Lecture 13

Orthogonality

Inner product and orthogonality

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e.g.: $\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle^{\frac{1}{2}} = \left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)^{\frac{1}{2}} = \sqrt{1^2 + 2^2 + 3^2}$

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is the distance from $\vec{0}$ to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$.

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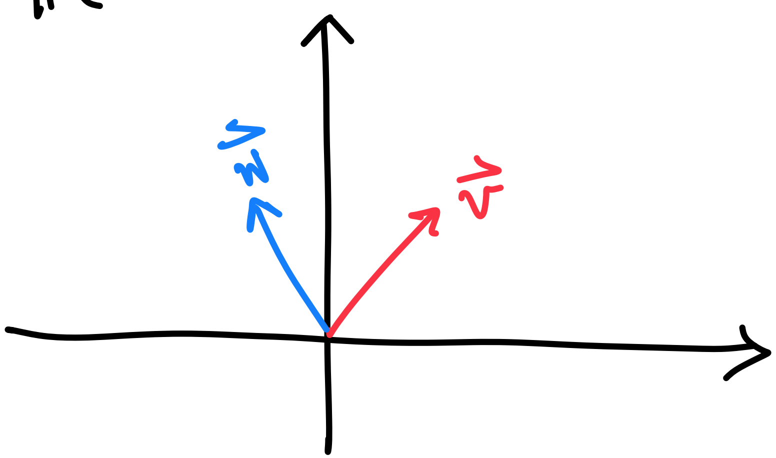

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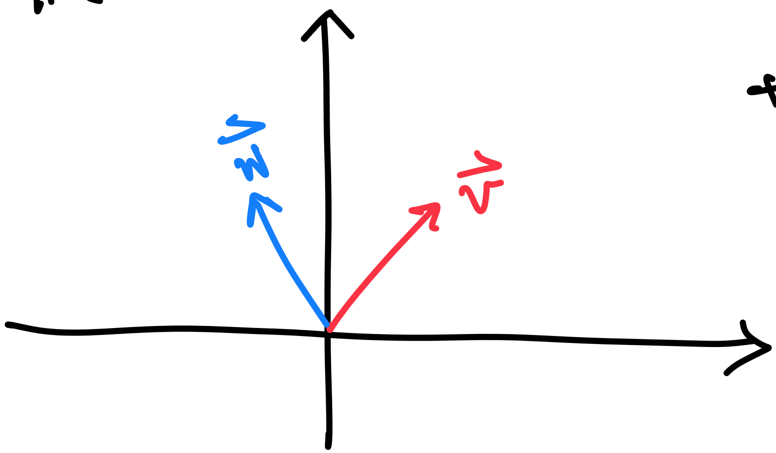

rotation + reflection : R.

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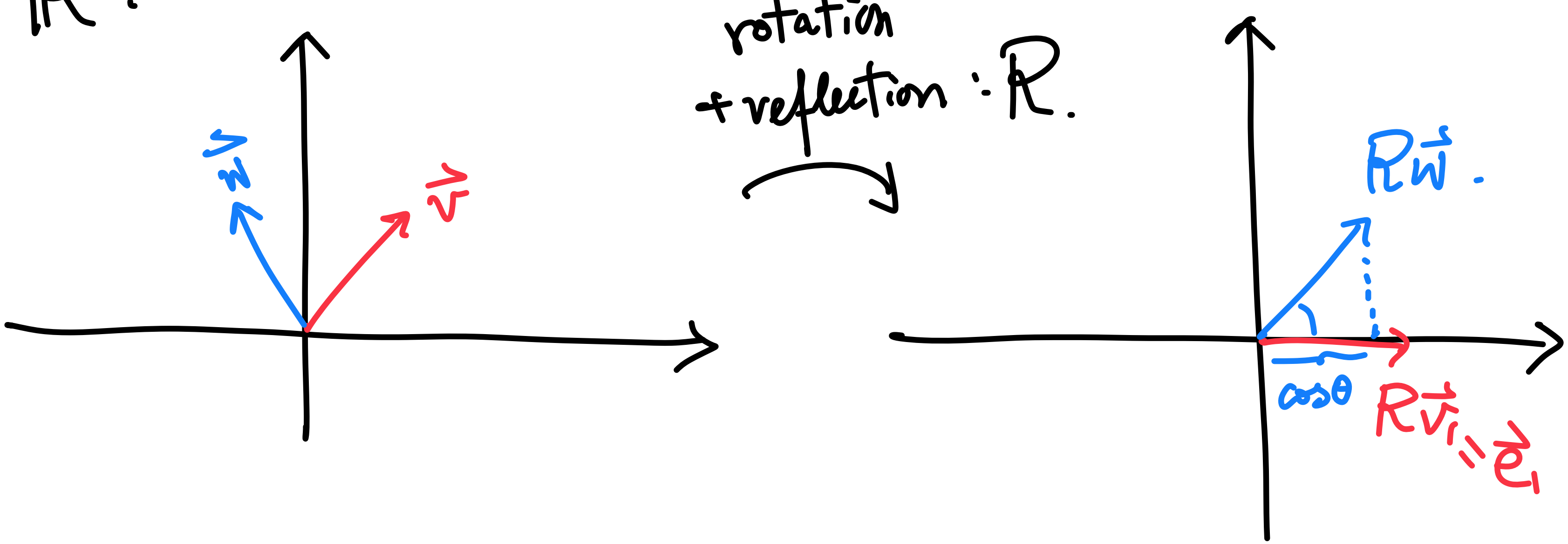

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- Any composition of rotations or reflection in $\mathbb{R}^2$ preserve $\langle \cdot, \cdot \rangle$.

