# Triangular factor and row operations

# Triangular factor and row operations

- Elementary matrices

  Type I: (permutation matrices)

# Triangular factor and row operations

- Elementary matrices

  Type I: (permutation matrices)

  obtained by interchanging two rows of $I$.

- eg. $P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

# Triangular factor and row operations

- Elementary matrices

Type I: (permutation matrices)

obtained by interchanging two rows of $I$.

- eg. $P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

- Effect: $P_{21} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \\ b_3 \end{bmatrix}$ ; $P_{31} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}$

# Triangular factor and row operations

- **Elementary matrices**

**Type I:** (permutation matrices)

obtained by interchanging two rows of $I$.

- eg. $P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

- **Effect:** $P_{21} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \\ b_3 \end{bmatrix}$ ; $P_{31} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}$

# Triangular factor and row operations

- Elementary matrices

Type I: (permutation matrices)

obtained by interchanging two rows of $I$.

- eg. $P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

- Effect: $P_{21}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \\ b_3 \end{bmatrix}$ ; $P_{31}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}$

# Elementary matrices

# Elementary matrices

- more generally:

$$P_{21} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

# Elementary matrices

- more generally:

$$P_{21}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- $$P\begin{bmatrix} | & | & | \\ a_1, & a_2, & a_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} P\begin{pmatrix} | \\ a_1 \\ | \end{pmatrix}, & P\begin{pmatrix} | \\ a_2 \\ | \end{pmatrix}, & P\begin{pmatrix} | \\ a_3 \\ | \end{pmatrix} \end{bmatrix}$$

# Elementary matrices

- more generally:

$$P_{21}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- $$P\begin{bmatrix} | & | & | \\ a_1, & a_2, & a_3 \\ | & | & | \end{bmatrix} = \left[ P\begin{pmatrix} | \\ a_1 \\ | \end{pmatrix}, P\begin{pmatrix} | \\ a_2 \\ | \end{pmatrix}, P\begin{pmatrix} | \\ a_3 \\ | \end{pmatrix} \right]$$

# Elementary matrices

# Elementary matrices

- Type II :

  obtained by multiplying a row of $I$ with $\lambda \neq 0$

# Elementary matrices

- Type II :

  obtained by multiplying a row of $I$ with $\lambda \neq 0$

- e.g. $C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , $C_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ \lambda b_2 \\ b_3 \end{bmatrix}$

# Elementary matrices

- Type II :

obtained by multiplying a row of $I$ with $\lambda \neq 0$

- e.g. $C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , $C_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ \lambda b_2 \\ b_3 \end{bmatrix}$

- e.g.: $C_2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ \lambda a_{21} & \lambda a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

# Elementary matrices

# Elementary matrices

- Type III :

  obtained from $I$ by adding $\ell$ multiple of one row to another.

# Elementary matrices

- Type III :

  obtained from $I$ by adding $l$ multiple of one row to another.

- e.g. $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l & 0 & 1 \end{bmatrix}$

$$E_{31} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} + l a_{11} & a_{32} + l a_{12} \end{bmatrix}$$

# Row vs Column operations

# Row vs Column operations

Q: What if we multiple E from the right?

# Row vs Column operations

Q: What if we multiple E from the right?

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \lambda a_{12} & a_{13} \\ a_{21} & \lambda a_{22} & a_{23} \end{bmatrix}$$

# Row vs Column operations

**Q:** What if we multiple E from the right?

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \lambda a_{12} & a_{13} \\ a_{21} & \lambda a_{22} & a_{23} \end{bmatrix}$$

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{bmatrix}$$

# Row vs Column operations

Q: What if we multiple E from the right?

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \lambda a_{12} & a_{13} \\ a_{21} & \lambda a_{22} & a_{23} \end{bmatrix}$$

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{bmatrix}$$

# Row vs Column operations

**Q:** What if we multiple $E$ from the right?

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \lambda a_{12} & a_{13} \\ a_{21} & \lambda a_{22} & a_{23} \end{bmatrix}$$

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{bmatrix}$$

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} + l a_{13} & a_{12} & a_{13} \\ a_{21} + l a_{23} & a_{22} & a_{23} \end{bmatrix}$$

# Row vs Column operations

**Q:** What if we multiple $E$ from the right?

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \lambda a_{12} & a_{13} \\ a_{21} & \lambda a_{22} & a_{23} \end{bmatrix}$$

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{bmatrix}$$

$\oplus$ $l$-times

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} + l a_{13} & a_{12} & a_{13} \\ a_{21} + l a_{23} & a_{22} & a_{23} \end{bmatrix}$$

# Row vs Column operations

# Row vs Column operations

Q: What if we multiple E from the right?

