Lecture 10

The 4 fundamental subspaces

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- Given a $m \times n$ matrix A, we can consider

1. null space of A: $N(A) = \{ x \mid x \in \mathbb{R}^n, Ax = 0 \}$

2. column space of A: $C(A) = \mathrm{Span}\{ \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \}$

3. Row space of $A := C(A^T)$

4. left nullspace of $A := N(A^T)$.

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- $N(A)$, $C(A^T)$ are subspaces of $\mathbb{R}^n$

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- What are the relation between them?

Recall: We have proven

$$\dim(C(A)) + \dim(N(A)) = n.$$

- Are there other relation between their dimension?

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$$A \longrightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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- $\dim(C(A^T))$ = # of pivot rows $= rk(A)$.

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$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix} \begin{bmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 3 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

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$$\begin{matrix} \| & \| & \| \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{matrix} \qquad\qquad \begin{matrix} \| & \| & \| \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{matrix}$$

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$$\Longrightarrow \left.\begin{array}{l} e_{11}\vec{a}_1 + e_{12}\vec{a}_2 + e_{13}\vec{a}_3 = \vec{u}_1 . \\ e_{21}\vec{a}_1 + e_{22}\vec{a}_2 + e_{23}\vec{a}_3 = \vec{u}_2 \\ e_{31}\vec{a}_1 + e_{32}\vec{a}_2 + e_{33}\vec{a}_3 = \vec{u}_3 \end{array}\right\}$$

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Columns: $\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3$ and $\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3$.

$$\Longrightarrow \left.\begin{aligned} e_{11}\vec{a}_1 + e_{12}\vec{a}_2 + e_{13}\vec{a}_3 &= \vec{u}_1 \\ e_{21}\vec{a}_1 + e_{22}\vec{a}_2 + e_{23}\vec{a}_3 &= \vec{u}_2 \\ e_{31}\vec{a}_1 + e_{32}\vec{a}_2 + e_{33}\vec{a}_3 &= \vec{u}_3 \end{aligned}\right\} \Rightarrow \begin{aligned} &\mathrm{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} \\ &\subseteq \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} \end{aligned}$$

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$\Rightarrow$ every column in AB lies in $C(A)$ ☒

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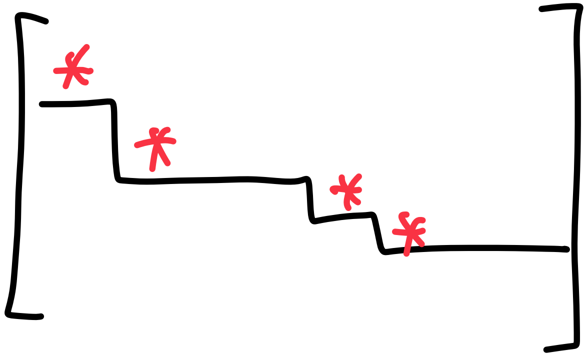

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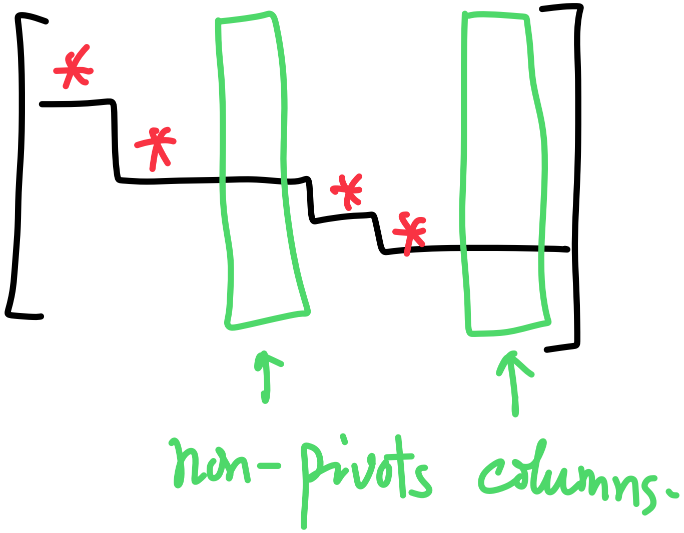

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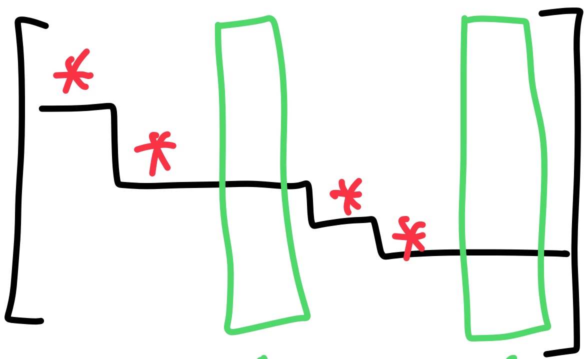

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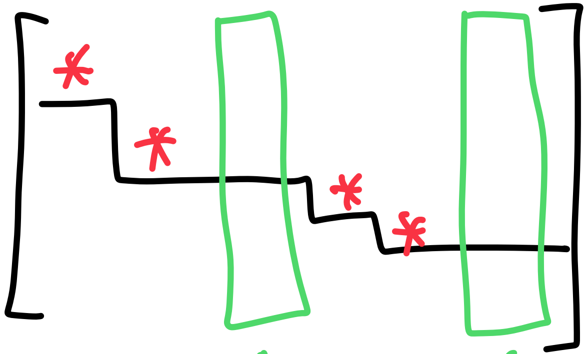
- $U =$ 

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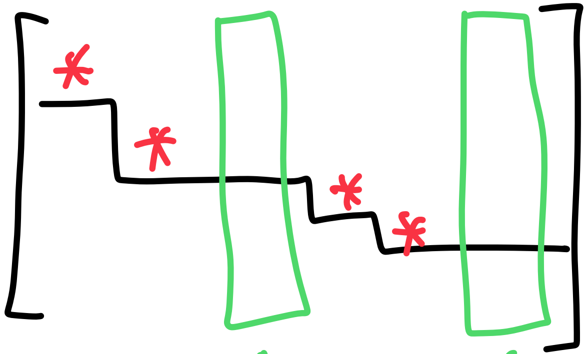
- $\mathcal{U} =$  then $\dim(N(A))$

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  $\dim = r$ $\dim = n-r$

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Pf:
- Write $A = \begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \end{bmatrix}$ as row.

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- $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in N(A)$, $\begin{bmatrix} - a_i - \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

  $\leftarrow \langle a_i^T, \vec{x} \rangle$ inner product

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$$a_1 w_1 + \dots + a_k w_k = -b_1 v_1 - \dots - b_\ell v_\ell .$$

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$$\underbrace{a_1 w_1 + \cdots + a_k w_k}_{\text{in } C(A^T)} = \underbrace{-b_1 v_1 - \cdots - b_\ell v_\ell}_{\text{in } N(A)} .$$

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- Take a basis $w_1, \ldots, w_k$ of $C(A^T)$, $v_1, \ldots, v_\ell$ of $N(A)$
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$$\underbrace{a_1 w_1 + \ldots + a_k w_k}_{\text{in } C(A^T)} = \underbrace{-b_1 v_1 - \ldots - b_\ell v_\ell}_{\text{in } N(A)} .$$

$\Rightarrow$ both sides are zero $\Rightarrow$ $a_1 = \ldots = a_k = 0 = b_1 = \ldots = b_\ell$.

