Lecture 14

Projection and least square

Projection

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$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

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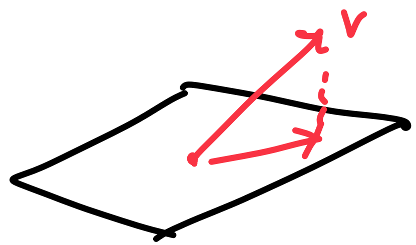
Q: $V \leq$ subspace, How we write down the orthogonal projection onto V?



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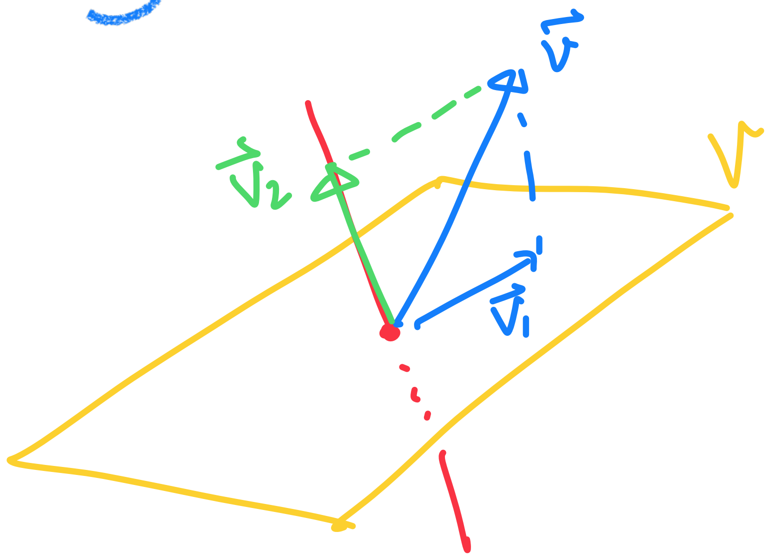

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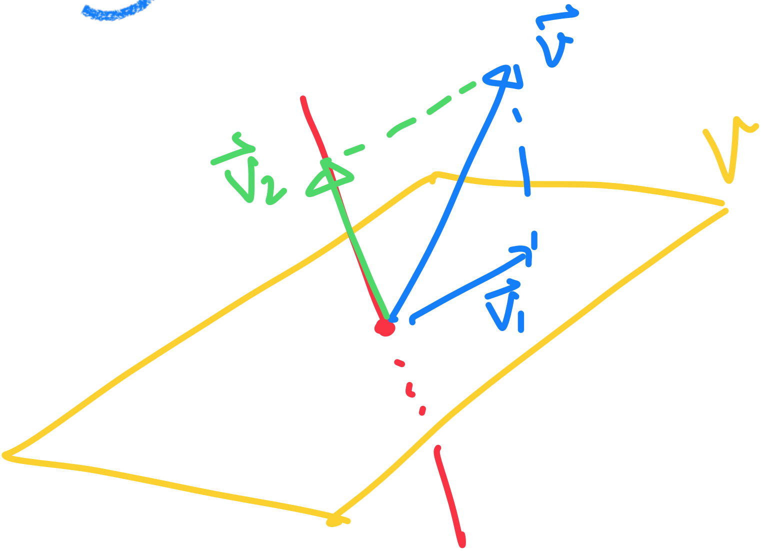

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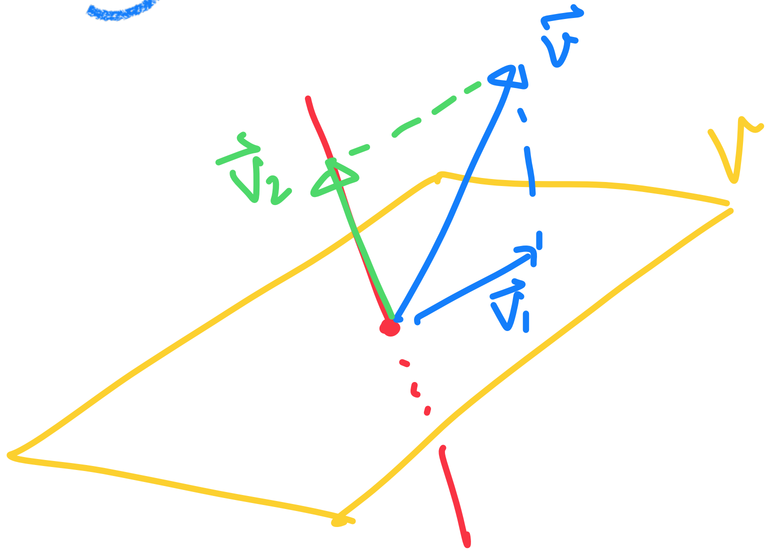

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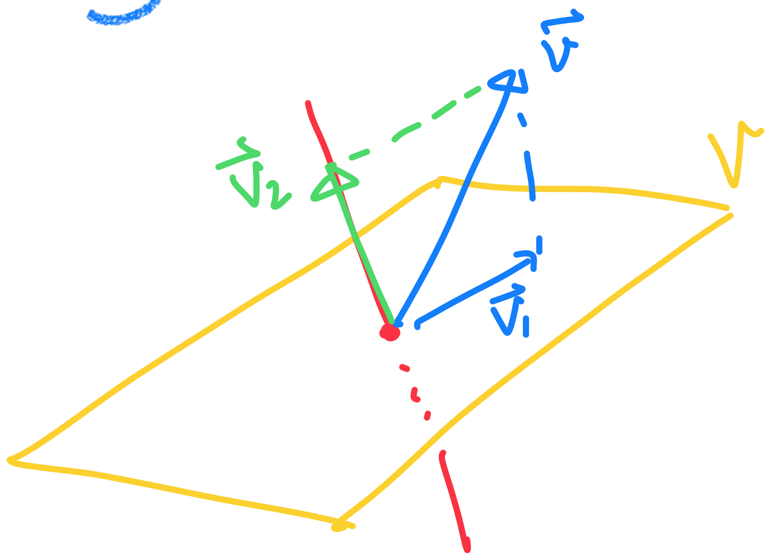

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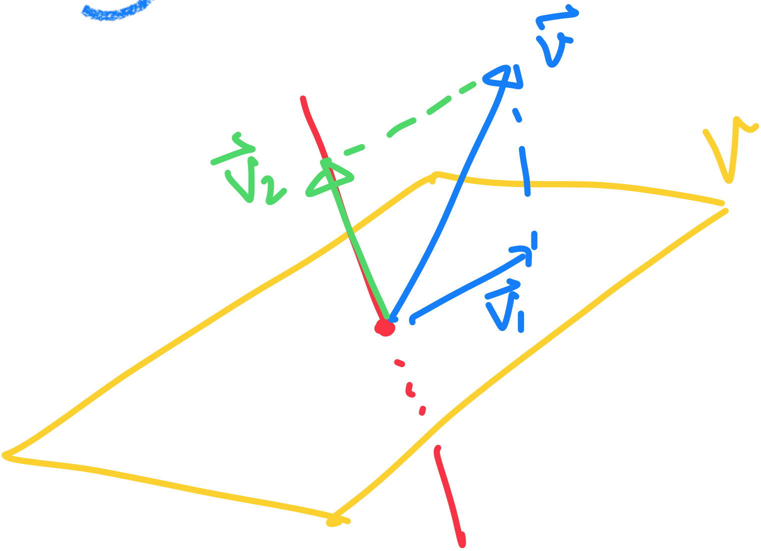

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Step 4: Transition matrix ${}_{e_i}[id]_{v_i}$

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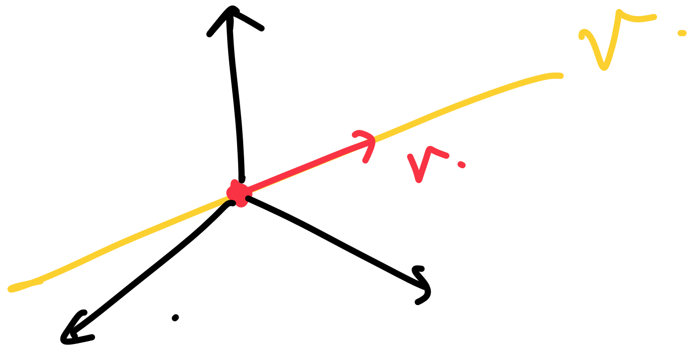

