Chapter 7: Quicksort

Introduction

- Quicksort is based on divide-and-conquer.
  For an array A[p..r],

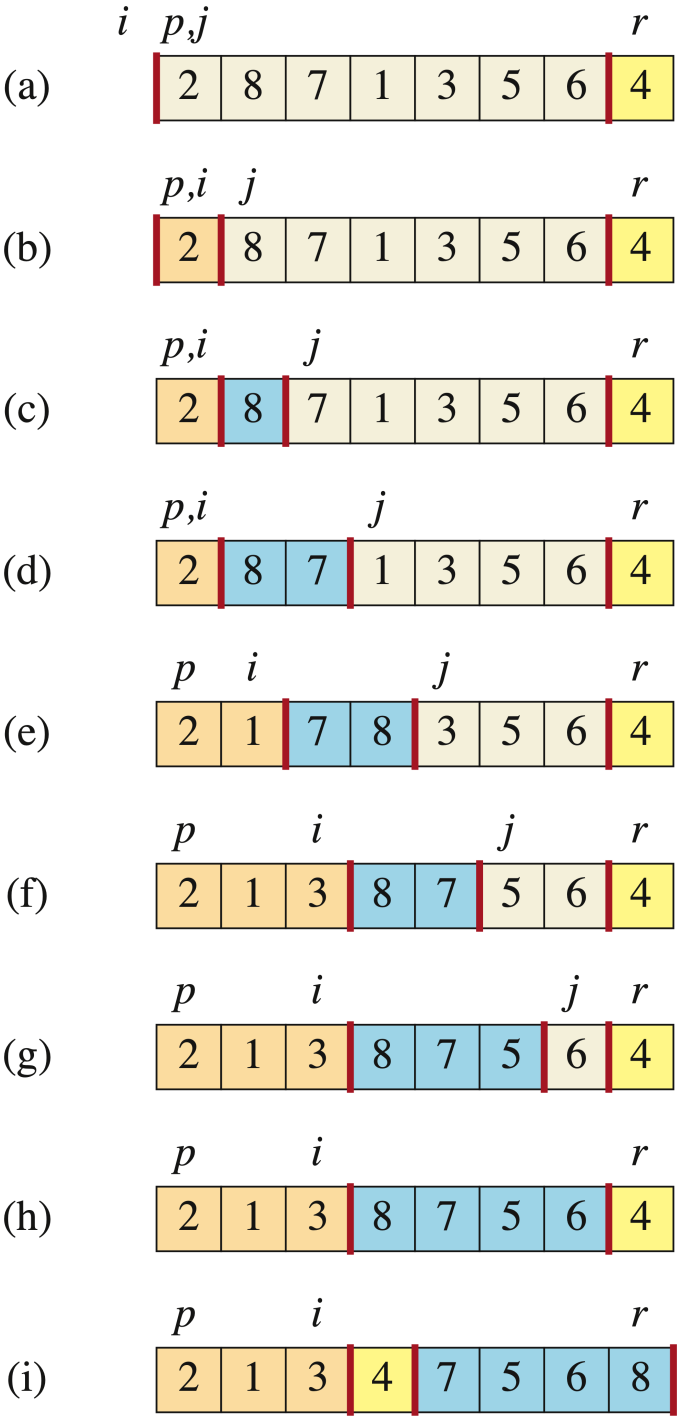
Divide:	Partition the array A[p..r] into two subarrays A[p..q-1] and A[q+1..r] (one of them could be empty) such that each element of A[p..q-1] $\leq$ A[q] and each element of A[q+1..r] $\geq$ A[q]. Index q will be computed and returned by this partition procedure (After partitioning, the element in A[q] is in its correct position)
Conquer:	Sort A[p..q-1] and A[q+1..r] recursively.
Combine:	No action for combine.

  Note that here the effort is in dividing the problem (unlike mergesort where dividing was simple but combining was more difficult).

```
Quicksort(A, p, r)
1. if p < r
2.    // partition the array around the pivot which ends up at A[q]
2.    q = Partition(A, p, r)
3.    Quicksort(A, p, q - 1) // recursively sort the low side
4.    Quicksort(A, q + 1, r) // recursively sort the high side

Partition(A, p, r)
1. x = A[r] // the pivot
2. i = p - 1 //highest index on the low side
3. for j = p to r - 1  //process each element other than the pivot
4.     if A[j] <= x    // does this element belong to the low side
5.          i = i + 1
6.          exchange A[i] and A[j]
7. exchange A[i+1] and A[r] //just to right of the low side
8. return i + 1 // the new index of the pivot
```