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- But $k+\ell = n \Rightarrow$ we cannot add more by dimension constraint.
- Hence $w_1, \ldots, w_k, v_1, \ldots, v_\ell$ is a basis for $\mathbb{R}^n$.

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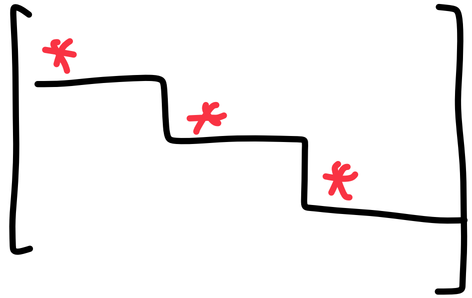
- If $A \xrightarrow{\text{row operation}} U =$ 

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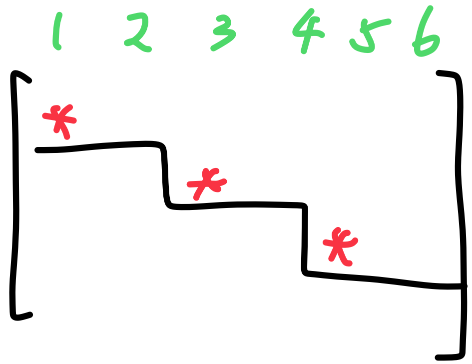

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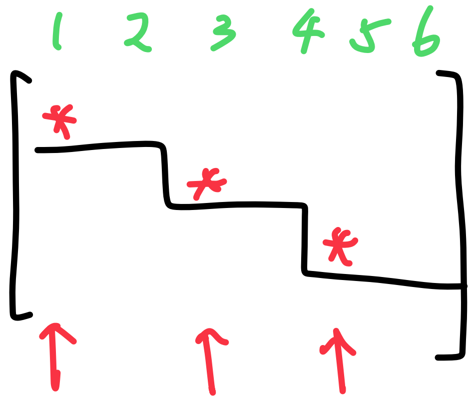

pivot columns.

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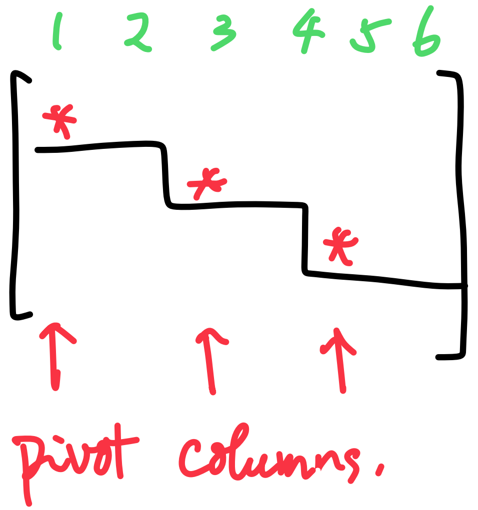

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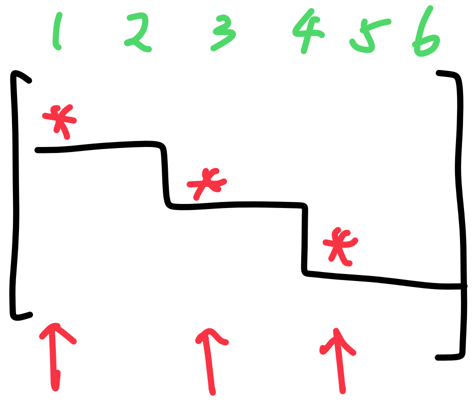
- If $A \xrightarrow{\text{row operation}} \mathcal{U} =$ 

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- In the above example, $\vec{a}_1, \vec{a}_3, \vec{a}_5$ is a basis for $C(A)$.

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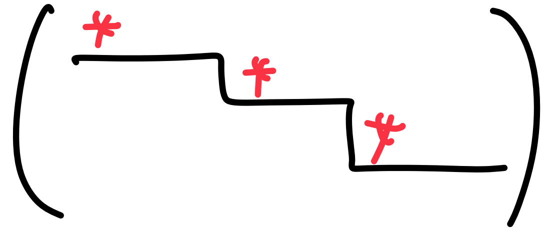
- Observe that for $U =$ 

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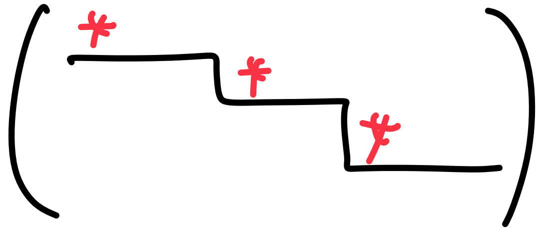
- Observe that for $U =$ 

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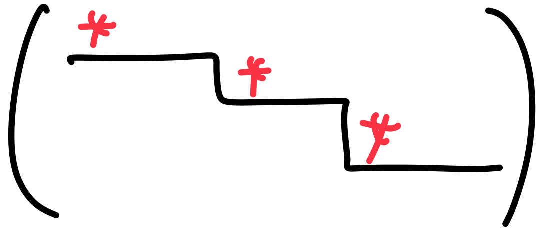
- Observe that for $\mathcal{U} =$ 

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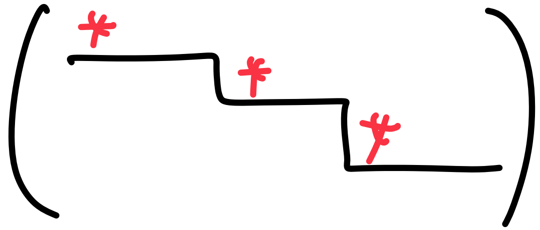
- Observe that for $\mathcal{U} =$ 

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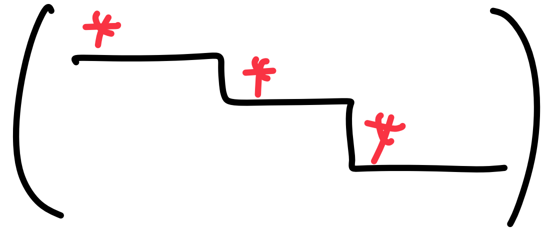
- Observe that for $U =$ 

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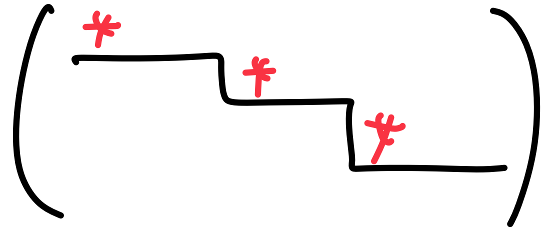
- Observe that for $U =$ 

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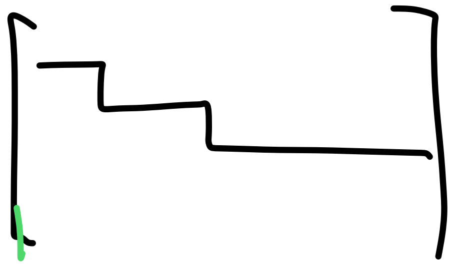

