned matrices

- <u>Case 4:</u>

$$\begin{array}{c c} & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{c c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} k_1 \\ k_2 \end{array} & \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \end{array} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Partitioned matrices

- <u>Case 4:</u>

$$
\begin{array}{cc} & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array}
\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} k_1 \\ k_2 \end{array} & \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \end{array}
=
\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \end{array}
$$

Partitioned matrices

- Case 4:

$$\begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{cc} k_1 & k_2 \\ \multicolumn{2}{c}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}} \end{array} \begin{array}{c} \\ k_1 \\ k_2 \end{array}\!\!\begin{array}{cc} n_1 & n_2 \\ \multicolumn{2}{c}{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}} \end{array} = \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{cc} n_1 & n_2. \\ \multicolumn{2}{c}{\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}} \end{array}$$

- The sizes of the matrices has to match.

Partitioned matrices

- Case 4:

$$
\begin{array}{cc} & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array}
\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} k_1 \\ k_2 \end{array} & \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \end{array}
=
\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \end{array}
$$

- The sizes of the matrices has to match.

- In general: whenever the partition into block has the right sizes block multiplication works!

Partitioned matrices

Partitioned matrices

- eg.:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 2 & 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

Partitioned matrices

- eg.:

$$A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

Partitioned matrices

- eg.:

A_{11}

$$A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

Partitioned matrices

- eg.:

$$A = \begin{array}{c} \begin{array}{cc} \quad A_{11} \quad & \quad A_{12} \quad \end{array} \\ \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \end{array} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

Partitioned matrices

- eg.:

A_{11} $\quad$ A_{12}

$$A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

A_{21}

Partitioned matrices

- eg.:

A_{11} $\quad$ A_{12}

$$A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

A_{21} $\quad$ A_{22}

Partitioned matrices

- eg.:

$$
\begin{array}{cc}
 & \begin{array}{cc} A_{11} & A_{12} \end{array} \\
\begin{array}{c} m_1=2 \\ \\ m_2=1 \end{array} & A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \\
 & \begin{array}{cc} A_{21} & A_{22} \end{array}
\end{array}
\qquad
B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}
$$

Partitioned matrices

- eg.:

$$
\begin{array}{cc}
 & \begin{array}{cc} k_1 = 2 & k_2 = 2 \end{array} \\
 & \begin{array}{cc} A_{11} & A_{12} \end{array} \\
\begin{array}{c} m_1 = 2 \\ \\ m_2 = 1 \end{array} & A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \\
 & \begin{array}{cc} A_{21} & A_{22} \end{array}
\end{array}
$$

$$
B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}
$$

Partitioned matrices

- eg.:

$$k_1 = 2 \qquad k_2 = 2$$

$$A_{11} \qquad A_{12}$$

$$\begin{array}{l} m_1 = 2 \\ \\ m_2 = 1 \end{array} \quad A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right]$$

$$A_{21} \qquad A_{22}$$

Partitioned matrices

- eg.:

$$\begin{array}{c c} & \begin{array}{cc} k_1=2 & k_2=2 \\ A_{11} & A_{12} \end{array} \\ \begin{array}{c} m_1=2 \\ \\ m_2=1 \end{array} & A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \\ & \begin{array}{cc} A_{21} & A_{22} \end{array} \end{array}$$

$$\begin{array}{c c} \begin{array}{c} k_1 \\ \\ k_2 \end{array} & B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right] \end{array}$$

Partitioned matrices

- eg.:

$$
\begin{array}{c c}
 & \begin{array}{cc} k_1=2 & k_2=2 \\ A_{11} & A_{12} \end{array} \\
\begin{array}{c} m_1=2 \\ \\ m_2=1 \end{array} & A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \\
 & \begin{array}{cc} A_{21} & A_{22} \end{array}
\end{array}
$$

$$
\begin{array}{c c}
 & \begin{array}{cc} n_1=3 & n_2=1. \end{array} \\
\begin{array}{c} k_1 \\ \\ k_2 \end{array} & B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right]
\end{array}
$$