# Row vs Column operations

Q: What if we multiple E from the right?

A: If we treat elementary matrices E as coming from applying column operation to I,

# Row vs Column operations

**Q:** What if we multiple $E$ from the right?

**A:** If we treat elementary matrices $E$ as coming from applying column operation to $I$,

- $AE$ = the matrix obtained from $A$ by applying corresponding column operation.

# Upper triangular matrix

# Upper triangular matrix

- We consider the equation: (when $A$ non-singular)

$$A\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \quad \text{as} \quad \text{matrix}$$

# Upper triangular matrix

- We consider the equation: (when A non-singular)

$$A\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \quad \text{as} \quad \text{matrix}$$

- $$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

# Upper triangular matrix

- We consider the equation: (when A non-singular)

$$A\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$ as matrix

- $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

# Upper triangular matrix

- We consider the equation: (when A non-singular)

$$A\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \text{ as matrix}$$

- $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$E_{21}A$$

$$\begin{gathered} E_{21}A \\ \shortparallel \\ \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix} \end{gathered}$$

$$E_{21}A$$
$$\parallel$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$E_{21}A$$

$$\parallel$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\overset{\overset{E_{21}A}{\shortparallel}}{\begin{bmatrix}2&1&1\\0&-8&-2\\-2&7&2\end{bmatrix}}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}5\\-12\\9\end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\overset{\displaystyle E_{21}A}{\overset{\parallel}{\begin{bmatrix}2&1&1\\0&-8&-2\\-2&7&2\end{bmatrix}}}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}5\\-12\\9\end{bmatrix}}$$

$$E_{31}E_{21}A = \begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}5\\-12\\14\end{bmatrix}$$

$$E_{21}A$$
$$\parallel$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}}_{E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}}_{\begin{bmatrix} 5 \\ -12 \\ 14 \end{bmatrix}}$$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -12 \\ 14 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\overbrace{\begin{bmatrix}2&1&1\\0&-8&-2\\-2&7&2\end{bmatrix}}^{\substack{E_{21}A\\ \parallel}}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}5\\-12\\9\end{bmatrix}}$$

$$E_{31}E_{21}A = \begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}5\\-12\\14\end{bmatrix}$$

$$\overbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}}^{E_{32}}\begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}5\\-12\\14\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\overset{\displaystyle E_{21}A}{\overset{\parallel}{\begin{bmatrix}2&1&1\\0&-8&-2\\-2&7&2\end{bmatrix}}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}5\\-12\\9\end{bmatrix}$$

$$E_{31}E_{21}A=\begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}5\\-12\\14\end{bmatrix}$$

$$\overbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}}^{E_{32}}\begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}5\\-12\\14\end{bmatrix}$$

$$\begin{bmatrix}2&1&1\\0&-8&-2\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}5\\-12\\2\end{bmatrix}$$

$$E_{21}A$$

$$\|$$

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}2&1&1\\0&-8&-2\\-2&7&2\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}5\\-12\\9\end{bmatrix}}$$

$$E_{31}E_{21}A = \begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}5\\-12\\14\end{bmatrix}$$

$$\underbrace{\overbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}}^{E_{32}}\begin{bmatrix}2&1&1\\0&-8&-2\\0&8&3\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}5\\-12\\14\end{bmatrix}$$

$$U\vec{x} = \begin{bmatrix}2&1&1\\0&-8&-2\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}5\\-12\\2\end{bmatrix}$$

$$U\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

$$U\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

- <u>Recall</u>: $U = E_{32} E_{31} E_{21} A.$

  is upper triangular

$$U\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

- Recall: $U = E_{32} E_{31} E_{21} A.$

  is upper triangular

$$U\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

- Recall: $U = E_{32} E_{31} E_{21} A.$

  is upper triangular

- $E_{32}$, $E_{31}$, $E_{21}$ are lower triangular

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

- Recall: $U = E_{32} E_{31} E_{21} A.$

  is upper triangular

- $$\begin{matrix} E_{32} & & E_{31} & & E_{21} \\ \| & & \| & & \| \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$ are lower triangular

$$U\vec{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

- Recall: $U = E_{32} E_{31} E_{21} A$.