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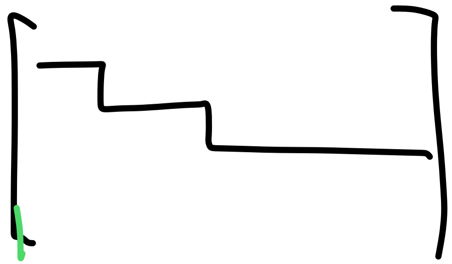
products of elementary matrices. $\rightarrow$ $EA = U =$ 

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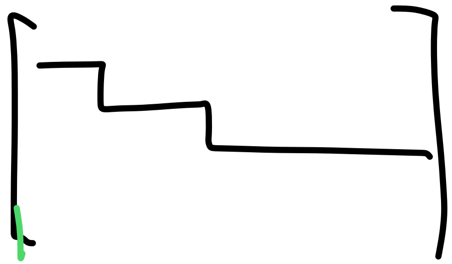

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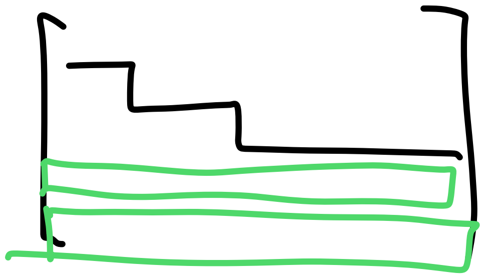

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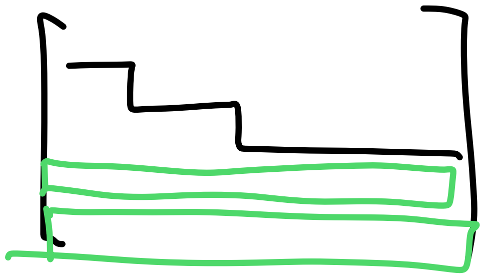

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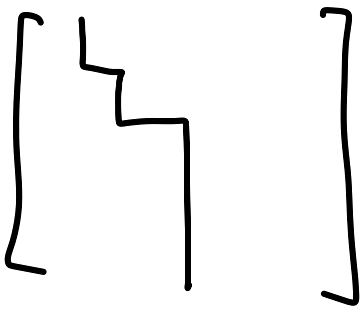

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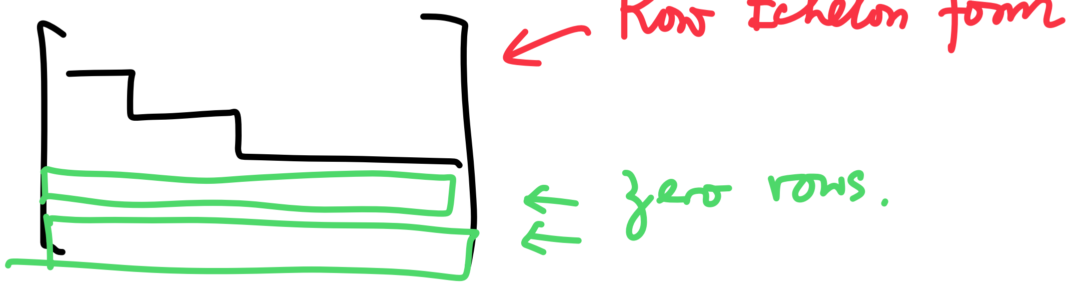

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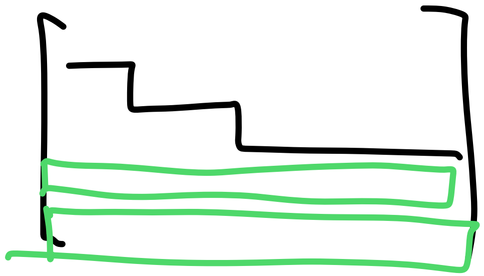

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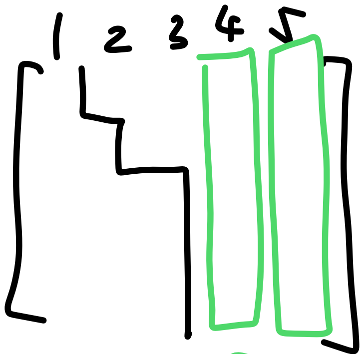

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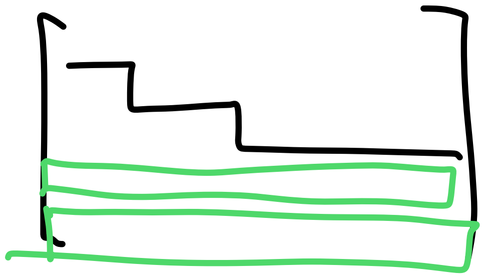

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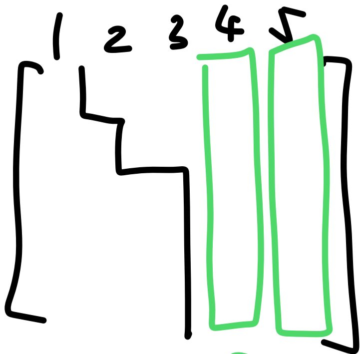

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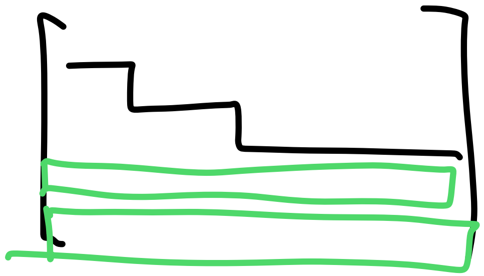

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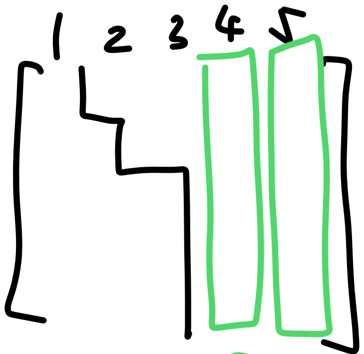

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basis for $N(A^T)$

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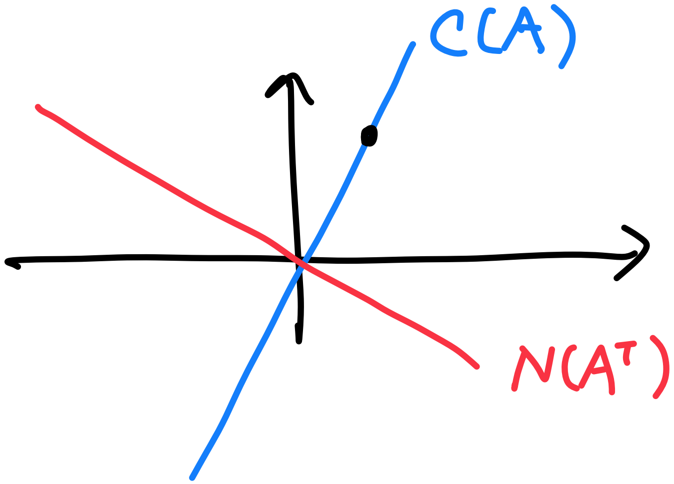