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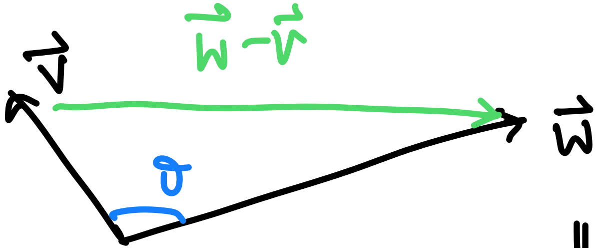

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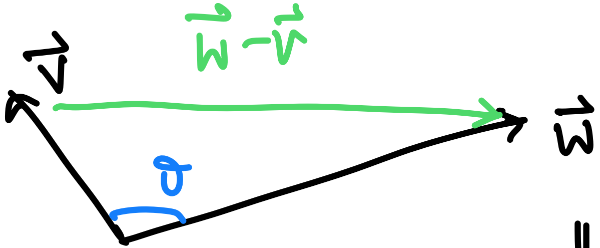

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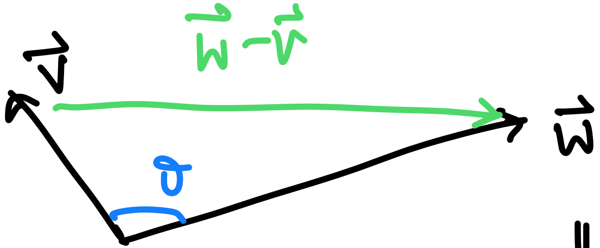

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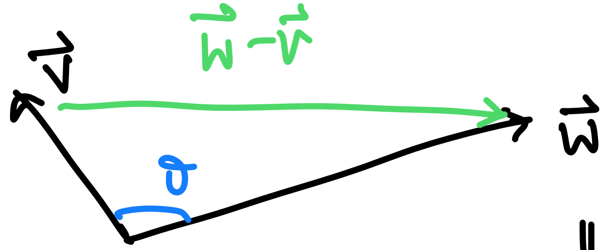

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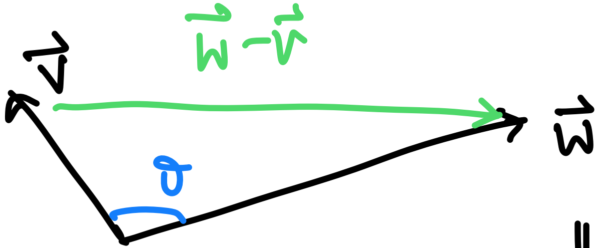

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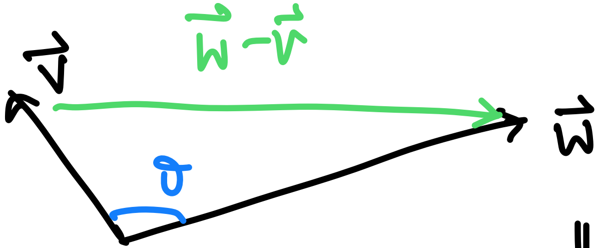

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e.g. $\vec{x} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ are orthogonal.

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This is for any $i = 1, \dots, k$.

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We look at $m \times n$ matrix A with associated $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

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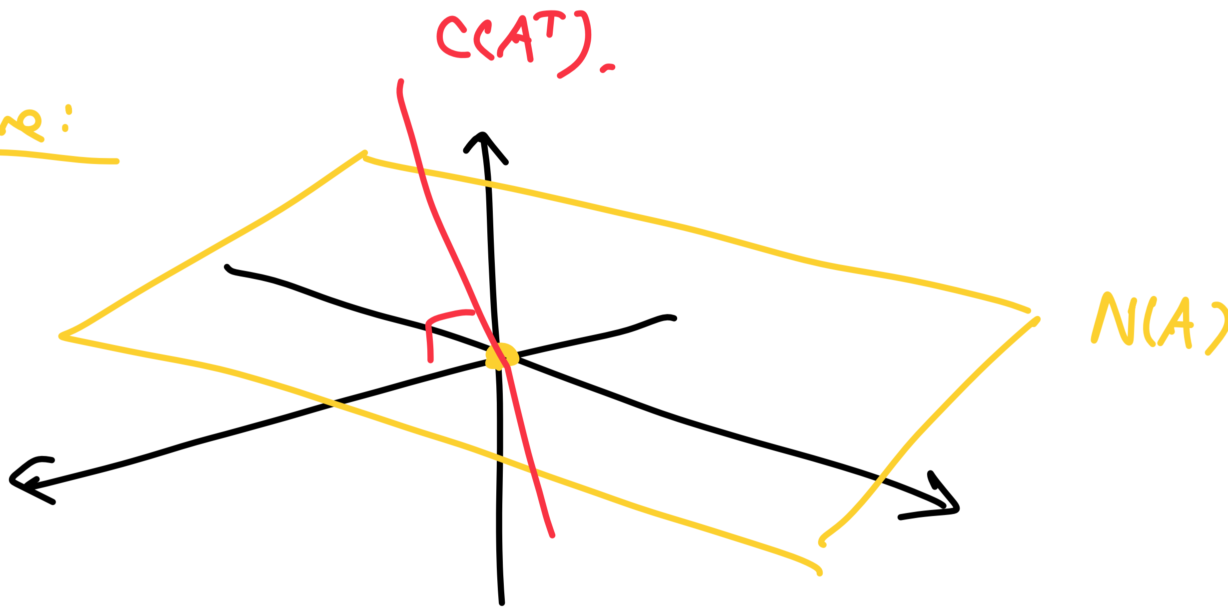
Picture:



orthogonality complement

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Def: $V \subseteq \mathbb{R}^n$ a subspace, the we define the orthogonal complement of V, denoted by $V^{\perp} = \{ \vec{w} \in \mathbb{R}^n \mid \langle \vec{v}, \vec{w} \rangle = 0 \text{ for all } \vec{v} \in V \}$.

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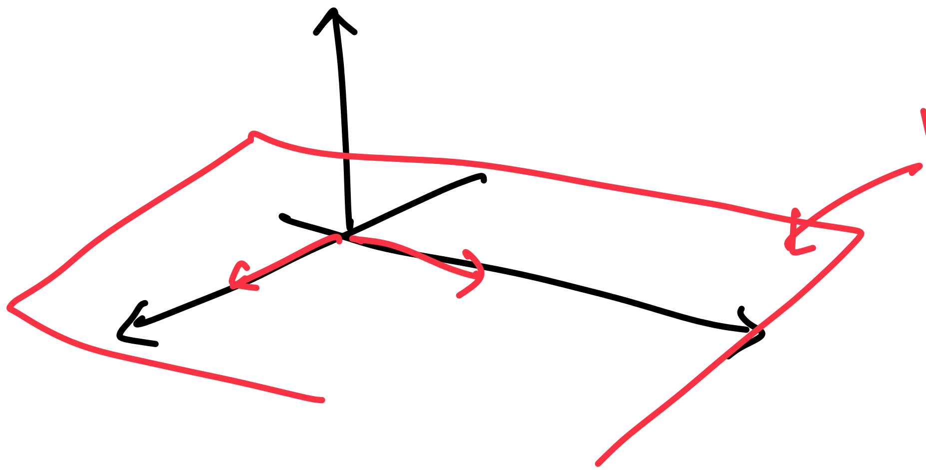

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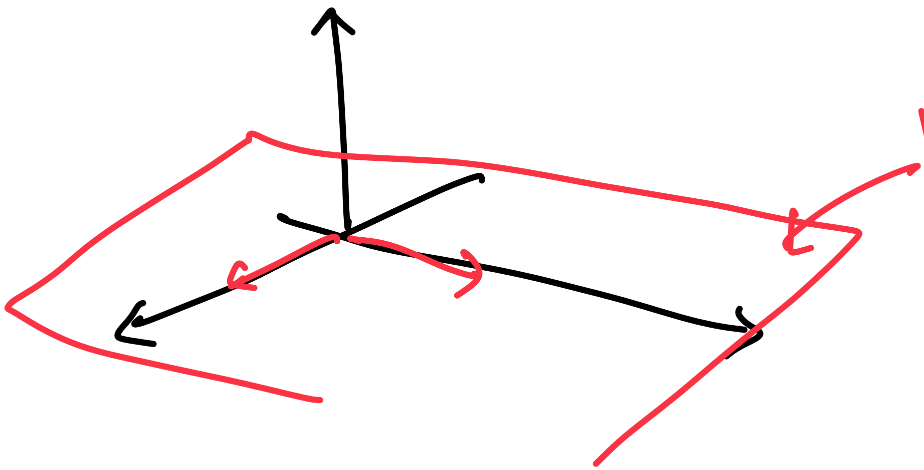

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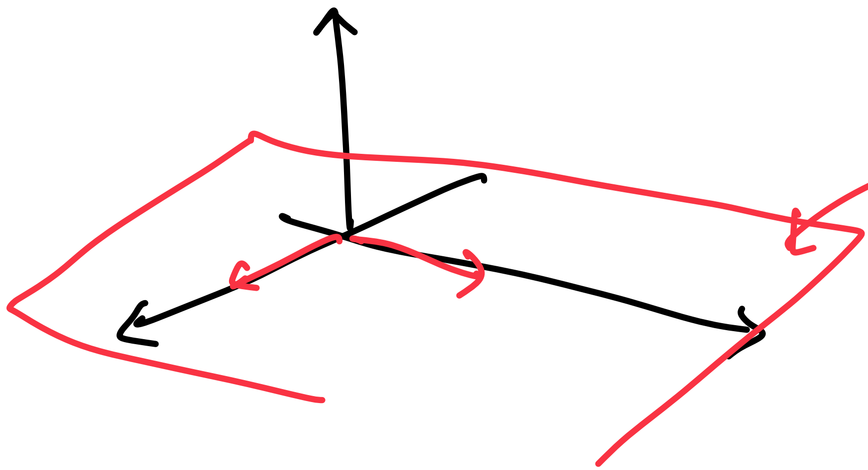