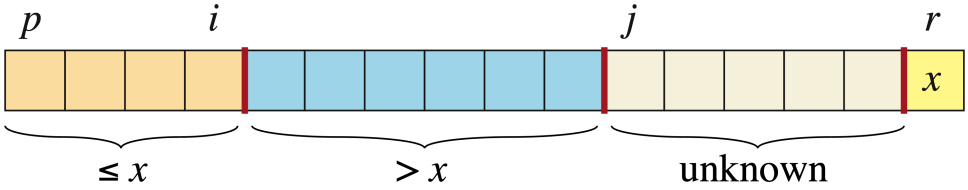
An example:


i p,j r
(a) 2 8 7 1 3 5 6 4
p,i j r
(b) 2 8 7 1 3 5 6 4
p,i j r
(c) 2 8 7 1 3 5 6 4
p,i j r
(d) 2 8 7 1 3 5 6 4
p i j r
(e) 2 1 7 8 3 5 6 4
p i j r
(f) 2 1 3 8 7 5 6 4
p i j r
(g) 2 1 3 8 7 5 6 4
p i r
(h) 2 1 3 8 7 5 6 4
p i r
(i) 2 1 3 4 7 5 6 8

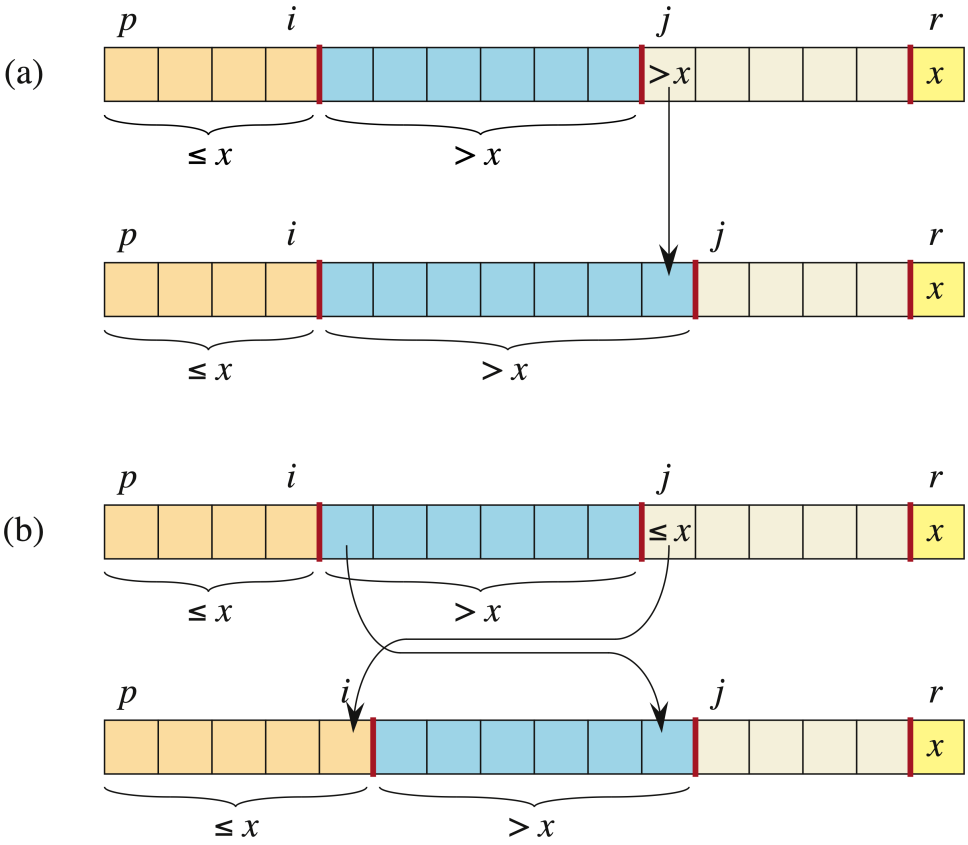
Loop invariant: During the execution of the Partition procedure,

1. Elements in the array before index i are less than or equal to the pivot x. That is,
   $A[k] \leq x$ if $p \leq k \leq i$
2. Elements in the array between the index $i+1$ and $j-1$ are larger than the pivot x. That is,
   $A[k] > x$ if $i+1 \leq k \leq j-1$
3. Elements in the array after index j are not yet compared.

The loop invariant can be illustrated by the following picture:



The following diagram shows the two cases to consider in the loop:



Recommended Exercises: Ex. 7.1-1, 7.1-2, 7.1-3, 7.1-4.

Performance of Quicksort

Depending on whether the partition is "balanced" or not
If balanced: asymptotically as fast as merge sort: $\Theta(n \log n)$.
If not balanced: asymptotically as slow as selection sort: $\Theta(n^2)$.