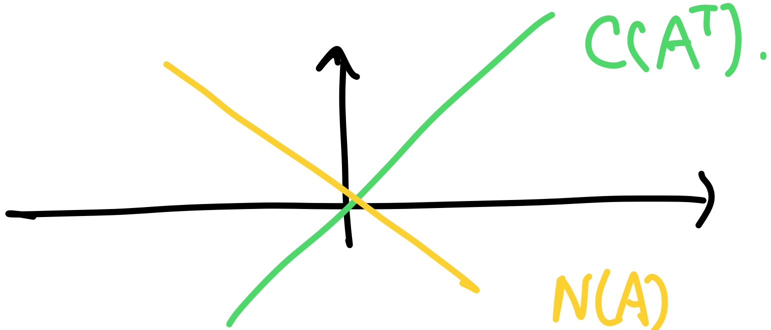

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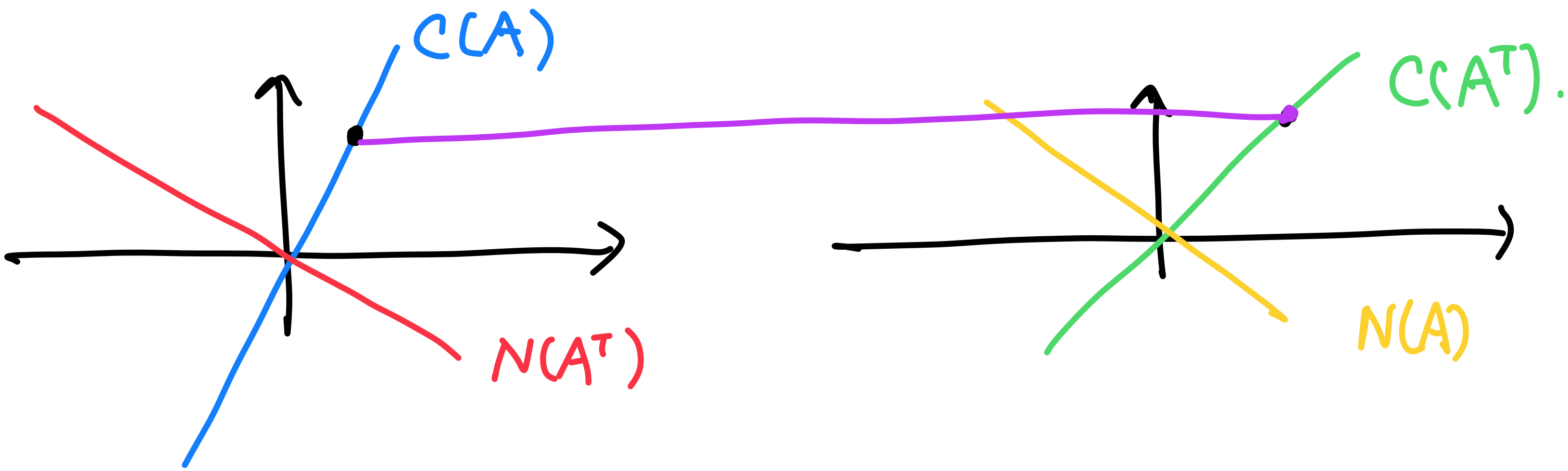

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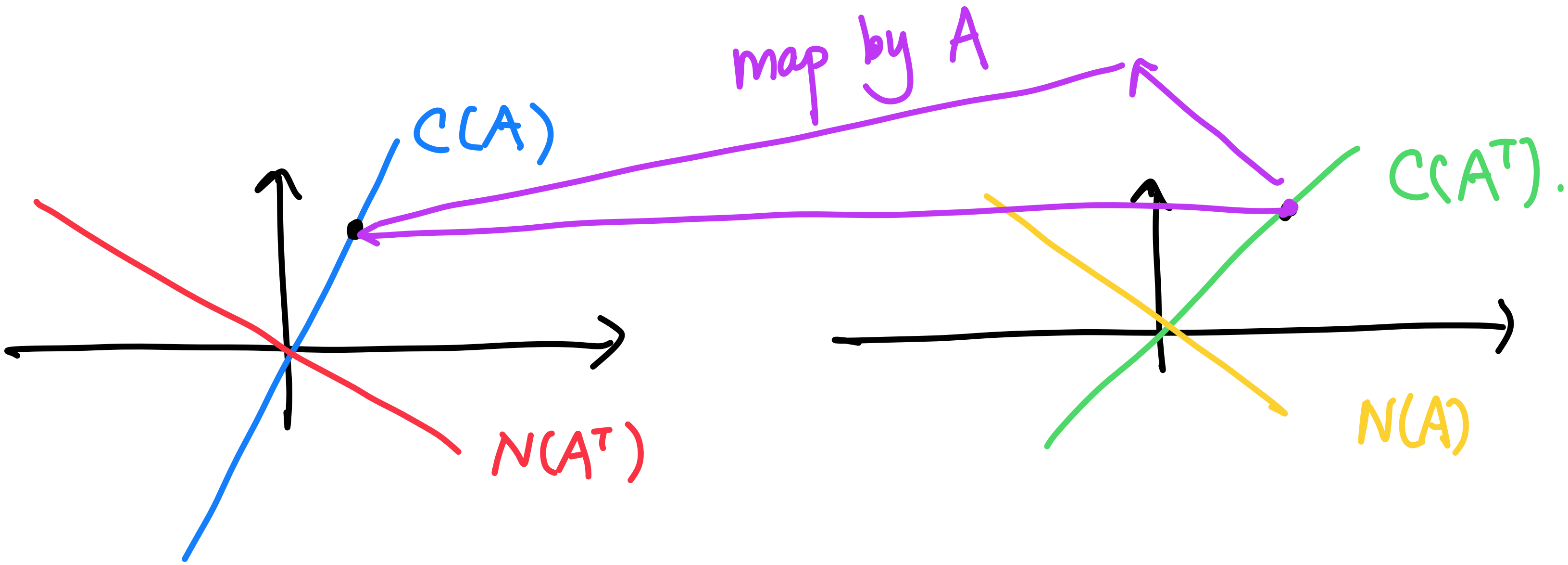

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and we say $B = A^{-1}$ (because inverse is unique)

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$$C = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix}$$

Left inverse and right inverse

- A way to find right inverse $C = \underbrace{A^T}_{n\times m}\underbrace{(AA^T)^{-1}}_{m\times m}$

- Symbolically: $AC = (AA^T)(AA^T)^{-1} = I_m$.

e.g.: $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$, $AA^T = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$

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is one of the right inverse.

Left inverse and right inverse

Left inverse and right inverse

Q: Why (AA^T) is invertible?

