Problem set 1

Linear algebra A-7

2025

1. Solve for a matrix X satisfying the equations:

   (a)
   $$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 4 & 7 \end{bmatrix}.$$

   (b)
   $$X \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 7 \end{bmatrix}.$$

2. Show that the system of linear equation

   $$\begin{aligned} x_1 - x_2 &= b_1 \\ x_2 - x_3 &= b_2 \\ x_3 - x_4 &= b_3 \\ x_4 - x_1 &= b_4 \end{aligned}$$

   is consistent if and only if $b_1 + b_2 + b_3 + b_4 = 0$.

3. Consider the following system of linear equations:

   $$\begin{aligned} \lambda x_1 + x_2 + x_3 &= 1 \\ x_1 + \lambda x_2 + x_3 &= \lambda \\ x_1 + x_2 + \lambda x_3 &= \lambda^2 \end{aligned}$$

   determine for which value of λ that the equation

   (a) has no solution;

   (b) has infinitely many solution.

4. Find the equation of sphere $x^2 + y^2 + z^2 + ax + by + cz + d = 0$ passing through the point $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 0, 0)$.

5. For the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

(a) show that $A^4 = 0$, and compute e^A if we define $e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots$;

(b) show that $e^A e^{-A} = I$ (the right hand side means multiple the two matrices e^A and e^{-A}), where I is the 4×4 identity matrix (Remark: $e^A e^B \neq e^{A+B}$ in general for matrices A, B);

(c) For $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

find e^A (no step is needed).