

# AVL Trees - a Balanced Binary Search Tree

## Balanced Binary Search Trees

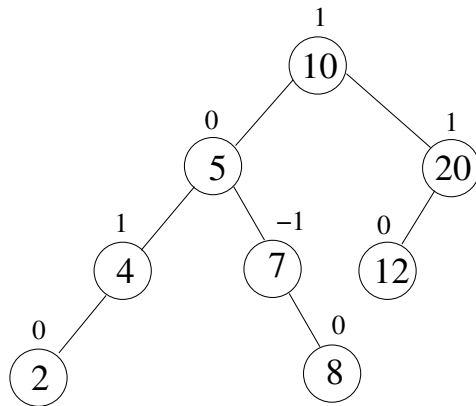
- **What is a Balanced Binary Search Trees:**

It is a binary search tree with height  $\Theta(\log n)$ , where  $n$  is the # of nodes in the tree.

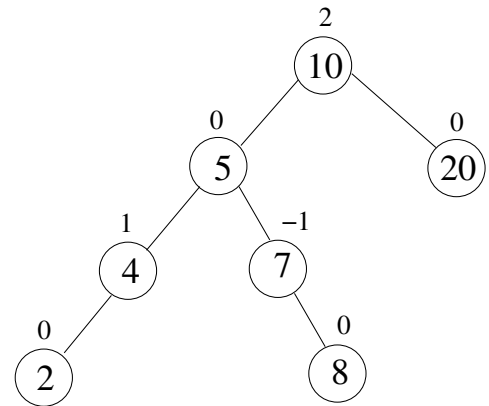
- **AVL Trees:**

- An AVL tree is a BST.
- For each node, the difference between the height of its left and right subtrees is either  $+1, 0$ , or  $-1$ .

Ex:



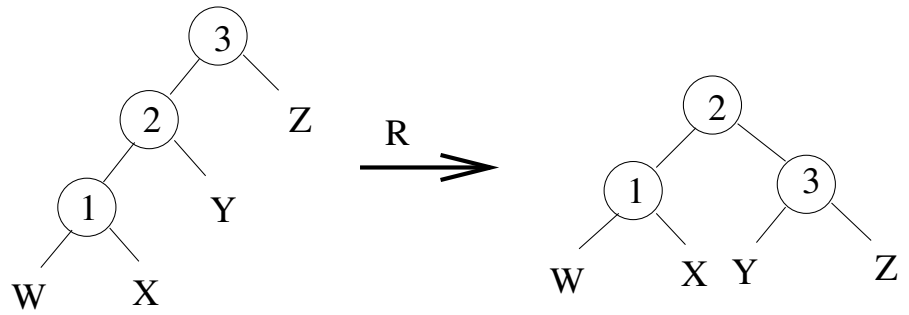
An AVL tree



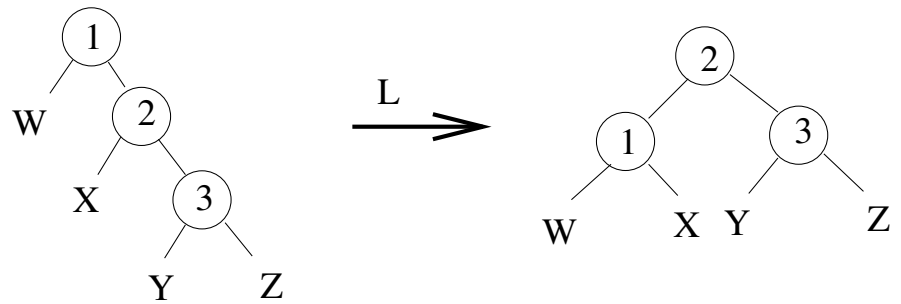
not an AVL tree

- Technique to maintain AVL tree: 4 different rotations.  
See next page.