Left inverse and right inverse

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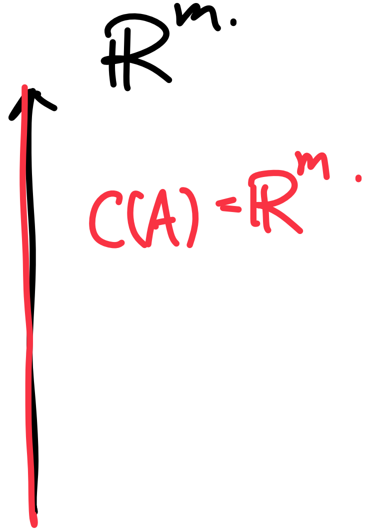
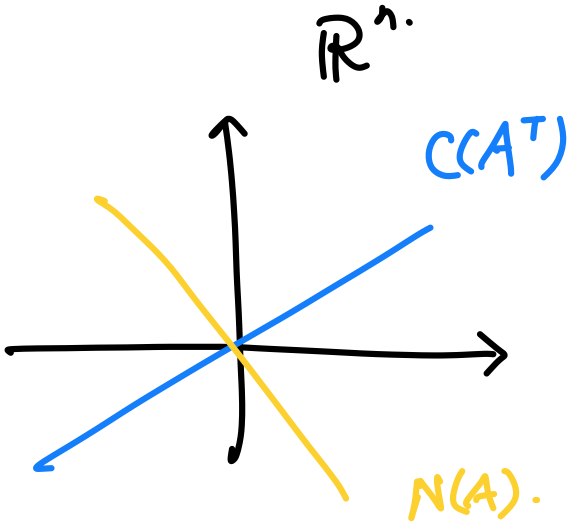
Pictorially:





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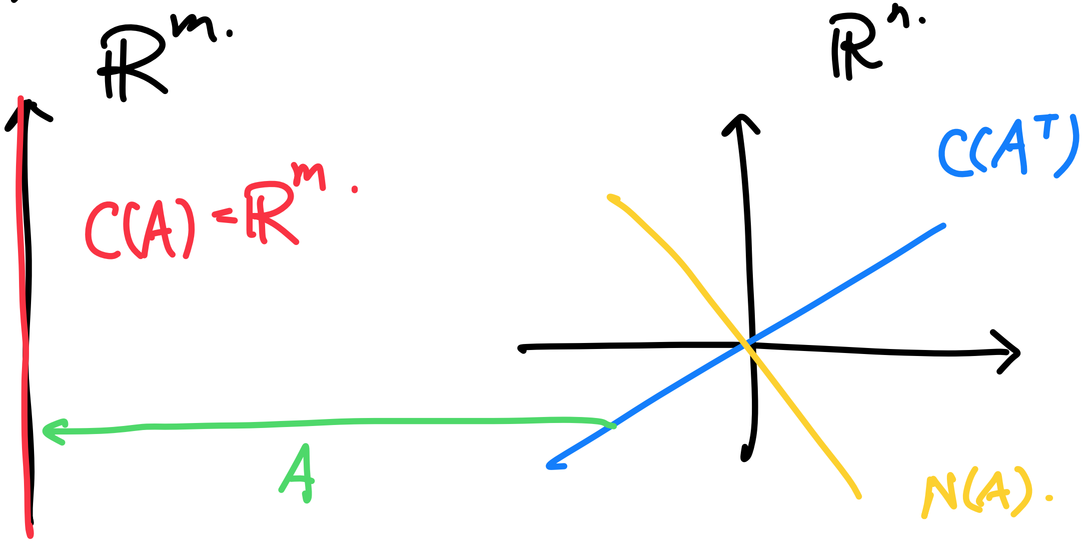

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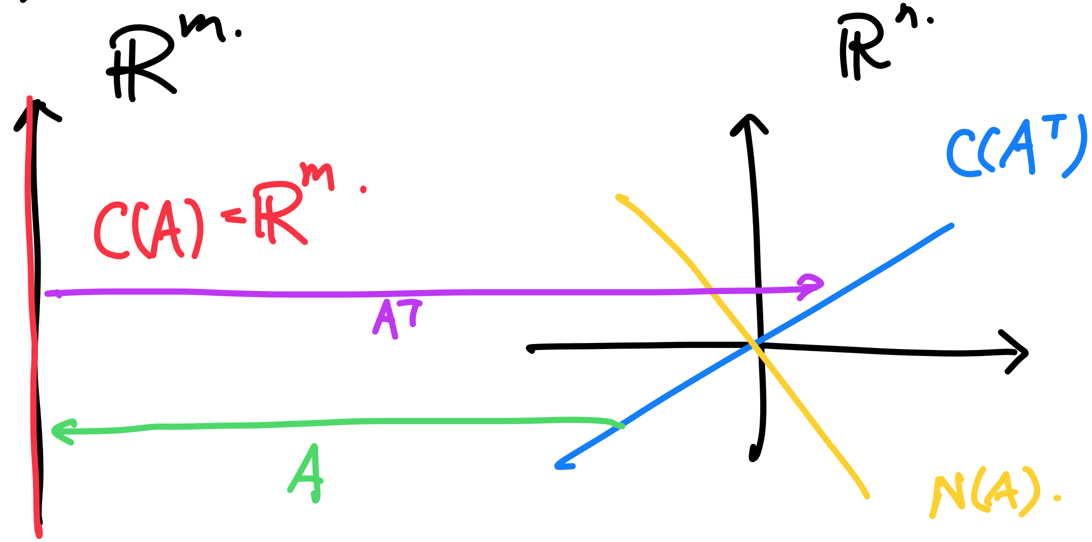

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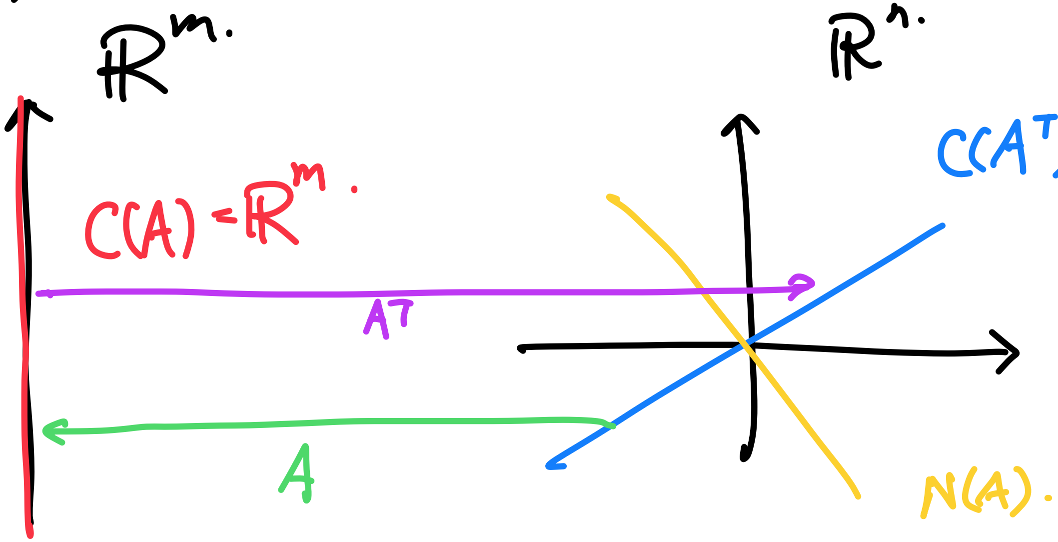

at least we see that $rk(AA^T) = m$ in this case.

