Lecture 9

Linear independent, basis and dimension

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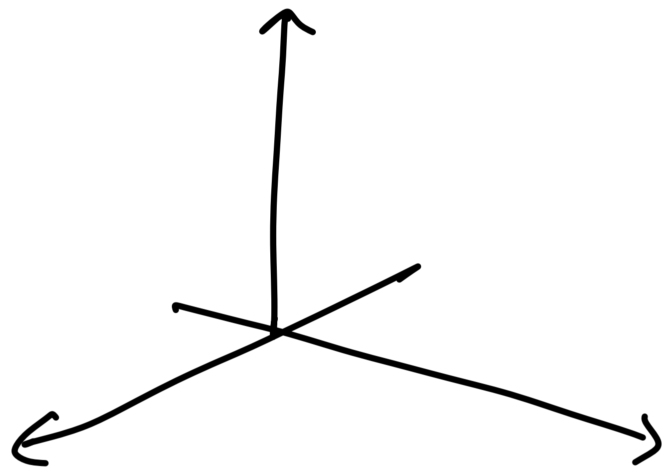

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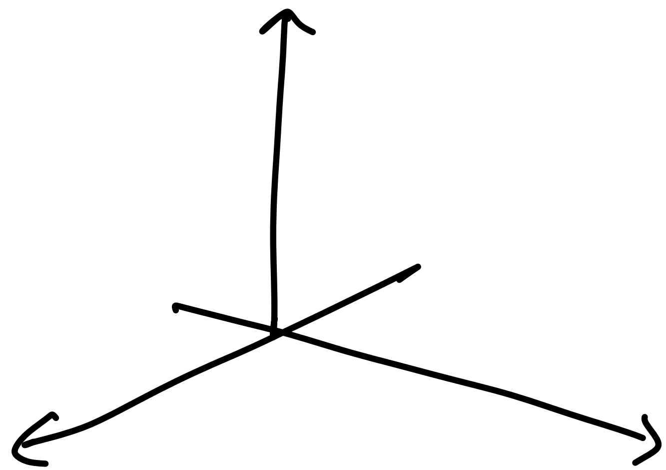

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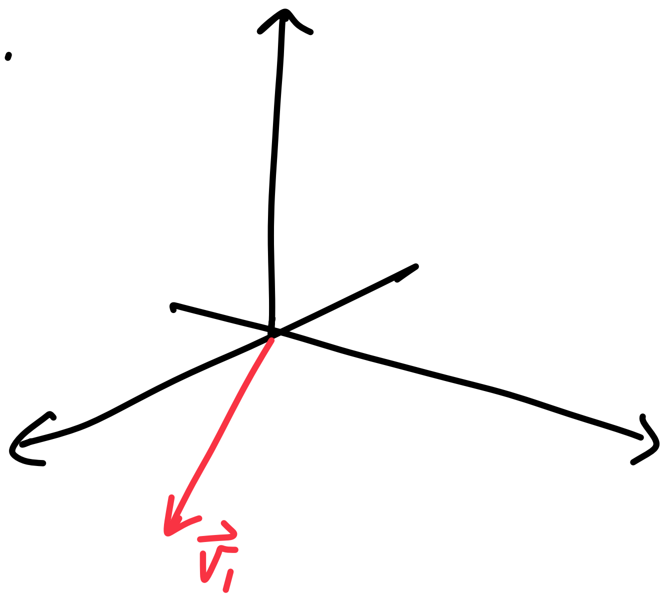

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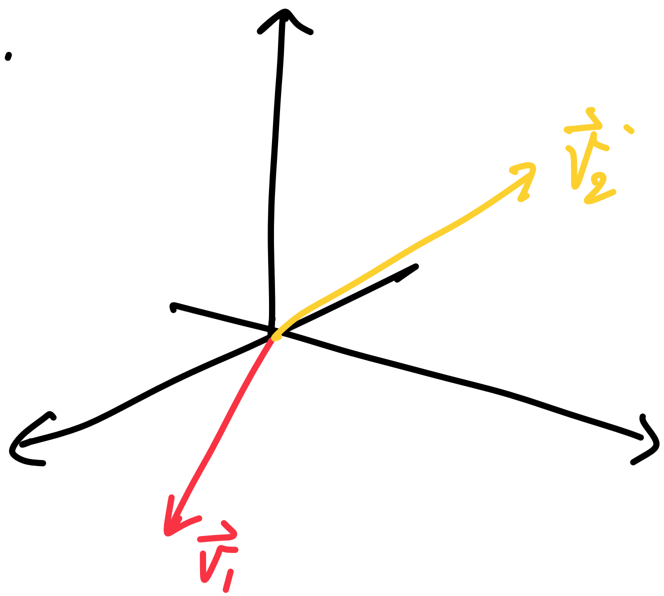