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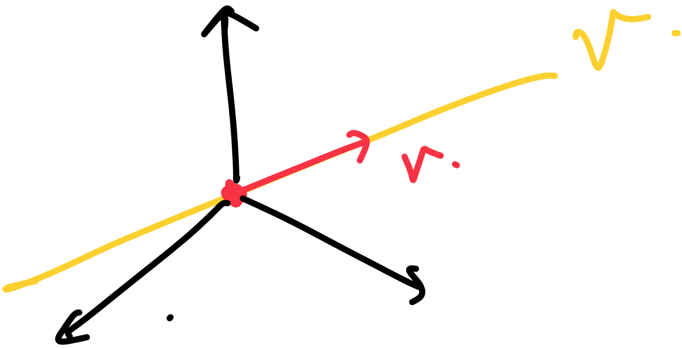

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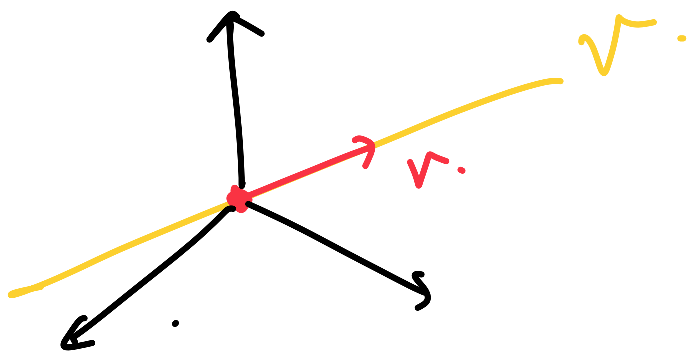

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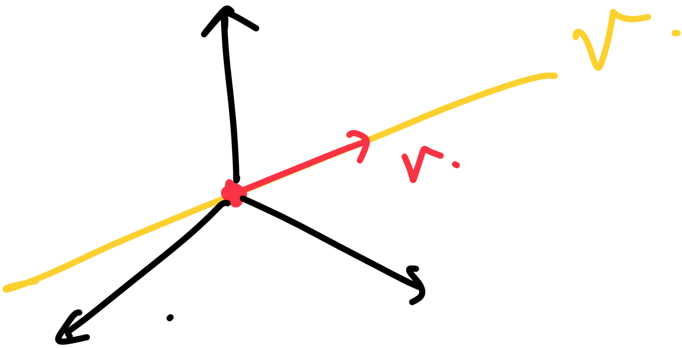

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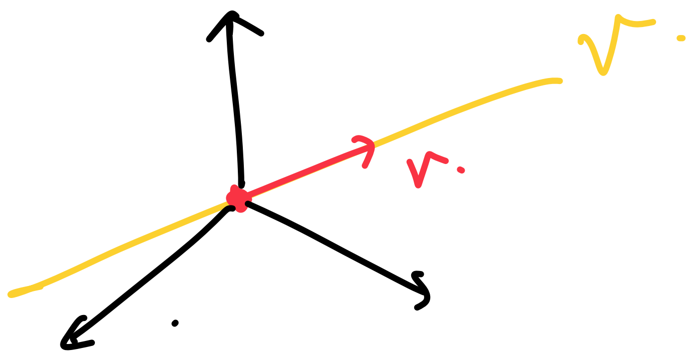

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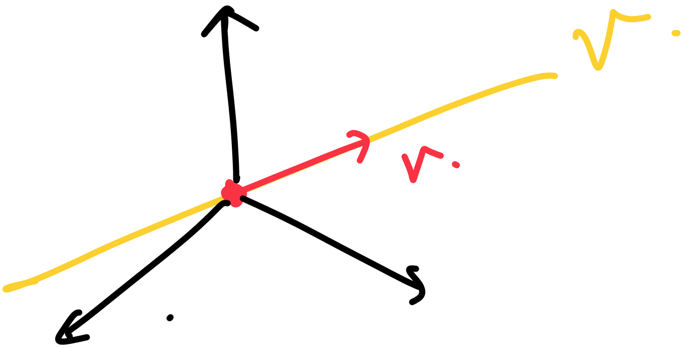

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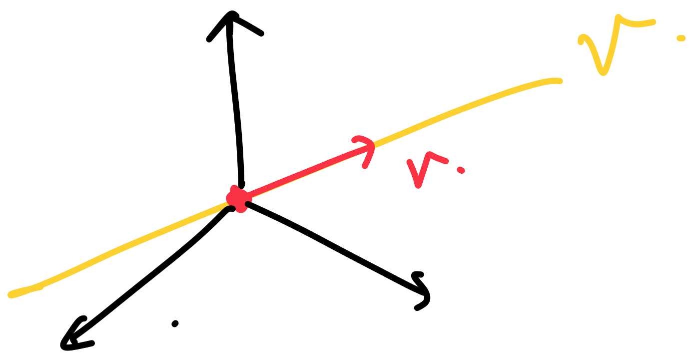

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- Guess a formula for c

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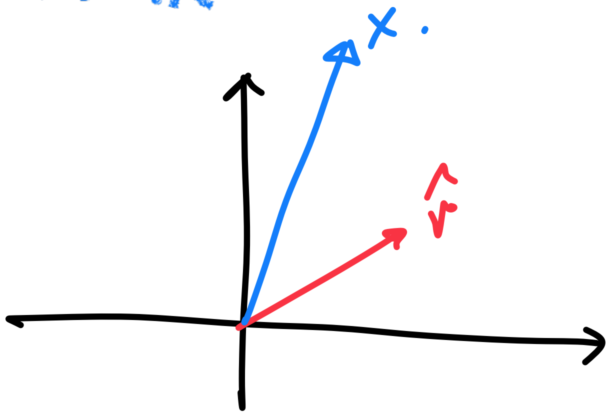

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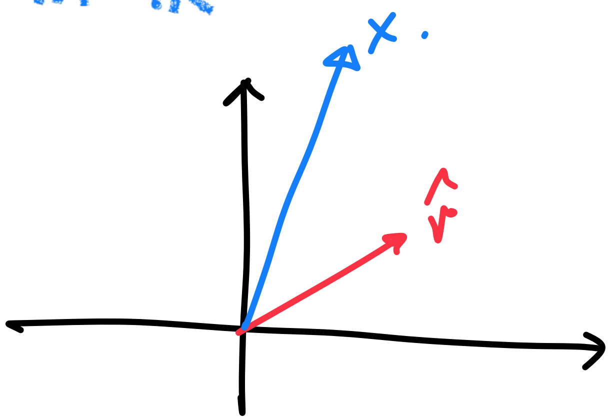

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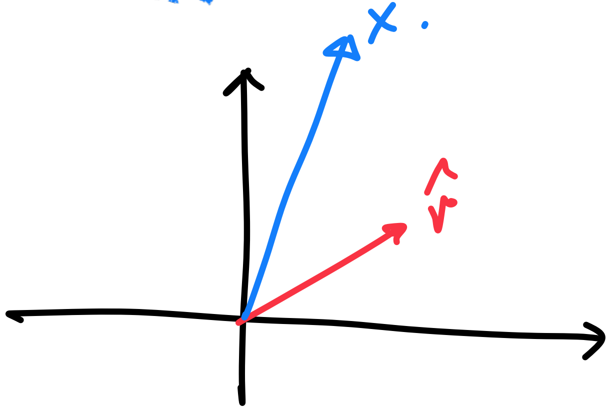

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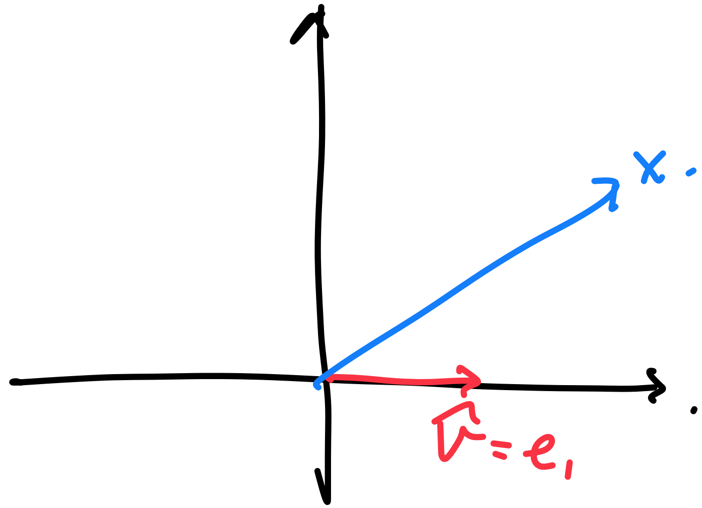