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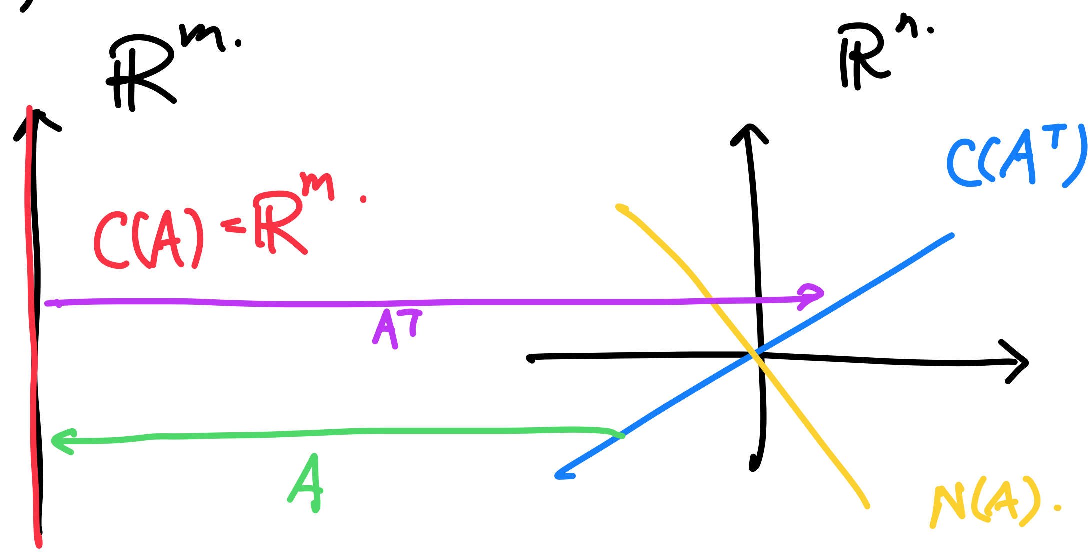

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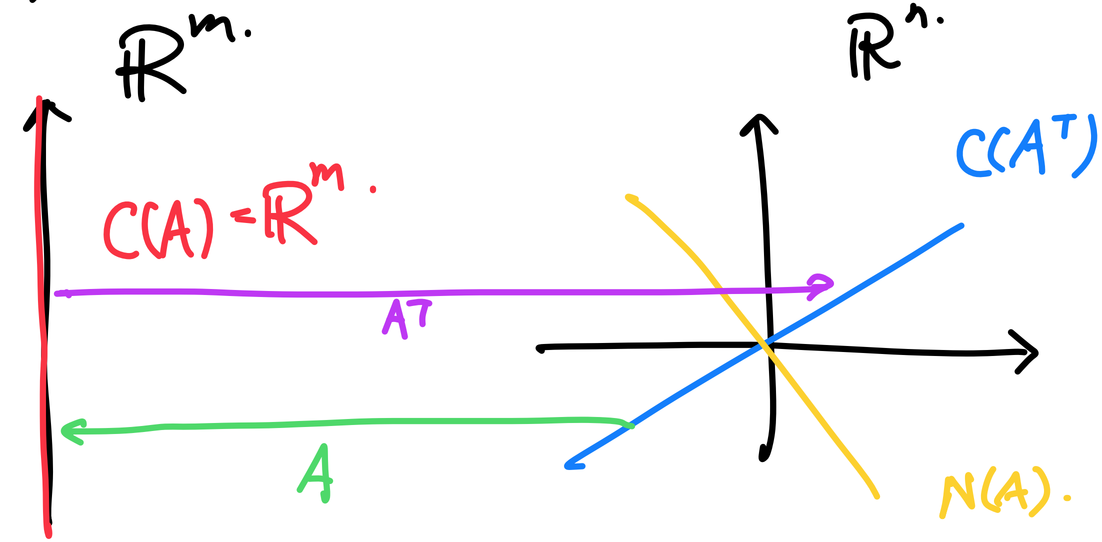

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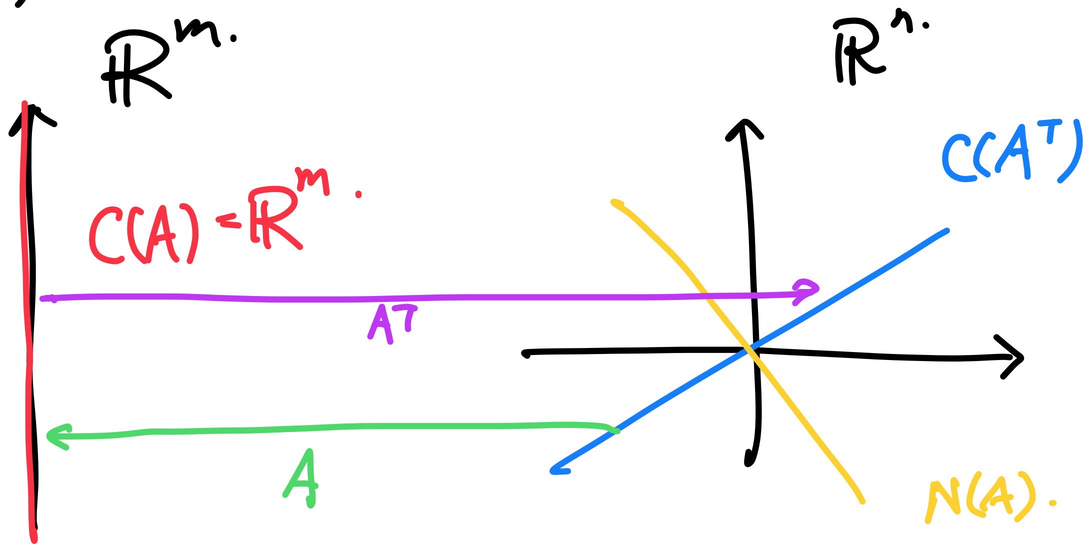

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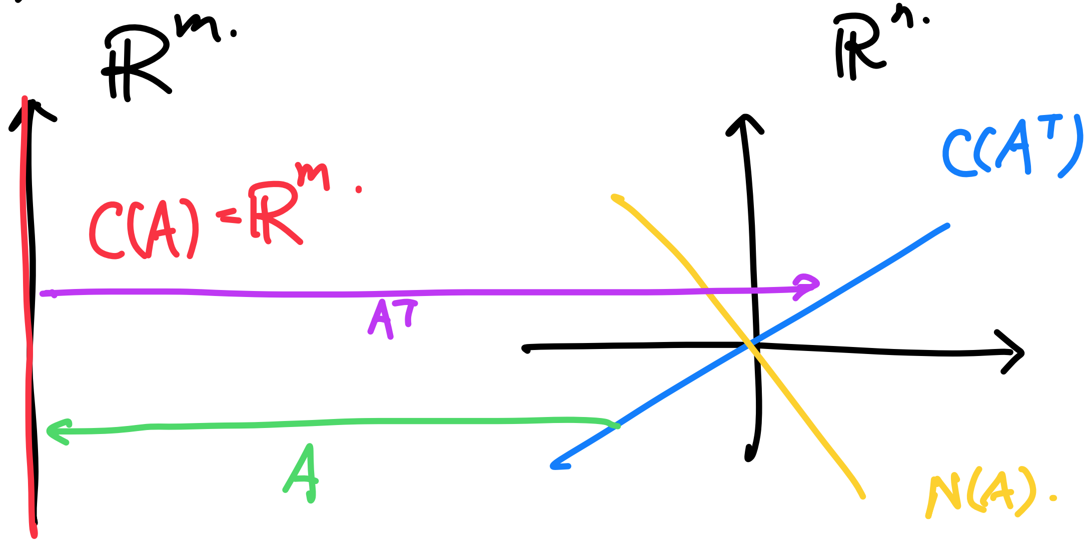