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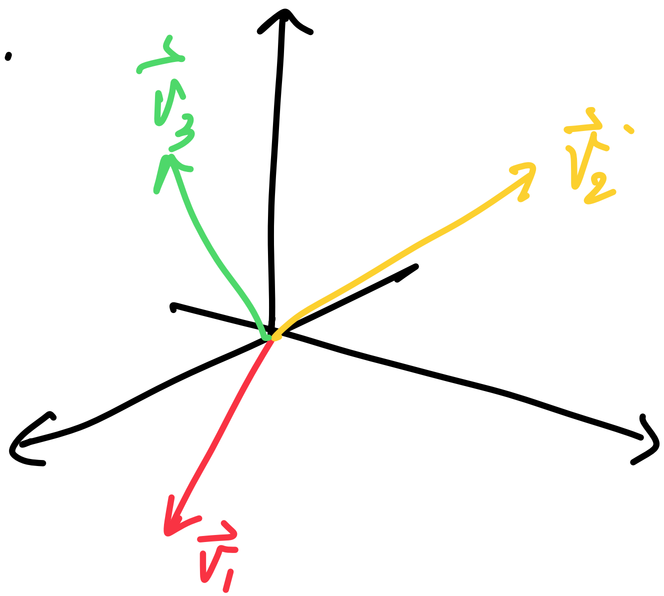

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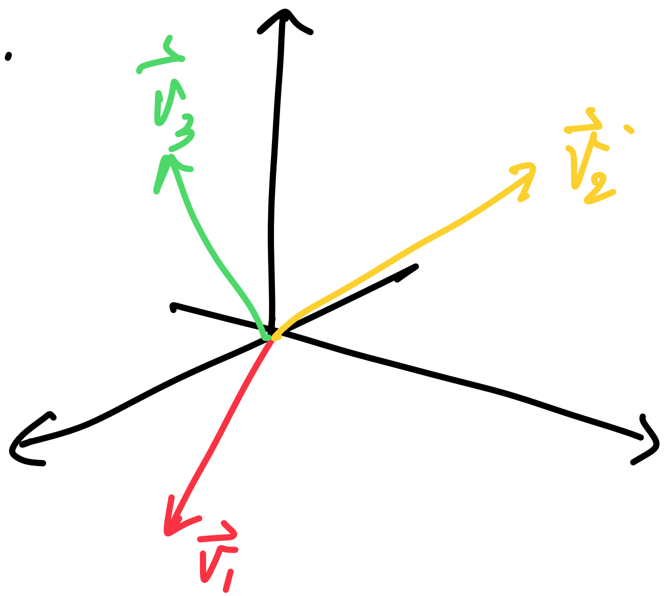

3. Three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are l.i. $\Longleftrightarrow$ not one is lying in the plane spanned by other two.

4. Four vectors cannot be l.i.

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$\Rightarrow$) similarly.

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Reason: $N(A) = 0$, $C(A) = \mathbb{R}^2$ with $A = [\vec{v}_1, \vec{v}_2]$.

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- $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is Not a basis for $\mathbb{R}^2$

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- $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ are l.i. as $N\left(\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \{0\}$.

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- Pivot columns is a set of basis for $C(\mathcal{U})$ in the case that $\mathcal{U}$ is in Row Echelon form.

