Lecture 6

Solving Ax=b

Ax=b and Ax=0

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- Recall that we discuss

  Solution to $Ax = 0$ is the null space $N(A)$.

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$$x_p + x_n \longleftrightarrow x_n$$

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a particular solution to $Ax=b$

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$Ax=b$ and $Ax=0$

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- i.e. to determine whether $Ax=b$ is consistent:

  geometrically we want to find $C(A)$.

Ax=b and Ax=0

- $Ax = b$ has a solution $\Longleftrightarrow$ $b \in C(A)$.

- i.e. to determine whether $Ax=b$ is consistent:

  geometrically we want to find $C(A)$.

- to find all solution $x_p + x_n$:

  geometrically we try to understand $N(A)$.

Null space

Null space

- Given any $m \times n$ matrix A

Null space

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- We can have $P = P_1 \cdots\cdots P_k$

  $L = E_1 \cdots\cdots E_s$

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Type 3 elementary matrices.

<u>s.t.</u> $PA = LU$

Null space

- Given any $m \times n$ matrix A

- We can have $P = P_1 \cdots P_k$ permutation matrices

  $L = E_1 \cdots E_s$ Type 3 elementary matrices.

  s.t. $PA = LU$

- We talked about the case that A is a $n \times n$ matrix the same method works here.

Null space

Null space

- U would be in the row echelon form

$$U = \begin{bmatrix} p_1 & * & * & * & * & * & * \\ 0 & 0 & p_2 & * & * & * & * \\ 0 & 0 & 0 & 0 & p_3 & * & * \\ 0 & 0 & 0 & 0 & 0 & p_4 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Null space

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Pivots

Null space

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Pivots

- We can further write

$$U = \begin{bmatrix} p_1 & & & & & & \\ & p_2 & & & & 0 & \\ & & p_3 & & & & \\ & & & p_4 & & & \\ & 0 & & & 1 & & \\ & & & & & \ddots & \end{bmatrix} \begin{bmatrix} 1 & * & * & * & * & * & * \\ & & 1 & * & * & * & * \\ & & & & 1 & * & * \\ & & 0 & & & 1 & * \end{bmatrix}$$

Null space

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Pivots

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Null space

Null space

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Null space

- Let's consider $\mathcal{U} = \begin{bmatrix} 1 & * & * & * & * & * & * \\ & & 1 & * & * & * & * \\ & & & & 1 & * & * \\ & \mathbf{0} & & & & 1 & * \\ & & & & & & \end{bmatrix}$

Null space

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- By further row operation, using $m \times m$ upper triangular matrices

Null space

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$$E\tilde{U} = \begin{bmatrix} 1 & * & 0 & * & 0 & 0 & * \\ & & 1 & * & 0 & 0 & * \\ & & & & 1 & 0 & * \\ & \text{\huge 0} & & & & 1 & * \\ & & & & & & \end{bmatrix}$$

Null space

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in Reduced Row Echelon form.

Null space

- Let's consider $\tilde{U} = \begin{bmatrix} 1 & * & * & * & * & * & * \\ & & 1 & * & * & * & * \\ & & & & 1 & * & * \\ & 0 & & & & 1 & * \end{bmatrix}$

- By further row operation, using $m \times m$ upper triangular matrices

$$E\tilde{U} = \underbrace{\begin{bmatrix} 1 & * & 0 & * & 0 & 0 & * \\ & & 1 & * & 0 & 0 & * \\ & & & & 1 & 0 & * \\ & 0 & & & & 1 & * \end{bmatrix}}_{R}$$

in Reduced Row Echelon form.

Null space

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Prop: $N(A) = N(\mathcal{U}) = N(\hat{\mathcal{U}}) = N(R)$

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Pf: • $PA = LU$

invertible $m \times m$ matrices.

Null space

Prop: $N(A) = N(U) = N(\hat{U}) = N(R)$

Pf:

- $PA = LU$

  invertible $m \times m$ matrices.

- then $Ax = 0$

Null space

Prop: $N(A) = N(U) = N(\hat{\tilde{U}}) = N(R)$

Pf: • $\underbrace{P}A = \underbrace{L}U$

invertible $m \times m$ matrices.

• then $Ax = 0 \iff (L^{-1}P)Ax = 0$

Null space

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• Similarily: $U = D\tilde{U}$

$E\tilde{U} = R$

Null space

Prop: $N(A) = N(U) = N(\tilde{U}) = N(R)$

Pf:

- $PA = LU$

  invertible $m \times m$ matrices.

