CHAPTER 15

Time Series Analysis and Forecasting

CONTENTS

A forecast is simply a prediction of what will happen in the future. Managers must learn to accept the fact that regardless of the technique used, they will not be able to develop perfect forecasts.

The purpose of this chapter is to provide an introduction to time series analysis and forecasting. Suppose we are asked to provide quarterly forecasts of sales for one of our company's products over the coming one-year period. Production schedules, raw material purchasing, inventory policies, and sales quotas will all be affected by the quarterly forecasts we provide. Consequently, poor forecasts may result in poor planning and increased costs for the company. How should we go about providing the quarterly sales forecasts? Good judgment, intuition, and an awareness of the state of the economy may give us a rough idea or "feeling" of what is likely to happen in the future, but converting that feeling into a number that can be used as next year's sales forecast is difficult.

Forecasting methods can be classified as qualitative or quantitative. Qualitative methods generally involve the use of expert judgment to develop forecasts. Such methods are appropriate when historical data on the variable being forecast are either not applicable or unavailable. Quantitative forecasting methods can be used when (1) past information about the variable being forecast is available, (2) the information can be quantified, and (3) it is reasonable to assume that the pattern of the past will continue into the future. We will focus exclusively on quantitative forecasting methods in this chapter.

If the historical data are restricted to past values of the variable to be forecast, the forecasting procedure is called a time series method and the historical data are referred to as a time series. The objective of time series analysis is to discover a pattern in the historical data or time series and then extrapolate the pattern into the future; the forecast is based solely on past values of the variable and/or on past forecast errors.

In Section 15.1 we discuss the various kinds of time series that a forecaster might be faced with in practice. These include a constant or horizontal pattern, trends, seasonal patterns, both a trend and a seasonal pattern, and cyclical patterns. In order to build a quantitative forecasting model, it is necessary to have a measurement of forecast accuracy. Different measurements, and their respective advantages and disadvantages, are discussed in Section 15.2. In Section 15.3 we consider the simplest case, which is a horizontal or constant pattern. For this pattern, we develop the classical moving average and exponential smoothing models. We show how the best parameters can be selected using an optimization model, which provides a good application of the optimization tools developed in Chapters 2 through 8. Many time series have a trend, and taking this trend into account is important. In Section 15.4 we give optimization models for finding the best model parameters when a trend is present. Finally, in Section 15.5 we show how to incorporate both a trend and seasonality into a forecasting model.

MANAGEMENT SCIENCE IN ACTION

FORECASTING ENERGY NEEDS IN THE UTILITY INDUSTRY*

Duke Energy is a diversified energy company with a portfolio of natural gas and electric businesses and an affiliated real estate company. In 2006, Duke Energy merged with Cinergy of Cincinnati, Ohio, to create one of North America's largest energy companies, with assets totaling more than $70 billion. As a result of this merger the Cincinnati Gas & Electric Company became part of Duke Energy. Today, Duke Energy services over 5.5 million retail electric and gas customers in North Carolina, South Carolina, Ohio, Kentucky, Indiana, and Ontario, Canada.

Forecasting in the utility industry offers some unique perspectives. Because electricity cannot take the form of finished goods or in-process inventories, this product must be generated to meet the instantaneous requirements of the customers. Electrical shortages are not just lost sales, but "brownouts" or "blackouts." This situation places an unusual burden on the utility forecaster. On the positive side, the demand for energy and the sale of energy are more predictable than for many other products. Also, unlike the situation in a multiproduct firm, a great amount of forecasting effort and expertise can be concentrated on the two products: gas and electricity.

The largest observed electric demand for any given period, such as an hour, a day, a month, or a year, is defined as the peak load. The forecast of the annual electric peak load guides the timing decision for constructing future generating units, and the financial impact of this decision is great. Obviously, a timing decision that leads to having the unit available no sooner than necessary is crucial.

The energy forecasts are important in other ways also. For example, purchases of coal as fuel for the generating units are based on the forecast levels of energy needed. The revenue from the electric operations of the company is determined from forecasted sales, which in turn enters into the planning of rate changes and external financing. These planning and decision-making processes are among the most important managerial activities in the company. It is imperative that the decision makers have the best forecast information available to assist them in arriving at these decisions.

*Based on information provided by Dr. Richard Evans of Cincinnati Gas & Electric Company, Cincinnati, Ohio.

15.1 TIME SERIES PATTERNS

A time series is a sequence of observations on a variable measured at successive points in time or over successive periods of time. The measurements may be taken every hour, day, week, month, or year, or at any other regular interval.[1] The pattern of the data is an important factor in understanding how the time series has behaved in the past. If such behavior can be expected to continue in the future, we can use it to guide us in selecting an appropriate forecasting method.

To identify the underlying pattern in the data, a useful first step is to construct a time series plot. A time series plot is a graphical presentation of the relationship between time and the time series variable; time is on the horizontal axis and the time series values are shown on the vertical axis. Let us review some of the common types of data patterns that can be identified when examining a time series plot.

Horizontal Pattern

A horizontal pattern exists when the data fluctuate around a constant mean. To illustrate a time series with a horizontal pattern, consider the 12 weeks of data in Table 15.1. These data show the number of gallons of gasoline sold by a gasoline distributor in Bennington, Vermont, over the past 12 weeks. The average value or mean for this time series is 19.25, or 19,250 gallons per week. Figure 15.1 shows a time series plot for these data. Note how the data fluctuate around the sample mean of 19,250 gallons. Although random variability is present, we would say that these data follow a horizontal pattern.

The term stationary time series[2] is used to denote a time series whose statistical properties are independent of time. In particular this means that

1. The process generating the data has a constant mean.
2. The variability of the time series is constant over time.

[1]We limit our discussion to time series in which the values of the series are recorded at equal intervals. Cases in which the observations are made at unequal intervals are beyond the scope of this text.

[2]For a formal definition of stationarity see Box, G. E. P., G. M. Jenkins, and G. C. Reinsell (1994), *Time Series Analysis: Forecasting and Control,* 3rd ed., Prentice-Hall, p. 23.

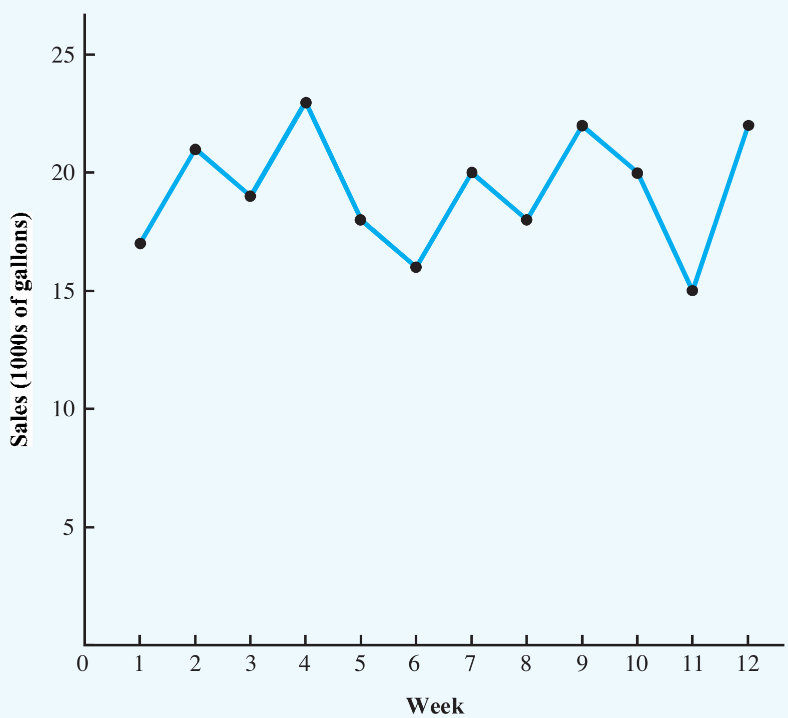
TABLE 15.1 GASOLINE SALES TIME SERIES

WEB file
Gasoline

Week	Sales (1000s of gallons)
1	17
2	21
3	19
4	23
5	18
6	16
7	20
8	18
9	22
10	20
11	15
12	22

A time series plot for a stationary time series will always exhibit a horizontal pattern. But simply observing a horizontal pattern is not sufficient evidence to conclude that the time series is stationary. More advanced texts on forecasting discuss procedures for determining whether a time series is stationary and provide methods for transforming a time series that is not stationary into a stationary series.

FIGURE 15.1 GASOLINE SALES TIME SERIES PLOT



Changes in business conditions can often result in a time series that has a horizontal pattern shifting to a new level. For instance, suppose the gasoline distributor signs a contract with the Vermont Sate Police to provide gasoline for state police cars located in southern Vermont. With this new contract, the distributor expects to see a major increase in weekly sales starting in week 13. Table 15.2 shows the number of gallons of gasoline sold for the original time series and the 10 weeks after signing the new contract. Figure 15.2 shows the corresponding time series plot. Note the increased level of the time series beginning in week 13. This change in the level of the time series makes it more difficult to choose an appropriate forecasting method. Selecting a forecasting method that adapts well to changes in the level of a time series is an important consideration in many practical applications.

Trend Pattern

Although time series data generally exhibit random fluctuations, a time series may also show gradual shifts or movements to relatively higher or lower values over a longer period of time. If a time series plot exhibits this type of behavior, we say that a trend pattern exists. Trend is usually the result of long-term factors such as population increases or decreases, changing demographic characteristics of the population, technology, and/or consumer preferences.