- Worst case partition:
  This case occurs when partition produces one subarray with $n-1$ elements.
  If this worst case partitioning occurs at each recursive step of the algorithm, then Quicksort takes $T(n) = \Theta(n^2)$ since
  there are n partitions altogether with n, $n-1$, $\ldots, 1$ elements. Thus, the **worst case** running time of Quicksort is $T(n) = n + (n-1) + \ldots + 1 = \Theta(n^2)$.
  For example, if the input array is already sorted in either order.
- Best case partition:
  This case occurs when partition produces two subarrays with $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil - 1$ elements. If this best case partitioning occurs at each recursive step of the algorithm, then Quicksort requires the following run time (we are simplifying $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil - 1$ to n/2 each):

  $$T(n) = 2T(n/2) + \Theta(n)$$

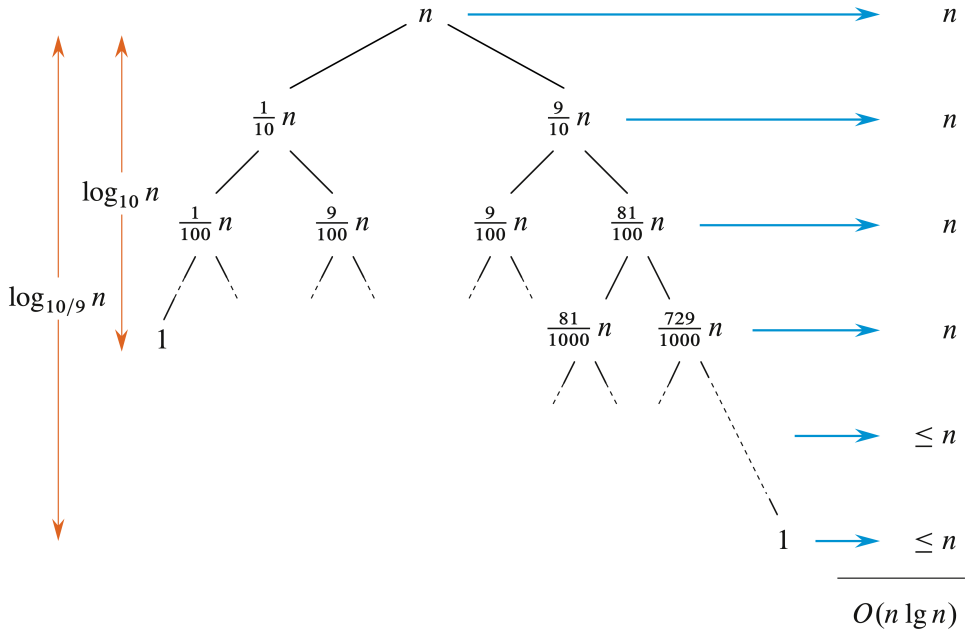
  This is the same as merge sort, so we have $T(n) = \Theta(n \lg n)$ (which we analyzed earlier using a recurrence tree).
  Let us also calculate the run-time directly using substitution.
  There is 1 partition with n elements, 2 partitions with $\frac{n}{2}$ elements, 4 partitions with $\frac{n}{4}$ elements, $\ldots$, n partitions with $\frac{n}{n}$ elements. Thus, the **best case** running time of Quicksort is

  $$\begin{aligned} T(n) &= 1 \cdot \tfrac{n}{1} + 2 \cdot \tfrac{n}{2} + \ldots + n \cdot \tfrac{n}{n} \\ &= 2^0 \cdot \tfrac{n}{2^0} + 2^1 \cdot \tfrac{n}{2^1} + \ldots + 2^{\log n} \cdot \tfrac{n}{2^{\log n}} \\ &= n + n + \ldots + n \quad (n \text{ adds itself } \log n \text{ times}) \\ &= n \log n \end{aligned}$$

- Average case running time of Quicksort:
  The **average case** running time of Quicksort is $\Theta(n \log n)$.
  The average case running time analysis of quicksort will be discussed in CS421 with the knowledge of divide-and-conquer.
  We will provide some intuition with an example of an uneven split. Suppose partition always splits the given subarray into a 9-1 proportional split. For n elements, one side is $9n/10$ and the other side is $n/10$. That gives us a run time:

  $$T(n) = T(9n/10) + T(n/10) + cn$$

The following figure shows the recurrence tree.



Any split of constant proportionality results in a running time of $O(n \lg n)$.
Recommended Exercises: Ex 7.2-1, 7.2-2, 7.2-3, 7.2-4.

Randomized Versions of Quicksort

For average-case analysis, we assume that all permutations of the input numbers are equally likely. However, this assumption is not realistic because:

- Two special inputs impose the worst-case split: sorted array in either order.
- These two inputs are very common for lots of applications.

We need to randomize the input array to reduce the probability of the worst-case. Two approaches to randomize the input array.

1. Before feeding the input to Quicksort algorithm, the input is randomly permuted.
2. Randomly choose the pivot element at each step in Quicksort.

The pseudocode for the 2nd approach.

```
Randomized-Partition(A, p, r)
1. i = Random(p, r)
2. exchange A[r] and A[i]
3. Partition(A, p, r)
```

```
Randomized-Quicksort(A, p, r)
1. if p < r
2.    q = Randomized-Partition(A, p, r)
3.    Randomized-Quicksort(A, p, q-1)
4.    Randomized-Quicksort(A, q+1, r)
```

Recommended Exercises

- Interesting **bolts and nuts problem:**
  There are n pairs of bolts and nuts mixed together. Each bolt has one, and only one, matching nut. Please suggest an efficient algorithm to match all bolts and nuts (A bolt cannot be compared to another bolt. Similarly, a nut cannot be compared to another nut).