- then $Ax = 0 \iff (L^{-1}P)Ax = 0$
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- Similarly: $U = D\tilde{U}$
  $E\tilde{U} = R$

  invertible $m \times m$ matrices

Null space

Null space

e.g.:

$$R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ solve } R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

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Pivots

- We solve from the bottom rows :

Null space

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- We solve from the bottom rows :

$$x_3 + x_4 = 0 \implies x_3 = -x_4$$

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- We solve from the bottom rows:

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- Then the rows above it:

$$x_1 + 3x_2 - x_4 = 0 \implies x_1 = -3x_2 + x_4$$

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i.e. $\vec{x} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$

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i.e. $\vec{x} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$ or $N(R) = \left\{ \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} \middle| \; x_2, x_4 \in \mathbb{R} \right\}$

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variables correspond to pivots are expressed in terms of free variables

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- We can write $\vec{x} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

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or $N(R) = \mathrm{Span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

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- minimal size of spanning set $=$ # of free variables

$= n -$ non-zero rows.

Null space

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e.g.: $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0.$

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$$x_4 + 3x_5 = 0 \implies x_4 = -3x_5$$

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- We start from the bottom:

$$x_4 + 3x_5 = 0 \Longrightarrow \quad x_4 = -3x_5$$

- then the next row:

$$x_2 + 2x_3 + 8x_5 = 0 \Longrightarrow x_2 = -2x_3 - 8x_5$$

Null space

e.g.: $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0.$

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- Finally: $x_1 + x_3 + 3x_5 = 0 \Rightarrow x_1 = -x_3 - 3x_5.$

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e.g.: $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$.

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- Finally: $x_1 + x_3 + 3x_5 = 0 \Rightarrow x_1 = -x_3 - 3x_5$.

$$N(A) = \left\{ \begin{bmatrix} -x_3 - 3x_5 \\ -2x_3 - 8x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} \middle| \, x_3, x_5 \in \mathbb{R} \right\}$$

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- Finally: $x_1 + x_3 + 3x_5 = 0 \Rightarrow x_1 = -x_3 - 3x_5.$

$$N(A) = \left\{ \begin{bmatrix} -x_3 - 3x_5 \\ -2x_3 - 8x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} \middle| \, x_3, x_5 \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Null space

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- $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

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- Maybe you realize a quick way to write down $N(A)$:

Null space

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  non-pivot columns $\longleftrightarrow$ free variables x_3 and x_5

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- In this case: # of free variables $= 2 =$ # non pivot columns

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$$N(A) = \text{Span}\left\{ \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \right\}$$

Null space

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non-pivot position

Null space

- $A = \begin{bmatrix} 1 & 0 & 1 & 0 & \boxed{3} \\ 0 & 1 & 2 & 0 & \boxed{8} \\ 0 & 0 & 0 & 1 & \boxed{3} \\ 0 & 0 & 0 & 0 & \boxed{0} \end{bmatrix}$

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$$N(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} \\ \\ 1 \\ \\ 0 \end{bmatrix} \right\}$$

non-pivot position

Null space

- $A = \begin{bmatrix} 1 & 0 & \boxed{1} & 0 & \boxed{3} \\ 0 & 1 & \boxed{2} & 0 & \boxed{8} \\ 0 & 0 & \boxed{0} & 1 & \boxed{3} \\ 0 & 0 & \boxed{0} & 0 & \boxed{0} \end{bmatrix}$

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non-pivot position

Ax=b

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e.g.:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Ax=b

e.g:

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- $$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & -3 & 3 & 4 & b_3 \end{array}\right]$$

Ax=b

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solvable $\Longleftrightarrow$ $b_3 - 2b_2 + 5b_1 = 0$.

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- say $x_2 = 0 = x_4$: $\begin{cases} 3x_3 = 3 \\ x_1 + 3x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_3 = 1 \\ x_1 = -2 \end{cases}$

Ax=b

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- As a conclusion: $x_p = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $N(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

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- Therefore $C(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \middle| \; b_3 - 2b_2 + 5b_1 = 0 \right\}$

Column space

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- Say $R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ in rref.

