LAw3

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· Vector spaces and subspaces

· Find a way to describe the solution set of $Ax=0$

$S=\{x: Ax=0\}$ common feature ?

1. if $Ax=0$. the $A(cx)=0$

2. if $Ax=0$. $Ay=0$, then $A(x+y)=0$

Recall . $x_1 \cdots x_k$ are solution to $Ax=0$

then $c_1\vec{x}_1+c_2\vec{x}_2+\cdots+c_k\vec{x}_k$ is again a solution .

Def. A vector space is a set V equipped with two operations

1. scalar multiplication $\cdot : \mathbb{R}\times V \to V$
   any vector $\vec{v}$ can be scaled by real number $c\vec{v}$
2. addition $+ : V\times V \to V$

any vectors $\vec{v}, \vec{w}$ can be added together to give $\vec{v}+\vec{w}$

Satisfy the 8 compatibility conditions.

1. $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

2. $(c_1+c_2)\vec{x} = c_1\vec{x}+c_2\vec{x}$

3. $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$

4. $\vec{x}+\vec{y} = \vec{y}+\vec{x}$

5. $c_1(c_2\vec{x}) = (c_1c_2)\vec{x}$

6. $1\cdot\vec{x} = \vec{x}$

7. $\exists\ \vec{0}$, s.t. $\vec{0}+\vec{x}=\vec{x}$ for all $\vec{x}$

8. for any $\vec{x}$, there exists $-\vec{x}$ s.t. $-\vec{x}+\vec{x}=0$

Prop. $\vec{0}$ is unique:

Pf. assume that there are 2 zeros $\vec{0}_1, \vec{0}_2$

$\vec{0}_1+\vec{0}_2 = \vec{0}_2 = \vec{0}_2+\vec{0}_1 = \vec{0}_1$

Prop. for any $\vec{v}$ in V. $-\vec{v}$ is unique

Pf. Assume there is $-\vec{v}_1$, $-\vec{v}_2$ for $\vec{v}$

$\vec{v} + -\vec{v}_1 = \vec{0}$ $\quad$ $\vec{v} + -\vec{v}_1 + -\vec{v}_2 = -\vec{v}_2$

$v + -\vec{v}_2 = 0$ $\quad$ $\vec{v} + -\vec{v}_2 + -\vec{v}_1 = -\vec{v}_1$

$\Rightarrow -\vec{v}_2 = -\vec{v}_1 \Rightarrow -\vec{v}$ is unique.

prop. for any $0 \cdot \vec{v} = \vec{0}$ for any $\vec{v}$

$0 \cdot \vec{v} + 0 \cdot \vec{v} = (0+0) \cdot \vec{v} = 0\vec{v}$

$\Rightarrow 0 \cdot \vec{v} + 0 \cdot \vec{v} + -0 \cdot \vec{v} = 0 \cdot \vec{v} + -0 \cdot \vec{v}$

$0 \cdot \vec{v} + \vec{0} = \vec{0} \Rightarrow 0\vec{v} = \vec{0}$

prop. $(-1) \cdot \vec{v} = -\vec{v}$

$1 \cdot \vec{v} + (-1) \cdot \vec{v} = (1 + -1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$

$1 \cdot \vec{v} + -\vec{v} = \vec{0}$

Given that $-\vec{v}$ is unique

$\Rightarrow (-1)\cdot\vec{v} = -\vec{v}$

Take $V = \mathbb{R}_+$. set .

$\circ: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ by

$\alpha\circ x := x^{\alpha}$

$+: \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ by

$x\oplus y := xy$

$\Rightarrow \mathbb{R}_+$ is a vector space under these operation

$$\begin{array}{ccc} V: \mathbb{R} & \xleftrightarrow{\text{一一对应}} & \mathbb{R}_+ \\ \Downarrow & & \\ v & \longrightarrow & e^{v} \\ \log x & \longleftarrow & x \end{array}$$

将实数加法与乘法，对应到正实数的加法和乘法

$\Rightarrow \mathbb{R}$与$\mathbb{R}^+$同构

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = -x_3 \quad x_2 = -x_3$$

$$\Rightarrow W = \left\{ \begin{bmatrix} -c \\ -c \\ c \end{bmatrix} \middle| \ c \in \mathbb{R} \right\}$$

Def.

For a $m \times n$ matrix A.

we define its nullspace to

$$N(A) := \{ \vec{x} \mid \vec{x} \in \mathbb{R}^n, A\vec{x} = 0 \}$$

e.g. 1. if $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $N(A) = \mathbb{R}^2$

2. if A non-singular $N(A) = \{ \vec{0} \}$

Def A vector subspace W of a vector space V is a non-empty subset of V s.t.

1) $c \cdot \vec{x}$ in W if $\vec{x}$ in W

2) $\vec{x} + \vec{y}$ in W if $\vec{x}, \vec{y}$ in W

e.g. $W = \{ A \mid A \in M_{2\times 2}(\mathbb{R}), A^T = A \}$

is a subspace of $M_{2\times2}(\mathbb{R})$

is a subspace of $M_{2\times2}(\mathbb{R})$

(1) $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T \Rightarrow \vec{0}$ in W

(2)

$(CA) = \begin{bmatrix} ca & cb \\ cb & cd \end{bmatrix} = (CA)^T \Rightarrow (CA)^T$ in W

(3) $(A+B)^T = A^T + B^T = A+B$

e.g.1. let $W = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R}_{\geq 0} \right\}$

then W is not a subspace of $\mathbb{R}^2$

$-1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is not in W

2. $W = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| ad - bc \neq 0 \right\}$ is Not

a subspace of $M_{2\times2}\mathbb{R}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ not in W

prop. of subspaces

1. If W is a subspace of V.

than $\vec{0}_V \in W$ $\to$ it is also $\vec{0}_W$

Pf: take $\vec{v} \in W$ W is not empty

$0 \cdot \vec{w}$ is in W

在 V 中. $0 \cdot \vec{w} = \vec{0}$. $\Rightarrow$ $\vec{0}$ is in W

$\Rightarrow$ if W is a subspace of V

then $\vec{0}_V = \vec{0}_W$

Some notation:

1. $\{ \cdots \}$ $\to$ a set of things

2. $\in$ means inside a set

3. $f: X \to Y$ function if it assign every element in X to an element in Y

$X \to$ set, $Y \to$ set

4. $W \subset V$. W is a subspace of V

$N(A) = \{ \vec{x} \mid \vec{x} \in \mathbb{R}^n, A\vec{x} = \vec{0} \}$

· $C(A)$.

def. Given a $m \times n$ matrix $A = [\vec{a}_1, \cdots \vec{a}_n]$

the column space of A is defined to be

$$C(A) = \{ c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n \mid c_i \in \mathbb{R} \}$$