Problem set 4

Linear algebra 06

October 11, 2025

1. The trace $\mathrm{tr}(A)$ for a $n \times n$ matrix $A = (a_{ij})_{nn}$ is defined to be the sum of entries on the diagonal, i.e.

$$\mathrm{tr}(A) := a_{11} + a_{22} + \cdots + a_{nn}.$$

   Prove that we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $n \times n$ matrices A, B.

2. Suppose we have the homogeneous equation of the form $A\vec{x} = 0$, with

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 5 & 0 & -1 \end{bmatrix},$$

   and another vector subspace

$$W := \mathrm{Span}\{(-1, 1, a+2, 2)^T, (-1, 2, 4, a+8)^T\},$$

   (a) Write down a basis for the nullspace $N(A)$,

   (b) find the value of a such that their intersection $N(A) \cap W$ contains a non zero element (here intersection of two subspaces $W_1 \cap W_2$ refers the set containing elements lying in both W_1 and W_2).

3. Given the matrix A and its reduced row echelon form R:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 & -1 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for $\mathcal{C}(A)$.

   (b) Find a basis for $\mathcal{C}(A^T)$.

   (c) Find a basis for $N(A)$.

4. Let P_4 be the vector space consisting of polynomials with degree ≤ 4, equipped with usual addition and scalar multiplication of polynomials. Let

$$V := \{f(x) \mid f(x) \in P_4,\ f(-1) = 0,\ f(1) = 0\}$$

   be the subset.

(a) Show that V is a subspace of P_4.

(b) Find a basis of V.

Let $W = \{A \mid A \in M_{3\times3}(\mathbb{R}),\ A^T = A\}$ be the subset of vector space V consisting of 3×3 matrices.

(a) Show that W is a subspace of V.

(b) Find a basis for W.