Lecture 11

Linear transformation

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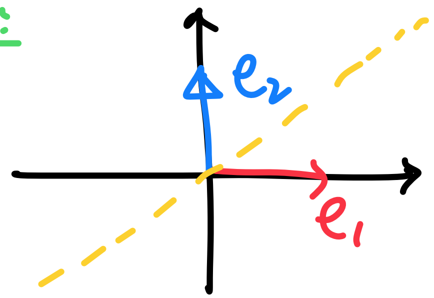

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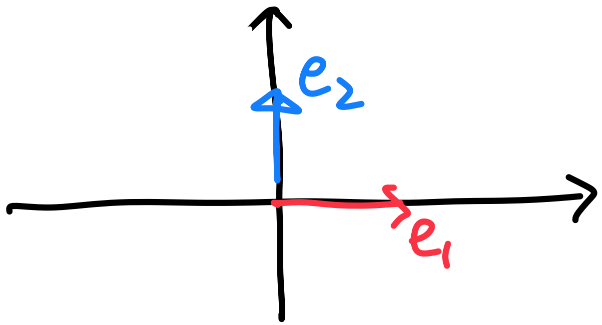

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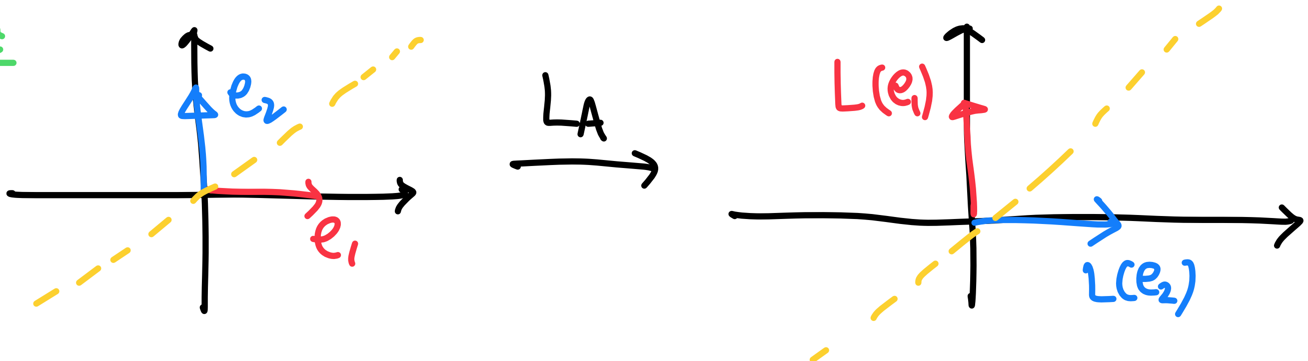

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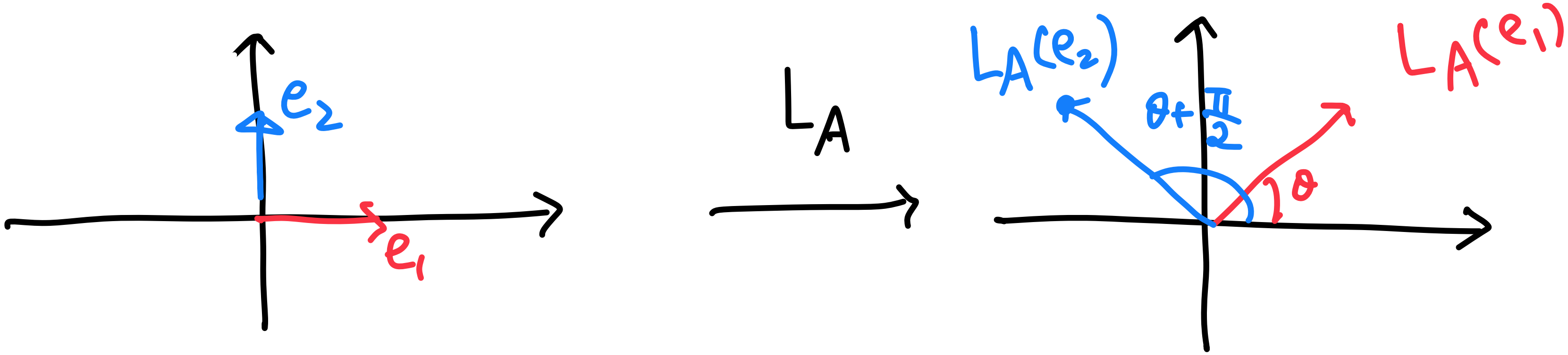

