

Chapter 23: Minimum Spanning Trees

Given a connected, undirected graph $G = \langle V, E \rangle$, each edge $(u, v) \in E$ has a weight $w(u, v)$. The minimum spanning tree T is an acyclic subset of E that connects all vertices of V and whose weight $w(T) = \sum_{(u,v) \in T} w(u, v)$ is minimized.

Since T is acyclic and connects all vertices, it is a tree. We call it minimum spanning tree.

• Growing a Minimum Spanning Tree:

- The generic algorithm:

Manage a set A of edges that is always a subset of some minimum spanning tree.

Initially $A = \emptyset$. At each step, we add an edge $(u, v) \in E$ into A , where (u, v) is **safe** to A .

An edge (u, v) is safe to A if $A \cup \{(u, v)\}$ is still a subset of some minimum spanning tree.

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Generic-MST( $G, w$ )
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1.  $A \leftarrow \{\}$ 
2. while  $A$  does not form a minimum spanning tree
3.     do find an edge  $(u, v)$  that is safe to  $A$ 
5.      $A \leftarrow A \cup \{(u, v)\}$  //  $\cup$ : union
6. return  $A$ 
```

- What kind of edges are safe to A ?

- * Let a **cut** $(S, V - S)$ of an undirected graph $G = \langle V, E \rangle$ is a partition of V .

- * An edge $(u, v) \in E$ **crossing the cut** $(S, V - S)$ if one of its end points is in S and the other one is in $V - S$.

- * A **cut respects a set A of edges** if no edge in A crosses the cut.

- * An edge is a **light edge crossing a cut** if its weight is the minimum among all edges crossing the cut.

Theorem: *If A is a subset of E that is included in some minimum spanning tree for G . Let $(S, V - S)$ be any cut of G that respects A , and let (u, v) be a light edge crossing $(S, V - S)$. Then, the edge (u, v) is safe to A .*

We ignore the proof of the theorem. The above theorem suggests us a way to find a minimum spanning tree.

Ex:

It is easy for human with eye vision to find a sequence of cuts that respects A . But for a computer algorithm, it is not easy. Any minimum spanning tree algorithm tries to suggest a sequence of cuts respects the growing A .

– Kruskal's Algorithm:

The safe edge added to A at each step is always the least-weight edge in the graph that

connects two distinct components.

The algorithm is greedy because at each step it adds to A an edge with the least possible weight.

MST-Kruskal(G, w)

```
1.  A  $\leftarrow \{\}$ 
2.  for each vertex  $v$  in  $V[G]$ 
3.      do Make-Set( $v$ )
4.  sort the edges of  $E$  by non-decreasing weight  $w$ 
5.  for each edge  $(u,v)$  in  $E$ , in order by non-decreasing weight
6.      do if Find-Set( $u$ )  $\neq$  Find-Set( $v$ )
7.          then A  $\leftarrow A \cup \{(u,v)\}$ 
8.              Union( $u,v$ )
9.  return A
```

Ex:

- Prim's Algorithm:

The set A maintained is a growing tree.

The safe edge added to A at each step is always the least-weight edge crossing the cut $(B, V - B)$, where B is the set of vertices connected by edges in A .

The algorithm is greedy because the tree is augmented at each step with an edge that contributes the minimal amount of possible weight.

```
MST-Prim( $G, w, r$ )  //  $r$ : source vertex
1.   $Q \leftarrow V[G]$       //  $Q$ : priority queue
2.  for each  $u$  in  $Q$ 
3.      do  $key[u] \leftarrow \text{infinity}$ 
4.   $key[r] \leftarrow 0$ 
5.   $p[r] \leftarrow \text{nil}$ 
6.  while  $Q \neq \{\}$ 
7.      do  $u \leftarrow \text{Extract-Min}(Q)$ 
8.          for each  $v$  in  $\text{Adj}[u]$ 
9.              do if  $v$  in  $Q$  and  $w(u, v) < key[v]$ 
10.                  then  $p[v] \leftarrow u$ 
11.                       $key[v] \leftarrow w(u, v)$ 
```

Prim's algorithm uses a priority queue Q based on a key field.

The queue Q maintains all vertices that are still not in the growing tree.

The key field for each vertex v in Q , $key[v]$, is the weight of the light edge connecting v to the tree.

Ex:

Actually, the sequence of cuts suggested by Prim is the sequence of cuts that separate the growing tree from the rest of vertices at each step.