  is upper triangular

- $E_{32}$, $E_{31}$, $E_{21}$ are lower triangular

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

is lower triangular with diagonal being 1

# LU factorization

# LU factorization

- $$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

# LU factorization

- $$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

  we found $E_{32} \cdot E_{32}^{-1} = I.$

# LU factorization

- $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

we found $E_{32} \cdot E_{32}^{-1} = I$.

similarily for $E_{31}$, $E_{21}$

# LU factorization

- $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

  we found $E_{32} \cdot E_{32}^{-1} = I$.

  similarily for $E_{31}$, $E_{21}$

- $$E_{32}\, E_{31}\, E_{21}\, A = U$$

# LU factorization

- $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

  we found $E_{32} \cdot E_{32}^{-1} = I$.

  similarily for $E_{31}$, $E_{21}$

- $E_{32}\, E_{31}\, E_{21}\, A = \qquad\qquad U$

# LU factorization

- $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

  we found $E_{32} \cdot E_{32}^{-1} = I$.

  similarily for $E_{31}$, $E_{21}$

- $E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} E_{32}\, E_{31}\, E_{21}\, A = \qquad\qquad \mathcal{U}$

# LU factorization

- $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

  we found $E_{32} \cdot E_{32}^{-1} = I$.

  similarily for $E_{31}$ , $E_{21}$

- $E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} E_{32} E_{31} E_{21} A = \overbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}^{L} \, U$

# LU factorization

- $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

  we found $E_{32} \cdot E_{32}^{-1} = I$.

  similarily for $E_{31}$, $E_{21}$

- $E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} E_{32} E_{31} E_{21} A = \overbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}^{L.} U$

- From $A$, we find lower triangular $L$ such that $LU = A$

# LU factorization

- $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

  we found $E_{32} \cdot E_{32}^{-1} = I$.

  similarily for $E_{31}$, $E_{21}$

- $E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} E_{32} E_{31} E_{21} A = \overbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}^{L.} \mathcal{U}$

- From $A$, we find lower triangular $L$ such that $L\mathcal{U} = A$

  lower triangular.

# LU factorization

- $$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

  we found $E_{32} \cdot E_{32}^{-1} = I$.

  similarily for $E_{31}$, $E_{21}$

- $$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}E_{32}E_{31}E_{21}A = \overbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}^{L.}\,U$$

- From $A$, we find lower triangular $L$ such that $LU = A$.

  lower triangular.

  upper triangular

# LU factorization

# LU factorization

eg:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

# LU factorization

eg:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_{43} E_{32} E_{21} A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# LU factorization

eg:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = E_{43} E_{32} E_{21} A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# LU factorization

eg:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = E_{43} E_{32} E_{21} A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# LU factorization

eg:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = E_{43} E_{32} E_{21} A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$u$

# LU factorization

eg:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = E_{43} E_{32} E_{21} A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{u}$$

# LU factorization

# LU factorization

**E.g 2:**

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}, \text{ find LU factorization}$$

# LU factorization

**E.g 2:** $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$, find LU factorization

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

# LU factorization

E.g 2:

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}, \text{ find LU factorization}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 4 & -1 & 9 \end{bmatrix}$$

# LU factorization

**E.g 2:**

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}, \text{ find LU factorization}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 4 & -1 & 9 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

# LU factorization

**E.g 2:** $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$, find LU factorization

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 4 & -1 & 9 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

# LU factorization

E.g 2: $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$, find LU factorization

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 4 & -1 & 9 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \quad u$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

# LU factorization

**E.g 2:** $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$, find LU factorization

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 4 & -1 & 9 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}}_{U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

# LU factorization

# LU factorization

- We observe that if $A = LU$ , then

# LU factorization

- We observe that if $A = LU$, then

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

# LU factorization

- We observe that if $A = LU$, then

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

where $-l_{ij}$ : the multiple of the $j$-th rows which is added to the $i$-th row.