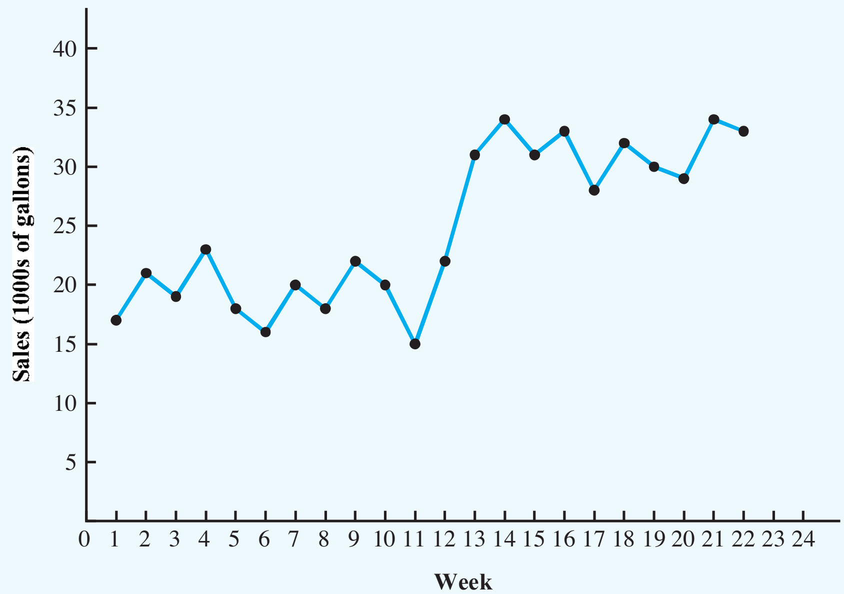
TABLE 15.2 GASOLINE SALES TIME SERIES AFTER OBTAINING THE CONTRACT WITH THE VERMONT STATE POLICE

WEB file

GasolineRevised

Week	Sales (1000s of gallons)
1	17
2	21
3	19
4	23
5	18
6	16
7	20
8	18
9	22
10	20
11	15
12	22
13	31
14	34
15	31
16	33
17	28
18	32
19	30
20	29
21	34
22	33

FIGURE 15.2 GASOLINE SALES TIME SERIES PLOT AFTER OBTAINING THE CONTRACT WITH THE VERMONT STATE POLICE



To illustrate a time series with a trend pattern, consider the time series of bicycle sales for a particular manufacturer over the past 10 years, as shown in Table 15.3 and Figure 15.3. Note that 21,600 bicycles were sold in year 1, 22,900 were sold in year 2, and so on. In year 10, the most recent year, 31,400 bicycles were sold. Visual inspection of the time series plot shows some up and down movement over the past 10 years, but the time series seems to also have a systematically increasing or upward trend.

The trend for the bicycle sales time series appears to be linear and increasing over time, but sometimes a trend can be described better by other types of patterns. For instance, the

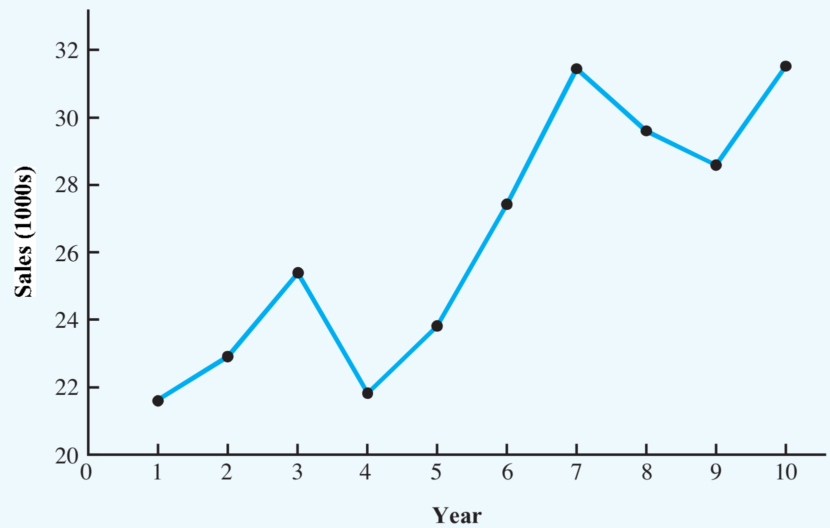
TABLE 15.3 BICYCLE SALES TIME SERIES

Year	Sales (1000s)
1	21.6
2	22.9
3	25.5
4	21.9
5	23.9
6	27.5
7	31.5
8	29.7
9	28.6
10	31.4

WEB file

Bicycle

FIGURE 15.3 BICYCLE SALES TIME SERIES PLOT



data in Table 15.4 and the corresponding time series plot in Figure 15.4 show the sales revenue for a cholesterol drug since the company won FDA approval for it 10 years ago. The time series increases in a nonlinear fashion; that is, the rate of change of revenue does not increase by a constant amount from one year to the next. In fact the revenue appears to be growing in an exponential fashion. Exponential relationships such as this are appropriate when the percentage change from one period to the next is relatively constant.

Seasonal Pattern

The trend of a time series can be identified by analyzing multiyear movements in historical data. Seasonal patterns are recognized by seeing the same repeating patterns over successive periods of time. For example, a manufacturer of swimming pools expects low sales

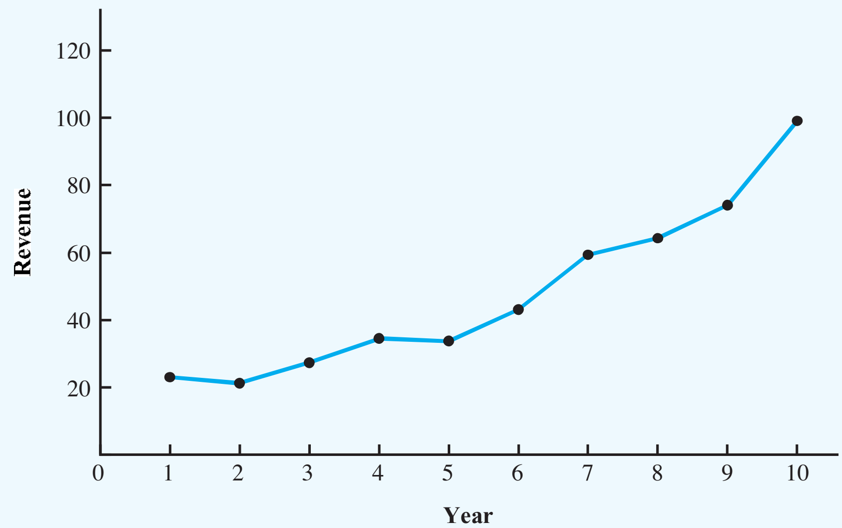
TABLE 15.4 CHOLESTEROL REVENUE TIME SERIES ($ MILLIONS)

WEB file

Cholesterol

Year	Revenue
1	23.1
2	21.3
3	27.4
4	34.6
5	33.8
6	43.2
7	59.5
8	64.4
9	74.2
10	99.3

FIGURE 15.4 CHOLESTEROL REVENUE TIMES SERIES PLOT ($ MILLIONS)



activity in the fall and winter months, with peak sales in the spring and summer months. Manufacturers of snow removal equipment and heavy clothing, however, expect just the opposite yearly pattern. Not surprisingly, the pattern for a time series plot that exhibits a repeating pattern over a one-year period due to seasonal influences is called a seasonal pattern. Although we generally think of seasonal movement in a time series as occurring within one year, time series data can also exhibit seasonal patterns of less than one year in duration. For example, daily traffic volume shows within-the-day "seasonal" behavior, with peak levels occurring during rush hours, moderate flow during the rest of the day and early evening, and light flow from midnight to early morning.

As an example of a seasonal pattern, consider the number of umbrellas sold at a clothing store over the past five years. Table 15.5 shows the time series and Figure 15.5 shows the corresponding time series plot. The time series plot does not indicate any long-term trend in sales. In fact, unless you look carefully at the data, you might conclude that the data follow a horizontal pattern. But closer inspection of the time series plot reveals a regular pattern in the data. That is, the first and third quarters have moderate sales, the second quarter has the highest sales, and the fourth quarter tends to have the lowest sales volume. Thus, we would conclude that a quarterly seasonal pattern is present.

Trend and Seasonal Pattern

Some time series include a combination of a trend and seasonal pattern. For instance, the data in Table 15.6 and the corresponding time series plot in Figure 15.6 show television set sales for a particular manufacturer over the past four years. Clearly, an increasing trend is present. But Figure 15.6 also indicates that sales are lowest in the second quarter of each year and increase in quarters 3 and 4. Thus, we conclude that a seasonal pattern also exists for television set sales. In such cases we need to use a forecasting method that has the capability to deal with both trend and seasonality.

TABLE 15.5 UMBRELLA SALES TIME SERIES

WEB file

Umbrella

Year	Quarter	Sales
1	1	125
	2	153
	3	106
	4	88
2	1	118
	2	161
	3	133
	4	102
3	1	138
	2	144
	3	113
	4	80
4	1	109
	2	137
	3	125
	4	109
5	1	130
	2	165
	3	128
	4	96

FIGURE 15.5 UMBRELLA SALES TIME SERIES PLOT

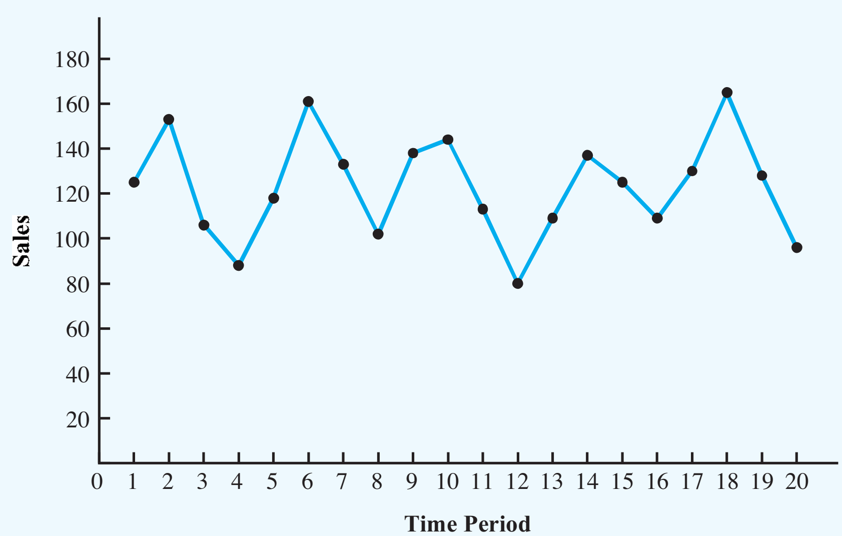
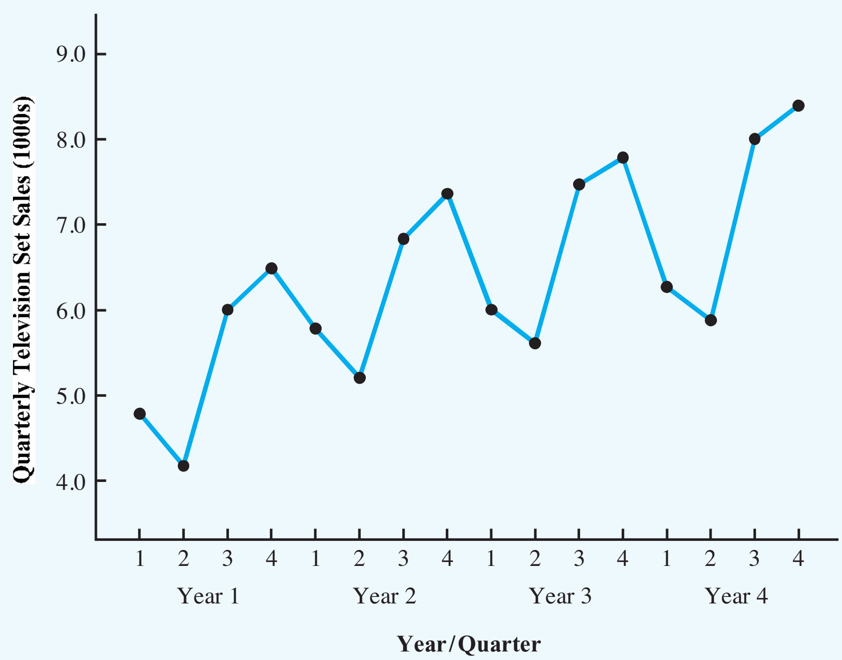


