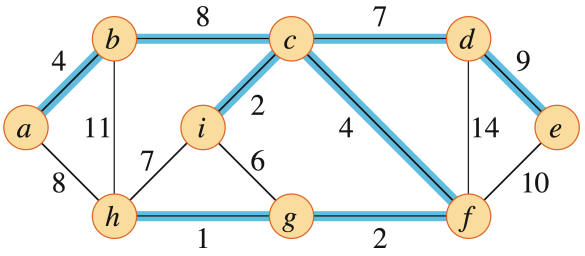
Chapter 21: Minimum Spanning Trees

Given a connected, undirected graph $G =< V, E >$, each edge $(u,v) \in E$ has a weight $w(u,v)$. The minimum spanning tree T is an acyclic subset of E that connects all vertices of V and whose weight $w(T) = \sum_{(u,v)\in V} w(u,v)$ is minimized.

Since T is acyclic and connects all vertices, it is a tree. We call it a minimum-weight spanning tree. It is usually just called a minimum spanning tree.

The two algorithms that we will study both use a greedy approach to the problem but they differ in how they apply this approach.



Growing a Minimum Spanning Tree

- The generic algorithm:

  Loop Invariant: Manage a set A of edges that is always a subset of some minimum spanning tree.

  Initially $A =$. At each step, we add an edge $(u,v) \in E$ into A, where (u,v) is **safe** to A.

  An edge (u,v) is **safe** to A if $A \cup \{(u,v)\}$ is still a subset of some minimum spanning tree.

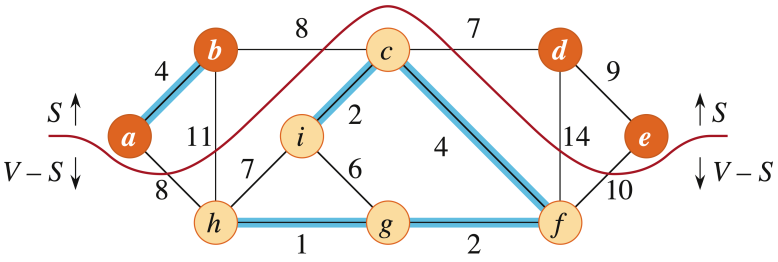
```
Generic-MST(G, w)
1.  A = {}  //empty set
2.  while A does not form a minimum spanning tree
3.        find an edge (u,v) that is safe to A
4.        A = A U {(u,v)}     // U: union
5.  return A
```

- What kind of edges are safe to A?
  - Let us define a **cut** $(S, V-S)$ of an undirected graph $G =< V, E >$ as a partition of V.
  - **An edge** $(u,v) \in E$ **crossing the cut** $(S, V-S)$ if one of its end points is in S and the other one is in $V-S$.
  - **A cut respects a set** A **of edges** if no edge in A crosses the cut.

– An edge is a **light edge crossing a cut** if its weight is the minimum among all edges crossing the cut.

Theorem: *If A is a subset of E that is included in some minimum spanning tree for G. Let $(S, V-S)$ be any cut of G that respects A, and let (u,v) be a light edge crossing $(S, V-S)$. Then, the edge (u,v) is safe to A.*

We will skip the proof of the theorem. The above theorem suggests a way to find a minimum spanning tree. For example:



It is easy for human with eye vision to find a sequence of cuts that respects A. But for a computer algorithm, it is not easy. Any minimum spanning tree algorithm tries to suggest a sequence of cuts respects the growing A.

Prim's Algorithm

The set A maintained is a growing tree. The safe edge added to A at each step is always the least-weight edge crossing the cut $(B, V-B)$, where B is the set of vertices connected by edges in A. The algorithm is greedy because the tree is augmented at each step with an edge that contributes the minimal amount of possible weight.

```
MST-PRIM(G, w, r)  // r: source vertex
1.  for each u in G.V
2.      u.key = infinity
3.      u.parent = NIL
4.  r.key = 0
5.  Q = {}      // Q: priority queue
6.  for each vertex u in G.V
7.      INSERT(Q, u)
8.  while Q != {}
9.      u = EXTRACT-MIN(Q)      // remove from PQ and add u to the tree
10.     for each v in G.Adj[u] //update keys of u's non-tree neighbors
11.        if v in Q and w(u,v) < v.key
12.           v.parent  = u
13.           v.key = w(u,v)
14.           DECREASE-KEY(Q, v, w(u,v))
```

Example:

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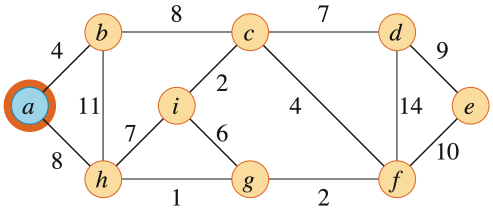