# LU factorization

- We observe that if $A = LU$, then

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

where $-l_{ij}$ : the multiple of the $j$-th rows which is added to the $i$-th row.

- The same holds for $n \times n$ case

# LU factorization

- We observe that if $A = LU$, then

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

where $-l_{ij}$ : the multiple of the $j$-th rows which is added to the $i$-th row.

- The same holds for $n \times n$ case

# LU factorization

# LU factorization

- One linear system = Two triangular system

# LU factorization

- One linear system = Two triangular system

$$Ax = b \iff LUx = b$$

$$\iff \begin{cases} Lc = b \\ Ux = c \end{cases}$$

# LU factorization

- One linear system = Two triangular system

$$Ax = b \iff LUx = b$$

$$\iff \begin{cases} Lc = b \\ Ux = c \end{cases}$$

# LU factorization

- One linear system = Two triangular system

$$Ax = b \iff LUx = b$$

$$\iff \begin{cases} Lc = b \\ Ux = c \end{cases}$$

- triangular system can be solved effectively.

$$Ux = c$$

# LU factorization

- one linear system = Two triangular system

$$Ax = b \iff LUx = b$$

$$\iff \begin{cases} Lc = b \\ Ux = c \end{cases}$$

- triangular system can be solved effectively.

$$Ux = c$$

back-ward substitution

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} \text{ steps!}$$

# LU factorization

# LU factorization

- e.g.:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# LU factorization

- e.g.:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ equivalent to solve:

# LU factorization

- e.g.:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ equivalent to solve:

$$\begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \end{cases}$$

# LU factorization

- e.g.:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ equivalent to solve:

$$\begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \end{cases}$$

$$\Rightarrow \quad c_1 = 1, \quad c_2 = 2, \quad c_3 = 3, \quad c_4 = 4.$$

# LU factorization

- e.g.:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ equivalent to solve:

$$\begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 1 \\ c_2 = 2 \\ c_3 = 3 \\ c_4 = 4. \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 10 \\ x_2 = 9 \\ x_3 = 7 \\ x_4 = 4 \end{matrix} \end{cases}$$

# LU factorization

# LU factorization

- take $U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$, we can rewrite it

# LU factorization

- take $U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$, we can rewrite it

$$U = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 \\ 0 & 1 & u_{23}/d_2 \\ 0 & 0 & 1 \end{bmatrix}$$

# LU factorization

- take $U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$ , we can rewrite it

$$U = \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D:\ \text{diagonal matrix}} \underbrace{\begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 \\ 0 & 1 & u_{23}/d_2 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

# LU factorization

- take $U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$, we can rewrite it

$$U = \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D\,:\,\text{diagonal matrix}} \underbrace{\begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 \\ 0 & 1 & u_{23}/d_2 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

- We can rewrite $A = LU$ into $A = LD\tilde{U}$

# LU factorization

- take $U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$, we can rewrite it

$$U = \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D \text{ : diagonal matrix}} \underbrace{\begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 \\ 0 & 1 & u_{23}/d_2 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

- We can rewrite $A = LU$ into $A = LD\tilde{U}$

has 1 on its diagonal

# LU factorization

- take $U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$, we can rewrite it

$$U = \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D\text{: diagonal matrix}} \underbrace{\begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 \\ 0 & 1 & u_{23}/d_2 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

- We can rewrite $A = LU$ into $A = LD\tilde{U}$

has 1 on its diagonal

$$A = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

# LU factorization

Prop: uniqueness If $A = L_1 D_1 U_1 = L_2 D_2 U_2$ be two such factorization, then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$.

Pf: we see that there always exist $L_1^{-1}, L_2^{-1}$ s.t. $L_1 L_1^{-1} = I$, $L_2 L_2^{-1} = I$.

$$L_1 D_1 U_1 = L_2 D_2 U_2.$$

$$\Rightarrow (L_2^{-1} L_1) D_1 U_1 = D_2 U_2.$$

$$\Rightarrow (L_2^{-1} L_1) D_1 = D_2 U_2 U_1^{-1}.$$

lower triangular upper triangular.