TABLE 15.6 QUARTERLY TELEVISION SET SALES TIME SERIES

WEB file
TVSales

Year	Quarter	Sales (1000s)
1	1	4.8
	2	4.1
	3	6.0
	4	6.5
2	1	5.8
	2	5.2
	3	6.8
	4	7.4
3	1	6.0
	2	5.6
	3	7.5
	4	7.8
4	1	6.3
	2	5.9
	3	8.0
	4	8.4

FIGURE 15.6 QUARTERLY TELEVISION SET SALES TIME SERIES PLOT



Cyclical Pattern

A cyclical pattern exists if the time series plot shows an alternating sequence of points below and above the trend line lasting more than one year. Many economic time series exhibit cyclical behavior with regular runs of observations below and above the trend line. Often, the cyclical component of a time series is due to multiyear business cycles. For example, periods of moderate inflation followed by periods of rapid inflation can lead to time series that alternate below and above a generally increasing trend line (e.g., a time series for housing costs). Business cycles are extremely difficult, if not impossible, to forecast. As a result, cyclical effects are often combined with long-term trend effects and referred to as trend-cycle effects. In this chapter we do not deal with cyclical effects that may be present in the time series.

Selecting a Forecasting Method

The underlying pattern in the time series is an important factor in selecting a forecasting method. Thus, a time series plot should be one of the first things developed when trying to determine which forecasting method to use. If we see a horizontal pattern, then we need to select a method appropriate for this type of pattern. Similarly, if we observe a trend in the data, then we need to use a forecasting method that has the capability to handle trends effectively. The next two sections illustrate methods that can be used in situations where the underlying pattern is horizontal; in other words, no trend or seasonal effects are present. We then consider methods appropriate when trend and or seasonality are present in the data.

15.2 FORECAST ACCURACY

In this section we begin by developing forecasts for the gasoline time series shown in Table 15.1, using the simplest of all the forecasting methods, an approach that uses the most recent week's sales volume as the forecast for the next week. For instance, the distributor sold 17 thousand gallons of gasoline in week 1; this value is used as the forecast for week 2. Next, we use 21, the actual value of sales in week 2, as the forecast for week 3, and so on. The forecasts obtained for the historical data using this method are shown in Table 15.7 in the column labeled Forecast. Because of its simplicity, this method is often referred to as a naïve forecasting method.

How accurate are the forecasts obtained using this naïve forecasting method? To answer this question, we will introduce several measures of forecast accuracy. These measures are used to determine how well a particular forecasting method is able to reproduce the time series data that are already available. By selecting the method that has the best accuracy for the data already known, we hope to increase the likelihood that we will obtain better forecasts for future time periods.

The key concept associated with measuring forecast accuracy is forecast error, defined as follows:

$$\text{Forecast Error} = \text{Actual Value} - \text{Forecast}$$

For instance, because the distributor actually sold 21 thousand gallons of gasoline in week 2 and the forecast, using the sales volume in week 1, was 17 thousand gallons, the forecast error in week 2 is

$$\text{Forecast Error in week 2} = 21 - 17 = 4$$

TABLE 15.7 COMPUTING FORECASTS AND MEASURES OF FORECAST ACCURACY USING THE MOST RECENT VALUE AS THE FORECAST FOR THE NEXT PERIOD

Week	Time Series Value	Forecast	Forecast Error	Absolute Value of Forecast Error	Squared Forecast Error	Percentage Error	Absolute Value of Percentage Error
1	17						
2	21	17	4	4	16	19.05	19.05
3	19	21	−2	2	4	−10.53	10.53
4	23	19	4	4	16	17.39	17.39
5	18	23	−5	5	25	−27.78	27.78
6	16	18	−2	2	4	−12.50	12.50
7	20	16	4	4	16	20.00	20.00
8	18	20	−2	2	4	−11.11	11.11
9	22	18	4	4	16	18.18	18.18
10	20	22	−2	2	4	−10.00	10.00
11	15	20	−5	5	25	−33.33	33.33
12	22	15	7	7	49	31.82	31.82
		Totals	5	41	179	1.19	211.69

The fact that the forecast error is positive indicates that in week 2 the forecasting method underestimated the actual value of sales. Next, we use 21, the actual value of sales in week 2, as the forecast for week 3. Because the actual value of sales in week 3 is 19, the forecast error for week 3 is $19 - 21 = -2$. In this case the negative forecast error indicates that in week 3 the forecast overestimated the actual value. Thus, the forecast error may be positive or negative, depending on whether the forecast is too low or too high. A complete summary of the forecast errors for this naïve forecasting method is shown in Table 15.7 in the column labeled Forecast Error.

A simple measure of forecast accuracy is the mean or average of the forecast errors. Table 15.7 shows that the sum of the forecast errors for the gasoline sales time series is 5; thus, the mean or average error is $5/11 = 0.45$. Note that although the gasoline time series consists of 12 values, to compute the mean error we divided the sum of the forecast errors by 11 because there are only 11 forecast errors. Because the mean forecast error is positive, the method is under-forecasting; in other words, the observed values tend to be greater than the forecasted values. Because positive and negative forecast errors tend to offset one another, the mean error is likely to be small; thus, the mean error is not a very useful measure of forecast accuracy.

The mean absolute error, denoted MAE, is a measure of forecast accuracy that avoids the problem of positive and negative forecast errors offsetting one another. As you might expect given its name, MAE is the average of the absolute values of the forecast errors. Table 15.7 shows that the sum of the absolute values of the forecast errors is 41; thus

$$\text{MAE} = \text{average of the absolute value of forecast errors} = \frac{41}{11} = 3.73$$

Another measure that avoids the problem of positive and negative errors offsetting each other is obtained by computing the average of the squared forecast errors. This measure of forecast accuracy, referred to as the mean squared error, is denoted MSE. From Table 15.7, the sum of the squared errors is 179; hence,

$$\text{MSE} = \text{average of the sum of squared forecast errors} = \frac{179}{11} = 16.27$$

The size of MAE and MSE depend upon the scale of the data. As a result it is difficult to make comparisons for different time intervals, such as comparing a method of forecasting monthly gasoline sales to a method of forecasting weekly sales, or to make comparisons across different time series. To make comparisons like these, we need to work with relative or percentage error measures. The mean absolute percentage error, denoted MAPE, is such a measure. To compute MAPE, we must first compute the percentage error for each forecast. For example, the percentage error corresponding to the forecast of 17 in week 2 is computed by dividing the forecast error in week 2 by the actual value in week 2 and multiplying the result by 100. For week 2 the percentage error is computed as follows:

$$\text{Percentage error for week 2} = \frac{4}{21}(100) = 19.05\%$$

Thus, the forecast error for week 2 is 19.05% of the observed value in week 2. A complete summary of the percentage errors is shown in Table 15.7 in the column labeled Percentage Error. In the next column, we show the absolute value of the percentage error.

Table 15.7 shows that the sum of the absolute values of the percentage errors is 211.69; thus

$$\begin{aligned} \text{MAPE} = \text{average of the absolute value of percentage} \\ \text{forecast errors} = \frac{211.69}{11} = 19.24\% \end{aligned}$$

Summarizing, using the naïve (most recent observation) forecasting method, we obtained the following measures of forecast accuracy:

$$\begin{aligned} \text{MAE} &= 3.73 \\ \text{MSE} &= 16.27 \\ \text{MAPE} &= 19.24\% \end{aligned}$$

These measures of forecast accuracy simply measure how well the forecasting method is able to forecast historical values of the time series. Now, suppose we want to forecast sales for a future time period, such as week 13. In this case the forecast for week 13 is 22, the actual value of the time series in week 12. Is this an accurate estimate of sales for week 13? Unfortunately, there is no way to address the issue of accuracy associated with forecasts for future time periods. However, if we select a forecasting method that works well for the historical data and we think that the historical pattern will continue into the future, we should obtain results that will ultimately be shown to be good.

Before closing this section, let us consider another method for forecasting the gasoline sales time series in Table 15.1. Suppose we use the average of all the historical data available as the forecast for the next period. We begin by developing a forecast for week 2. Because there is only one historical value available prior to week 2, the forecast for week 2 is just the time series value in week 1; thus, the forecast for week 2 is 17 thousand gallons of gasoline. To compute the forecast for week 3, we take the average of the sales values in weeks 1 and 2. Thus,

$$\text{Forecast for week 3} = \frac{17 + 21}{2} = 19$$

Similarly, the forecast for week 4 is

$$\text{Forecast for week 4} = \frac{17 + 21 + 19}{3} = 19$$

The forecasts obtained using this method for the gasoline time series are shown in Table 15.8 in the column labeled Forecast. Using the results shown in Table 15.8, we obtained the following values of MAE, MSE, and MAPE:

$$\text{MAE} = \frac{26.81}{11} = 2.44$$

$$\text{MSE} = \frac{89.07}{11} = 8.10$$

$$\text{MAPE} = \frac{141.34}{11} = 12.85\%$$

We can now compare the accuracy of the two forecasting methods we have considered in this section by comparing the values of MAE, MSE, and MAPE for each method.

	Naïve Method	**Average of Past Values**
MAE	3.73	2.44
MSE	16.27	8.10
MAPE	19.24%	12.85%

For every measure, the average of past values provides more accurate forecasts than using the most recent observation as the forecast for the next period. In general, if the underlying time series is stationary, the average of all the historical data will always provide the best results.