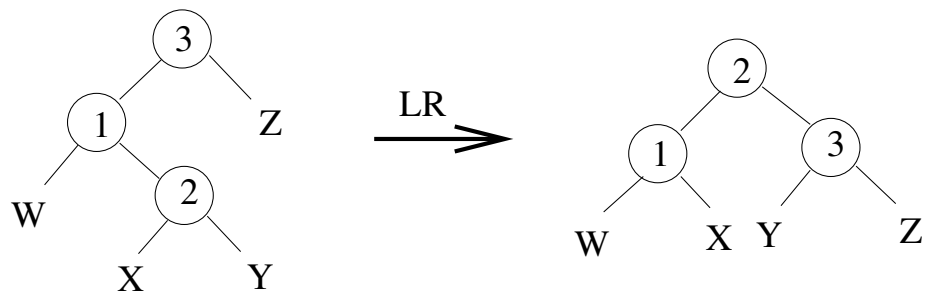
R Rotation:



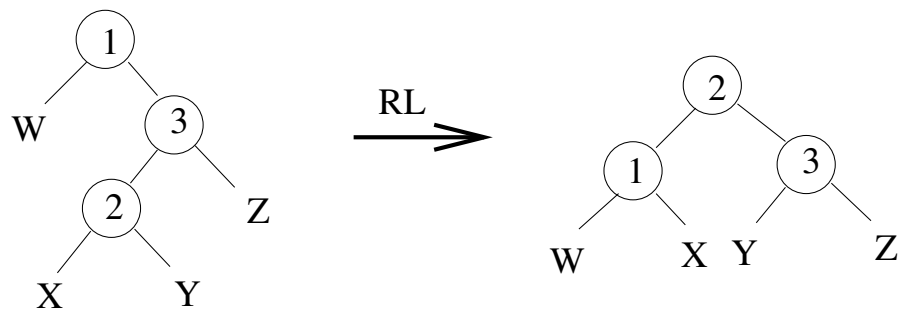
L Rotations:



LR Rotation:



RL Rotation:



– **AVL Insertion:** Two steps.

1. Perform the  $\text{BST-Insert}(T, z)$ .

2. Perform re-structuring through rotations if necessary.

\* A rotation is performed when a subtree rooted at a node whose **balanced factor** becomes  $+2$  or  $-2$  after  $\text{BST-Insert}$ . If there are several such unbalanced nodes, we rotate at a node  $A$  that is closest to the newly inserted leaf.

\* To decide which rotation to use, we start from the node  $A$  and test the balanced factor in the following way.

$+2 \xrightarrow{L} +1/0 \xrightarrow{L}$ : R Rotation.

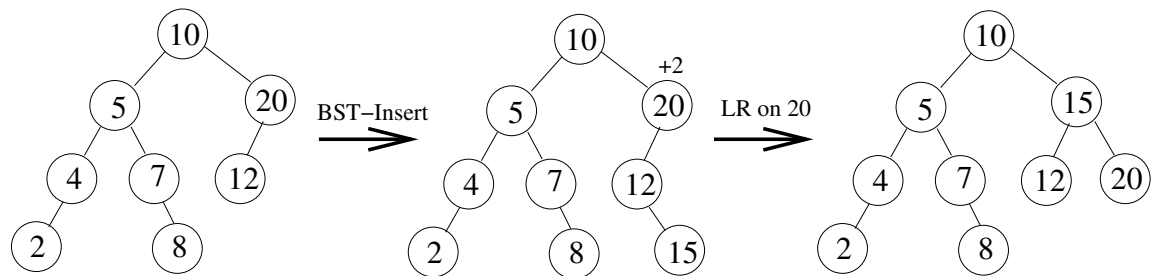
$+2 \xrightarrow{L} -1 \xrightarrow{R}$ : LR Rotation.

$-2 \xrightarrow{R} +1 \xrightarrow{L}$ : RL Rotation.