# LU factorization

# LU factorization

Say $n=3$:

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} \tilde{d}_1 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 \\ 0 & 0 & \tilde{d}_3 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & \underline{1} \end{bmatrix}$$

# LU factorization

Say $n=3$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L_2^{-1}L_1} \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D_1} = \begin{bmatrix} \tilde{d}_1 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 \\ 0 & 0 & \tilde{d}_3 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

# LU factorization

Say $n=3$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L_2^{-1}L_1} \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D_1} = \underbrace{\begin{bmatrix} \tilde{d}_1 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U_2 U_1^{-1}}$$

# LU factorization

Say $n=3$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L_2^{-1}L_1} \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D_1} = \underbrace{\begin{bmatrix} \tilde{d}_1 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U_2U_1^{-1}}$$

$$\| $$

$$\begin{bmatrix} d_1 & 0 & 0 \\ d_1 l_{21} & d_2 & 0 \\ d_1 l_{31} & d_2 l_{32} & d_3 \end{bmatrix}$$

# LU factorization

Say $n=3$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L_2^{-1}L_1} \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D_1} = \underbrace{\begin{bmatrix} \tilde{d}_1 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U_2 U_1^{-1}}$$

$$\begin{bmatrix} d_1 & 0 & 0 \\ d_1 l_{21} & d_2 & 0 \\ d_1 l_{31} & d_2 l_{32} & d_3 \end{bmatrix} \qquad \qquad \begin{bmatrix} \tilde{d}_1 & \tilde{d}_1 u_{12} & \tilde{d}_1 u_{13} \\ 0 & \tilde{d}_2 & \tilde{d}_2 u_{23} \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}$$

# LU factorization

Say $n=3$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L_2^{-1}L_1} \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D_1} = \underbrace{\begin{bmatrix} \tilde{d}_1 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U_2U_1^{-1}}$$

$$\begin{bmatrix} d_1 & 0 & 0 \\ d_1 l_{21} & d_2 & 0 \\ d_1 l_{31} & d_2 l_{32} & d_3 \end{bmatrix} = \begin{bmatrix} \tilde{d}_1 & \tilde{d}_1 u_{12} & \tilde{d}_1 u_{13} \\ 0 & \tilde{d}_2 & \tilde{d}_2 u_{23} \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}$$

# LU factorization

Say $n=3$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L_2^{-1}L_1} \underbrace{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}}_{D_1} = \underbrace{\begin{bmatrix} \tilde{d}_1 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U_2 U_1^{-1}}$$

$$\begin{bmatrix} d_1 & 0 & 0 \\ d_1 l_{21} & d_2 & 0 \\ d_1 l_{31} & d_2 l_{32} & d_3 \end{bmatrix} = \begin{bmatrix} \tilde{d}_1 & \tilde{d}_1 u_{12} & \tilde{d}_1 u_{13} \\ 0 & \tilde{d}_2 & \tilde{d}_2 u_{23} \\ 0 & 0 & \tilde{d}_3 \end{bmatrix}$$

$\Longrightarrow \quad d_1 = \tilde{d}_1, \ d_2 = \tilde{d}_2, \ d_3 = \tilde{d}_3$ and $L_2^{-1}L_1 = I = U_2 U_1^{-1}$

# LU factorization

# LU factorization

**Prop:** 1. Product of $n \times n$-lower triangular matrices is lower triangular

# LU factorization

**Prop:** 1. Product of $n \times n$-lower triangular matrices is lower triangular

2. Furthermore, if diagonal entries are all 1's then so is the product

# LU factorization

**Prop:** 1. Product of $n \times n$-lower triangular matrices is lower triangular

2. Furthermore, if diagonal entries are all 1's then so is the product

**Pf:** • (for $n=3$)

# LU factorization

**Prop:** 1. Product of $n \times n$-lower triangular matrices is lower triangular

2. Furthermore, if diagonal entries are all 1's then so is the product

Pf: • (for $n=3$)

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

# LU factorization

**Prop:** 1. Product of $n \times n$-lower triangular matrices is lower triangular

2. Furthermore, if diagonal entries are all 1's then so is the product

Pf: • (for $n=3$)

$$A = \begin{pmatrix} \boxed{a_{11} \quad 0 \quad 0} \\ a_{21} \quad a_{22} \quad 0 \\ a_{31} \quad a_{32} \quad a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \boxed{\begin{matrix} 0 \\ b_{22} \\ b_{32} \end{matrix}} & \begin{matrix} 0 \\ 0 \\ b_{33} \end{matrix} \end{pmatrix}$$

