Chapter 2.3: Designing Algorithms and Merge sort

Techniques for Designing Algorithms

- Insertion sort, selection sort, linear search use an incremental algorithm design techniques. These usually result in *iterative* algorithms.
- Recursive algorithms are useful and common and provide a different way of tackling problems. It goes hand-in-hand with the divide-and-conquer algorithm design technique.
- Divide-and-Conquer is an useful recursive technique for designing algorithms. It consists of three steps:

  Divide the problem into a number of subproblems that are smaller instances of the same problem.

  Conquer the subproblems by solving them recursively. If the subproblems are small enough, just solve the problems directly.

  Combine the solutions of the subproblems into the solution for the original problem.

Merge sort: a divide and conquer algorithm

- **Merge sort** is an example of a divide-and-conquer algorithm.

  Divide: the n-element sequence to be sorted into two subsequences of $n/2$ elements each.

  Conquer: sort the two subsequences recursively using merge sort.

  Combine: by merging the two sorted subsequences to produce a sorted answer.

- The base case for the recursion is when the sequence to be sorted has length 1.
- The key part is to merge to two sorted subsequence. Let's examine the merge procedure shown below.

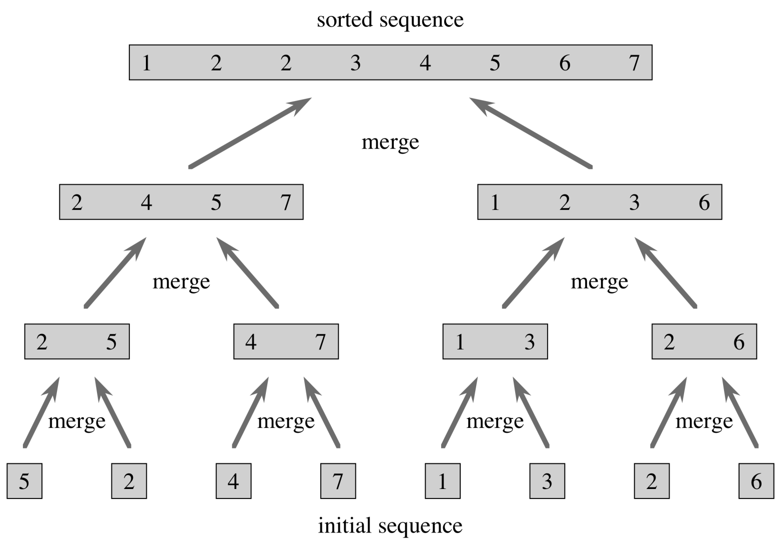
```
MERGE(A, p, q, r)
n_L = q - p + 1          // length of A[p : q]
n_R = r - q              // length of A[q + 1 : r]
let L[0 : n_L - 1] and R[0 : n_R - 1] be new arrays
for i = 0 to n_L - 1     // copy A[p : q] into L[0 : n_L - 1]
    L[i] = A[p + i]
for j = 0 to n_R - 1     // copy A[q + 1 : r] into R[0 : n_R - 1]
    R[j] = A[q + j + 1]
i = 0                    // i indexes the smallest remaining element in L
j = 0                    // j indexes the smallest remaining element in R
k = p                    // k indexes the location in A to fill
// As long as each of the arrays L and R contains an unmerged element,
//     copy the smallest unmerged element back into A[p : r].
while i < n_L and j < n_R
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
    k = k + 1
// Having gone through one of L and R entirely, copy the
//     remainder of the other to the end of A[p : r].
while i < n_L
    A[k] = L[i]
    i = i + 1
    k = k + 1
while j < n_R
    A[k] = R[j]
    j = j + 1
    k = k + 1
```

- Observations about the Merge procedure
  - The worst-case run-time is $\Theta(n)$.
  - The algorithm is **oblivious** in that its run-time doesn't change due to the instance of the problem.

- Now we can write out the pseudo-code for the merge sort algorithm:

```
MERGE-SORT(A, p, r)
if p ≥ r                        // zero or one element?
    return
q = ⌊(p + r)/2⌋                 // midpoint of A[p : r]
MERGE-SORT(A, p, q)             // recursively sort A[p : q]
MERGE-SORT(A, q + 1, r)         // recursively sort A[q + 1 : r]
// Merge A[p : q] and A[q + 1 : r] into A[p : r].
MERGE(A, p, q, r)
```

- Here is an example of running merge sort on the input $A = < 5, 2, 4, 6, 1, 3, 2, 6 >$



- Try Exercise 2.3-1 (on your own): Try merge sort on the input $A = \langle 3, 41, 52, 26, 38, 57, 9, 49 \rangle$.