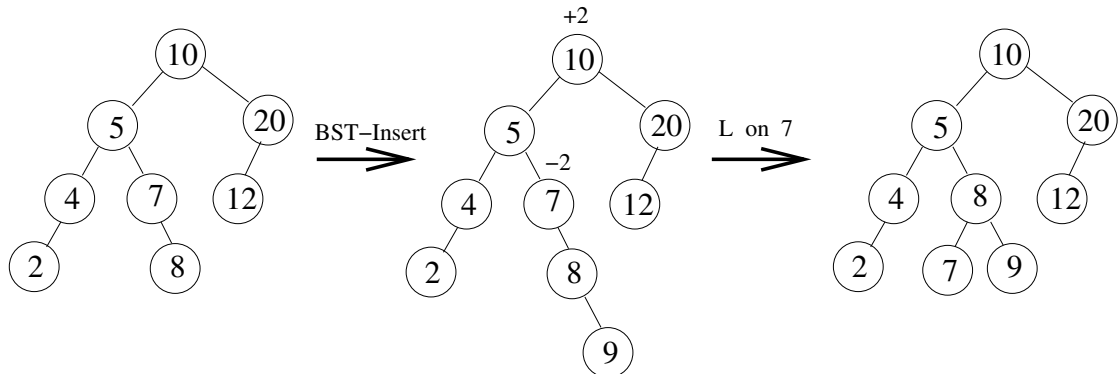
$-2 \xrightarrow{R} -1/0 \xrightarrow{R}$ : L Rotation.

Ex:

Insert 15:



Insert 9:



– **AVL Deletion:** Two steps.

1. Perform the BST-Delete( $T, z$ ).

2. Perform re-structuring through rotations if necessary.

\* A rotation is performed when a subtree rooted at a node whose **balanced factor** becomes  $+2$  or  $-2$  after BST-Insert. If there are several such unbalanced nodes, we rotate at a node  $A$  that is closest to the newly deleted node (i.e., the node has been physically removed).

\* To decide which rotation to use, we start from the node  $A$  and test the balanced factor in the following way.

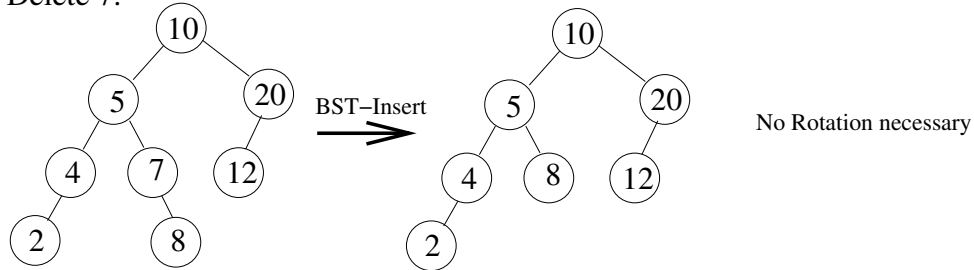
$+2 \xrightarrow{L} +1/0 \xrightarrow{L}$ : R Rotation.

$+2 \xrightarrow{L} -1 \xrightarrow{R}$ : LR Rotation.

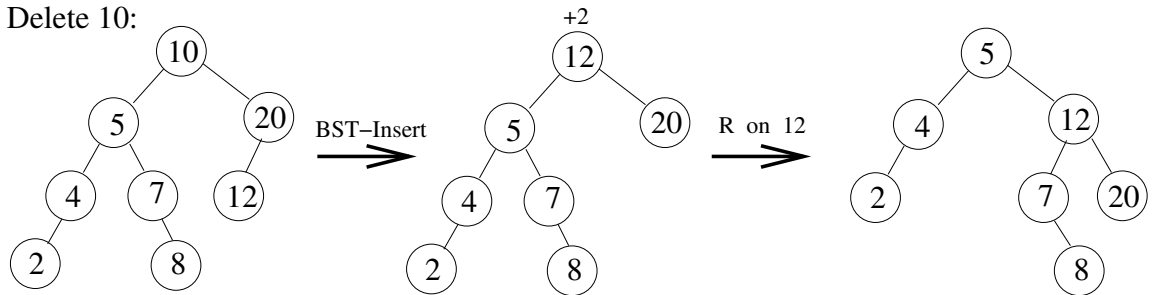
$-2 \xrightarrow{R} +1 \xrightarrow{L}$ : RL Rotation.

