Problem set 2

Linear algebra 06

September 17, 2025

1. Given any $n \times n$ matrix A, if we let

$$B = A + A^T, \qquad C = A - A^T,$$

show that:

   (a) B is symmetric (here symmetric means $B^T = B$) and C is skew-symmetric (here skew-symmetric means $C^T = -C$);

   (b) A can be written as a sum of a symmetric matrix and a skew symmetric matrix;

   (c) show that $\vec{x}^T C \vec{x} = 0$ for any n vector $\vec{x}$.

2. Let A be the 4×4 matrix

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix},$$

where θ, λ are real numbers, compute A^n for some positive integer n.

3. (a) Let A be a 4×4 matrix, prove that: if $AB = BA$ for any 4×4 matrix B, then $A = \lambda I$ for some real number λ. (Hints: Set B to be the matrix with only non-zero entry at (i, j) entry with value 1, and compute AB and BA.)

   (b) What happen for $n \times n$ matrices?

4. Suppose $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, compute the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{2025} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{2026}.$$

5. Let A, B be 4×4 matrices, determine whether the following statement is true or not. Prove it if it is true, otherwise give a counter example.

   (a) If $A^2 = A$ then $A = 0$ or $A = I$;

   (b) If A, B are invertible, then $ABA^{-1}B^{-1} = I$.