Analysis of Divide and Conquer Algorithms

- The run time of a divide-and-conquer algorithm can be described by a recurrence equation (or just recurrence). Here is the general form:

  $$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq c, \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

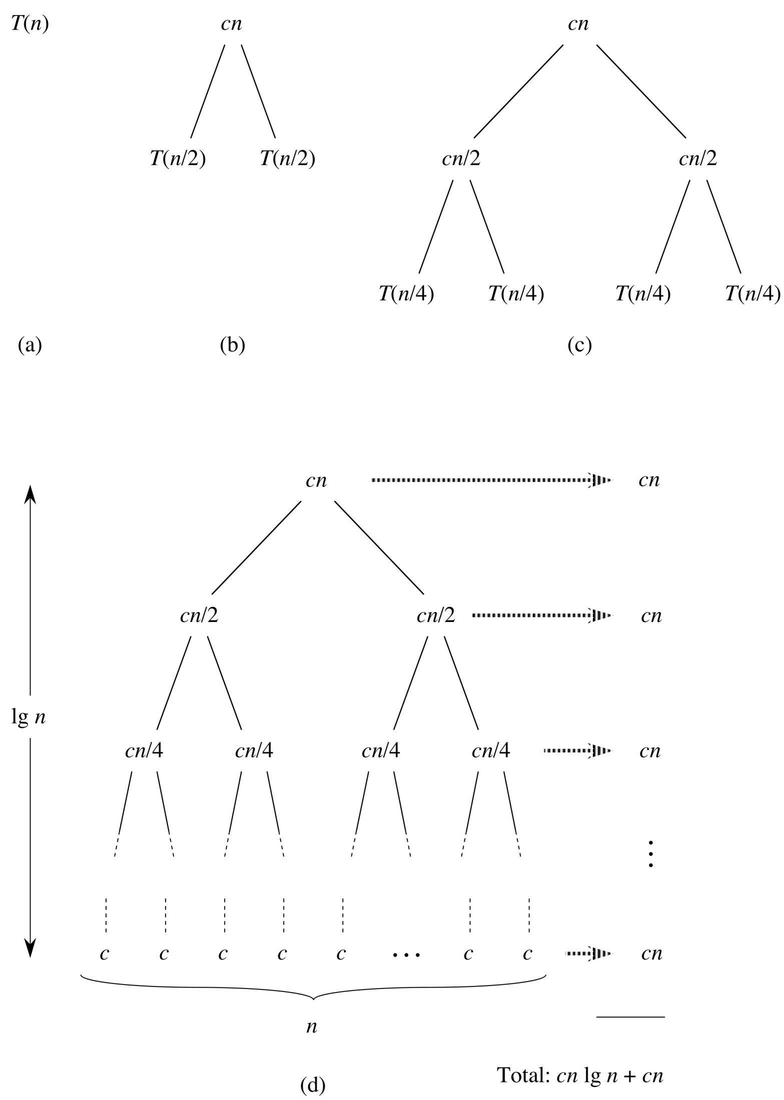
- For merge sort, a = 2 and b = 2.

  $$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1, \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

- Techniques for solving recurrence equations:

  - *guess* and then prove by mathematical induction
  - *substitution method* $\rightarrow$ keep expanding the recurrence until we can come up with the closed form. To be thorough, we would also have to prove using induction that the closed form is correct.
  - *draw a recurrence tree* and then use the tree to add up the run-time

- Running time analysis, by drawing the recurrence tree:



- $\lceil \log_2 n \rceil$ levels of merging in the tree
- Each level takes linear (to n) time
- Thus, total running time is $\Theta(n \log n)$

- **Recommended Exercises**:
  - Ex 2.3-6: Binary Search
  - Ex 2.3-7. Insertion sort combined with binary search
  - Problem 2.1: Merge sort combined with insertion sort

- **Extra reading (or listening!)**: Check out the song "*There's a hole in my bucket*" for recursion without a base case! It even has a wikipedia page for it: `https://en.wikipedia.org/wiki/There%27s_a_Hole_in_My_Bucket`o

Here is a classic performance of the song!

https://www.youtube.com/watch?v=xVAvMIhvqfk&ab_channel=HarryBelafonteTe