$-2 \xrightarrow{R} -1/0 \xrightarrow{R}$ : L Rotation.

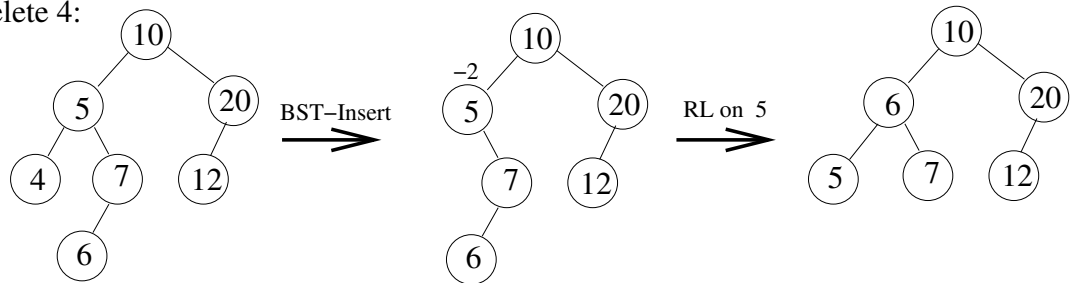
Delete 7:



Delete 10:



Delete 4:



Ex:

– **Height of AVL Trees:**  $\Theta(\log n)$

Given an AVL tree with height  $h$ .

The maximum # of nodes: the tree is full.

$$\begin{aligned} n &\leq 2^0 + 2^1 + \dots + 2^h = \sum_{i=0}^h 2^i = 2^{h+1} - 1 \\ \implies n &\leq 2^{h+1} - 1 \\ \implies h &\geq \log(n+1) - 1 \\ &= \Omega(\log n) \end{aligned}$$

The minimum # of nodes:

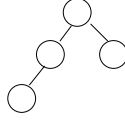
$h = 0$



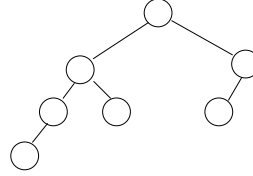
$h = 1$



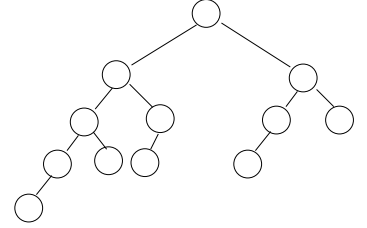
$h = 2$



$h = 3$



$h = 4$



Let  $n_h$  be the minimum # of nodes of an AVL tree with height  $h$

$$n_h = \begin{cases} 1 & \text{if } h = 0 \\ 2 & \text{if } h = 1 \\ n_{h-1} + n_{h-2} + 1 & \text{if } h \geq 2 \end{cases}$$

Recall the Fibonacci numbers as follows.

$$F_h = \begin{cases} 0 & \text{if } h = 0 \\ 1 & \text{if } h = 1 \\ F_{h-1} + F_{h-2} & \text{if } h \geq 2 \end{cases}$$

Thus, we have  $n_h > F_h, \forall h \geq 0$ .

Since  $F_h = \frac{\phi^h - \bar{\phi}^h}{\sqrt{5}}$ , where  $\phi = 1.61803$  and  $\bar{\phi} = -0.61803$ .

$$\begin{aligned} n_h &> F_h = \frac{\phi^h - \bar{\phi}^h}{\sqrt{5}} \simeq \frac{\phi^h}{\sqrt{5}} \quad \text{if } h \text{ is large} \\ \implies n &\geq n_h > \frac{\phi^h}{\sqrt{5}} \\ \implies \log_{\phi}(\sqrt{5} \cdot n) &> h \\ \implies h &< \log_{\phi} \sqrt{5} + \log_{\phi} n \\ &= O(\log n) \end{aligned}$$

Therefore, the height of any AVL tree is  $\Theta(\log n)$ .