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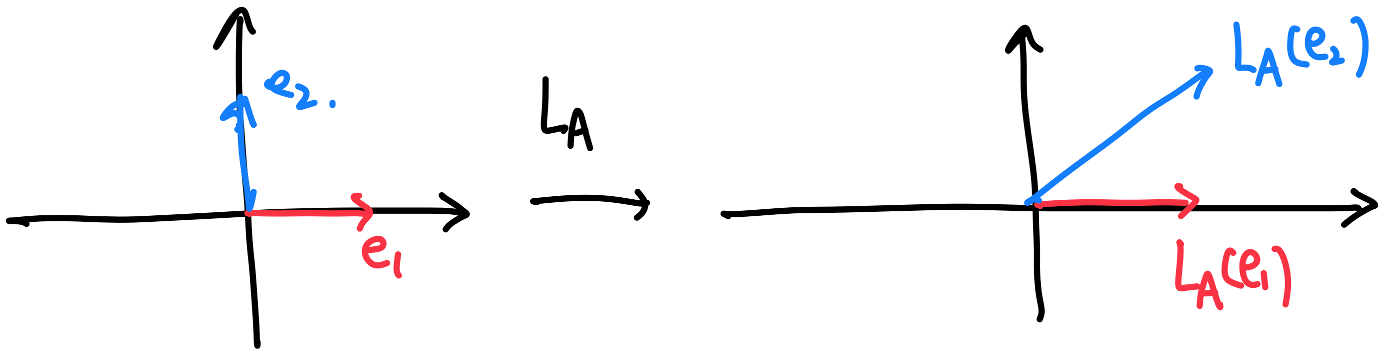

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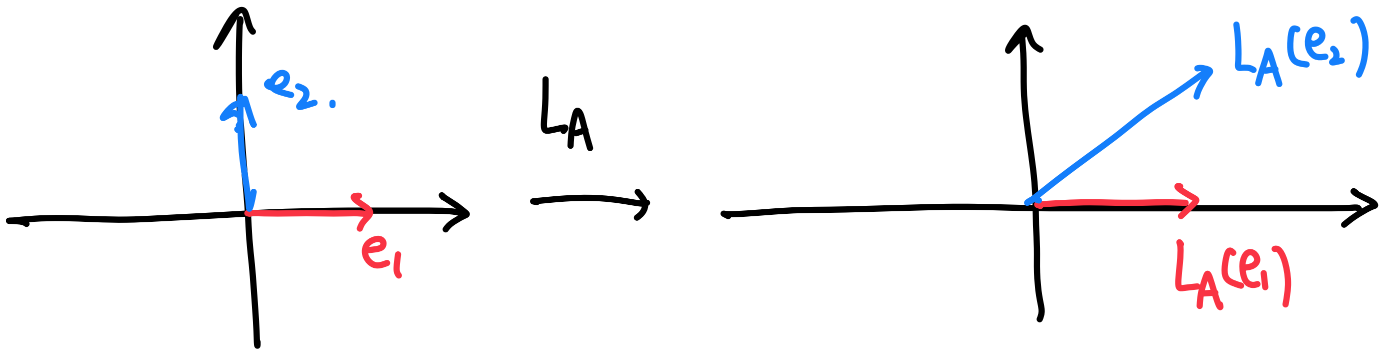

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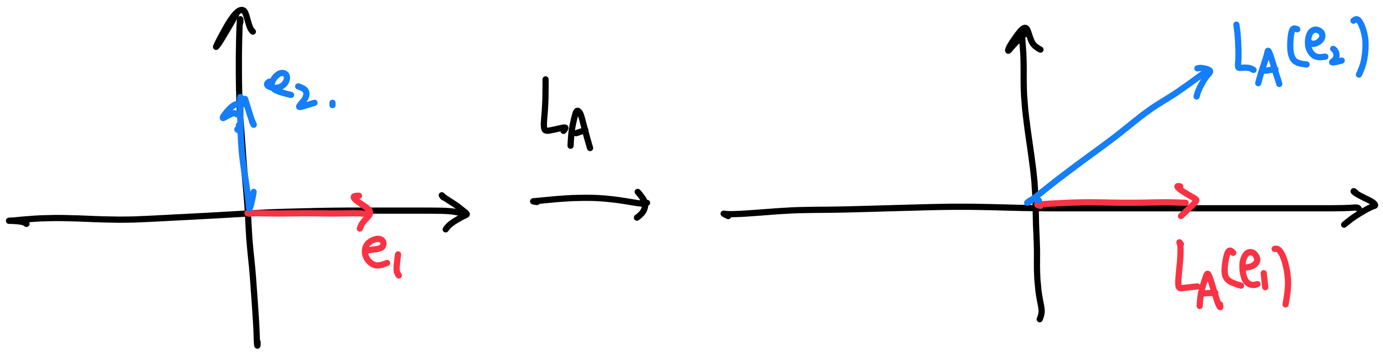