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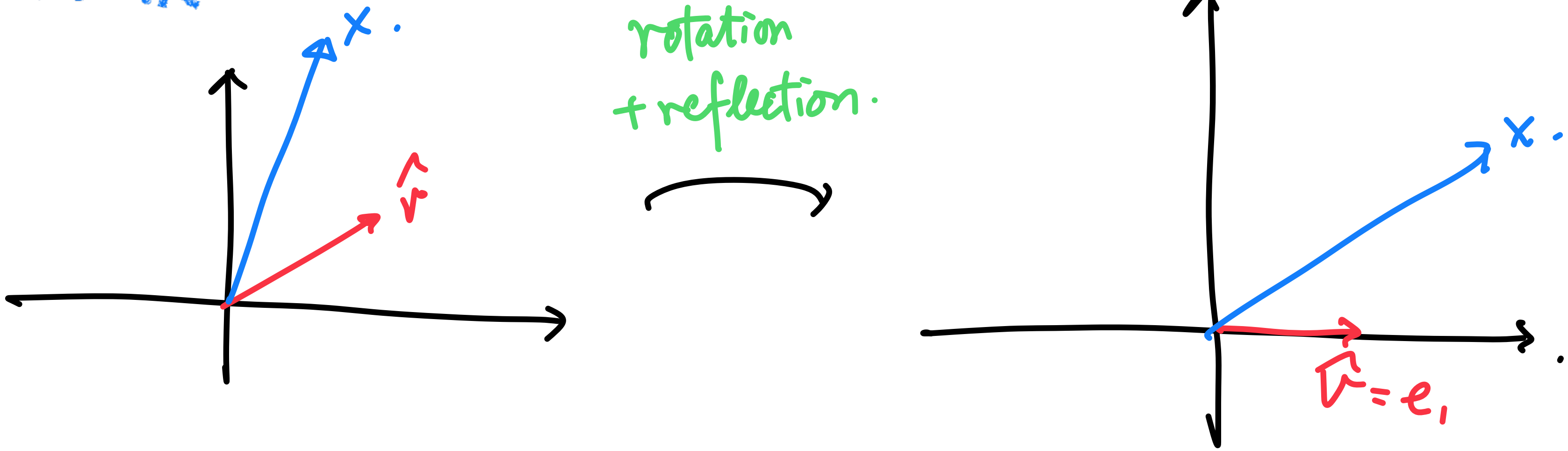

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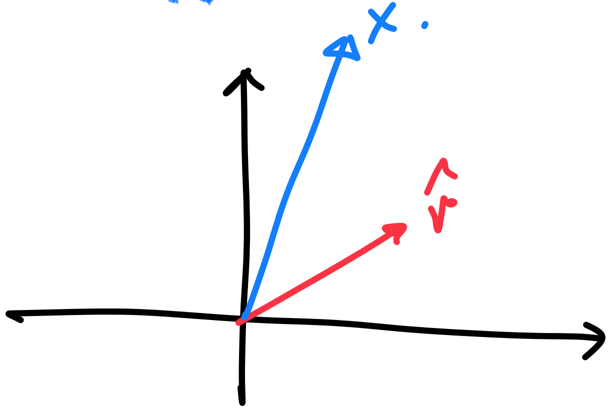

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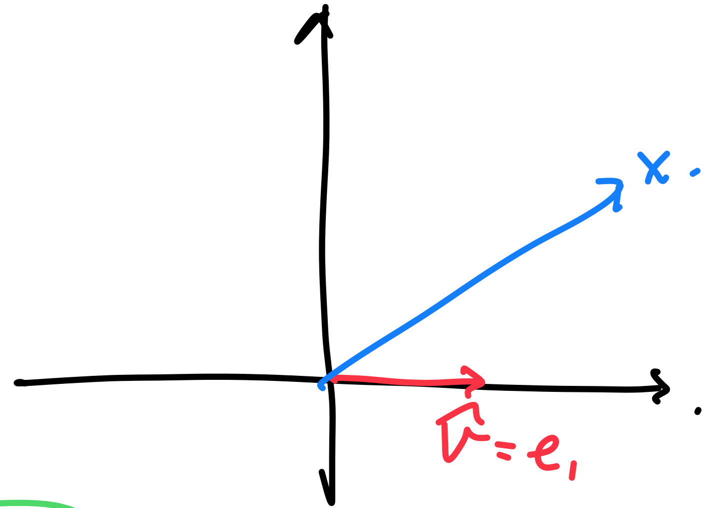

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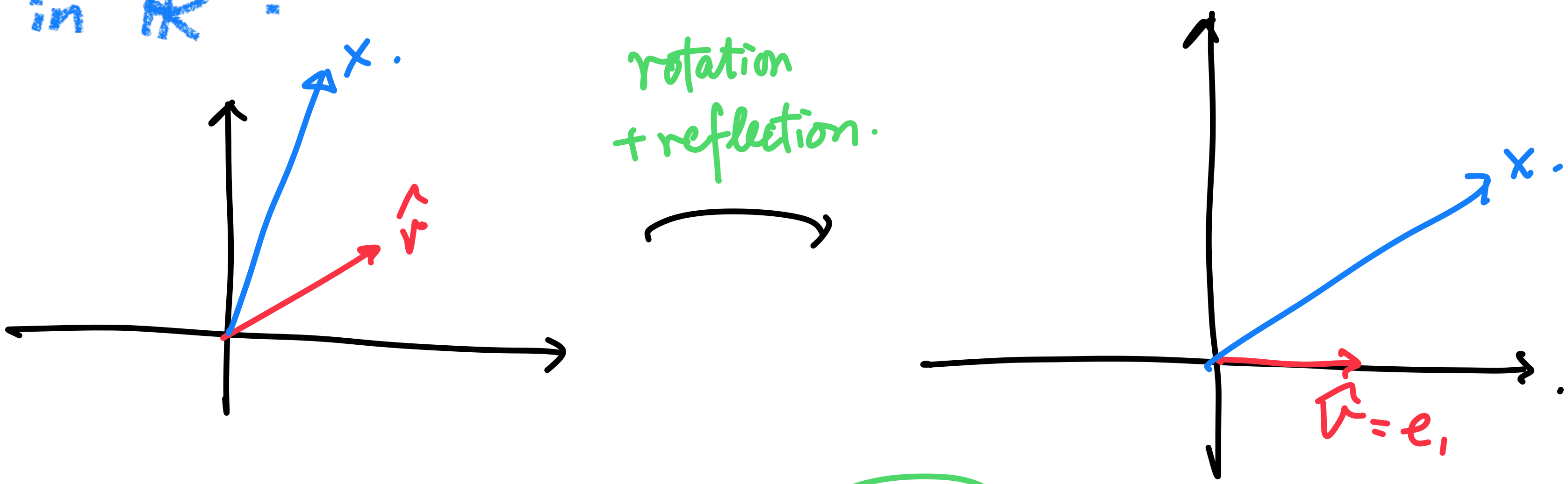

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Observation: This formula holds for any dimension.