TABLE 15.8 COMPUTING FORECASTS AND MEASURES OF FORECAST ACCURACY USING THE AVERAGE OF ALL THE HISTORICAL DATA AS THE FORECAST FOR THE NEXT PERIOD

Week	Time Series Value	Forecast	Forecast Error	Absolute Value of Forecast Error	Squared Forecast Error	Percentage Error	Absolute Value of Percentage Error
1	17						
2	21	17.00	4.00	4.00	16.00	19.05	19.05
3	19	19.00	0.00	0.00	0.00	0.00	0.00
4	23	19.00	4.00	4.00	16.00	17.39	17.39
5	18	20.00	−2.00	2.00	4.00	−11.11	11.11
6	16	19.60	−3.60	3.60	12.96	−22.50	22.50
7	20	19.00	1.00	1.00	1.00	5.00	5.00
8	18	19.14	−1.14	1.14	1.31	−6.35	6.35
9	22	19.00	3.00	3.00	9.00	13.64	13.64
10	20	19.33	0.67	0.67	0.44	3.33	3.33
11	15	19.40	−4.40	4.40	19.36	−29.33	29.33
12	22	19.00	3.00	3.00	9.00	13.64	13.64
		Totals	4.52	26.81	89.07	2.75	141.34

But suppose that the underlying time series is not stationary. In Section 15.1 we mentioned that changes in business conditions can often result in a time series that has a horizontal pattern shifting to a new level. We discussed a situation in which the gasoline distributor signed a contract with the Vermont Sate Police to provide gasoline for state police cars located in southern Vermont. Table 15.2 shows the number of gallons of gasoline sold for the original time series and the 10 weeks after signing the new contract, and Figure 15.2 shows the corresponding time series plot. Note the change in level in week 13 for the resulting time series. When a shift to a new level like this occurs, it takes a long time for the forecasting method that uses the average of all the historical data to adjust to the new level of the time series. But in this case the simple naïve method adjusts very rapidly to the change in level because it uses the most recent observation available as the forecast.

Measures of forecast accuracy are important factors in comparing different forecasting methods; but we have to be careful to not rely upon them too heavily. Good judgment and knowledge about business conditions that might affect the forecast also have to be carefully considered when selecting a method. In addition, historical forecast accuracy is not the only consideration, especially if the time series is likely to change in the future.

In the next section we will introduce more sophisticated methods for developing forecasts for a time series that exhibits a horizontal pattern. Using the measures of forecast accuracy developed here, we will be able to determine whether such methods provide more accurate forecasts than we obtained using the simple approaches illustrated in this section. The methods that we will introduce also have the advantage that they adapt well in situations where the time series changes to a new level. The ability of a forecasting method to adapt quickly to changes in level is an important consideration, especially in short-term forecasting situations.

15.3 MOVING AVERAGES AND EXPONENTIAL SMOOTHING

In this section we discuss three forecasting methods that are appropriate for a time series with a horizontal pattern: moving averages, weighted moving averages, and exponential smoothing. These methods also adapt well to changes in the level of a horizontal pattern such as what we saw with the extended gasoline sales time series (Table 15.2 and Figure 15.2). However, without modification they are not appropriate when significant trend, cyclical, or seasonal effects are present. Because the objective of each of these methods is to "smooth out" the random fluctuations in the time series, they are referred to as smoothing methods. These methods are easy to use and generally provide a high level of accuracy for short-range forecasts, such as a forecast for the next time period.

Moving Averages

The moving averages method uses the average of the most recent k data values in the time series as the forecast for the next period. Mathematically, a moving average forecast of order k is as follows:

$$F_{t+1} = \frac{\sum(\text{most recent } k \text{ data values})}{k} = \frac{Y_1 + Y_{t-1} + \cdots + Y_{t-k+1}}{k} \tag{15.1}$$

where

$$F_{t+1} = \text{forecast of the time series for period } t + 1$$

The term *moving* is used because every time a new observation becomes available for the time series, it replaces the oldest observation in the equation and a new average is computed. As a result, the average will change, or move, as new observations become available.

To illustrate the moving averages method, let us return to the gasoline sales data in Table 15.1 and Figure 15.1. The time series plot in Figure 15.1 indicates that the gasoline sales time series has a horizontal pattern. Thus, the smoothing methods of this section are applicable.

To use moving averages to forecast a time series, we must first select the order, or number of time series values, to be included in the moving average. If only the most recent values of the time series are considered relevant, a small value of k is preferred. If more past values are considered relevant, then a larger value of k is better. As mentioned earlier, a time series with a horizontal pattern can shift to a new level over time. A moving average will adapt to the new level of the series and resume providing good forecasts in k periods. Thus, a smaller value of k will track shifts in a time series more quickly. But larger values of k will be more effective in smoothing out the random fluctuations over time. So managerial judgment based on an understanding of the behavior of a time series is helpful in choosing a good value for k.

To illustrate how moving averages can be used to forecast gasoline sales, we will use a three-week moving average ($k = 3$). We begin by computing the forecast of sales in week 4 using the average of the time series values in weeks 1–3.

$$F_4 = \text{average of weeks } 1\text{–}3 = \frac{17 + 21 + 19}{3} = 19$$

Thus, the moving average forecast of sales in week 4 is 19, or 19,000 gallons of gasoline. Because the actual value observed in week 4 is 23, the forecast error in week 4 is $23 - 19 = 4$.

Next, we compute the forecast of sales in week 5 by averaging the time series values in weeks 2–4.

$$F_5 = \text{average of weeks } 2\text{–}4 = \frac{21 + 19 + 23}{3} = 21$$

Hence, the forecast of sales in week 5 is 21 and the error associated with this forecast is $18 - 21 = -3$. A complete summary of the three-week moving average forecasts for the gasoline sales time series is provided in Table 15.9. Figure 15.7 shows the original time series plot and the three-week moving average forecasts. Note how the graph of the moving average forecasts has tended to smooth out the random fluctuations in the time series.

To forecast sales in week 13, the next time period in the future, we simply compute the average of the time series values in weeks 10, 11, and 12.

$$F_{13} = \text{average of weeks } 10\text{–}12 = \frac{20 + 15 + 22}{3} = 19$$

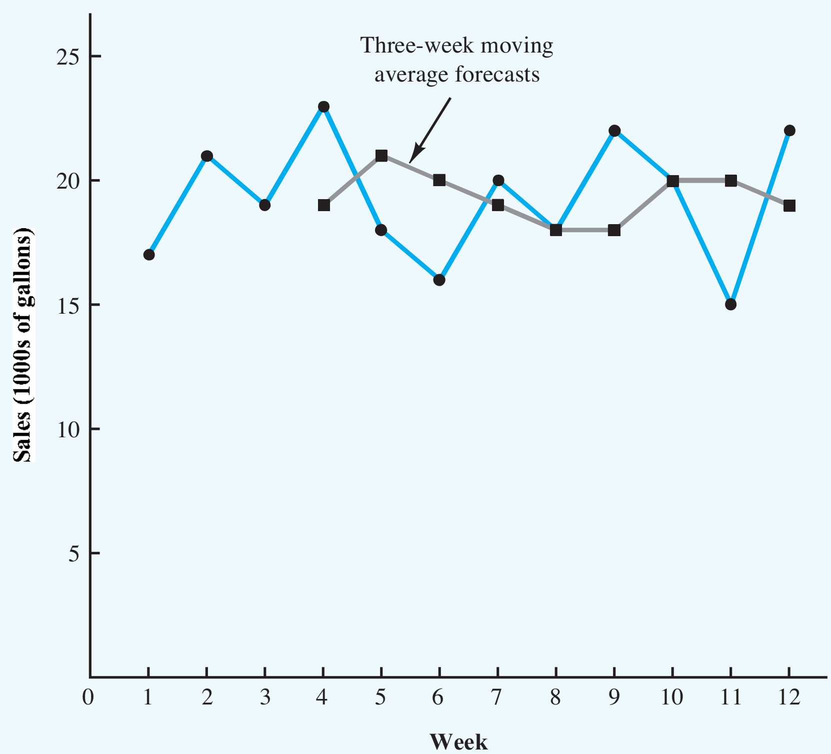
Thus, the forecast for week 13 is 19, or 19,000 gallons of gasoline.

Forecast Accuracy In Section 15.2 we discussed three measures of forecast accuracy: mean absolute error (MAE), mean squared error (MSE), and mean absolute percentage error (MAPE). Using the three-week moving average calculations in Table 15.9, the values for these three measures of forecast accuracy are

TABLE 15.9 SUMMARY OF THREE-WEEK MOVING AVERAGE CALCULATIONS

Week	Time Series Value	Forecast	Forecast Error	Absolute Value of Forecast Error	Squared Forecast Error	Percentage Error	Absolute Value of Percentage Error
1	17						
2	21						
3	19						
4	23	19	4	4	16	17.39	17.39
5	18	21	−3	3	9	−16.67	16.67
6	16	20	−4	4	16	−25.00	25.00
7	20	19	1	1	1	5.00	5.00
8	18	18	0	0	0	0.00	0.00
9	22	18	4	4	16	18.18	18.18
10	20	20	0	0	0	0.00	0.00
11	15	20	−5	5	25	−33.33	33.33
12	22	19	3	3	9	13.64	13.64
		Totals	0	24	92	−20.79	129.21

FIGURE 15.7 GASOLINE SALES TIME SERIES PLOT AND THREE-WEEK MOVING AERAGE FORECASTS



$$\text{MAE} = \frac{24}{9} = 2.67$$

$$\text{MSE} = \frac{92}{9} = 10.22$$

$$\text{MAPE} = \frac{129.21}{9} = 14.36\%$$

In situations where you need to compare forecasting methods for different time periods, such as comparing a forecast of weekly sales to a forecast of monthly sales, relative measures such as MAPE are preferred.

In Section 15.2 we showed that using the most recent observation as the forecast for the next week (a moving average of order $k = 1$) resulted in values of MAE $= 3.73$; MSE $= 16.27$; and MAPE $= 19.24\%$. Thus, in each case the three-week moving average approach provided more accurate forecasts than simply using the most recent observation as the forecast.

To determine whether a moving average with a different order k can provide more accurate forecasts, we recommend using trial and error to determine the value of k that minimizes MSE. For the gasoline sales time series, it can be shown that the minimum value of MSE corresponds to a moving average of order $k = 6$ with MSE $= 6.79$. If we are willing to assume that the order of the moving average that is best for the historical data will also be best for future values of the time series, the most accurate moving average forecasts of gasoline sales can be obtained using a moving average of order $k = 6$.