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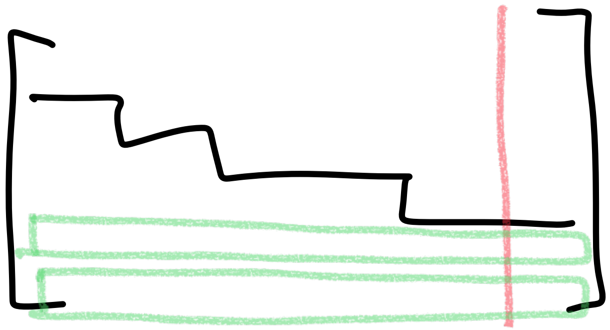

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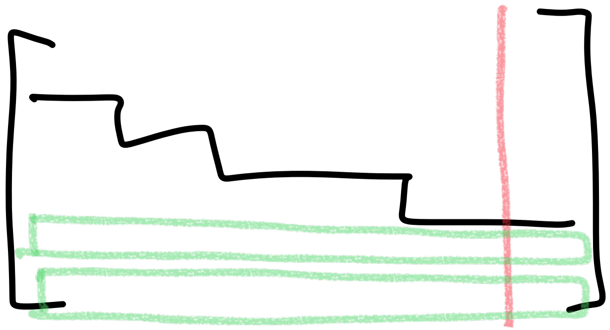

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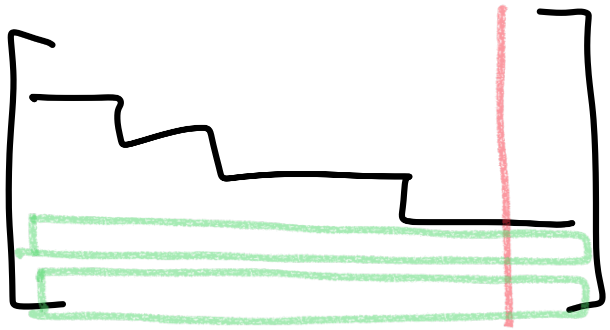

This gives equation on $\vec{b}$ in $C(A)$.

- Say $b_1 + 2b_2 + 5b_3 = 0$. is one of the equation.
  $[1, 2, 5]^T$ is a vector in $N(A^T)$.

orthogonality complement

orthogonality complement

Picture: $\mathbb{R}^m$: $\mathbb{R}^n$:

orthogonality complement

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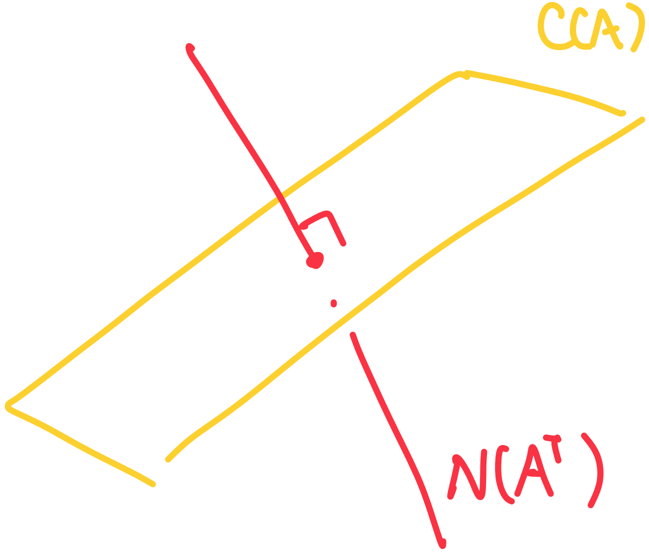
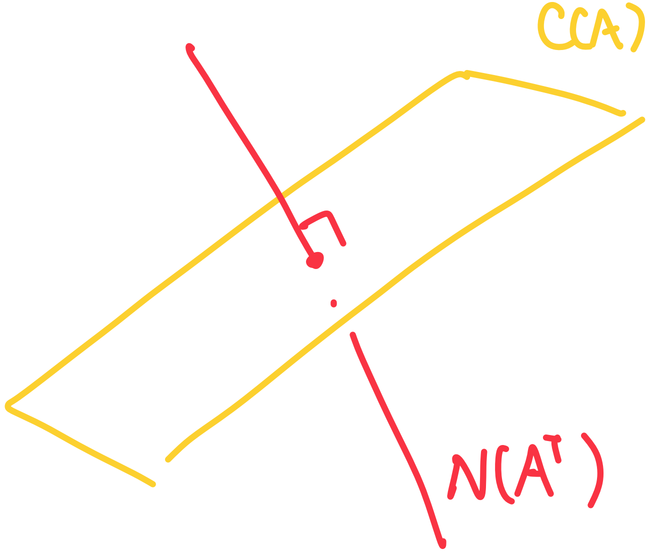