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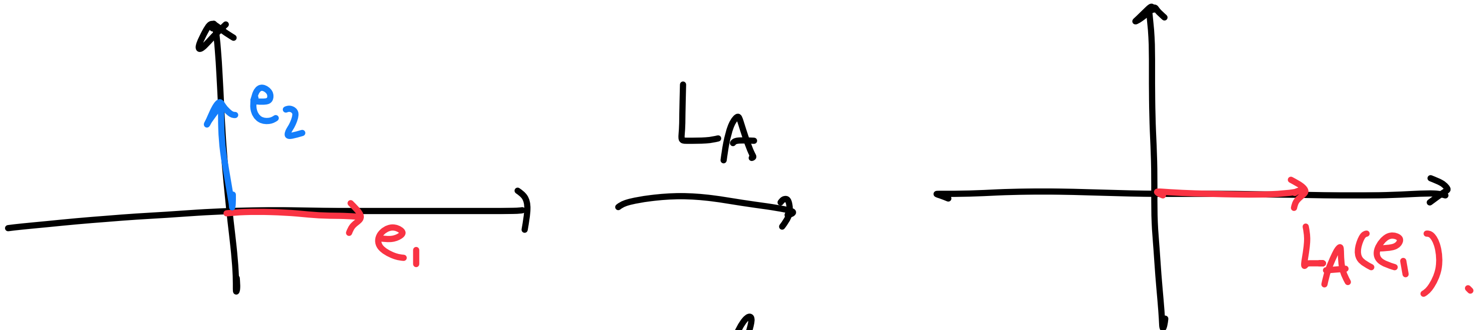

is the projection along y-axis.

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3. $L_A(\vec{c}) \in C(A)$ for any $\vec{c} \in \mathbb{R}^n$.

   $C(A)$ is also called the image of L_A

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$$\begin{aligned} L_A(a\vec{x} + b\vec{y}) &= A(a\vec{x} + b\vec{y}) \\ &= aA\vec{x} + bA\vec{y} \\ &= aL_A(\vec{x}) + bL_A(\vec{y}). \end{aligned}$$

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$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = \frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3)$$

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- It is a linear tranformation:

$$\begin{cases} D(cf(x)) = cD(f(x)) \\ D(f(x) + g(x)) = D(f(x)) + D(g(x)). \end{cases}$$

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  $$= x\,(2a_2 + 6a_3 x) = 2a_2 x + 6a_3 x^2.$$

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$$I(f(x)+g(x)) = \int_0^x (f(t)+g(t))\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$$

$$= I(f(x)) + I(g(x)).$$

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• $g \circ f$ is defined as $g \circ f(\vec{v}) = g(f(\vec{v}))$.

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then from the above proposition, we can deduce linearity of D from that of $\frac{d}{dx}$ and g.

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Set $A = [L(\vec{e}_1), L(\vec{e}_2), L(\vec{e}_3)]$ be the matrix.

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- L is uniquely determined by its value on a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

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this we call it the matrix representing L.

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- Fixing basis $\vec{v}_i$'s for V, and $\vec{w}_i$'s for W. The matrix $[L]$ encode all information about L.

- If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

$$L(\vec{v}) = c_1 L(\vec{v}_1) + c_2 L(\vec{v}_2) + c_3 L(\vec{v}_3)$$

$$= [L(\vec{v}_1), L(\vec{v}_2), L(\vec{v}_3)] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= [\vec{w}_1, \vec{w}_2] [L] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

this usual matrix multiplication tell us how to find $L(\vec{v})$.