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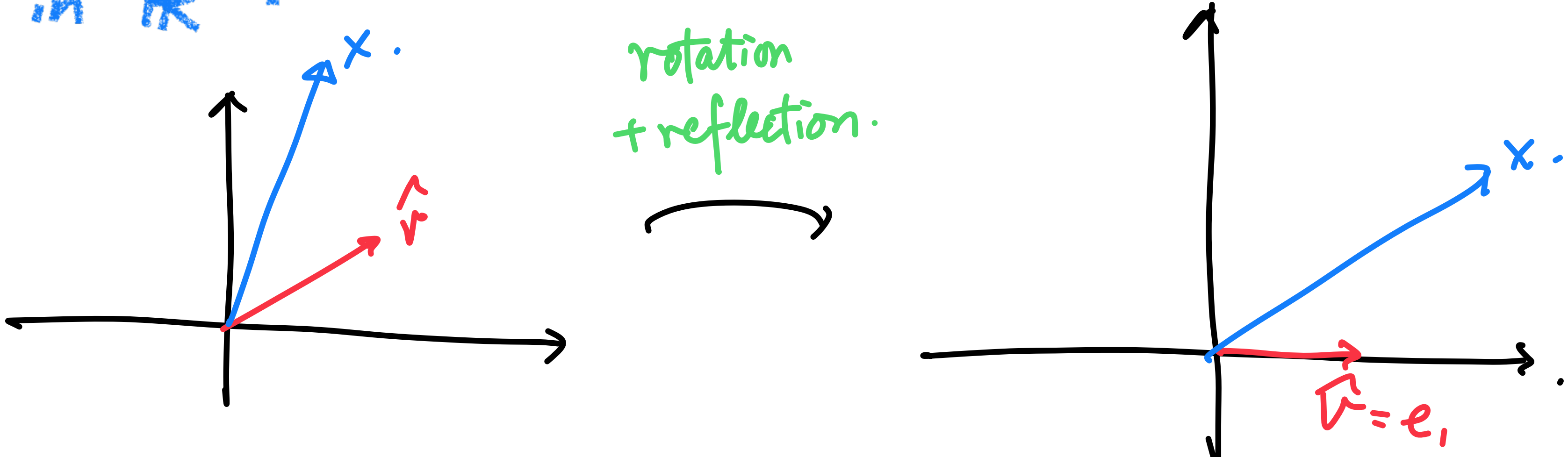

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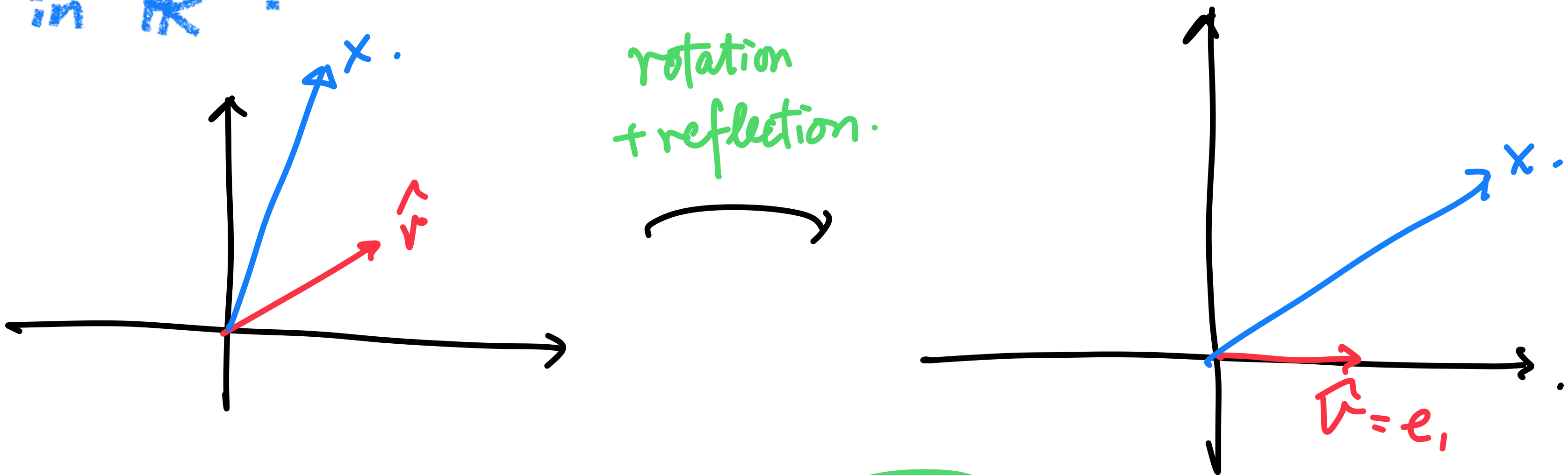

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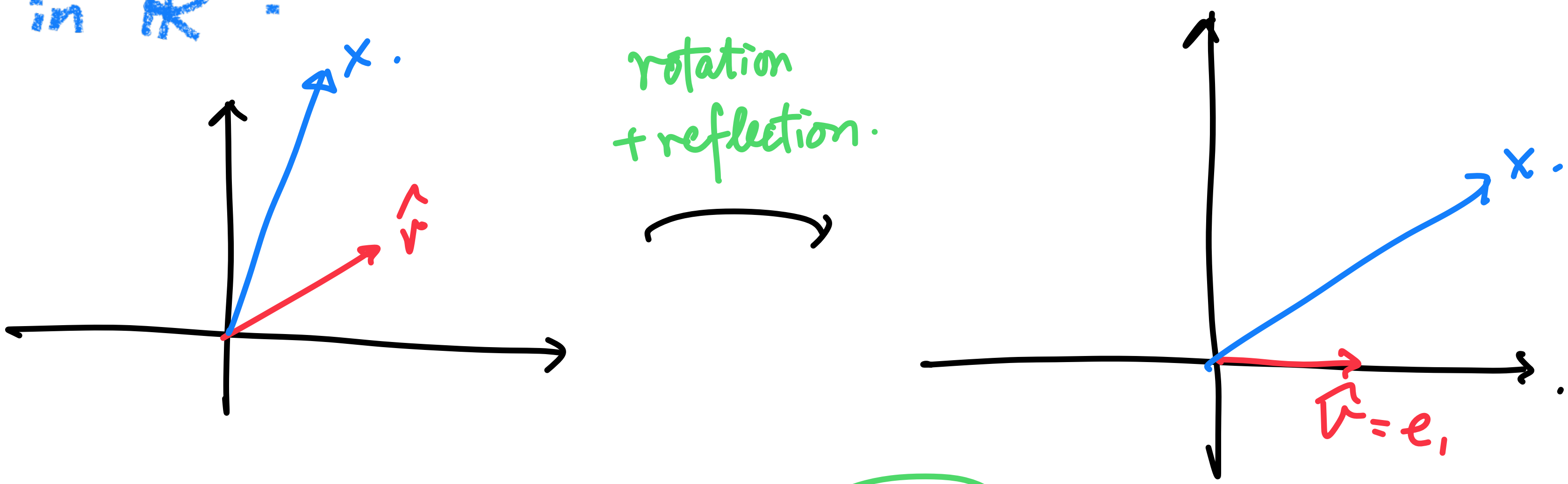

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then $[P] = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Projection

Observation: For the projection P onto a line:

$[P]$ is a symmetric matrix:

$$[P]^T = \left(\frac{1}{\|v\|^2} v \cdot v^T\right)^T = \frac{1}{\|v\|^2} v \cdot v^T = [P].$$

- Indeed: Any projection P onto V has its matrix representation under e_i's being symmetric (prove later).

e.g. We may check the formula for projection onto the θ-line in $\mathbb{R}^2$.

Projection

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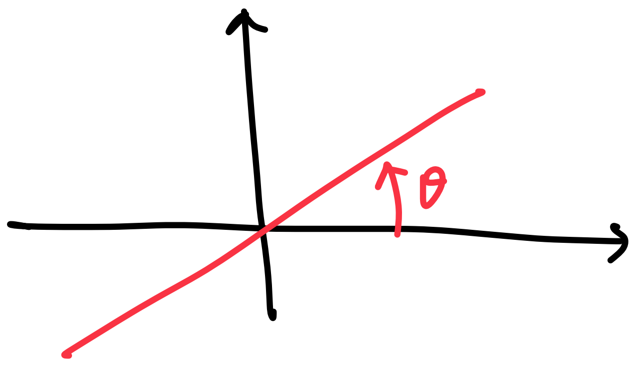

Find projection P onto the θ-line?