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$$\vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

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$$= (x_1 + 3x_2 + 2x_4) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (x_3 + 3x_4) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

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- $C(R) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

Column space

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- Another view on $C(A)$:

$$E^{-1}A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

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invertible

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- $$C(A) = \left\{ x_1 E\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2 E\begin{pmatrix}3\\0\\0\end{pmatrix} + x_3 E\begin{pmatrix}0\\1\\0\end{pmatrix} + x_4 E\begin{pmatrix}2\\3\\0\end{pmatrix} \right\}$$

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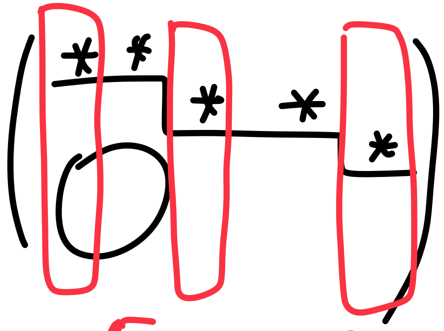
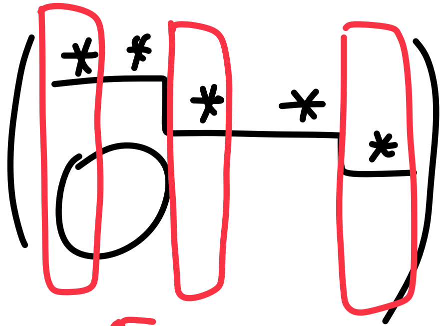


pivot columns.

Column space

- i.e. If $L^{-1}PA = \mathcal{U} =$ 

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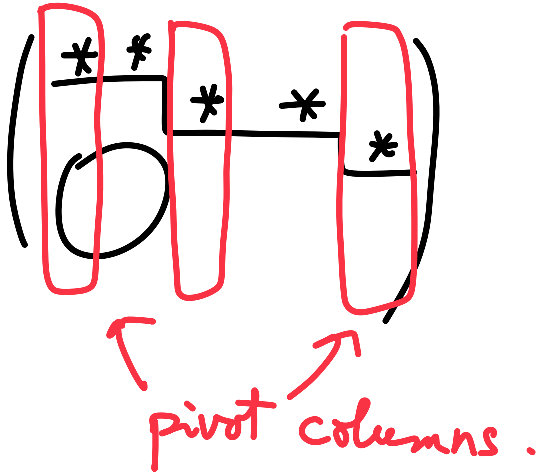
- i.e. If $L^{-1}PA = U =$ 

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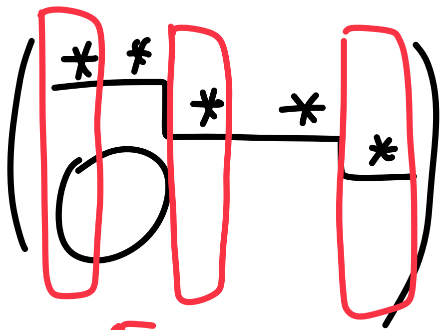
- i.e. If $L^{-1}PA = U =$ 

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- minimal size of spanning set $=$ # of pivots columns.

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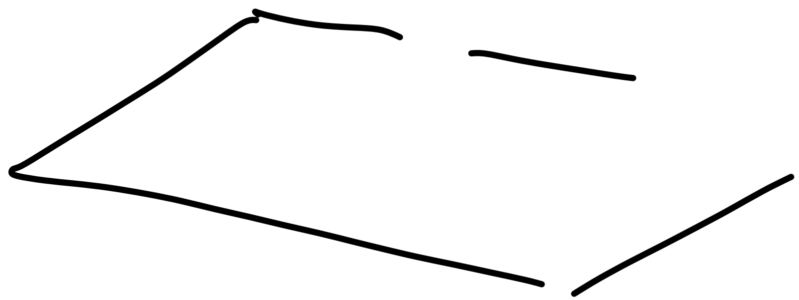

Column space

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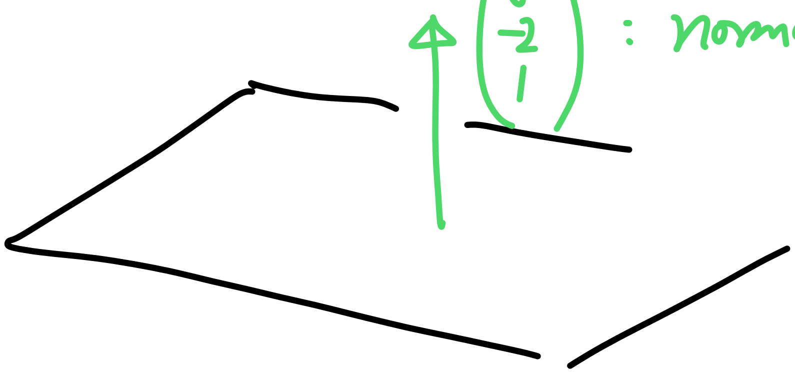
$$C(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

- $= \left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \middle| \; b_3 - 2b_2 + 5b_1 = 0 \right\}.$

- <u>Pictorially:</u>

$\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$: normal to the plane



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