Problem set 5

Linear algebra 06

October 15, 2025

1. If v_1, v_2, v_3 are linearly independent in a vector space V, determine whether $w_1 = 2v_1 + v_2 - 3v_3$, $w_2 = 2v_2 - v_3$ and $w_3 = v_2 - 2v_3$ are linear independent or not.

2. Suppose w_1, w_2, w_3 is a basis for a vector space V, show that any basis v_1, v_2, v_3 is related to w_1, w_2, w_3 by
$$v_j = \sum_{i=1}^{3} a_{ij} w_i$$
for each j, and some invertible 3×3 matrix $A = (a_{ij})$.

3. Suppose A, B are $n \times n$ matrices satisfying $A^2 = A$, $B^2 = B$ and $I_{n\times n} - (A + B)$ is invertible, show that $\mathrm{rk}(A) = \mathrm{rk}(B)$.

4. Given two vector subspaces W_1 and W_2 of a vector space V. Show that
   (a) $W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2\}$ is the smallest vector subspace of V containing $W_1 \cup W_2$, i.e. if W_3 is a subspace containing $W_1 \cup W_2$, then $W_1 + W_2 \subset W_3$,
   (b) show that
   $$\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2).$$

5. Let P_4 be the vector space of polynomials with degree less than or equal to 4, and let
$$V = \Big\{ f(x) \mid f(x) \in P_4,\ f(2) = 0,\ f(1) = f(-1) \Big\}.$$
   (a) Show that V is a subspace of P_4.
   (b) Writing $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$, show that $f(x) \in V$ if and only if
   $$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \end{bmatrix}^T \in N\left(\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix} \right).$$
   (hints: write down the equation $f(2) = 0$ and $f(1) = f(-1)$ in terms of a_0, a_1, a_2, a_3, a_4)
   (c) Find a basis for V and hence its dimension.