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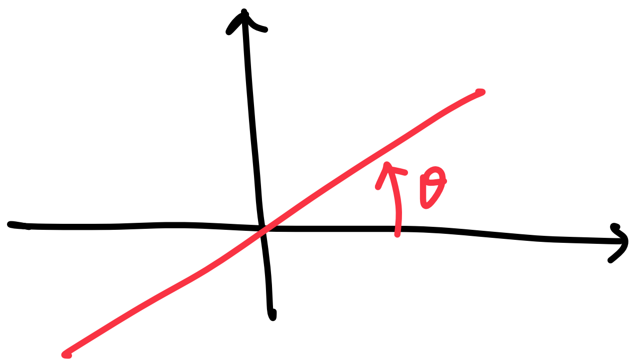

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$$v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

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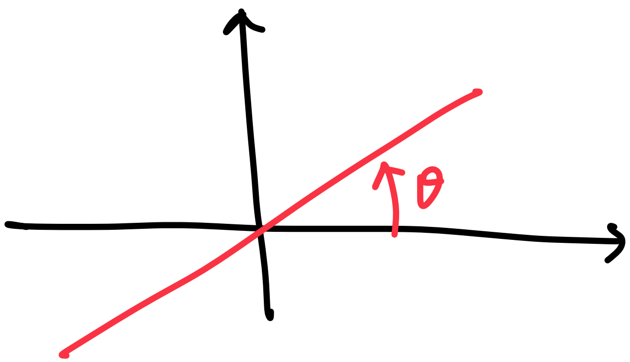

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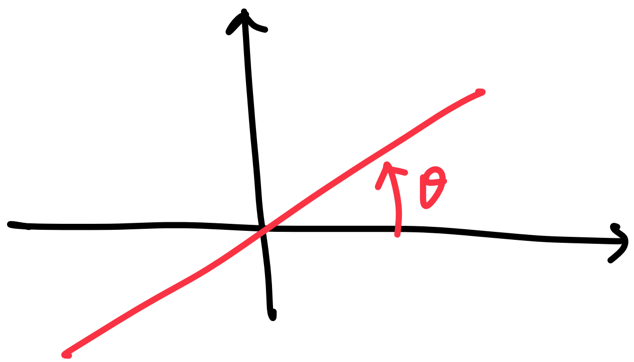

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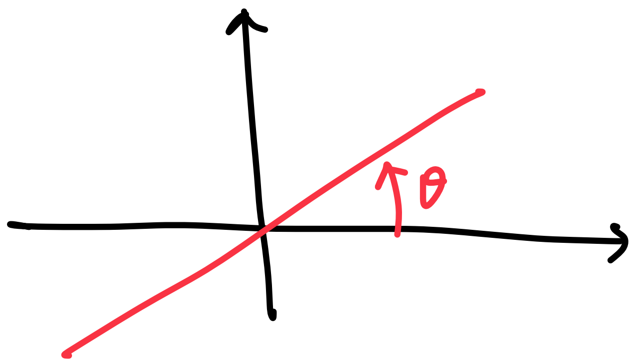

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which agrees with the one we get using transition matrix.

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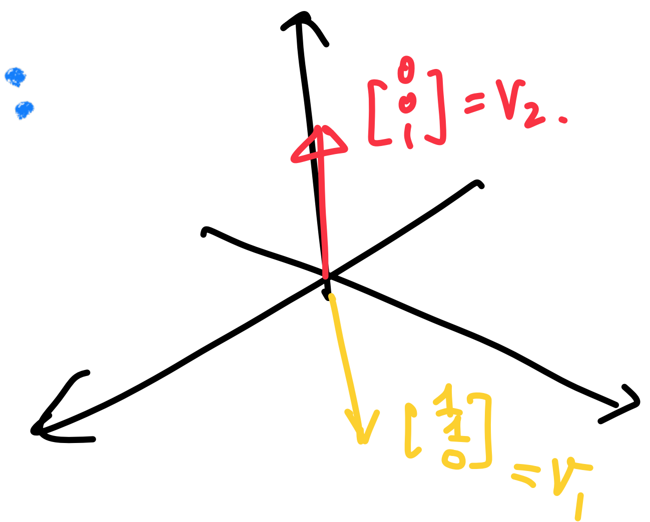

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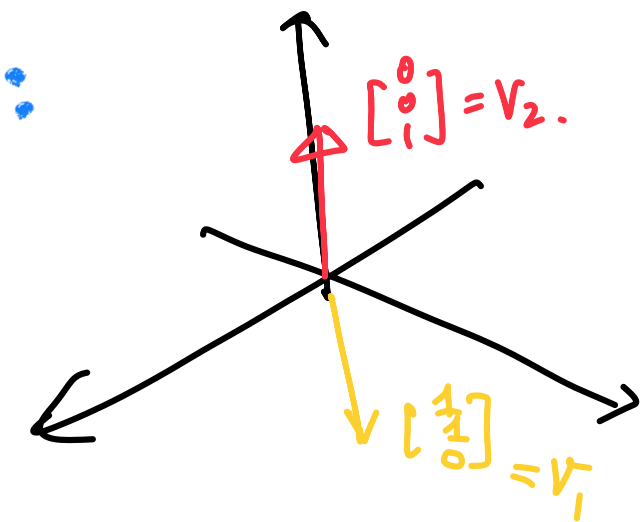

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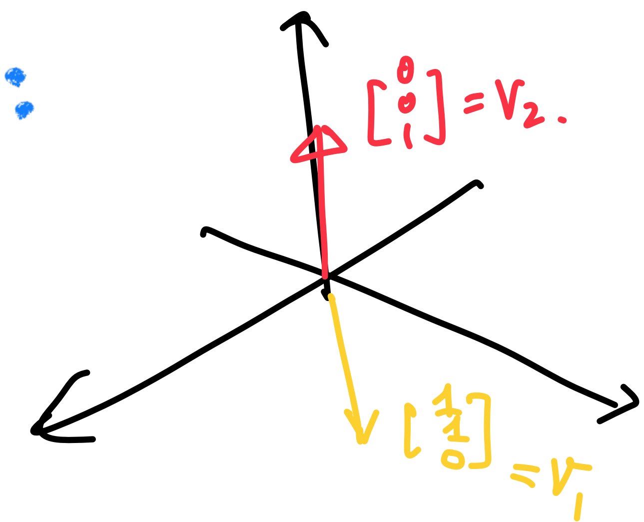

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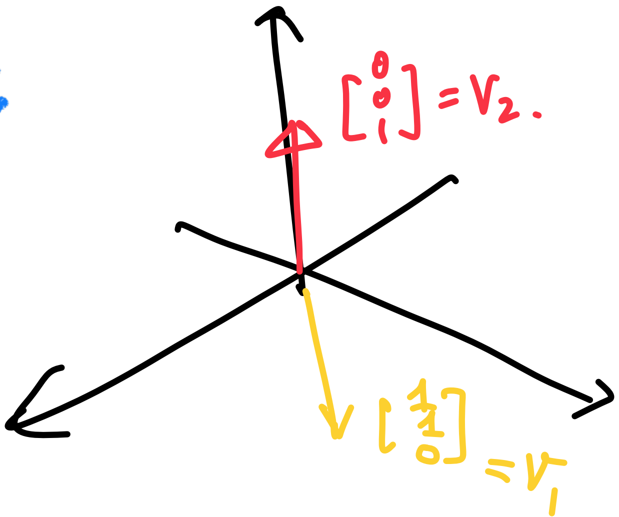

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- $[P] = q_1 q_1^T + q_2 q_2^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Transpose