Partitioned matrices

- eg.:

$$
\begin{array}{ccc}
 & k_1=2 & k_2=2 \\
 & A_{11} & A_{12} \\
\begin{array}{c} m_1=2 \\ \\ m_2=1 \end{array} & A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] & \\
 & A_{21} & A_{22}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cc} n_1=3 & n_2=1. \end{array} \\
\begin{array}{c} k_1 \\ \\ k_2 \end{array} & B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right]
\end{array}
$$

or

$$
A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 2 & 2 & 2 \end{bmatrix}
\qquad
B = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ \hline 3 & 2 & 1 & 2 \end{array}\right]
$$

Partitioned matrices

- eg.:

$$\begin{array}{cc} & \begin{array}{cc} k_1=2 & k_2=2 \\ A_{11} & A_{12} \end{array} \\ \begin{array}{c} m_1=2 \\ \\ m_2=1 \end{array} & A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \\ & \begin{array}{cc} A_{21} & A_{22} \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} n_1=3 & n_2=1. \end{array} \\ \begin{array}{c} k_1 \\ \\ k_2 \end{array} & B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right] \end{array}$$

or

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \qquad B = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ \hline 3 & 2 & 1 & 2 \end{array}\right]$$

Partitioned matrices

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$$\begin{array}{cc} & \begin{array}{cc} k_1 = 2 & k_2 = 2 \\ A_{11} & A_{12} \end{array} \\ \begin{array}{c} m_1 = 2 \\ \\ m_2 = 1 \end{array} & A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 2 & 2 & 2 \end{array}\right] \\ & \begin{array}{cc} A_{21} & A_{22} \end{array} \end{array}$$

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- with A_{ii}, B_{ii} are square matrices.

Inverse matrices

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- Recall that when $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l & 0 & 1 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{pmatrix}$

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e.g.: $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = I \Rightarrow P^{-1} = P$.

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Prop: if A, B are invertible (可逆), then so is AB.
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Prop: A $n \times n$ matrix A is invertible if and only if the elimination process gives n pivots.
(i.e.) non-singular .

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P: permutation matrices' product

L: elementary matrices product.

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$$EU = U \Rightarrow E = I \quad \text{contradiction}$$

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$$= \begin{pmatrix} 1 & -u_{12} & -u_{13} + u_{12} u_{23} \\ 0 & 1 & -u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

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- In particular: $$A[B, C] = [AB, AC]$$

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- **Recall:** $$A\begin{bmatrix} | & | & & | \\ b_1, & b_2, & \dots, & b_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Ab_1, & \dots, & Ab_n \\ | & & | \end{bmatrix}$$

  i.e. multiplication can be done column by column.

- In particular: $$A[B, C] = [AB, AC]$$

  B, C: matrices with same no. of rows

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Say 3×3 case:

$$A\begin{bmatrix} | & | & | \\ x_1, & x_2, & x_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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- We need to do Gaussian elimination for $[A \,|\, \vec{e}_i]$:
- Two cases:
  - if A is singular, elimination fails
    no inverse
  - if A non-singular, find unique $\vec{x}_i$'s

The Gauss-Jordan Method

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e.g.: $[A \mid \quad , \quad , \quad] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & \\ 4 & -6 & 0 & \\ -2 & 7 & 2 & \end{array}\right]$

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- Use row operation to turn it into the form.

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- If elimination fails to bring L.H.S to I, A is not invertible!

The Gauss-Jordan Method

$$\left[\begin{array}{c|c} — a_1 — & 0 \\ \vdots & \vdots \\ — a_i — & \vdots \\ \vdots & \vdots \\ — a_n — & 0 \end{array}\right]$$

The Gauss-Jordan Method

- The cost: $L^{-1}A = U$

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observe only row operations of adding a_j $(j \geq i)$ to other rows will have effects on R.H.S.

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$$\Longrightarrow \quad (n-i)+(n-i-1)+\dots+1 = \frac{(n-i)(n-i+1)}{2}.$$

steps taken on the R.H.S.

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obtain from $\vec{e}_i$ via row operation

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- The we need to solve $U\vec{x}_i = \vec{c}_i$ by backward substitution.

obtain from $\vec{e}_i$ via row operation

For each i: the cost is $1+2+\dots+n-1 = \frac{n(n-1)}{2}$.

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- Use Gauss Elimination directly.

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- We write A^T for the transpose matrix

Def: Let $A = (a_{ij})$ be $m \times n$ matrix, we let an $n \times m$ matrix $A^T = (a^T_{ji})$ by $a^T_{ji} = a_{ij}$.

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e.g.

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Pf: 1 exercise.

Transpose matrix

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Pf: 2.