Weighted Moving Averages

In the moving averages method, each observation in the moving average calculation receives the same weight. One variation, known as weighted moving averages, involves selecting a different weight for each data value and then computing a weighted average of the most recent k values as the forecast. In most cases, the most recent observation receives the most weight, and the weight decreases for older data values. Let us use the gasoline sales time series to illustrate the computation of a weighted three-week moving average. We assign a weight of $\frac{3}{6}$ to the most recent observation, a weight of $\frac{2}{6}$ to the second most recent observation, and a weight of $\frac{1}{6}$ to the third most recent observation. Using this weighted average, our forecast for week 4 is computed as follows:

$$\text{Forecast for week 4} = \tfrac{1}{6}(17) + \tfrac{2}{6}(21) + \tfrac{3}{6}(19) = 19.33$$

Note that for the weighted moving average method the sum of the weights is equal to 1.

A moving average forecast of order $k = 3$ is just a special case of the weighted moving averages method in which each weight is equal to $\frac{1}{3}$.

Forecast Accuracy To use the weighted moving averages method, we must first select the number of data values to be included in the weighted moving average and then choose weights for each of the data values. In general, if we believe that the recent past is a better predictor of the future than the distant past, larger weights should be given to the more recent observations. However, when the time series is highly variable, selecting approximately equal weights for the data values may be best. The only requirements in selecting the weights are that they be nonnegative and that their sum must equal 1. To determine whether one particular combination of number of data values and weights provides a more accurate forecast than another combination, we recommend using MSE as the measure of forecast accuracy. That is, if we assume that the combination that is best for the past will also be best for the future, we would use the combination of the number of data values and weights that minimized MSE for the historical time series to forecast the next value in the time series.

Exponential Smoothing

There are a number of exponential smoothing procedures. Because it has a single smoothing constant α, the method presented here is often referred to as single exponential smoothing.

Exponential smoothing also uses a weighted average of past time series values as a forecast; it is a special case of the weighted moving averages method in which we select only one weight—the weight for the most recent observation. The weights for the other data values are computed automatically and become smaller as the observations move farther into the past. The exponential smoothing model follows:

$$F_{t+1} = \alpha Y_t + (1 - \alpha)F_t \tag{15.2}$$

where

$$\begin{aligned} F_{t+1} &= \text{forecast of the time series for period } t + 1 \\ Y_t &= \text{actual value of the time series in period } t \\ F_t &= \text{forecast of the time series for period } t \\ \alpha &= \text{smoothing constant } (0 \leq \alpha \leq 1) \end{aligned}$$

Equation (15.2) shows that the forecast for period $t + 1$ is a weighted average of the actual value in period t and the forecast for period t. The weight given to the actual value in period t is the smoothing constant α and the weight given to the forecast in period t is $1 - \alpha$. It turns out that the exponential smoothing forecast for any period is actually a weighted average of *all the previous actual values* of the time series. Let us illustrate by working with a time series involving only three periods of data: Y_1, Y_2, and Y_3.

To initiate the calculations, we let F_1 equal the actual value of the time series in period 1; that is, $F_1 = Y_1$. Hence, the forecast for period 2 is

$$\begin{aligned} F_2 &= \alpha Y_1 + (1 - \alpha)F_1 \\ &= \alpha Y_1 + (1 - \alpha)Y_1 \\ &= Y_1 \end{aligned}$$

We see that the exponential smoothing forecast for period 2 is equal to the actual value of the time series in period 1.

The forecast for period 3 is

$$F_3 = \alpha Y_2 + (1-\alpha)F_2 = \alpha Y_2 + (1-\alpha)Y_1$$

Finally, substituting this expression for F_3 into the expression for F_4, we obtain

$$\begin{aligned} F_4 &= \alpha Y_3 + (1-\alpha)F_3 \\ &= \alpha Y_3 + (1-\alpha)[\alpha Y_2 + (1-\alpha)Y_1] \\ &= \alpha Y_3 + \alpha(1-\alpha)Y_2 + (1-\alpha)^2 Y_1 \end{aligned}$$

The term exponential smoothing comes from the exponential nature of the weighting scheme for the historical values.

We now see that F_4 is a weighted average of the first three time series values. The sum of the coefficients, or weights, for Y_1, Y_2, and Y_3 equals 1. A similar argument can be made to show that, in general, any forecast F_{t+1} is a weighted average of all the previous time series values.

Despite the fact that exponential smoothing provides a forecast that is a weighted average of all past observations, all past data do not need to be saved to compute the forecast for the next period. In fact, Equation (15.2) shows that once the value for the smoothing constant α is selected, only two pieces of information are needed to compute the forecast: Y_t, the actual value of the time series in period t; and F_t, the forecast for period t.

To illustrate the exponential smoothing approach to forecasting, let us again consider the gasoline sales time series in Table 15.1 and Figure 15.2. As indicated previously, to start the calculations we set the exponential smoothing forecast for period 2 equal to the actual value of the time series in period 1. Thus, with $Y_1 = 17$, we set $F_2 = 17$ to initiate the computations. Referring to the time series data in Table 15.1, we find an actual time series value in period 2 of $Y_2 = 21$. Thus, period 2 has a forecast error of $21 - 17 = 4$.

Continuing with the exponential smoothing computations using a smoothing constant of $\alpha = 0.2$, we obtain the following forecast for period 3:

$$F_3 = 0.2Y_2 + 0.8F_2 = 0.2(21) + 0.8(17) = 17.8$$

Once the actual time series value in period 3, $Y_3 = 19$, is known, we can generate a forecast for period 4 as follows:

$$F_4 = 0.2Y_3 + 0.8F_3 = 0.2(19) + 0.8(17.8) = 18.04$$

Continuing the exponential smoothing calculations, we obtain the weekly forecast values shown in Table 15.10. Note that we have not shown an exponential smoothing forecast or a forecast error for week 1 because no forecast was made. For week 12, we have $Y_{12} = 22$ and $F_{12} = 18.48$. We can use this information to generate a forecast for week 13.

$$F_{13} = 0.2Y_{12} + 0.8F_{12} = 0.2(22) + 0.8(18.48) = 19.18$$

Thus, the exponential smoothing forecast of the amount sold in week 13 is 19.18, or 19,180 gallons of gasoline. With this forecast, the firm can make plans and decisions accordingly.

TABLE 15.10 SUMMARY OF THE EXPONENTIAL SMOOTHING FORECASTS AND FORECAST ERRORS FOR THE GASOLINE SALES TIME SERIES WITH SMOOTHING CONSTANT $\alpha = 0.2$

Week	Time Series Value	Forecast	Forecast Error	Squared Forecast Error
1	17			
2	21	17.00	4.00	16.00
3	19	17.80	1.20	1.44
4	23	18.04	4.96	24.60
5	18	19.03	–1.03	1.06
6	16	18.83	–2.83	8.01
7	20	18.26	1.74	3.03
8	18	18.61	–0.61	0.37
9	22	18.49	3.51	12.32
10	20	19.19	0.81	0.66
11	15	19.35	–4.35	18.92
12	22	18.48	3.52	12.39
		Totals	10.92	98.80

FIGURE 15.8 ACTUAL AND FORECAST GASOLINE TIME SERIES WITH SMOOTHING CONSTANT $\alpha = 0.2$

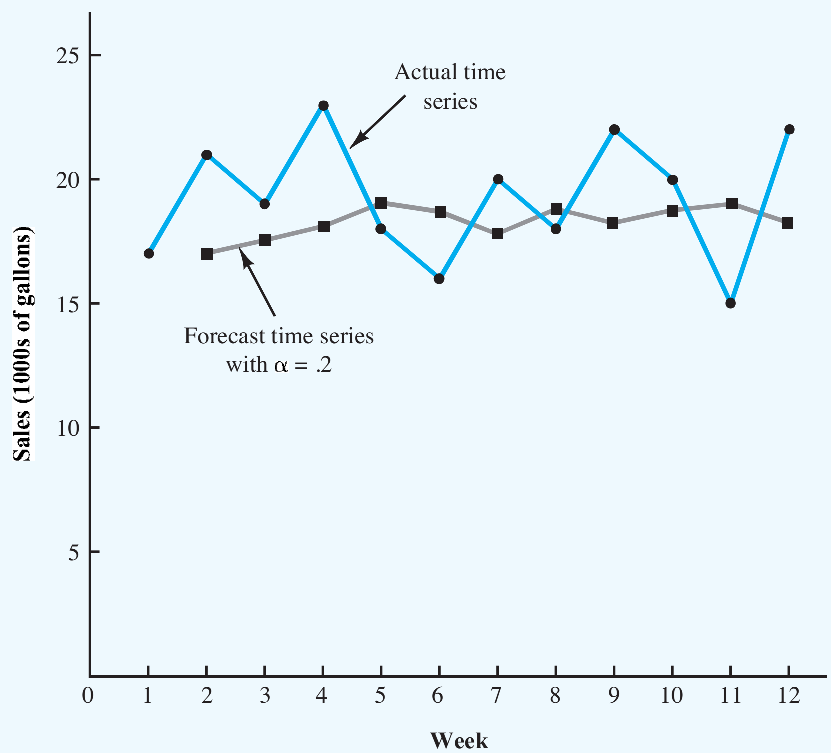


Figure 15.8 shows the time series plot of the actual and forecast time series values. Note in particular how the forecasts "smooth out" the irregular or random fluctuations in the time series.

Forecast Accuracy In the preceding exponential smoothing calculations, we used a smoothing constant of $\alpha = 0.2$. Although any value of α between 0 and 1 is acceptable, some values will yield better forecasts than others. Insight into choosing a good value for α can be obtained by rewriting the basic exponential smoothing model as follows:

$$F_{t+1} = \alpha Y_t + (1 - \alpha)F_t$$
$$F_{t+1} = \alpha Y_t + F_t - \alpha F_t$$
$$F_{t+1} = F_t + \alpha(Y_t - F_t) \quad (15.3)$$

Thus, the new forecast F_{t+1} is equal to the previous forecast F_t plus an adjustment, which is the smoothing constant α times the most recent forecast error, $Y_t - F_t$. That is, the forecast in period $t + 1$ is obtained by adjusting the forecast in period t by a fraction of the forecast error. If the time series contains substantial random variability, a small value of the smoothing constant is preferred. The reason for this choice is that if much of the forecast error is due to random variability, we do not want to overreact and adjust the forecasts too quickly. For a time series with relatively little random variability, forecast errors are more likely to represent a change in the level of the series. Thus, larger values of the smoothing constant provide the advantage of quickly adjusting the forecasts; this allows the forecasts to react more quickly to changing conditions.

