LAw4

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$Ax=b$ and $Ax=0$

$Ax=b$ has a solution when $b \in C(A)$

Given any $m\times n$ matrix A

$P = P_1, \cdots, P_k$

$L = E_1, \cdots E_s$

$P(A) = LU \rightarrow$ rowechlon form

$PA=LU$

$U = D\tilde{U} \Rightarrow$ for non-square matrices.

$\tilde{U} = ER$ $\quad$ U is ref (row echelon form)

Prop: $N(A) = N(U) = N(\tilde{U}) = N(R)$

Pf: $PA = LU$

$Ax=0 \Leftrightarrow (L^{-1}P)Ax=0$

$$\Leftrightarrow \quad Ux = 0$$

similarly: $U = D\tilde{U}$

$$E\tilde{U} = R$$

e.g. $R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ solve $R\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

solve from bottom:

$$x_3 + x_4 = 0 \Rightarrow x_3 = -x_4$$

$$x_1 + 3x_2 - x_4 = 0 \Rightarrow x_1 = -3x_2 + x_4$$

ie. $\vec{x} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$

or:

$$N(R) = \left\{ \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} \middle| \ x_2, x_4 \in \mathbb{R} \right\}$$

or:

$$\vec{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

or

$$N(R) = \text{span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

minimal size of spanning set = # of free variables

= n - (non-zero rows)

$$C(A) = \left\{ aE\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bE\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

null space: $N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0 \}$

is a subspace of $\mathbb{R}^n$

column space: $C(A) = \{ \vec{b} \in \mathbb{R}^n \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n \}$