# LU factorization

**Prop:** 1. Product of $n \times n$-lower triangular matrices is lower triangular

2. Furthermore, if diagonal entries are all 1's then so is the product

Pf: • (for $n=3$)

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$(AB)_{12} = \begin{pmatrix} a_{11} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ b_{22} \\ b_{32} \end{pmatrix} = 0.$$

# LU factorization

# LU factorization

**Part I:** $A = (a_{ik})$ , $B = (b_{kl})$

# LU factorization

**Part I:** $A = (a_{ik})$ , $B = (b_{kl})$

satisfying $\quad a_{ik} = 0 \quad \text{if} \quad k > i$

$\qquad\qquad\quad b_{kl} = 0 \quad \text{if} \quad l > k.$

# LU factorization

Part I: $A = (a_{ik})$ , $B = (b_{kl})$

satisfying $a_{ik} = 0$ if $k > i$

$b_{kl} = 0$ if $l > k$.

then if $l > i$, we have

$$(AB)_{il} = a_{i1} b_{1l} + a_{i1} b_{2l} + \cdots + a_{ii} b_{il}$$
$$+ a_{i(i+1)} b_{(i+1)l} + \cdots + a_{in} b_{nl}$$

# LU factorization

**Part I:** $A = (a_{ik})$ , $B = (b_{k\ell})$

satisfying $a_{ik} = 0$ if $k > i$

$b_{k\ell} = 0$ if $\ell > k$.

then if $\ell > i$ , we have

$$(AB)_{i\ell} = a_{i1}\cancelto{0}{b_{1\ell}} + a_{i1}\cancelto{0}{b_{2\ell}} + \dots + a_{ii}\cancelto{0}{b_{i\ell}}$$

$$+ a_{i(i+1)} b_{(i+1)\ell} + \dots + a_{in} b_{n\ell}$$

# LU factorization

**Part I:** $A = (a_{ik})$ , $B = (b_{kl})$

satisfying $a_{ik} = 0$ if $k > i$

$b_{kl} = 0$ if $l > k$.

then if $l > i$, we have

$$(AB)_{il} = a_{i1}\cancel{b_{1l}}^{0} + a_{i1}\cancel{b_{2l}}^{0} + \cdots + a_{ii}\cancel{b_{il}}^{0}$$

$$+ \cancel{a_{i(i+1)}}^{0} b_{(i+1)l} + \cdots + \cancel{a_{in}}^{0} b_{nl}$$

# LU factorization

Part I: $A = (a_{ik})$ , $B = (b_{k\ell})$

satisfying $a_{ik} = 0$ if $k > i$

$b_{k\ell} = 0$ if $\ell > k$.

then if $\ell > i$ , we have

$$(AB)_{i\ell} = a_{i1} \cancel{b_{1\ell}}^{0} + a_{i1} \cancel{b_{2\ell}}^{0} + \cdots + a_{ii} \cancel{b_{i\ell}}^{0}$$

$$+ \cancel{a_{i(i+1)}}^{0} b_{(i+1)\ell} + \cdots + \cancel{a_{in}}^{0} b_{n\ell}$$

$$= 0 .$$

# LU factorization

**Part I:** $A = (a_{ik})$ , $B = (b_{kl})$

satisfying $\quad a_{ik} = 0 \quad \text{if} \quad k > i$

$\qquad\qquad\quad b_{kl} = 0 \quad \text{if} \quad l > k.$

then if $l > i$, we have

$$(AB)_{il} = a_{i1}\cancel{b_{1l}}^{0} + a_{i1}\cancel{b_{2l}}^{0} + \cdots + a_{ii}\cancel{b_{il}}^{0}$$

$$+ \cancel{a_{i(i+1)}}^{0} b_{(i+1)l} + \cdots + \cancel{a_{in}}^{0} b_{nl}$$

$$= 0.$$

**Part II:** • if $a_{ii} = 1 = b_{ii}$ for all $i$.

then $(AB)_{ii} = a_{ii} b_{ii} = 1.$

# Row exchanges and permutation matrix

# Row exchanges and permutation matrix

- Recall that we may have:

$$Ax = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

# Row exchanges and permutation matrix

- Recall that we may have:

$$Ax = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- The LU decomposition for A fails, we need to permute the rows.