The criterion we will use to determine a desirable value for the smoothing constant α is the same as the criterion we proposed for determining the number of periods of data to include in the moving averages calculation. That is, we choose the value of α that minimizes the mean squared error (MSE). A summary of the MSE calculations for the exponential smoothing forecast of gasoline sales with $\alpha = 0.2$ is shown in Table 15.10. Note that there is one less squared error term than the number of time periods because we had no past values with which to make a forecast for period 1. The value of the sum of squared forecast errors is 98.80; hence MSE $= 98.80/11 = 8.98$. Would a different value of α provide better results in terms of a lower MSE value? Determining the value of α that minimizes MSE is a nonlinear optimization problem, as discussed in Chapter 8 (see Problem 8.12). These types of optimization models are often referred to as *curve-fitting* models.

The objective is to minimize the sum of the squared error (note that this is equivalent to minimizing MSE), subject to the smoothing parameter requirement, $0 \leq \alpha \leq 1$. The smoothing parameter α is treated as a variable in the optimization model. In addition, we define a set of variables F_t, the forecast for period t, for $t = 1, \ldots, 12$. The objective of minimizing the sum of squared error is then

$$\begin{aligned}\text{Minimize } \{&(21 - F_2)^2 + (19 - F_3)^2 + (23 - F_4)^2 + (18 - F_5)^2 + (16 - F_6)^2 \\ &+ (20 - F_7)^2 + (18 - F_8)^2 + (22 - F_9)^2 + (20 - F_{10})^2 \\ &+ (15 - F_{11})^2 + (22 - F_{12})^2\}\end{aligned}$$

The first set of constraints defines the forecasts as a function of observed and forecasted values as defined by equation (15.2). Recall that we set the forecast in period 1 to the observed time series value in period 1:

$$\begin{aligned}F_1 &= 17 \\ F_2 &= \alpha 17 + (1-\alpha)F_1 \\ F_3 &= \alpha 21 + (1-\alpha)F_2 \\ F_4 &= \alpha 19 + (1-\alpha)F_3 \\ F_5 &= \alpha 23 + (1-\alpha)F_4 \\ F_6 &= \alpha 18 + (1-\alpha)F_5 \\ F_7 &= \alpha 16 + (1-\alpha)F_6 \\ F_8 &= \alpha 20 + (1-\alpha)F_7 \\ F_9 &= \alpha 18 + (1-\alpha)F_8 \\ F_{10} &= \alpha 22 + (1-\alpha)F_9 \\ F_{11} &= \alpha 20 + (1-\alpha)F_{10} \\ F_{12} &= \alpha 15 + (1-\alpha)F_{11}\end{aligned}$$

Finally, the value of α is restricted to

$$0 \leq \alpha \leq 1$$

The complete nonlinear curve-fitting optimization model is:

$$\text{Minimize } \{(21 - F_2)^2 + (19 - F_3)^2 + (23 - F_4)^2 + (18 - F_5)^2 + (16 - F_6)^2 + (20 - F_7)^2 + (18 - F_8)^2 + (22 - F_9)^2 + (20 - F_{10})^2 + (15 - F_{11})^2 + (22 - F_{12})^2\}$$

s.t.

$$\begin{aligned}
F_1 &= 17 \\
F_2 &= \alpha 17 + (1-\alpha)F_1 \\
F_3 &= \alpha 21 + (1-\alpha)F_2 \\
F_4 &= \alpha 19 + (1-\alpha)F_3 \\
F_5 &= \alpha 23 + (1-\alpha)F_4 \\
F_6 &= \alpha 18 + (1-\alpha)F_5 \\
F_7 &= \alpha 16 + (1-\alpha)F_6 \\
F_8 &= \alpha 20 + (1-\alpha)F_7 \\
F_9 &= \alpha 18 + (1-\alpha)F_8 \\
F_{10} &= \alpha 22 + (1-\alpha)F_9 \\
F_{11} &= \alpha 20 + (1-\alpha)F_{10} \\
F_{12} &= \alpha 15 + (1-\alpha)F_{11} \\
0 \le &\ \alpha \le 1
\end{aligned}$$

WEB file

Gasoline_ES

We may use Excel Solver or LINGO to solve for the best value of α. The optimal value of $\alpha = 0.17439$ with a sum of squared error of 98.56 and an MSE of 98.56/11 = 8.96. So, our initial value of $\alpha = .2$ is very close to the best we can do to minimize MSE. It will not always be the case that our guess will be so close to optimal, so we recommend you solve the nonlinear optimization for the best value of α.

The general optimization problem for exponential smoothing with n time periods and observed values Y_t is

$$\text{Min} \sum_{t=2}^{n} (Y_t - F_t)^2 \qquad (15.4)$$

s.t.

$$F_t = \alpha Y_{t-1} + (1 - \alpha)F_{t-1} \qquad t = 2, 3, \ldots, n \qquad (15.5)$$

$$F_1 = Y_1 \qquad (15.6)$$

$$0 \le \alpha \le 1 \qquad (15.7)$$

The objective function (equation 15.4) is to minimize the sum of the squared errors. As in Table 15.10, we have errors (observed data – forecast) only for time periods 2 through n, and we initialize F_1 to Y_1. The optimal value of α can be used in the exponential smoothing model to provide forecasts for the future. At a later date, after new time series observations are obtained, we may analyze the newly collected time series data to determine whether the smoothing constant should be revised to provide better forecasting results. Revised forecasts may be obtained by solving the model in (15.4)–(15.7), including any new observations.

NOTES AND COMMENTS

1. Spreadsheet packages are an effective tool for implementing exponential smoothing. With the time series data and the forecasting formulas in a spreadsheet as shown in Table 15.10, you can solve the nonlinear model described by equations (15.4)–(15.7) using Solver. Notice that in equation set (15.5) each forecast variable F_t is defined in terms of the smoothing parameter α and the previous periods forecast variable. Thus, these are what we have called definitional variables. In the Solver spreadsheet model, only α needs to be declared a decision variable. The forecast variables F_t are simply calculatons in the spreadsheet. We give details for doing this for the gasoline data in Appendix 15.1.
2. We presented the moving average and exponential smoothing methods in the context of a stationary time series. These methods can also be used to forecast a nonstationary time series which shifts in level, but exhibits no trend or seasonality. Moving averages with small values of k adapt more quickly than moving averages with larger values of k. Exponential smoothing models with smoothing constants closer to one adapt more quickly than models with smaller values of the smoothing constant.

15.4 TREND PROJECTION

We present two forecasting methods in this section that are appropriate for a time series exhibiting a trend pattern. First, we show how curve fitting may be used to forecast a time series with a linear trend. Second, we show how curve fitting can also be used to forecast time series with a curvilinear or nonlinear trend.

Linear Trend

In Section 15.1 we used the bicycle sales time series in Table 15.3 and Figure 15.3 to illustrate a time series with a trend pattern. Let us now use this time series to illustrate how curve fitting can be used to forecast a time series with a linear trend. The data for the bicycle time series are repeated in Table 15.11 and Figure 15.9.

Although the time series plot in Figure 15.9 shows some up and down movement over the past 10 years, we might agree that the linear trend line shown in Figure 15.10 provides

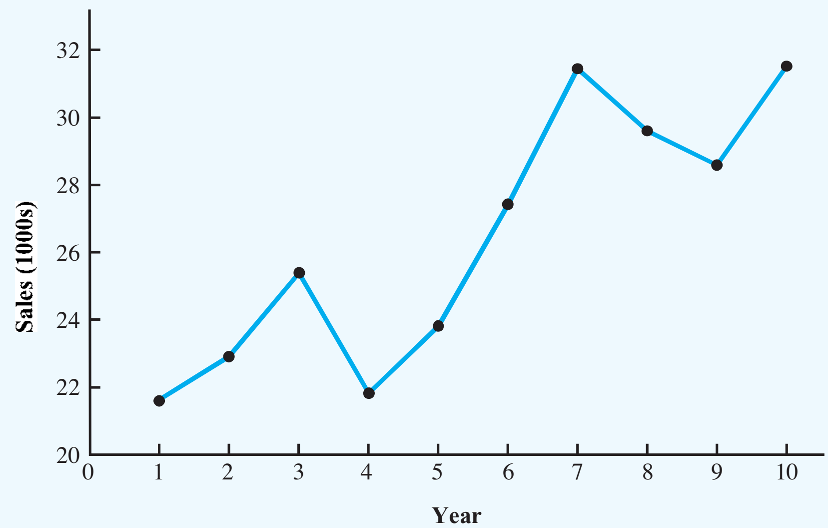
WEB file

Bicycle

TABLE 15.11 BICYCLE SALES TIME SERIES

Year	Sales (1000s)
1	21.6
2	22.9
3	25.5
4	21.9
5	23.9
6	27.5
7	31.5
8	29.7
9	28.6
10	31.4

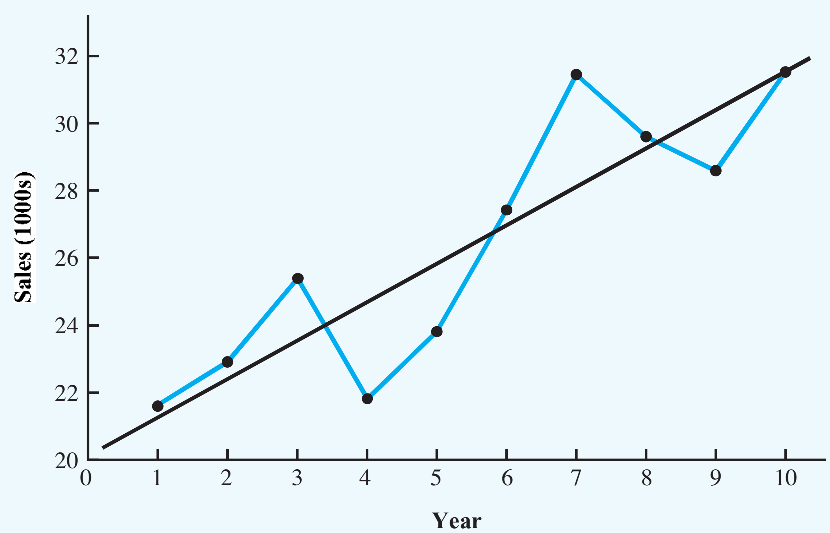
FIGURE 15.9 BICYCLE SALES TIME SERIES PLOT



a reasonable approximation of the long-run movement in the series. We can use curve fitting to develop such a linear trend line for the bicycle sales time series.