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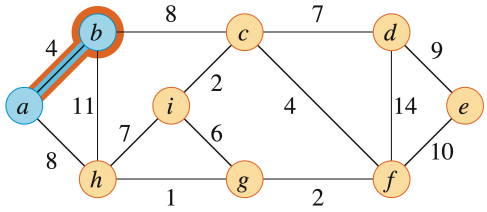

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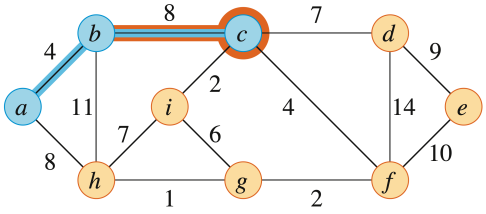

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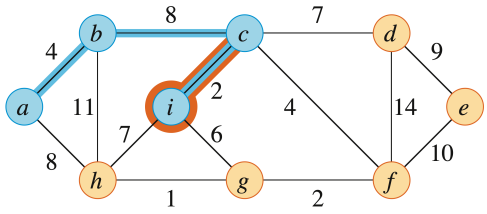

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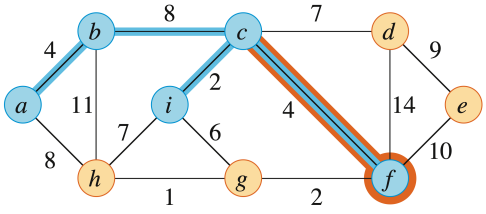

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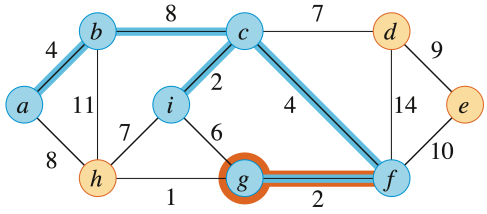

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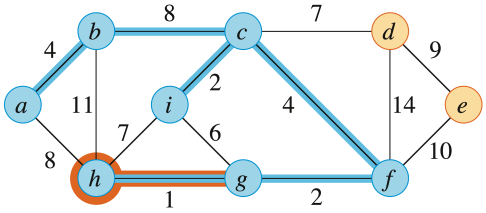

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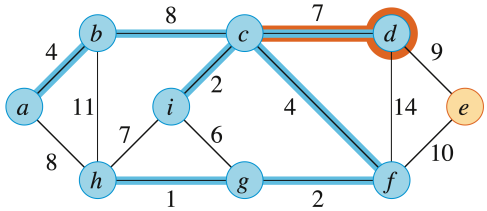

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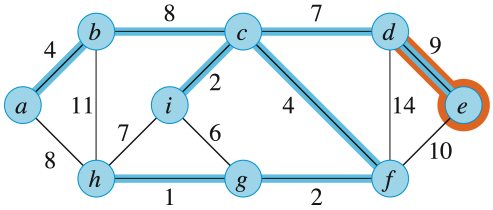


Prim's algorithm uses a priority queue Q based on a key field.

The queue Q maintains all vertices that are still not in the growing tree.

The *key* field for each vertex v in Q, $v.key$, is the weight of the light edge connecting v to the tree.

Actually, the sequence of cuts suggested by Prim is the sequence of cuts that separate the growing tree from the rest of vertices at each step.

Kruskal's Algorithm

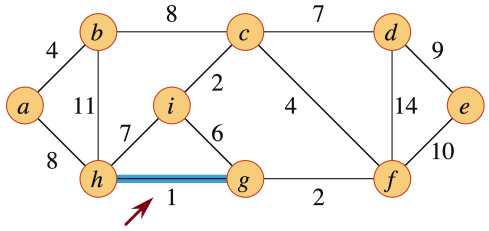
The safe edge added to A at each step is always the least-weight edge in the graph that connects two distinct components.