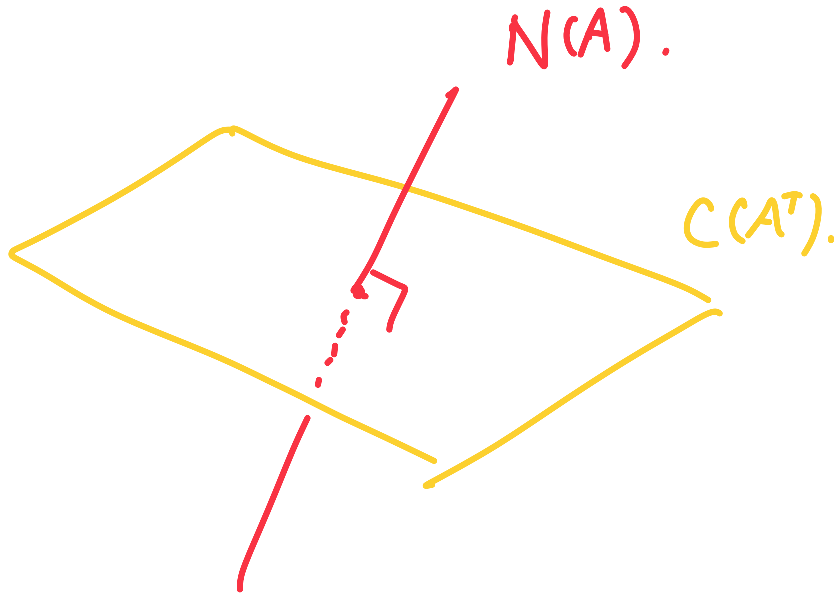
orthogonality complement

Picture:

$\mathbb{R}^m$:

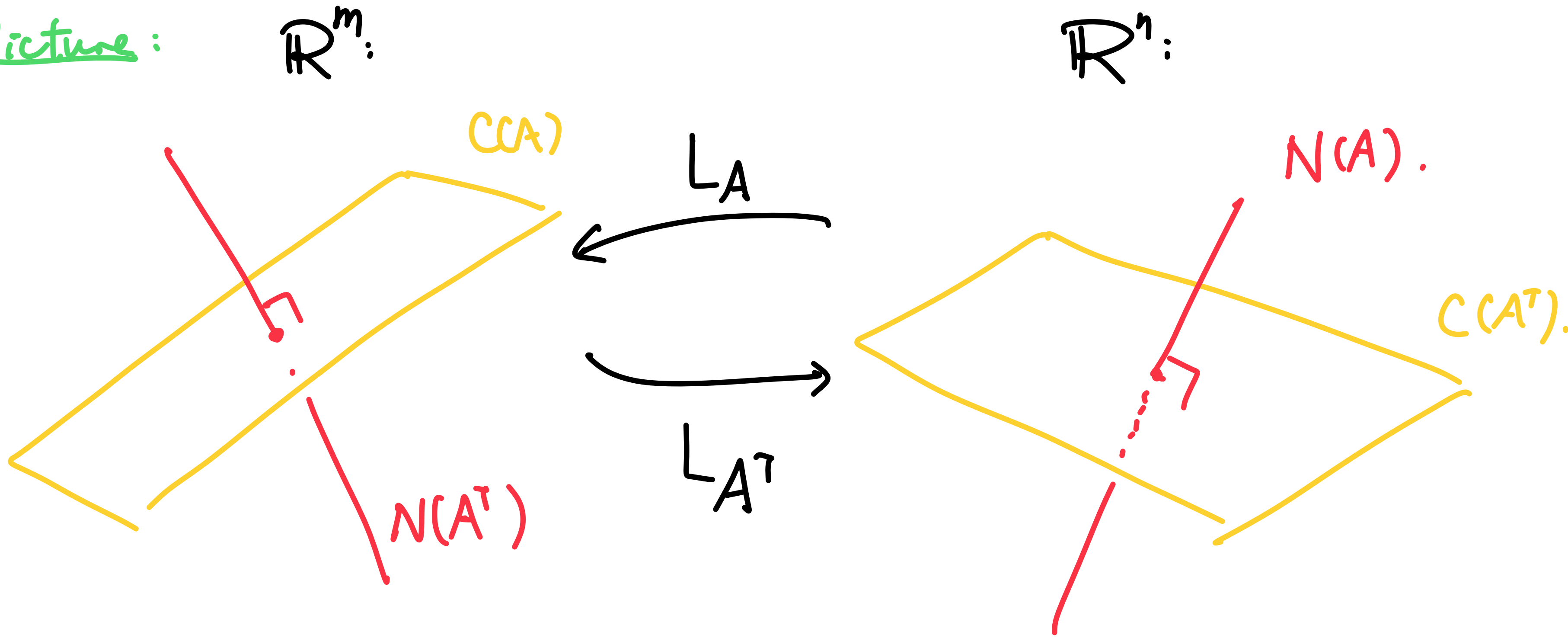


$\mathbb{R}^n$:



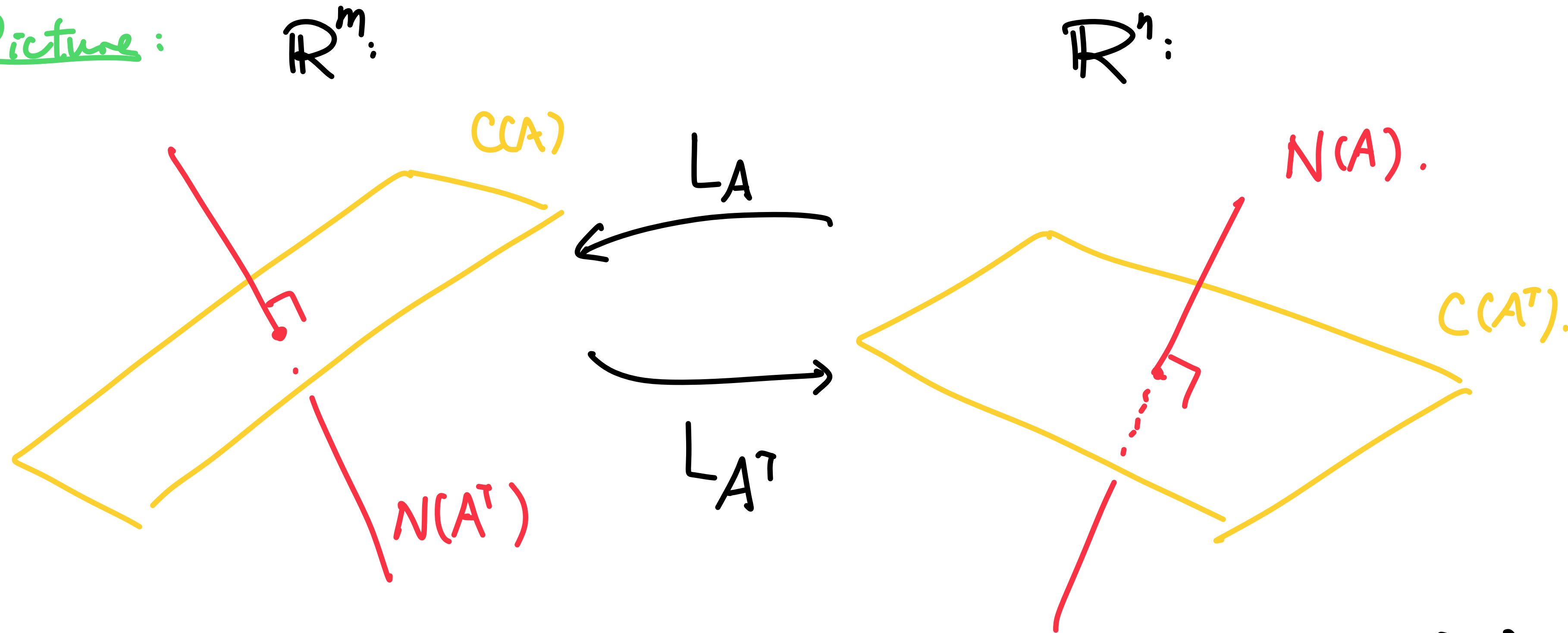
orthogonality complement

Picture:



orthogonality complement

Picture:



- L_A send $N(A)$ to $\{\vec{0}\}$, L_{A^T} send $N(A^T)$ to $\{\vec{0}\}$.