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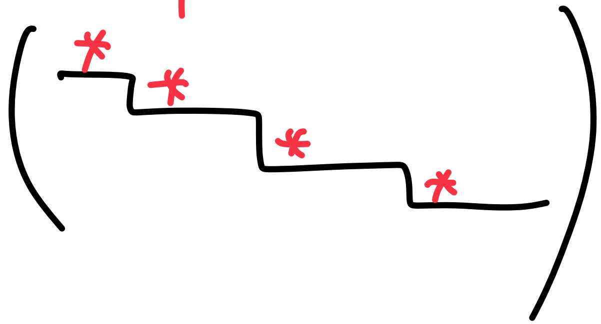

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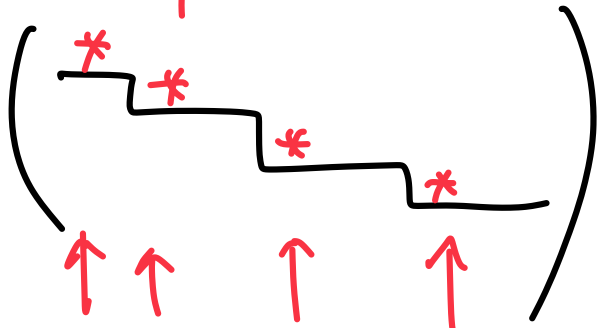

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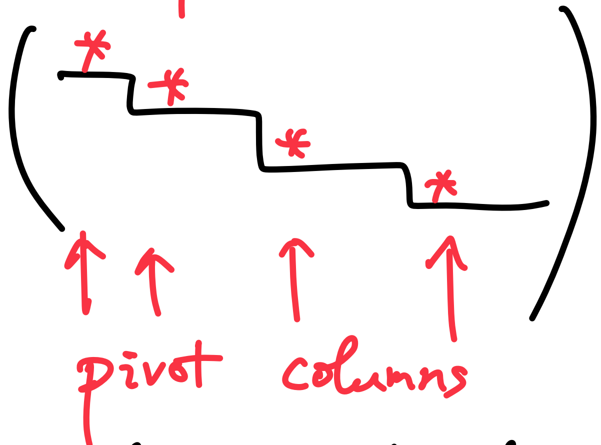

- The columns in A correspond to the pivot column form a basis of $C(A)$.

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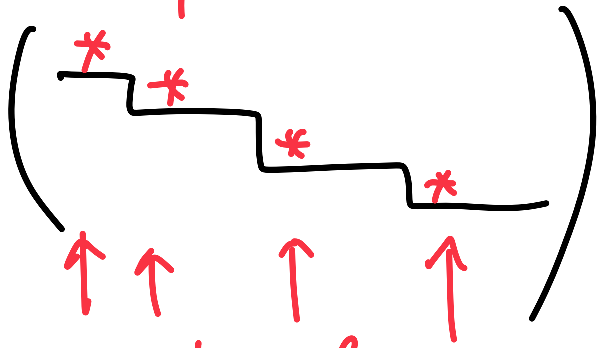

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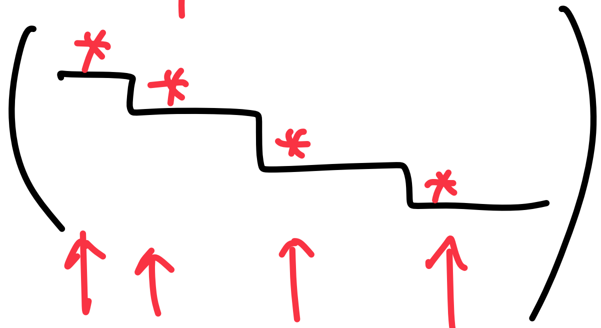

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fact: $\vec{v}_1, \ldots, \vec{v}_k$ basis for $W \Longleftrightarrow E\vec{v}_1, \ldots, E\vec{v}_k$ basis for $E(W)$ for invertible E.

Basis

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e.g. 1 Let $V = \{ a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{R} \}$.

then $1, x, x^2, x^3$ for a basis for V.

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2. take $M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$

then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

for a basis for $M_{2\times 2}(\mathbb{R})$.

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for a basis for $M_{2\times 2}(\mathbb{R})$.

3. Let $S_{2\times 2}(\mathbb{R}) = \{ A \mid A \in M_{2\times 2}(\mathbb{R}), A^T = A \}$

then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for a basis

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     $\therefore$ Say $\vec{v}_1$: if $\text{Span}\{\vec{v}_2, \dots, \vec{v}_k\} = V$.