Matrix representation of Linear transformation

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- Let's consider $D = \frac{d}{dx} : P_3 \longrightarrow P_3$ be the linear transformation.
- We fix a basis $\overset{\substack{\vec{v}_1 \\ \vec{w}_1}}{1}$ $\overset{\substack{\vec{v}_2 \\ \vec{w}_2}}{x}$, $\overset{\substack{\vec{v}_3 \\ \vec{w}_3}}{x^2}$, $\overset{\substack{\vec{v}_4 \\ \vec{w}_4}}{x^3}$ for P_3.

Matrix representation of Linear transformation

e.g.:
- Let's consider $D = \frac{d}{dx} : P_3 \longrightarrow P_3$ be the linear transformation.

$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 \end{array}$$

- We fix a basis $1, x, x^2, x^3$ for P_3.

(both $\vec{v}_i$'s and $\vec{w}_i$'s are chosen to be this)

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e.g.: • Let's consider $D=\frac{d}{dx}: P_3 \longrightarrow P_3$ be the linear transformation.

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(both $\vec{v}_i$'s and $\vec{w}_i$'s are chosen to be this)

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• $D(\vec{v}_1) = 0 = 0 \cdot \vec{w}_1 + 0 \cdot \vec{w}_2 + 0 \cdot \vec{w}_3 + 0 \cdot \vec{w}_4 = [\vec{w}_1 \ldots \vec{w}_4] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

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$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4$

$\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3 \quad \vec{w}_4$

• We fix a basis $1, x, x^2, x^3$ for P_3.

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$D(\vec{v}_2) = 1 = 1 \cdot \vec{w}_1 + 0 \cdot \vec{w}_2 + 0 \cdot \vec{w}_3 + 0 \cdot \vec{w}_4 = [\vec{w}_1, \dots, \vec{w}_4] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

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$$D(\vec{V}_3) = 2x = 0\cdot\vec{W}_1 + 2\cdot\vec{W}_2 + 0\cdot\vec{W}_3 + 0\cdot\vec{W}_4 = [\vec{W}_1, \dots, \vec{W}_4]\begin{bmatrix}0\\2\\0\\0\end{bmatrix}$$

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$$D(\vec{v}_1) = 0 = 0 \cdot \vec{w}_1 + 0 \cdot \vec{w}_2 + 0 \cdot \vec{w}_3 + 0 \cdot \vec{w}_4 = [\vec{w}_1 \dots \vec{w}_4] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(\vec{v}_2) = 1 = 1 \cdot \vec{w}_1 + 0 \cdot \vec{w}_2 + 0 \cdot \vec{w}_3 + 0 \cdot \vec{w}_4 = [\vec{w}_1, \dots, \vec{w}_4] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(\vec{v}_3) = 2x = 0 \cdot \vec{w}_1 + 2 \cdot \vec{w}_2 + 0 \cdot \vec{w}_3 + 0 \cdot \vec{w}_4 = [\vec{w}_1, \dots, \vec{w}_4] \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$D(\vec{v}_4) = 3x^2 = 0 \cdot \vec{w}_1 + 0 \cdot \vec{w}_2 + 3 \vec{w}_3 + 0 \cdot \vec{w}_4 = [\vec{w}_1, \dots, \vec{w}_4] \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Matrix representation of Linear transformation

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- We collect the column vector representing

$$L(\vec{v}_i) = [\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4] \begin{bmatrix} | \\ a_i \\ | \end{bmatrix}$$

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- We put $\begin{bmatrix} | \\ a_i \\ | \end{bmatrix}$ as columns of $[L]$: $[L] = \begin{bmatrix} | & | & | & | \\ a_1, & a_2, & a_3, & a_4 \\ | & | & | & | \end{bmatrix}$

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- In the example: $[L] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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- Let say we want to find $D(1+x-3x^2+5x^3)$

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We can write $1+x-3x^2+5x^3 = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4] \begin{bmatrix} 1 \\ 1 \\ -3 \\ 5 \end{bmatrix}$

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$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 15 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{bmatrix}\begin{bmatrix}1\\1\\-3\\5\end{bmatrix} = \begin{bmatrix}1\\-6\\15\\0\end{bmatrix}$$

- Then $[\vec{W}_1, \vec{W}_2, \vec{W}_3, \vec{W}_4]\begin{bmatrix}1\\-6\\15\\0\end{bmatrix} = 1-6x+15x^2$ is the resulting vector.