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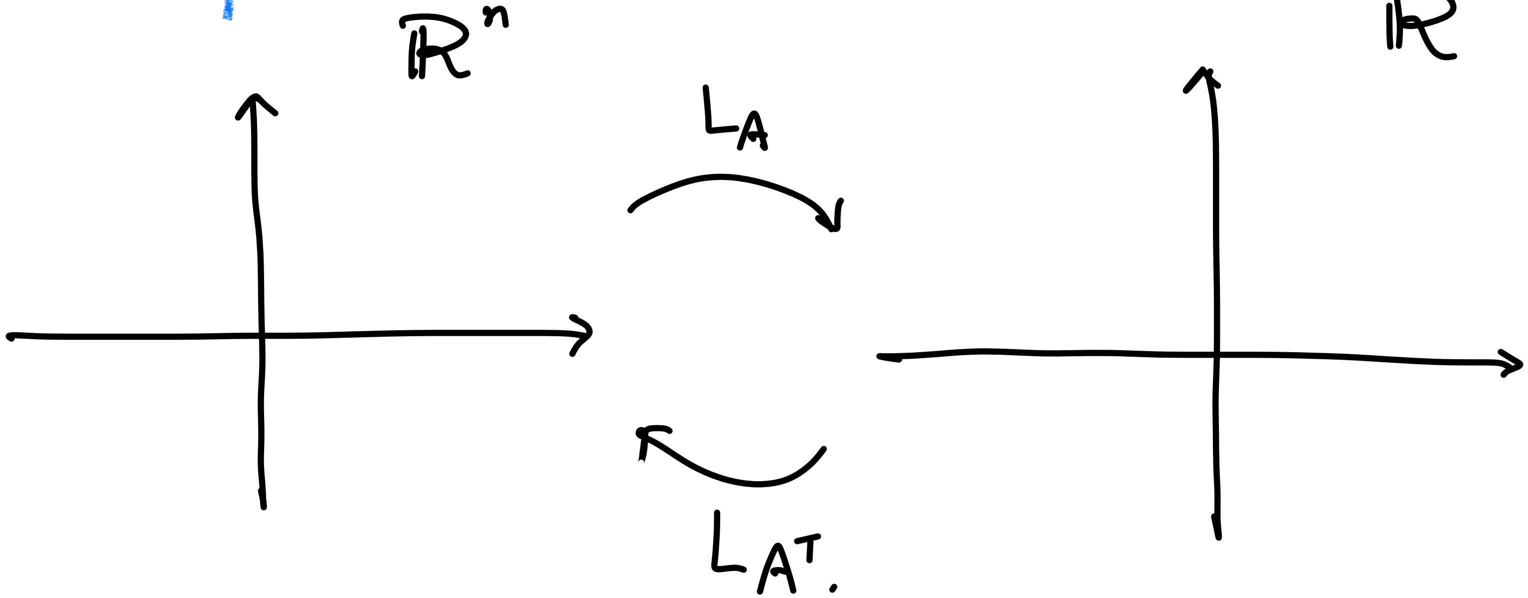
What is Transpose?

A: $m \times n$ matrix.



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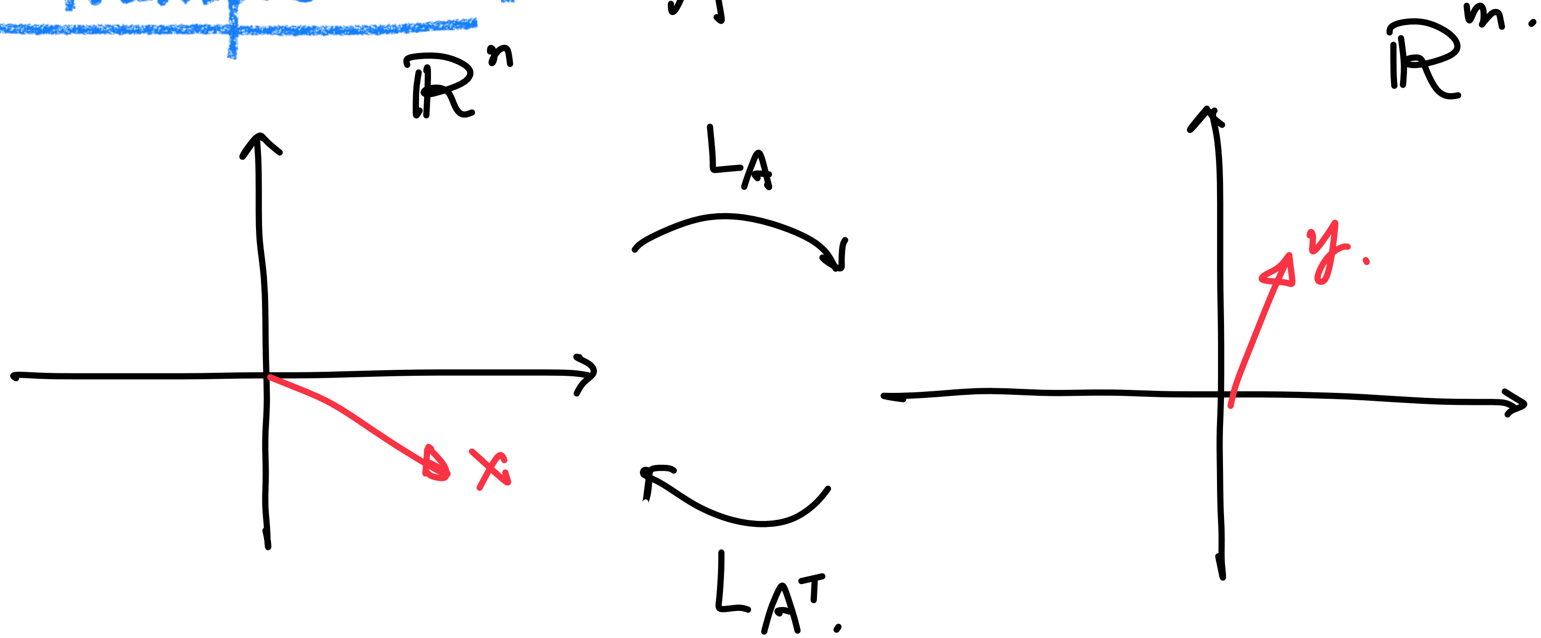
<u>What is Transpose?</u>

A : $m \times n$ matrix.



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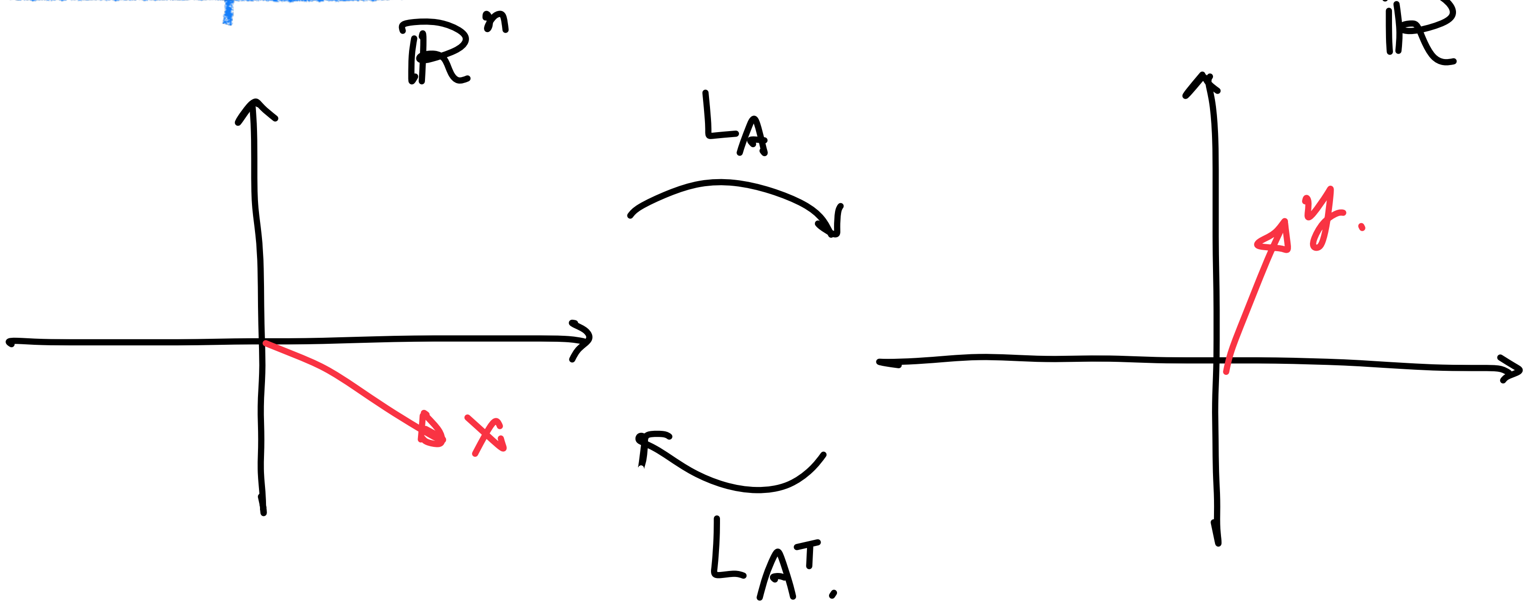