The algorithm is greedy because at each step it adds to A an edge with the least possible weight.

```
MST-KRUSKAL(G, w)
1.  A = {}
2.  for each vertex v in G.V
3.      do MAKE-SET(v)
4.  create a single list of the edges in G.E
5.  sort the list of edges monotonically increasing weight w
6.  for each edge (u,v) taken from the sorted list in order
7.      do if FIND-SET(u) != FIND-SET(v)
8.            then A = A U {(u,v)}
9.                 UNION(u,v)
10. return A
```

Example:

(a)



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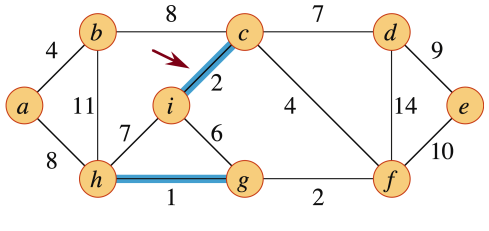

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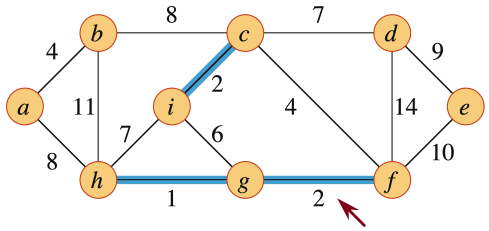

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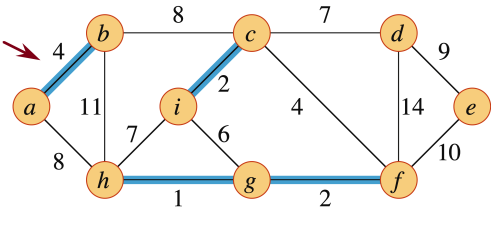

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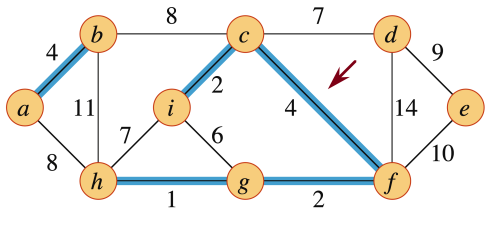

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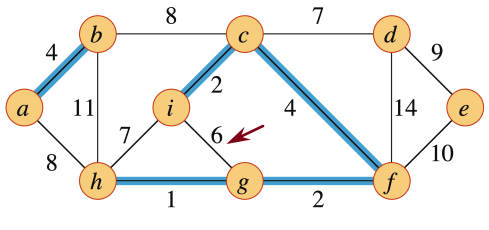

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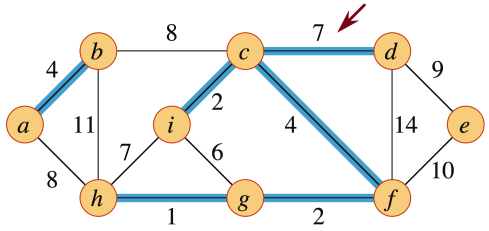

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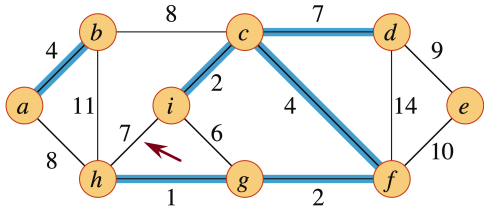

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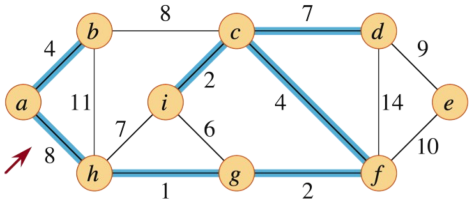

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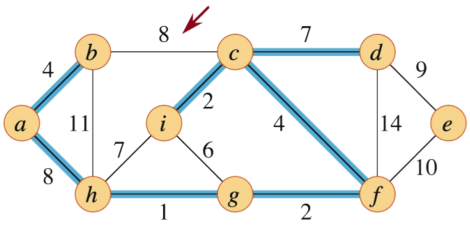

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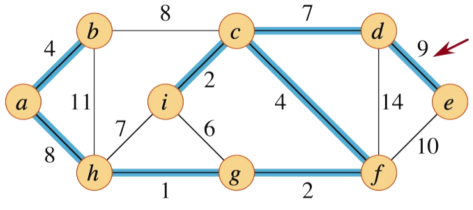

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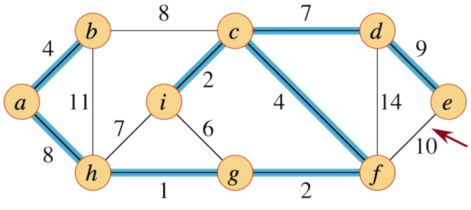

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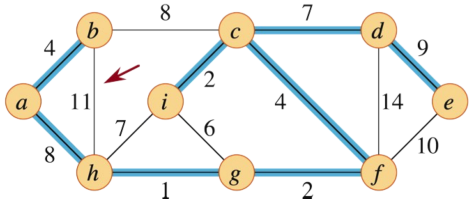

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