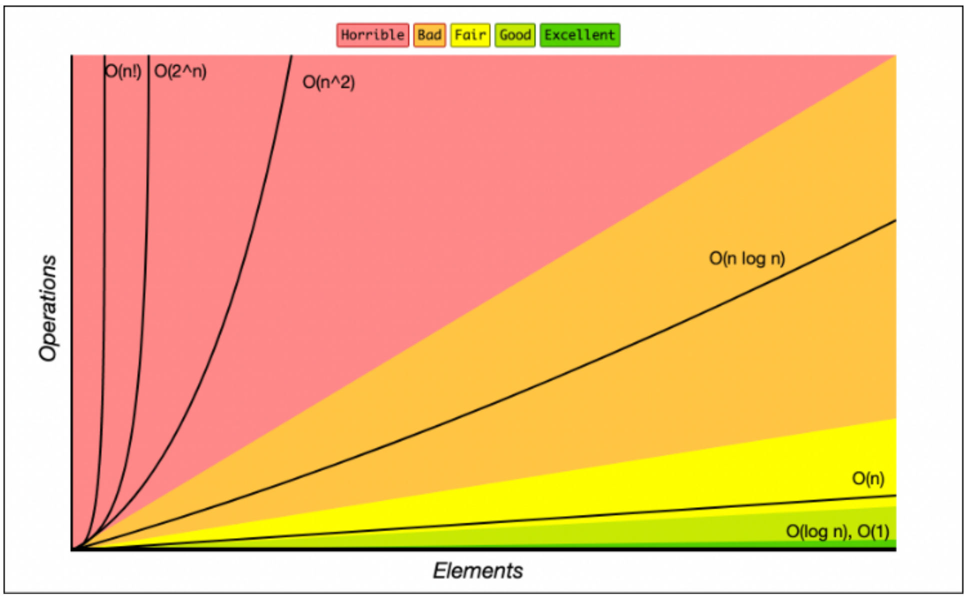
Chapter 3: Characterizing Running Times

Introduction

When we look at input sizes large enough to make only the order of growth of the running time relevant, we are studying **asymptotic** efficiency of algorithms. That is, we are concerned with how the running time increases with the size of the input *in the limit*, as the size of the input increases without bound.

Usually, an algorithm that is asymptotically more efficient will be the best choice for all but very small inputs.



Asymptotic Notation: informal introduction

- ***O*-notation** (big-Oh): The O-notation characterizes an upper bound on the asymptotic behavior of a function. It says that a function grows no faster than a certain rate, based upon its highest order term. Example: $7n^3 + 100n^2 - 20n + 6$ would be $O(n^3)$. It is also $O(n^4)$.

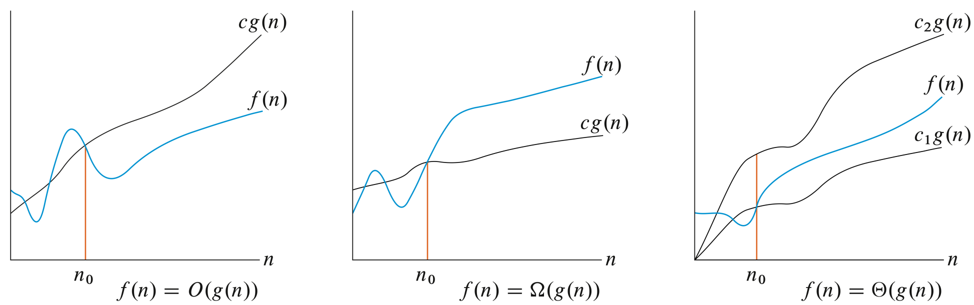
- **Ω-notation** (big-Omega) The Ω-notation characterizes a lower bound on the asymptotic behavior of a function. It says that a function grows at least as fast as a certain rate, based (again) upon its highest order term. Example: $7n^3 + 100n^2 - 20n + 6$ would be $\Omega(n^3)$. It is also $\Omega(n^2)$, and $\Omega(n)$.

- **Θ-notation** (big-Theta) The Θ-notation characterizes a tight bound on the asymptotic behavior of a function. It says that a function grows precisely at a certain rate, based on its highest order term. Example: $7n^3 + 100n^2 - 20n + 6$ would be $\Theta(n^3)$. However, it isn't $\Theta(n^2)$ or $\Theta(n^4)$.

Note: If a function is both $O((f(n))$ and $\Omega(f(n))$ for some function $f(n)$, then we have shown that the function is $\Theta(f(n))$.

Example: Do an informal asymptotic analysis of insertion sort to show that its worst-case running time is (n^2) and $\Omega(n^2)$ and thus is $\Theta(n^2)$.