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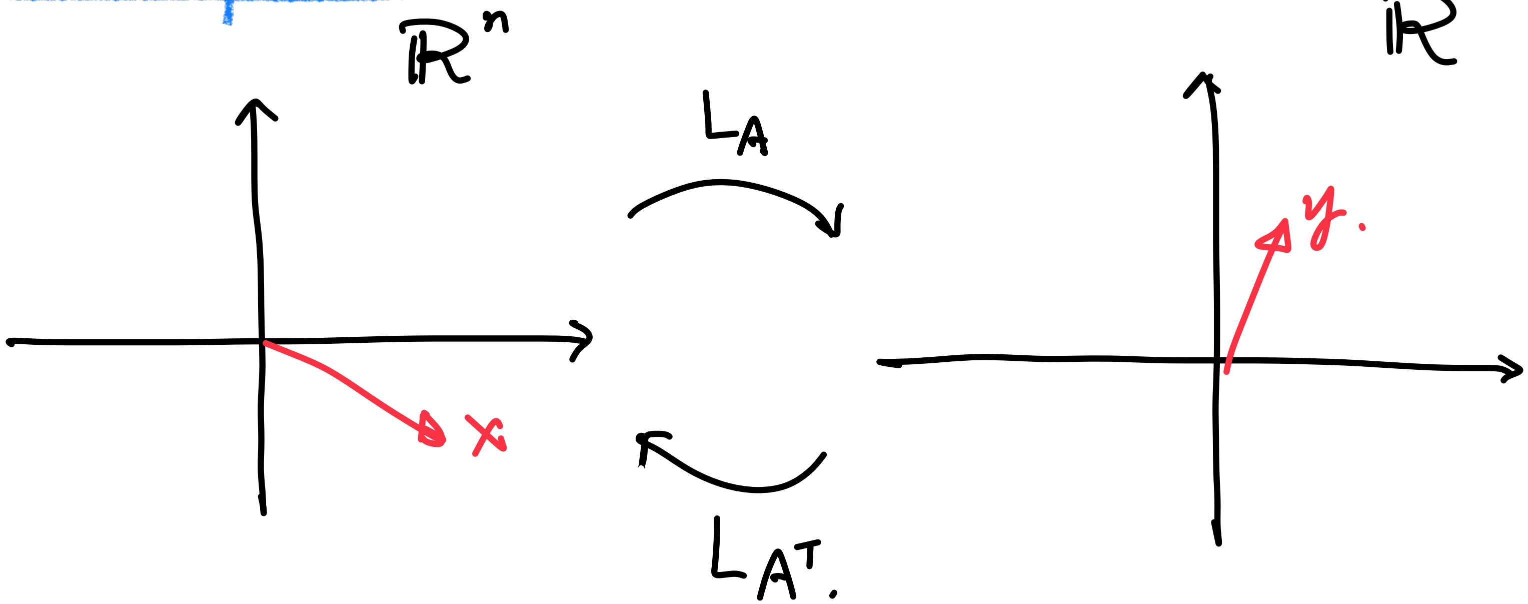

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Why: L.H.S. $= (Ax)^T y = x^T (A^T y) = \langle x, A^T y \rangle$

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by this property.

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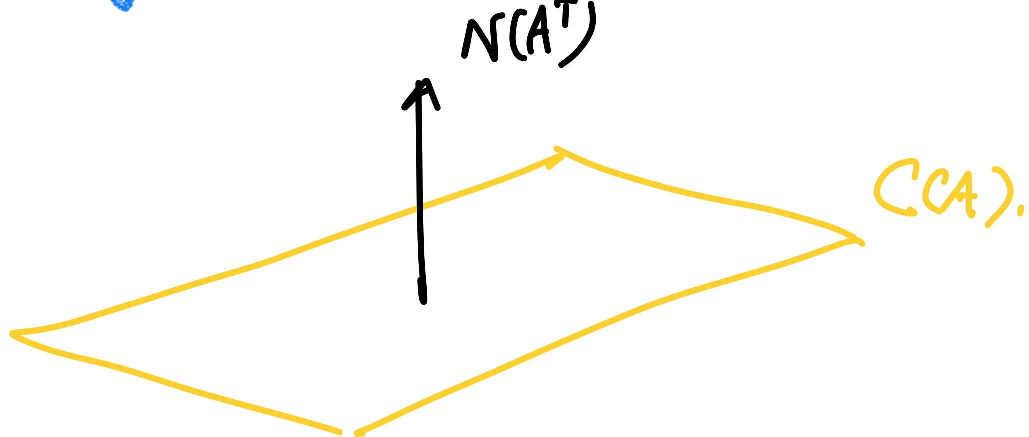

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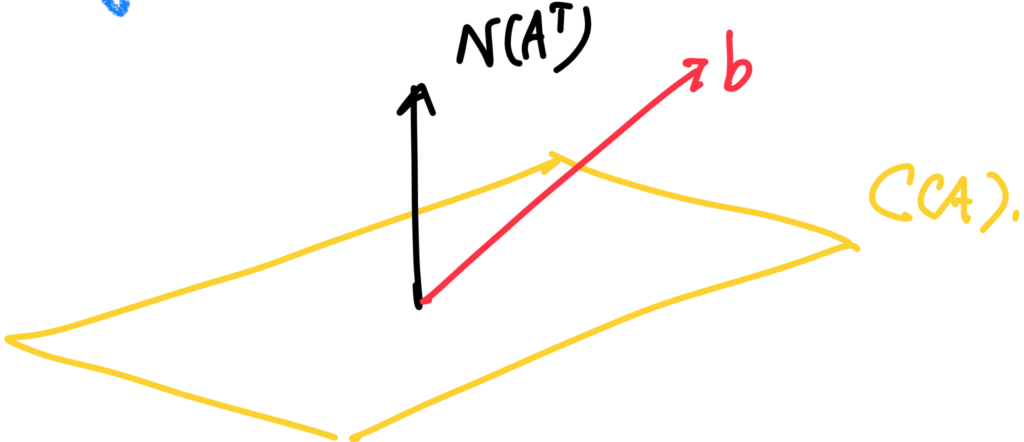

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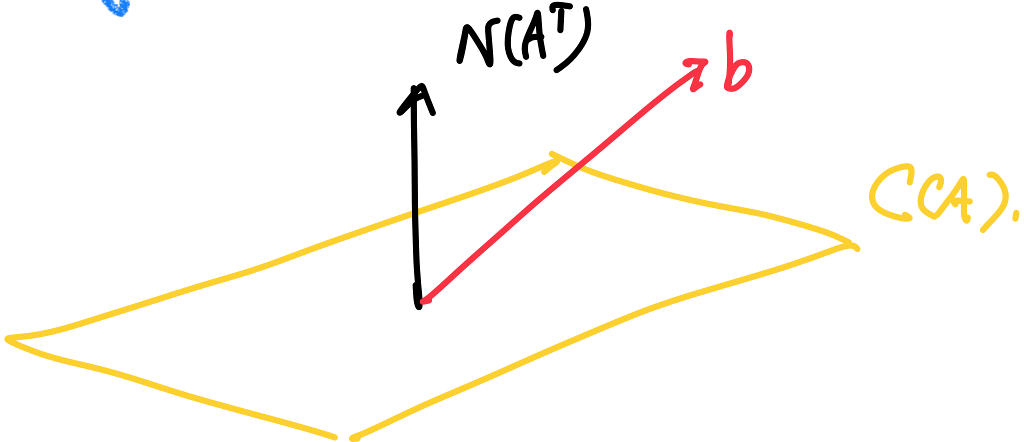