orthogonality complement

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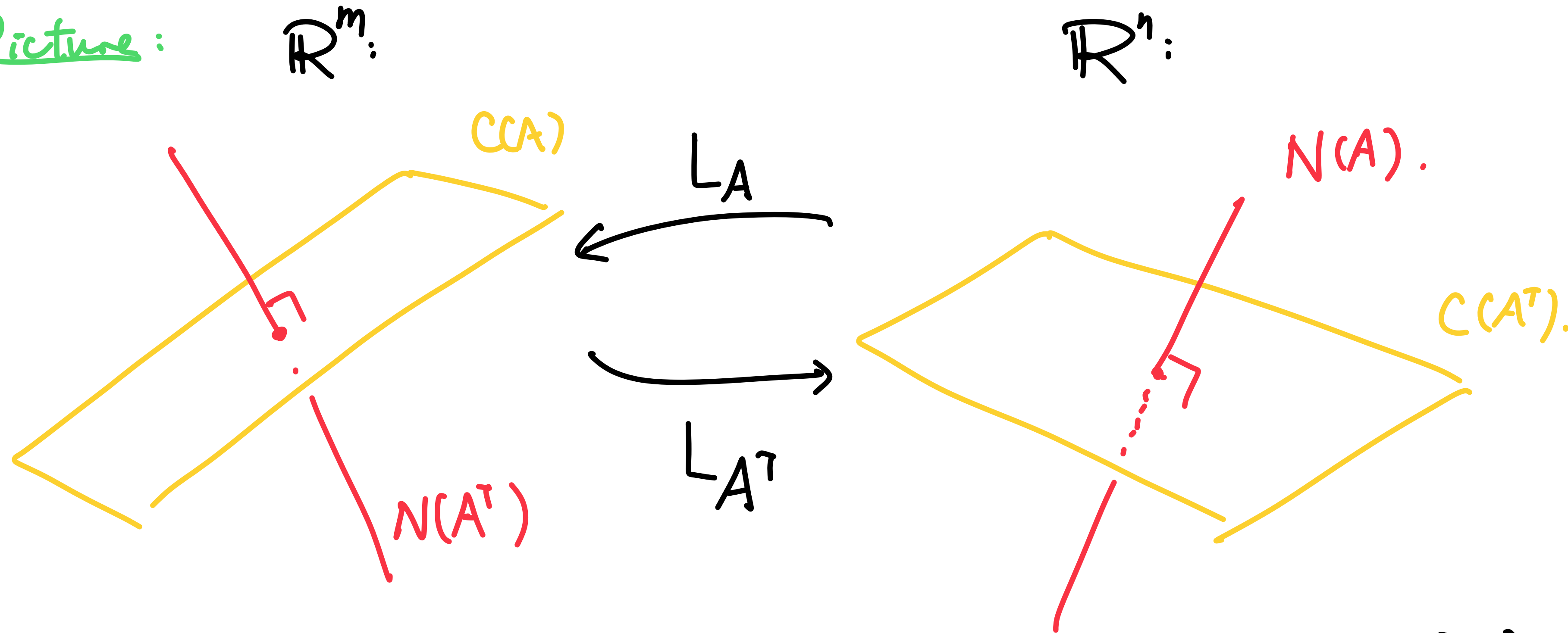

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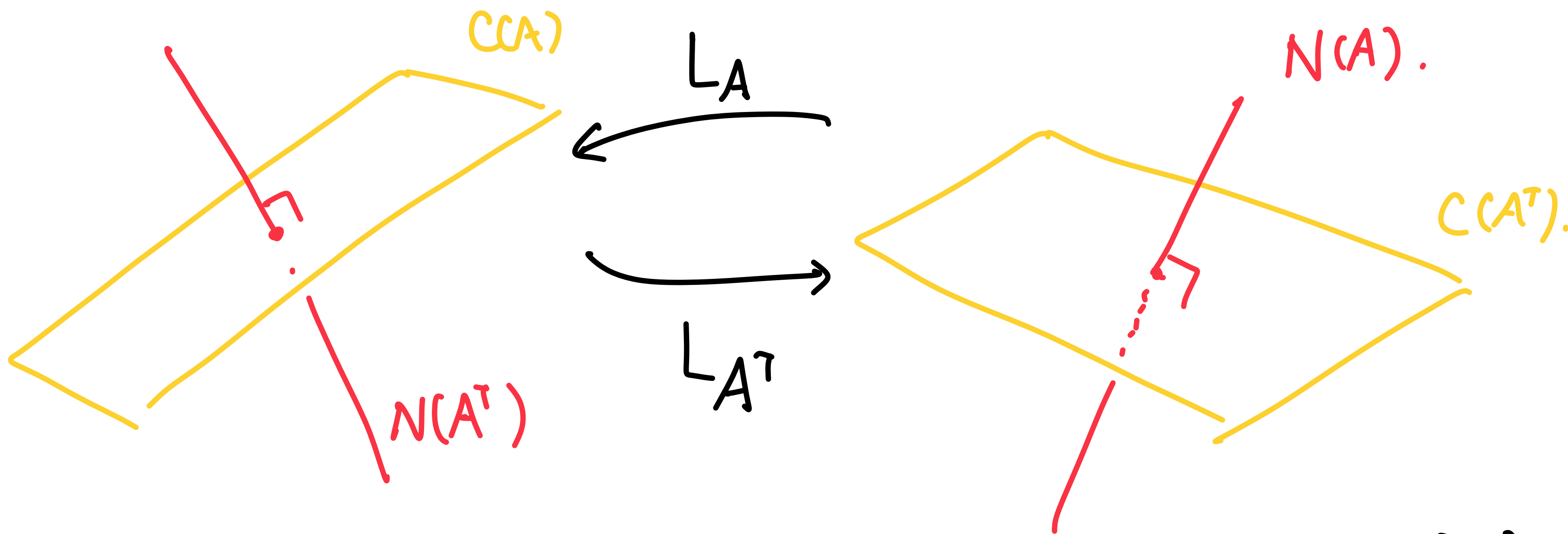