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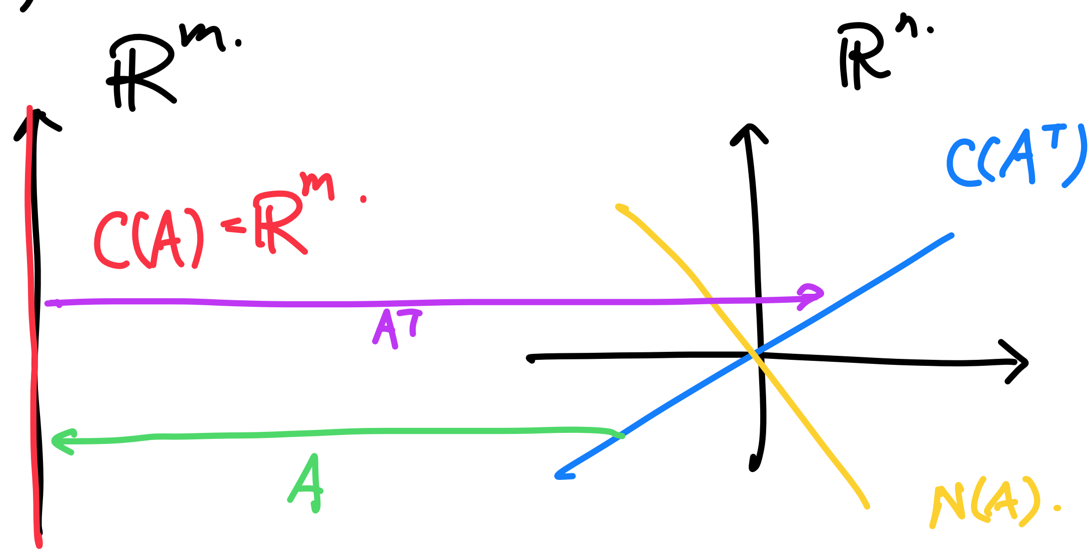

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$(N(A^T) = \{0\}) \Rightarrow \vec{x} = 0.$

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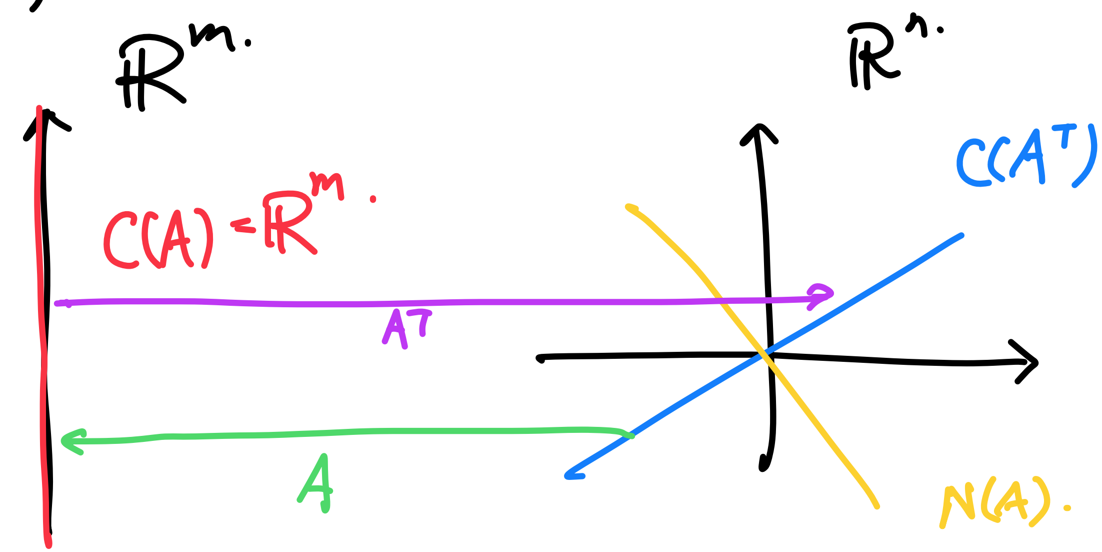

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- $N(AA^T) = 0$ and it is a square matrix $\Rightarrow$ it is invertible.

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  with $m > n = rk(A)$.

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6. It has left inverse
7. It has right inverse.

⋮

Vandermonde matrix

Vandermonde matrix

- Let consider the following problem:

  find a degree n polynomial $p(t) = a_0 + a_1 t + \dots + a_n t^n$

  passing through $(n+1)$-given points (t_i, b_i)

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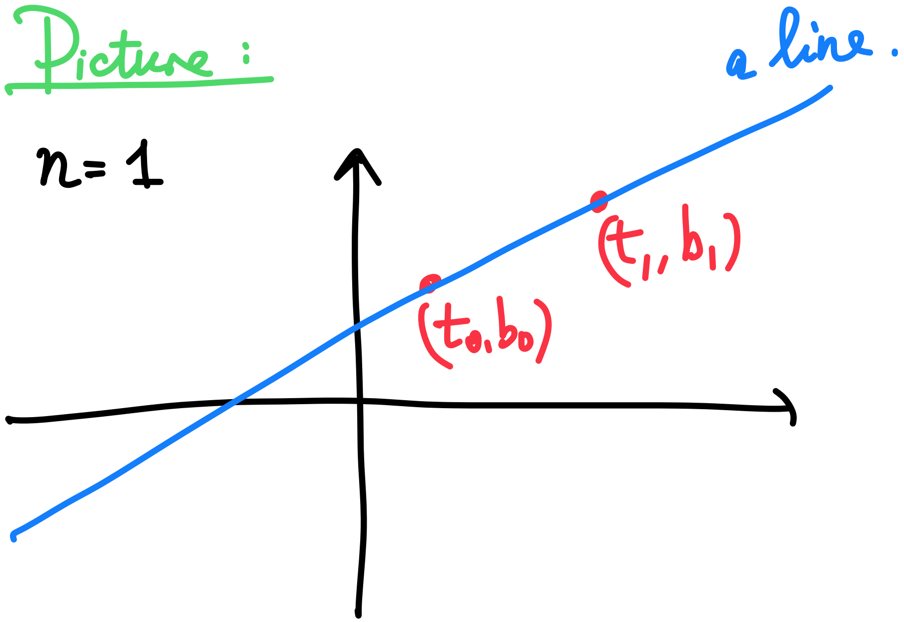
Picture:

$n = 1$



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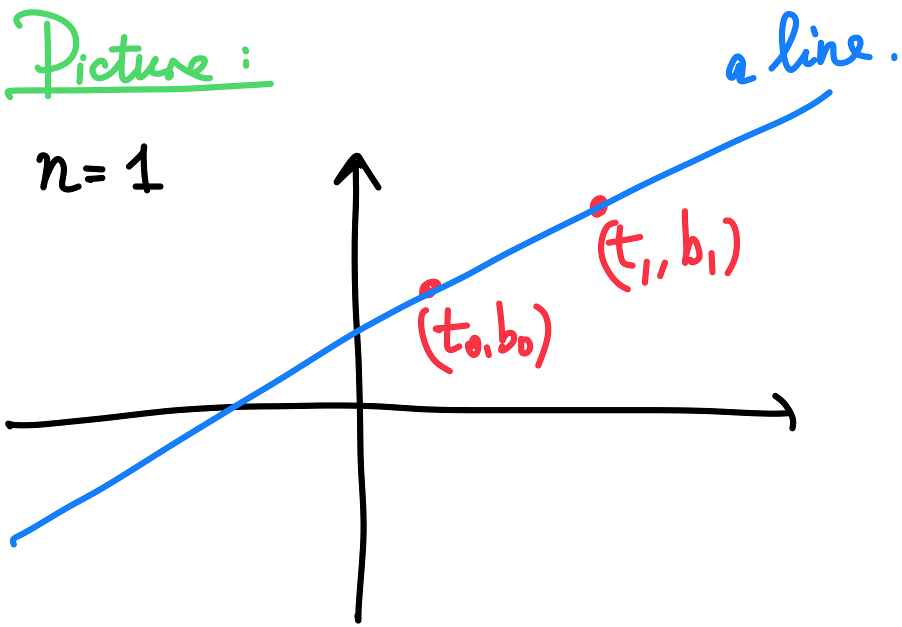

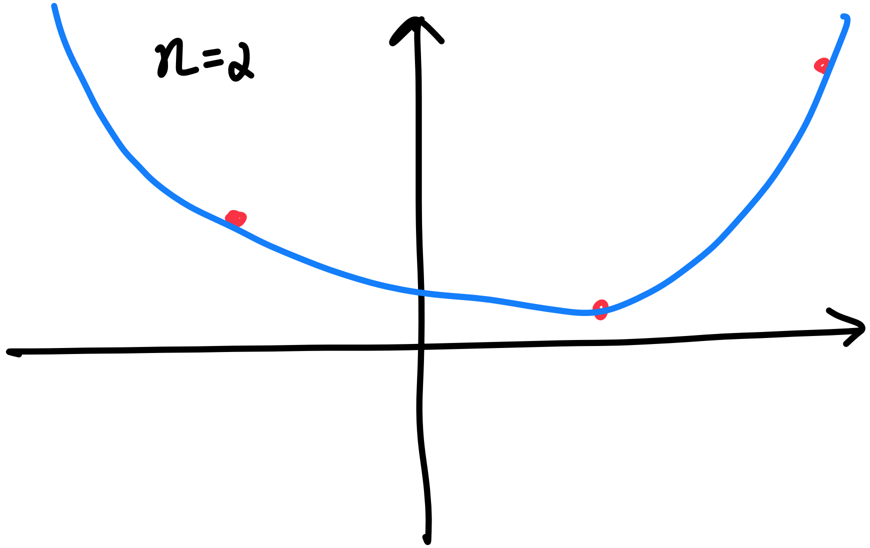

Vandermonde matrix

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$$a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 = b_0$$

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$$\begin{cases} a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 = b_0 \\ a_0 + a_1 t_1 + a_2 t_1^2 + a_3 t_1^3 = b_1 \\ a_0 + a_1 t_2 + a_2 t_2^2 + a_3 t_2^3 = b_2 \\ a_0 + a_1 t_3 + a_2 t_3^2 + a_3 t_3^3 = b_3 \end{cases}$$

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Solve for a_0, a_1, a_2, a_3.

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Solve for a_0, a_1, a_2, a_3.

or in matrix

$$V = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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$$\begin{cases} a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 = b_0 \\ a_0 + a_1 t_1 + a_2 t_1^2 + a_3 t_1^3 = b_1 \\ a_0 + a_1 t_2 + a_2 t_2^2 + a_3 t_2^3 = b_2 \\ a_0 + a_1 t_3 + a_2 t_3^2 + a_3 t_3^3 = b_3 \end{cases}$$

Solve for a_0, a_1, a_2, a_3.

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$$V = \underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{bmatrix}}_{\text{Vandermonde matrix : invertible if all } t_i \text{ are distinct!}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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- V is invertible if all t_i are distinct.

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Pf:

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & t_1-t_0 & t_1^2-t_0^2 & t_1^3-t_0^3 \\ 0 & t_2-t_0 & t_2^2-t_0^2 & t_2^3-t_0^3 \\ 0 & t_3-t_0 & t_3^2-t_0^2 & t_3^3-t_0^3 \end{bmatrix}$$

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$\tilde{V}_1 =$

ask whether it is invertible.

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$\tilde{V}_1 =$ the lower-right 3×3 block; ask whether it is invertible.

$$\tilde{V}_1 = \begin{bmatrix} t_1-t_0 & 0 & 0 \\ 0 & t_2-t_0 & 0 \\ 0 & 0 & t_3-t_0 \end{bmatrix} \begin{bmatrix} 1 & t_1+t_0 & t_1^2+t_1t_0+t_0^2 \\ 1 & t_2+t_0 & t_2^2+t_2t_0+t_0^2 \\ 1 & t_3+t_0 & t_3^2+t_3t_0+t_0^2 \end{bmatrix}$$

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invertible?

Vandermonde matrix

- V is invertible if all t_i are distinct.

Pf:

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$\tilde{V}_1 =$ the boxed 3×3 block: ask whether it is invertible.

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invertible?

$\tilde{V}_1$ is invertible

Vandermonde matrix

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Pf:

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invertible?

$\tilde{V}_1$ is invertible $\Longleftrightarrow$

Vandermonde matrix

- V is invertible if all t_i are distinct.

Pf:

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & \boxed{\begin{matrix} t_1-t_0 & t_1^2-t_0^2 & t_1^3-t_0^3 \\ t_2-t_0 & t_2^2-t_0^2 & t_2^3-t_0^3 \\ t_3-t_0 & t_3^2-t_0^2 & t_3^3-t_0^3 \end{matrix}} \end{bmatrix}$$

$\tilde{V}_1$ ask whether it is invertible.

$$\tilde{V}_1 = \begin{bmatrix} t_1-t_0 & 0 & 0 \\ 0 & t_2-t_0 & 0 \\ 0 & 0 & t_3-t_0 \end{bmatrix} \begin{bmatrix} 1 & t_1+t_0 & t_1^2+t_1t_0+t_0^2 \\ 1 & t_2+t_0 & t_2^2+t_2t_0+t_0^2 \\ 1 & t_3+t_0 & t_3^2+t_3t_0+t_0^2 \end{bmatrix}$$

invertible?

$\tilde{V}_1$ is invertible $\Longleftrightarrow$ $V_1 = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix}$ is invertible

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$$\Rightarrow \quad A = \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} \begin{bmatrix} 1 & c_2 & c_3 \end{bmatrix}$$

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e.g.:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

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- $m \times n$ A has rank 1 $\Longleftrightarrow$ $A = u v^T$
  for some non-zero $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$.