# Row exchanges and permutation matrix

- Recall that we may have:

$$Ax = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- The LU decomposition for A fails, we need to permute the rows.

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4/3 \\ 0 & 1 \end{bmatrix}$$

# Row exchanges and permutation matrix

- Recall that we may have:

$$Ax = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- The LU decomposition for A fails, we need to permute the rows.

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4/3 \\ 0 & 1 \end{bmatrix}$$

- <u>i.e.</u> $PA = LDU$ in this example.

# Row exchanges and permutation matrix

- Recall that we may have:

$$Ax = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- The LU decomposition for A fails, we need to permute the rows.

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4/3 \\ 0 & 1 \end{bmatrix}$$

- <u>i.e.</u> $PA = LDU$ in this example.

- If $A$ is non-singular, then we can find $P$ s.t. $PA = LDU$.

# Row exchanges and permutation matrix

# Row exchanges and permutation matrix

e.g.:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

# Row exchanges and permutation matrix

e.g.:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

# Row exchanges and permutation matrix

e.g.:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

# Row exchanges and permutation matrix

e.g.:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

# Row exchanges and permutation matrix

e.g.:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

- Consider $PA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

# Row exchanges and permutation matrix

e.g.:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

- In general, we need to find the correct $P$ by going through the elimination process.

- Consider $PA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

# Row exchanges and permutation matrix

# Row exchanges and permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

# Row exchanges and permutation matrix

Conclusion: $A$ : $n \times n$ matrix

if non-singular : there exists $P = P_k \cdots P_1$

# Row exchanges and permutation matrix

permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

if non-singular : there exists $P = P_k \cdots P_1$

# Row exchanges and permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

permutation matrix

if non-singular : there exists $P = P_k \cdots P_1$

s.t. $PA = LDU$

# Row exchanges and permutation matrix

permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

if non-singular : there exists $P = P_k \cdots P_1$

s.t. $PA = LDU$

lower triangular

# Row exchanges and permutation matrix

permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

if non-singular : there exists $P = P_k \cdots P_1$

s.t. $PA = LDU$. upper triangular.

lower triangular

# Row exchanges and permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

permutation matrix

if non-singular : there exists $P = P_k \cdots P_1$

diagonal

s.t. $PA = LDU$. upper triangular.

lower triangular

# Row exchanges and permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

permutation matrix

if non-singular : there exists $P = P_k \cdots P_1$

diagonal.

s.t. $PA = LDU$. upper triangular.

lower triangular

**Rk:** · $P \longleftrightarrow$ row exchange in elimination process

# Row exchanges and permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

if non-singular : there exists $P = P_k \cdots P_1$

permutation matrix

s.t. $PA = LDU$.

diagonal

upper triangular.

lower triangular

**Rk:**
- $P \longleftrightarrow$ row exchange in elimination process
- $L \longleftrightarrow$ type III elementary matrices.

# Row exchanges and permutation matrix

**Conclusion:** $A$ : $n \times n$ matrix

permutation matrix

if non-singular : there exists $P = P_k \cdots P_1$

diagonal

s.t. $PA = LDU$ ← upper triangular.

lower triangular

**Rk:**
- $P \longleftrightarrow$ row exchange in elimination process
- $L \longleftrightarrow$ type III elementary matrices
- $L, D, U$ are unique when $P$ fixed.

# Row exchanges and permutation matrix

# Row exchanges and permutation matrix

- if singular : there exists $P$ s.t.

$$PA = LU$$

# Row exchanges and permutation matrix

- if singular : there exists $P$ s.t.

$$PA = LU$$ → without full set of pivots.

# Row exchanges and permutation matrix

- if singular : there exists $P$ s.t.

$$PA = LU$$ → without full set of pivots.

i.e.

e.g. $$U = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

# Row exchanges and permutation matrix

- if singular : there exists $P$ s.t.

$$PA = LU$$ → without full set of pivots.

i.e.

e.g. $$U = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

in this case : we don't have uniqueness for $L$