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  $\because$ $Ker(L_A) = \{\vec{0}\}$, $Im(L_A) = C(A)$

orthogonality complement

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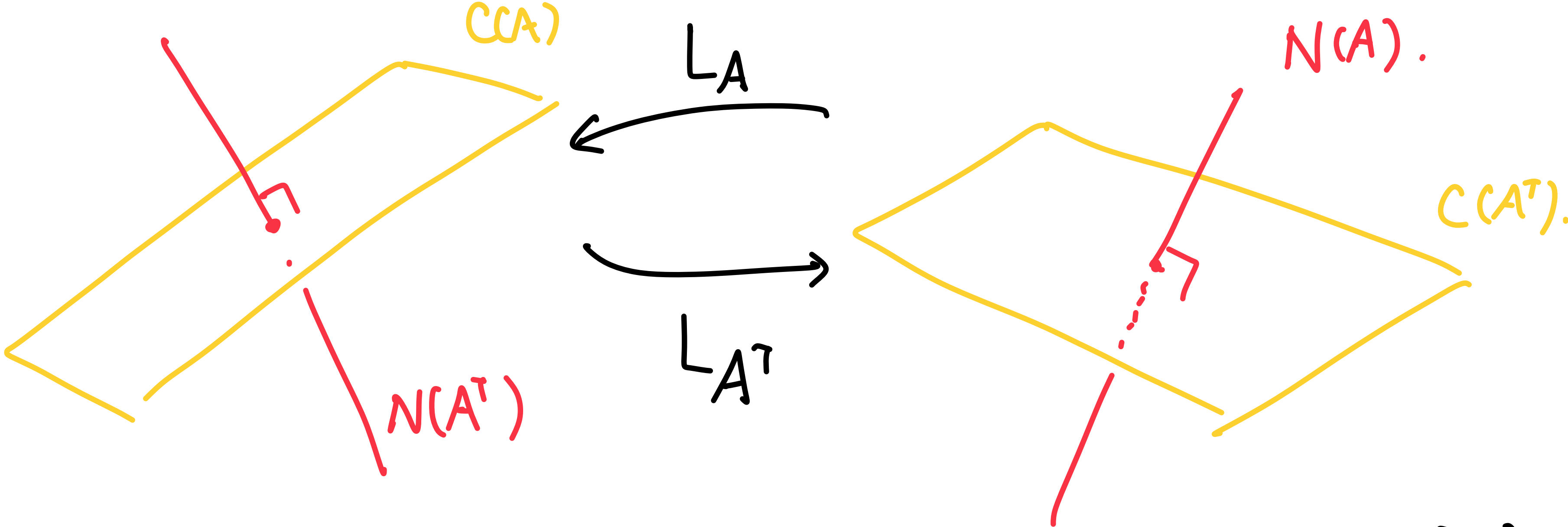

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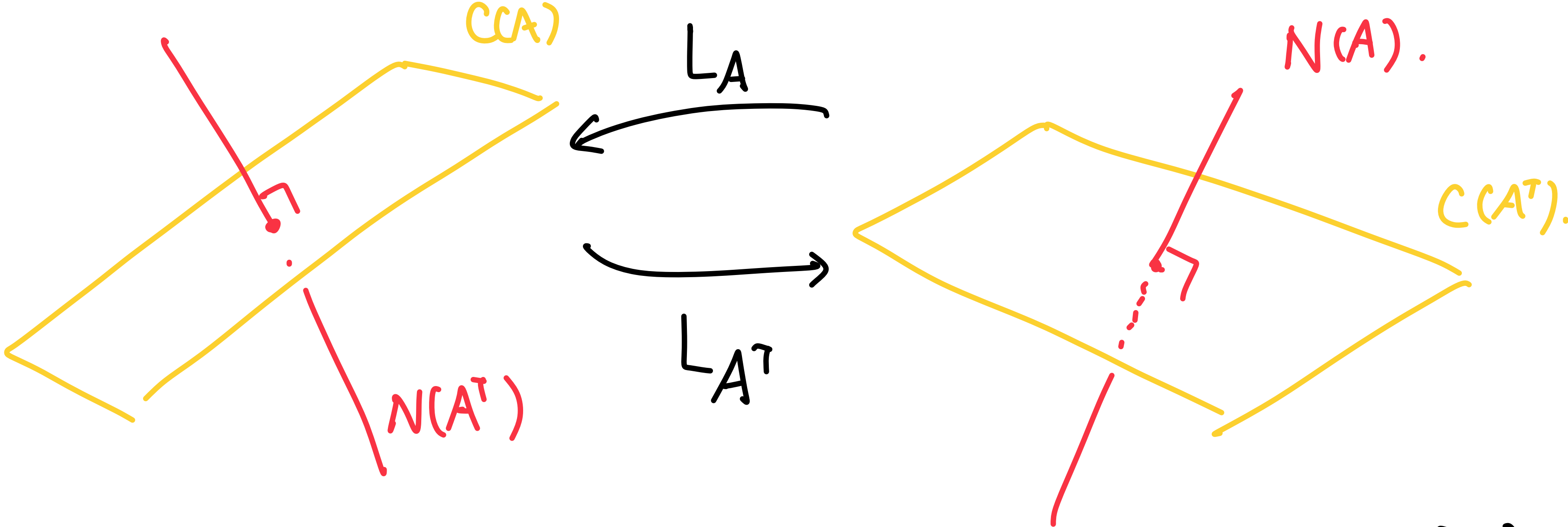

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Warning: $L_{A^T} \circ L_A = \mathrm{id}$ they are not inverse.