Curve fitting can be used to find a best-fitting line to a set of data that exhibits a linear trend. The criterion used to determine the best-fitting line is one we used in the previous section. Curve fitting minimizes the sum of squared error between the observed and fitted time series data where the model is a trend line. We build a nonlinear optimization model

FIGURE 15.10 TREND REPRESENTED BY A LINEAR FUNCTION FOR THE BICYCLE SALES TIME SERIES



that is similar to the model we used to find the best value of α for exponential smoothing. In the case of a straight line $y = a + mx$, our objective is to find the best values of parameters a and m, so that the line provides forecasts that minimize sum of squared error. For estimating the linear trend in a time series, we will use the following notation for a line:

$$T_t = b_0 + b_1 t \tag{15.8}$$

where

T_t = linear trend forecast in period t
b_0 = the intercept of the linear trend line
b_1 = the slope of the linear trend line
t = the time period

In equation (15.8), the time variable begins at $t = 1$ corresponding to the first time series observation (year 1 for the bicycle sales time series) and continues until $t = n$ corresponding to the most recent time series observation (year 10 for the bicycle sales time series). Thus, for the bicycle sales time series $t = 1$ corresponds to the oldest time series value and $t = 10$ corresponds to the most recent year.

Let us formulate the curve-fitting model that will give us the best values of b_0 and b_1 in equation (15.8) for the bicycle sales data. The objective is to minimize the sum of squared error between the observed values of the time series given in Table 15.11 and the forecasted values for each period:

$$\text{Min } \{(21.6 - T_1)^2 + (22.9 - T_2)^2 + (22.5 - T_3)^2 + (21.9 - T_4)^2 + (23.9 - T_5)^2$$
$$(27.5 - T_6)^2 + (31.5 - T_7)^2 + (29.7 - T_8)^2 + (28.6 - T_9)^2 + (31.4 - T_{10})^2\}$$

The only constraints then are to define the forecasts as a linear function of parameters b_0 and b_1 as described by equation (15.8):

$$T_1 = b_0 + b_1 1$$
$$T_2 = b_0 + b_1 2$$
$$T_3 = b_0 + b_1 3$$
$$T_4 = b_0 + b_1 4$$
$$T_5 = b_0 + b_1 5$$
$$T_6 = b_0 + b_1 6$$
$$T_7 = b_0 + b_1 7$$
$$T_8 = b_0 + b_1 8$$
$$T_9 = b_0 + b_1 9$$
$$T_{10} = b_0 + b_1 10$$

The entire nonlinear curve-fitting optimization model is:

$$\text{Min } \{(21.6 - T_1)^2 + (22.9 - T_2)^2 + (22.5 - T_3)^2 + (21.9 - T_4)^2 + (23.9 - T_5)^2$$
$$(27.5 - T_6)^2 + (31.5 - T_7)^2 + (29.7 - T_8)^2 + (28.6 - T_9)^2 + (31.4 - T_{10})^2\}$$

s.t.

$$T_1 = b_0 + b_1 1$$
$$T_2 = b_0 + b_1 2$$
$$T_3 = b_0 + b_1 3$$
$$T_4 = b_0 + b_1 4$$
$$T_5 = b_0 + b_1 5$$
$$T_6 = b_0 + b_1 6$$
$$T_7 = b_0 + b_1 7$$
$$T_8 = b_0 + b_1 8$$
$$T_9 = b_0 + b_1 9$$
$$T_{10} = b_0 + b_1 10$$

WEB file

Bicycle_Linear

Note that b_0, b_1, and T_t are decision variables and that none are restricted to be nonnegative.

The solution to this problem may be obtained using Excel Solver or LINGO. The solution is $b_0 = 20.4$ and $b_1 = 1.1$ with a sum of squared error of 30.7. Therefore, the trend equation is

$$T_t = 20.4 + 1.1t$$

The slope of 1.1 indicates that over the past 10 years the firm experienced an average growth in sales of about 1100 units per year. If we assume that the past 10-year trend in sales is a good indicator of the future, this trend equation can be used to develop forecasts for future time periods. For example, substituting $t = 11$ into the equation yields next year's trend projection or forecast, T_{11}.

$$T_{11} = 20.4 + 1.1(11) = 32.5$$

Thus, using trend projection, we would forecast sales of 32,500 bicycles for year 11.

Table 15.12 shows the computation of the minimized sum of squared errors for the bicycle sales time series. As previously noted, minimizing sum of squared error also minimizes the commonly used measure of accuracy, mean squared error (MSE). For the bicycle sales time series

$$\text{MSE} = \frac{\sum_{t=1}^{n}(Y_t - F_t)^2}{n} = \frac{30.7}{10} = 3.07$$

We may write a general optimization curve-fitting model for linear trend curve fitting for a time series with n data points. Let Y_t = the observed value of the time series in period t. The general model is

$$\text{Min} \sum_{t=1}^{n}(Y_t - T_t)^2 \qquad (15.9)$$

s.t.

$$T_t = b_0 + b_1 t \qquad t = 1,2,3 \ldots . n \qquad (15.10)$$

TABLE 15.12 SUMMARY OF THE LINEAR TREND FORECASTS AND FORECAST ERRORS FOR THE BICYCLE SALES TIME SERIES

Week	Sales (1000s) Y_t	Forecast T_t	Forecast Error	Squared Forecast Error
1	21.6	21.5	0.1	0.01
2	22.9	22.6	0.3	0.09
3	25.5	23.7	1.8	3.24
4	21.9	24.8	−2.9	8.41
5	23.9	25.9	−2.0	4.00
6	27.5	27.0	0.5	0.25
7	31.5	28.1	3.4	11.56
8	29.7	29.2	0.5	0.25
9	28.6	30.3	−1.7	2.89
10	31.4	31.4	0.0	0.00
			Total	30.70

The decision variables in this optimization model are b_0 the intercept and b_1 the slope of the line. The variables T_t, the fitted forecast for period t, are definitional variables, as discussed in Chapter 5. Note that none of these are restricted to be nonnegative. This model will have $n + 2$ decision variables and n constraints, one for each data point in the time series.

NOTES AND COMMENTS

1. The optimization model given by equations (15.9) and (15.10) is easily generalized for other types of models. Given the objective is to minimize the sum of squared errors, to test a different forecasting model, you only need to change the form of equation (15.10). We will see an example of this in the forthcoming seciton on nonlinear trend. Examples of both LINGO and Excel Solver models are provided in the appendices to this chapter.
2. Statistical packages such as Minitab and SAS, as well as Excel have routines to perform curve fitting under the label regression analysis. Regression analysis solves the curve-fitting problem of minimizing the sum of squared error, but also under certain assumptions, allows the analyst to make statistical statements about the parameters and the forecasts.

Nonlinear Trend

The use of a linear function to model trend is common. However, as we discussed previously, sometimes time series have a curvilinear or nonlinear trend. As an example, consider the annual revenue in millions of dollars for a cholesterol drug for the first ten years of sales. Table 15.13 shows the time series and Figure 15.11 shows the corresponding time series plot. For instance, revenue in year 1 was \$23.1 million; revenue in year 2 was \$21.3 million; and so on. The time series plot indicates an overall increasing or upward trend. But, unlike the bicycle sales time series, a linear trend does not appear to be appropriate. Instead, a curvilinear function appears to be needed to model the long-term trend.

TABLE 15.13 CHOLESTEROL REVENUE TIME SERIES ($ MILLIONS)

WEB file

Cholesterol

Year	Revenue ($ millions)
1	23.1
2	21.3
3	27.4
4	34.6
5	33.8
6	43.2
7	59.5
8	64.4
9	74.2
10	99.3

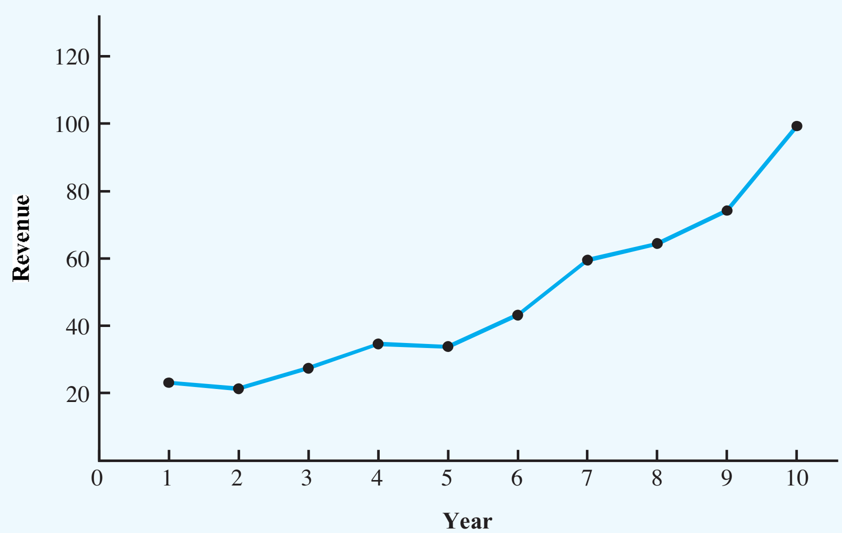
Quadratic Trend Equation A variety of nonlinear functions can be used to develop an estimate of the trend for the cholesterol time series. For instance, consider the following quadratic trend equation:

$$T_t = b_0 + b_1 t + b_2 t^2 \tag{15.11}$$

For the cholesterol time series, $t = 1$ corresponds to year 1, $t = 2$ corresponds to year 2, and so on.

Let us construct the optimization model to find the values of b_0, b_1, and b_2 that minimize the sum of squared errors. Note that we need the value of t and the value of t^2 for each period.

FIGURE 15.11 CHOLESTEROL REVENUE TIMES SERIES PLOT ($ MILLIONS)



The model to find the best values of b_0, b_1, and b_2 so that the sum of squared error is minimized is as follows:

$$\text{Min } \{(23.1 - T_1)^2 + (21.3 - T_2)^2 + (27.4 - T_3)^2 + (34.6 - T_4)^2 + (33.8 - T_5)^2$$
$$(43.2 - T_6)^2 + (59.5 - T_7)^2 + (64.4 - T_8)^2 + (74.2 - T_9)^2 + (99.3 - T_{10})^2\}$$

s.t.