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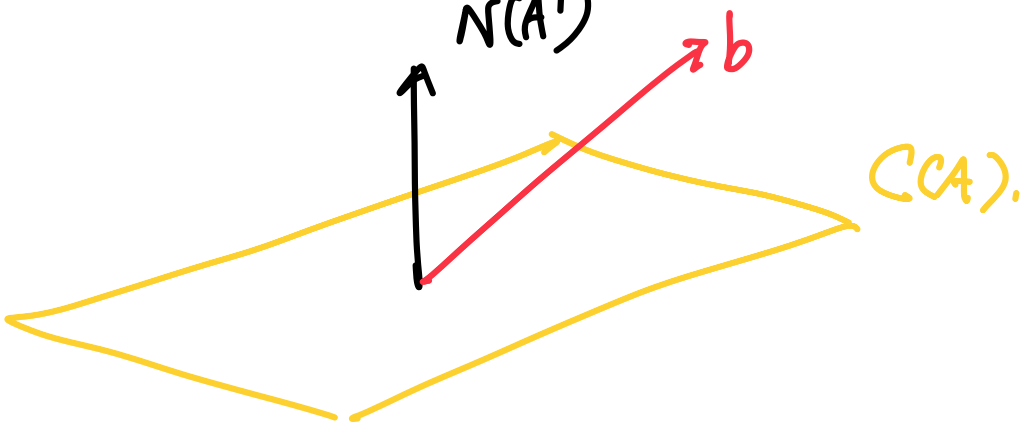

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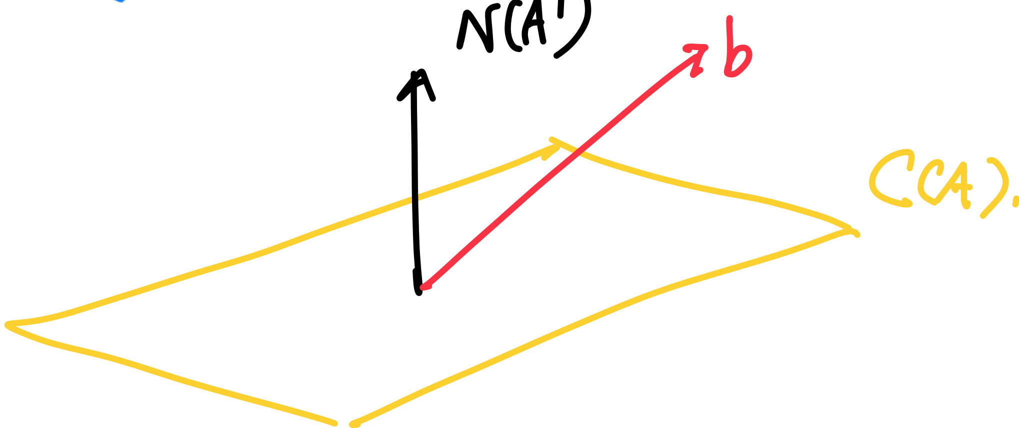

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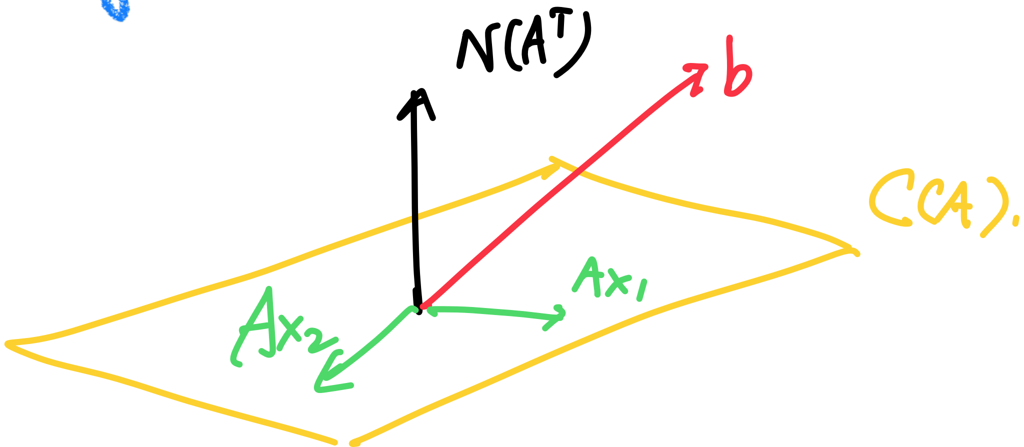

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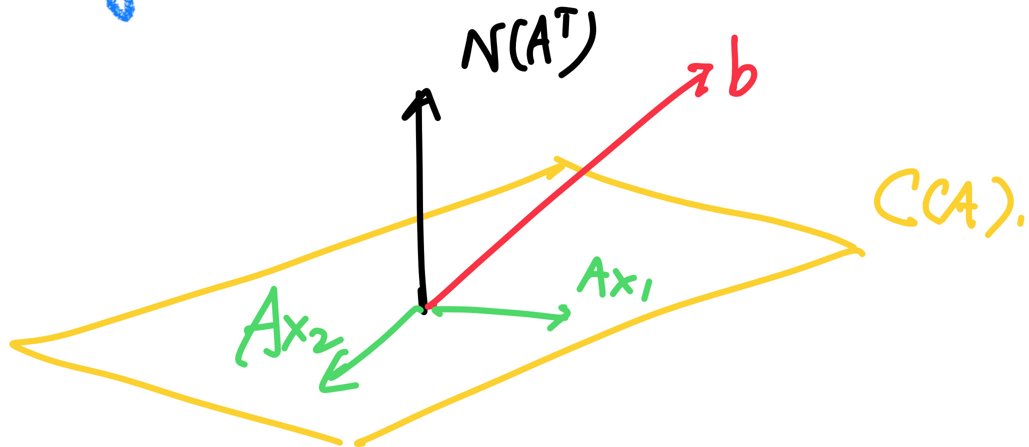

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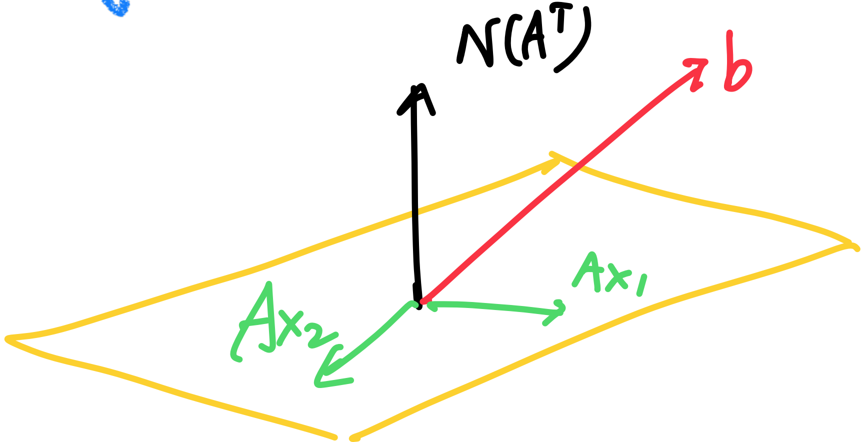

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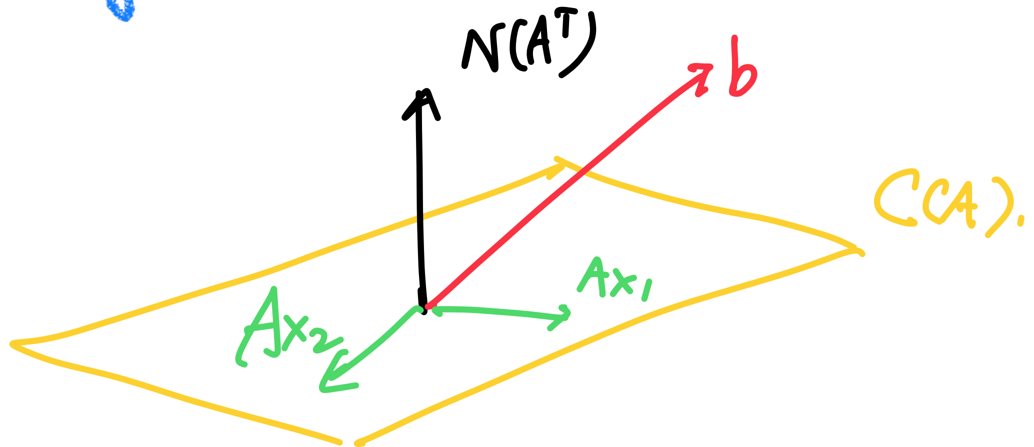

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- Then $b - A\hat{x}$ is in $C(A)^\perp = N(A^T)$.

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Therefore P is a projection.