Asymptotic Notation: formal definition



(a) (b) (c)

- ***O*-notation: asymptotically upper bound**
  - *Definition*: For a given function $g(n)$, we denote by $O(g(n))$ to be the *set of functions*:

    $O(g(n))$ = $\{f(n)$: there exist positive constants c, and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$
  - *Alternative Definition*: for two given functions $f(n)$ and $g(n)$:

    $f(n) = O(g(n)) \Longleftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = c \text{ } or \text{ } \infty$
  - We can use the alternative definition to show

$$\begin{aligned} f(n) = \tfrac{1}{2}n^2 - 4n + 100 &= O(n^2) \quad \text{since } \lim_{n\to\infty} \frac{n^2}{\frac{1}{2}n^2-4n+100} = 2 \\ &= O(n^3) \quad \text{since } \lim_{n\to\infty} \frac{n^3}{\frac{1}{2}n^2-4n+100} = \infty \\ &= O(\infty) \quad \text{since } \lim_{n\to\infty} \frac{\infty}{\frac{1}{2}n^2-4n+100} = \infty \end{aligned}$$

  - Another example:

$$f(n) = 10 + \frac{1}{n} \quad = \quad O(1) \text{ since } \lim_{n\to\infty}(10 + \frac{1}{n}) = 10 \qquad (1)$$

$$(2)$$

- **Ω-notation: asymptotically lower bound**

– *Definition*: For a given function $g(n)$, we denote by $\Omega(g(n))$ to be the *set of functions*:

$\Omega(g(n)) = \{f(n)$: there exist positive constants c, and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0\}$

– *Alternative Definition*: for two given functions $f(n)$ and $g(n)$:

$f(n) = \Omega(g(n)) \Longleftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = c \text{ or } 0$

– We can use the definition to show

$$\begin{aligned} f(n) = \tfrac{1}{2}n^2 - 4n + 100 &= \Omega(n^2) \quad \text{since } \lim_{n\to\infty} \frac{n^2}{\frac{1}{2}n^2-4n+100} = 2 \\ &= \Omega(n) \quad \text{since } \lim_{n\to\infty} \frac{n}{\frac{1}{2}n^2-4n+100} = 0 \\ &= \Omega(1) \quad \text{since } \lim_{n\to\infty} \frac{1}{\frac{1}{2}n^2-4n+100} = 0 \end{aligned}$$

– For example:

$$\begin{aligned} \text{INSERTION-SORT(n)} &= \Omega(1) \\ &= \Omega(n) \\ &\neq \Omega(n^2), \text{ since best-case takes time proportional to } n \end{aligned}$$

- **Θ-notation: asymptotically tight bound**

– *Definition*: For a given function $g(n)$, we denote by $\Theta(g(n))$ to be the *set of functions*:

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$ for all $n \geq n_0\}$

– *Alternative Definition*: For two given functions $f(n)$ and $g(n)$:

$f(n) = \Theta(g(n)) \Longleftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = c$

– We can use the definition to show $f(n) = \frac{1}{2}n^2 - 4n + 100 = \Theta(n^2)$, since

$$\lim_{n\to\infty} \frac{n^2}{\frac{1}{2}n^2 - 4n + 100} = 2$$

– Without using definition, we can determine the Θ-notation (as well as other notations) of a function $f(n)$ by throwing away lower-order terms and ignores the leading coefficient of the highest-order term.
For example,

$f(n) = \frac{1}{2}n^2 - 4n + 100 = \Theta(n^2)$

– People often use big-O notation to describe the running time of an algorithm rather than using Θ-notation (but they often mean to use Θ). However, Θ-notation tells people more exact running time of an algorithm. For example:

INSERTION-SORT(A) $= O(n^2)$ for all input n
INSERTION-SORT(A) $\neq \Theta(n^2)$
$\neq \Theta(n)$
The worst case of INSERTION-SORT(A) $= \Theta(n^2)$

- *Theorem: for any two functions $f(n)$ and $g(n)$,*
  $f(n) = \Theta(g(n)) \Longrightarrow f(n) = O(g(n))$ *and* $f(n) = \Omega(g(n))$

- **Properties of asymptotics**

  - **Transitivity**
    $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$
    $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$
    $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$
  - **Reflexivity**
    $f(n) = \Theta(f(n))$
    $f(n) = O(f(n))$
    $f(n) = \Omega(f(n))$
  - **Symmetry**
    $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
  - **Transpose Symmetry**
    $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$
  - **Trichotomy**: For any two numbers a and b, exactly one of the following must hold: $a < b, a = b, a > b$. Not all functions are asymptotically comparable. For example: n and $n^{1+\sin n}$ cannot be asymptotically compared.
  - Based on the above properties, we can draw an analogy to comparing two numbers a and b:
    $f(n) = O(g(n))$ is similar to $a \leq b$
    $f(n) = \Omega(g(n))$ is similar to $a \geq b$
    $f(n) = \Theta(g(n))$ is similar to $a = b$

- **Asymptotic notations in equations**

  - $2n^2 + 3n + 1 = 2n^2 + \Theta(n) \Longrightarrow 2n^2 + 3n + 1 = 2n^2 + f(n)$, where $f(n) = \Theta(n)$. In this case, $f(n) = 3n + 1$
    This example shows that we can mix asympotics in equations. For example, we replaced lower order terms above by $\Theta(n)$ so we could focus on the more important terms.