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- A basis $\vec{v}_1, \dots, \vec{v}_k$ for V is an optimal spanning set of V:

<u>1.</u> take any $\vec{v}_i$ away will not span V

$\because$ Say $\vec{v}_1$: if $\operatorname{Span}\{\vec{v}_2, \dots, \vec{v}_k\} = V$.

then $\vec{v}_1 \in \operatorname{Span}\{\vec{v}_2, \dots, \vec{v}_k\}$

$\Rightarrow \vec{v}_1, \dots, \vec{v}_k$ are l.d.

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  1. take any $\vec{v}_i$ away will not span V

     $\because$ Say $\vec{v}_1$: if $\text{Span}\{\vec{v}_2, \dots, \vec{v}_k\} = V$.

     then $\vec{v}_1 \in \text{Span}\{\vec{v}_2, \dots, \vec{v}_k\}$

     $\Rightarrow \vec{v}_1, \dots, \vec{v}_k$ are l.d.

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     $\vec{v}_{k+1} \in V = \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$.

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e.g.: 4. Let $sl_2(\mathbb{R}) := \{ A \in M_{2\times 2}(\mathbb{R}) \mid tr(A) = 0 \}$.

then we have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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6. Let $\underline{so}(3) = \{ A \mid A \in M_{3\times 3}(\mathbb{R}),\ A^T = -A \}$

$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ is a basis.

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3. Repeat 1. until it stop.

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In fact: One can prove that any vector space has a basis, but require the use of Zorn's lemma / axiom of choice.

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Write:

$$w_1 = a_{11} v_1 + a_{21} v_2$$

$$w_2 = a_{12} v_1 + a_{22} v_2$$

$$w_3 = a_{13} v_1 + a_{23} v_2.$$

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Formally we write:

$$[W_1, W_2, W_3] = [V_1, V_2] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

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$$[W_1, W_2, W_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \implies c_1 W_1 + c_2 W_2 + c_3 W_3 = 0$$

$\rightarrow\!\leftarrow$.

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$$\dim(C(\mathcal{U})) = rk(\mathcal{U}) = \#\text{ of pivots columns.}$$

2. If $A = E\mathcal{U}$ for invertible E.

$$\Rightarrow C(A) = \{ E\vec{u} \mid \vec{u} \in C(\vec{u}) \}.$$

and $E\vec{u}_1, \ldots, E\vec{u}_r$ is a basis for $C(A)$ if $\vec{u}_1, \ldots, \vec{u}$ is for $C(\mathcal{U})$.

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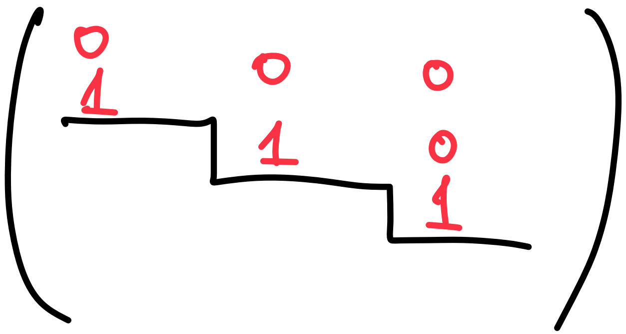
Pf: • Let say $U =$ 

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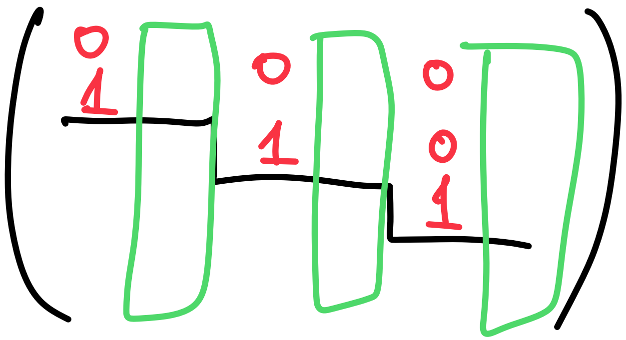

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Pf: • Let say

$$U = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \\ \begin{pmatrix} 0 & & 0 & & 0 & \\ 1 & & 1 & & 0 & \\ & & & & 1 & \end{pmatrix} \end{array}$$

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- $\mathcal{U} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

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- These three vectors are l.i. !

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• So the word "four dimensional vector" here refer to its ambient space