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<u>Pf:</u> 2. <u>e.g.:</u>

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$m \times k$ $k \times n$ matrices

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only need to know $(A\vec{b}_i)^T = \vec{b}_i^T \cdot A^T$.

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we conclude A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

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Thm: Suppose there is a LU factorization for non-singular symmetric $A = LDU$, then $L^T = U$.

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- So for symmetric matrices: elimination cost $\sim \frac{n^3}{6}$.

Inner and outer product

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- Inner product

For $\vec{x}, \vec{y} \in \mathbb{R}^n$ be column vector

we define: $\langle \vec{x}, \vec{y} \rangle := \vec{x}^T y \in \mathbb{R}$.

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<u>e.g:</u> $\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$

$= 32.$

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- length/Norm: we let $\|\vec{x}\| = \langle \vec{x}, \vec{x} \rangle^{1/2}$.

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- the columns are multiple of $\vec{x}$.

  the rows are multiple of $\vec{y}$.

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$\vec{x}^T =$

observe:

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Inner and outer product

e.g.: compute $\left(\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{x}^T =} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2020}$

observe:

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \langle \vec{x}, \vec{x} \rangle \begin{bmatrix} 1, 2, 3 \end{bmatrix}$$

Inner and outer product

e.g.: compute $\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^{2020}$

$\vec{x}^T = \begin{bmatrix}1 & 2 & 3\end{bmatrix}$

observe:

$$\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^2 = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

scalar!

$$= \begin{bmatrix}1\\2\\3\end{bmatrix}\langle \vec{x}, \vec{x}\rangle \begin{bmatrix}1, & 2, & 3\end{bmatrix}$$

Inner and outer product

e.g.: compute $\left(\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{} \begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^{2020}$

$\vec{x}^T =$

observe:

$$\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^2 = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

scalar!

$$= \begin{bmatrix}1\\2\\3\end{bmatrix} \langle \vec{x}, \vec{x} \rangle \begin{bmatrix}1, & 2, & 3\end{bmatrix}$$

$$= \langle \vec{x}, \vec{x} \rangle \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

Inner and outer product

e.g.: compute $\vec{x}^T = \left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^{2020}$

observe:

$$\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^{2} = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

scalar!

$$= \begin{bmatrix}1\\2\\3\end{bmatrix}\langle \vec{x}, \vec{x}\rangle \begin{bmatrix}1, & 2, & 3\end{bmatrix}$$

$$= \langle \vec{x}, \vec{x}\rangle \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

$$\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^{3} = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

Inner and outer product

e.g.: compute $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2020}$

$\vec{x}^T = $

observe: $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

scalar!

$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \langle \vec{x}, \vec{x} \rangle \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$

$= \langle \vec{x}, \vec{x} \rangle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Inner and outer product

e.g.: compute $\vec{x}^T = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2020}$

observe:

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

scalar!

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \langle \vec{x}, \vec{x} \rangle \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$$

$$= \langle \vec{x}, \vec{x} \rangle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= \langle \vec{x}, \vec{x} \rangle^2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Inner and outer product

Inner and outer product

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{20200}$$

Inner and outer product

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2020} = \langle \vec{x}, \vec{x} \rangle^{2019} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= 14^{2019} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 9 \end{bmatrix}$$

Inner and outer product

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2020} = \langle \vec{x}, \vec{x} \rangle^{2019} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= 14^{2019} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 9 \end{bmatrix}$$

e.g.: Compute the outer product expansion of XY^T

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$$

Inner and outer product

$$\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right)^{2020} = \langle \vec{x}, \vec{x} \rangle^{2019} \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

$$= 14^{2019} \begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 4 & 9\end{bmatrix}$$

e.g.: Compute the outer product expansion of XY^T

$$X = \begin{bmatrix}3 & 1\\2 & 4\\1 & 2\end{bmatrix}, \quad Y = \begin{bmatrix}1 & 2\\2 & 4\\3 & 1\end{bmatrix}$$

$$XY^T = \begin{bmatrix}3 & 1\\2 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\2 & 4 & 1\end{bmatrix}$$

Inner and outer product

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \right)^{2020} = \langle \vec{x}, \vec{x} \rangle^{2019} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= 14^{2019} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 9 \end{bmatrix}$$

e.g.: Compute the outer product expansion of XY^T

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$XY^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$$