– $T(n) = O(n) + \Theta(n) + \Omega(n) \Longrightarrow T(n) = \Omega(n)$
This example shows how to combine three asymptotic components in an equation. $T(n)$ consists of the sum of the following three terms:

* $O(n)$ which is growing at a rate $\leq n$,
* $\Theta(n)$ so growing at the same rate as n, and
* $\Omega(n)$ so growing at a rate greater than n.

If we combine those three, the best we can say is $\Omega(n)$ since that will be the dominant term for $T(n)$. This is because something growing faster than n will be growing bigger than two other terms growing at a rate equal to or less than n.

Categories of functions

growth rate	slowest	$\rightarrow$	$\rightarrow$	$\rightarrow$	fastest
run time	fastest	$\leftarrow$	$\leftarrow$	$\leftarrow$	slowest
categories	constant	logarithms	polynomials	exponentials	super exponentials
examples	5 1 10000	$\log_2 n$ $\log_{10} n$ $100\log_e n$	n^2 n^3 $n^{0.1}$	$2^{n/2}$ 2^n 3^n	$(\log n)^n$ $n!$ n^n

- For logarithm functions, the base does not matter. $\log_{10} n = \Theta(\log_2 n)$ or $\log_e n = \Theta(\log_2 n)$
- For exponential functions, the base matters. $2^{n/2} = \sqrt{2^n} = O(2^n)$, but $2^{n/2} \neq \Theta(2^n)$
- Comparison between polynomials and exponentials: n^a and b^n
  No matter how big the a is and how small the $b > 1$ is, the exponential b^n will always out-grow the polynomial n^a if n approaches ∞.

  For example, 1.000001^n will out-grow $n^{1,000,000}$ when $n \rightarrow \infty$
- **Review Section 3.3 from the textbook**: Discrete mathematics that is useful for analysis. Floors and ceilings, Modular arithmetic, Polynomials, Exponentials, Logarithms, Factorials (pay attention to Stirling's Approximation), and Fibonacci numbers.

  – **Exponentials**: For all real $a > 0$, m, and n

$$\begin{aligned} a^0 &= 1 \\ a^1 &= a \\ a^{-1} &= \frac{1}{a} \\ (a^m)^n &= a^{mn} \\ (a^m)^n &= (a^n)^m \end{aligned}$$

$$a^m a^n = a^{m+n}$$

- **Logarithms**: Useful properties to remember:

$$\log(ab) = \log a + \log b$$
$$\log(\frac{a}{b}) = \log a - \log b$$
$$\log(a^b) = b \log a$$

- Logarithms and Exponentials are inverse functions:

$$b^{\log_b a} = a$$
$$\log_b(b^a) = a$$

- **Change of base formula**:

$$\log_b a = \frac{\log_k a}{\log_k b}$$

  This shows that logarithms in different bases differ only by a constant factor.

- We often use the base-2 logarithm in computer science since computers are binary.

$$\lg n = log_2 n$$
$$\ln n = log_e n$$
$$\log n = log_{10} n$$
$$\log^k n = \; = (\log n)^k$$
$$\lg \lg n = \lg(\lg n)$$

- **Factorials and Stirling's Approximation**:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

- This allows us to show that $\lg(n!) = \Theta(n \lg n)$, which is useful in the analysis of several algorithms.

• **Review Calculus**: The derivative rule and *L'Hôpital's rule* are useful for obtaining limits of more complex functions.

- **Summations**: Arithmetic series and Geometric series sums are commonly found during analysis of algorithms.
  - **Simple Arithmetic series**:

    $$1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

    Here is a common variation:

    $$1+2+3+\ldots+n-1 = \frac{n(n-1)}{2}$$

  - **General Arithmetic Series:** Each term a_i is separated from the previous term a_{i-1} by a constant.

    $$a_1+a_2+\ldots+a_n = \frac{(a_1+a_n)}{2}n$$

  - **Simple Geometric Series**:

    $$1+2+2^2+2^3+\ldots+2^n = 2^{n+1}-1$$

    Here is a common variation:

    $$1+2+2^2+2^3+\ldots+2^{n-1} = 2^n-1$$

  - **General Geometric Series**:
    For $x \neq 1$, we have:

    $$1+x+x^2+x^3+\ldots+x^n = \frac{x^{n+1}-1}{x-1}$$

    If $\mid x \mid < 1$, then the sum is $\frac{1}{x-1}$.

Practice Problems

- Exercise 3.2-3. Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?
- Order the following functions by growth rate: n, n^2, $\sqrt{n}$, $n^{1.5}$, 2^n, $n!$, $(\log n)^2$, 3^n, $\log n^2$.

Math review

- Derivative rules: https://www.mathsisfun.com/calculus/derivatives-rules.html
- *L'Hôpital's rule*: https://www.mathsisfun.com/calculus/l-hopitals-rule.html