WEB file

Cholesterol_Quad

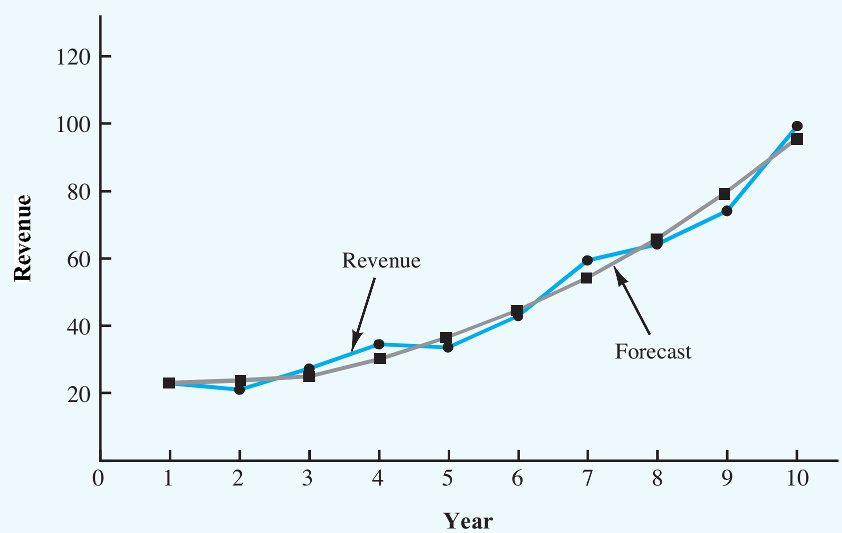
$$T_1 = b_0 + b_1 1 + b_2 1$$
$$T_2 = b_0 + b_1 2 + b_2 4$$
$$T_3 = b_0 + b_1 3 + b_2 9$$
$$T_4 = b_0 + b_1 4 + b_2 16$$
$$T_5 = b_0 + b_1 5 + b_2 25$$
$$T_6 = b_0 + b_1 6 + b_2 36$$
$$T_7 = b_0 + b_1 7 + b_2 49$$
$$T_8 = b_0 + b_1 8 + b_2 64$$
$$T_9 = b_0 + b_1 9 + b_2 81$$
$$T_{10} = b_0 + b_1 10 + b_2 100$$

This model may be solved with Excel Solver or LINGO. The optimal values from this optimization are $b_0 = 24.182$, $b_1 = -2.11$, and $b_2 = 0.922$ with a sum of squared errors of 110.65 and an MSE = 110.65/10 = 11.065. The fitted curve is therefore

$$T_t = 24.182 - 2.11\, t + 0.922\, t^2$$

Figure 15.12 shows the observed data along with this curve.

FIGURE 15.12 TIME SERIES QUADRATIC TREND FOR THE CHOLESTEROL SALES TIME SERIES



Exponential Trend Equation Another alternative that can be used to model the nonlinear pattern exhibited by the cholesterol time series is to fit an exponential model to the data. For instance, consider the following exponential growth trend equation:

$$T_t = b_0 (b_1)^t \tag{15.12}$$

Like equation (15.11), this model is a nonlinear function of period t. As with the quadratic case, we can update equation (15.12) to yield the values of b_0 and b_1 that minimize the sum of squared errors. For the cholesterol sales data, minimizing the sum of squared errors yields the following curve-fitting model:

$$\begin{aligned}&\text{Min } \{(23.1 - T_1)^2 + (21.3 - T_2)^2 + (27.4 - T_3)^2 + (34.6 - T_4)^2 + (33.8 - T_5)^2\\ &\quad (43.2 - T_6)^2 + (59.5 - T_7)^2 + (64.4 - T_8)^2 + (74.2 - T_9)^2 + (99.3 - T_{10})^2\}\\ &\text{s.t.}\end{aligned}$$

WEB file
Cholesterol_Exp

$$\begin{aligned}T_1 &= b_0\, b_1{}^1\\ T_2 &= b_0\, b_1{}^2\\ T_3 &= b_0\, b_1{}^3\\ T_4 &= b_0\, b_1{}^4\\ T_5 &= b_0\, b_1{}^5\\ T_6 &= b_0\, b_1{}^6\\ T_7 &= b_0\, b_1{}^7\\ T_8 &= b_0\, b_1{}^8\\ T_9 &= b_0\, b_1{}^9\\ T_{10} &= b_0\, b_1{}^{10}\end{aligned}$$

This may be solved with LINGO or Excel Solver. The optimal values are $b_0 = 15.42$ and $b_1 = 1.2$ with a sum of squared errors of 123.12 and an MSE $= 123.12/10 = 12.312$. Based on MSE, the quadratic model provides a better fit than the exponential model.

NOTES AND COMMENTS

The exponential model (15.12) is nonlinear and the curve-fitting optimization based on it can be difficult to solve. We suggest using a number of different starting values to ensure that the solution found is a global optimum. Also, we found it helpful to bound b_0 and b_1 away from zero (add constraints $b_0 \geq 0.01$ and $b_1 \geq 0.01$).

15.5 SEASONALITY

In this section we show how to develop forecasts for a time series that has a seasonal pattern. To the extent that seasonality exists, we need to incorporate it into our forecasting models to ensure accurate forecasts. We begin the section by considering a seasonal time series with no trend and then discuss how to model seasonality with trend.

Seasonality Without Trend

As an example, consider the number of umbrellas sold at a clothing store over the past five years. Table 15.14 shows the time series and Figure 15.13 shows the corresponding time series plot. The time series plot does not indicate any long-term trend in sales. In fact, unless you look carefully at the data, you might conclude that the data follow a horizontal pattern and that single exponential smoothing could be used to forecast sales. However, closer inspection of the time series plot reveals a pattern in the data. That is, the first and third quarters have moderate sales, the second quarter has the highest sales, and the fourth quarter tends to be the lowest quarter in terms of sales volume. Thus, we would conclude that a quarterly seasonal pattern is present.

We can model a time series with a seasonal pattern by treating the season as a *categorical variable*. Categorical variables are data used to categorize observations of data. When a categorical variable has k levels, $k - 1$ dummy or 0-1 variables are required. So, if there are four seasons, we need three dummy variables. For instance, in the umbrella sales time series the quarter each observation corresponds to is treated as a season; it is a categorical variable with four levels: Quarter 1, Quarter 2, Quarter 3, and Quarter 4. Thus, to model the seasonal effects in the umbrella time series we need $4 - 1 = 3$ dummy variables. The three dummy variables can be coded as follows:

$$\text{Qtr1} = \begin{cases} 1 \text{ if Quarter 1} \\ 0 \text{ otherwise} \end{cases} \qquad \text{Qtr2} = \begin{cases} 1 \text{ if Quarter 2} \\ 0 \text{ otherwise} \end{cases} \qquad \text{Qtr3} = \begin{cases} 1 \text{ if Quarter 3} \\ 0 \text{ otherwise} \end{cases}$$

Using F_t to denote the forecasted value of sales for period t, the general form of the equation relating the number of umbrellas sold to the quarter the sale takes place follows:

$$F_t = b_0 + b_1 \text{ Qtr1}_t + b_2 \text{ Qtr2}_t + b_3 \text{ Qtr3}_t$$

TABLE 15.14 UMBRELLA SALES TIME SERIES

Year	Quarter	Sales
1	1	125
	2	153
	3	106
	4	88
2	1	118
	2	161
	3	133
	4	102
3	1	138
	2	144
	3	113
	4	80
4	1	109
	2	137
	3	125
	4	109
5	1	130
	2	165
	3	128
	4	96

WEB file
Umbrella

FIGURE 15.13 UMBRELLA SALES TIME SERIES PLOT

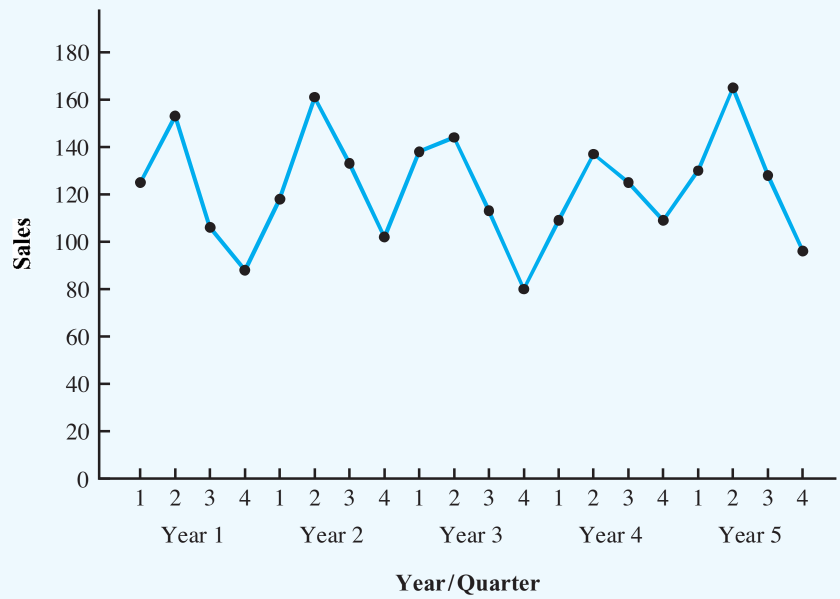


Table 15.15 is the umbrella sales time series with the coded values of the dummy variables shown. We may use an optimization model to find the values of b_0, b_1, b_2, and b_3 that minimize the sum of squared error. The model is as follows:

WEB file
Umbrella_Seas

Note that we have numbered the observations in Table 15.15 as periods 1–20. For example, year 3, quarter 3 is observation 11.

$$\text{Min } \{(125 - F_1)^2 + (153 - F_2)^2 + (106 - F_3)^2 \cdots + (96 - F_{20})^2\}$$

s.t.

$$\begin{aligned}
F_1 &= b_0 + 1b_1 + 0b_2 + 0b_3 \\
F_2 &= b_0 + 0b_1 + 1b_2 + 0b_3 \\
F_3 &= b_0 + 0b_1 + 0b_2 + 1b_3 \\
F_4 &= b_0 + 0b_1 + 0b_2 + 0b_3 \\
&\vdots \\
F_{17} &= b_0 + 1b_1 + 0b_2 + 0b_3 \\
F_{18} &= b_0 + 0b_1 + 1b_2 + 0b_3 \\
F_{19} &= b_0 + 0b_1 + 0b_2 + 1b_3 \\
F_{20} &= b_0 + 0b_1 + 0b_2 + 0b_3
\end{aligned}$$

This model may be solved with LINGO or Excel Solver. Using the data in Table 15.15 and the above optimization model, we obtained the following equation:

$$F_t = 95.0 + 29.0 \text{ Qtr1}_t + 57.0 \text{ Qtr2}_t + 26.0 \text{ Qtr3}_t \qquad (15.13)$$

TABLE 15.15 UMBRELLA SALES TIME SERIES WITH DUMMY VARIABLES

Period	Year	Quarter	Qtr1	Qtr2	Qtr3	Sales
1	1	1	1	0	0	125
2		2	0	1	0	153
3		3	0	0	1	106
4		4	0	0	0	88
5	2	1	1	0	0	118
6		2	0	1	0	161
7		3	0	0	1	133
8		4	0	0	0	102
9	3	1	1	0	0	138
10		2	0	1	0	144
11		3	0	0	1	113
12		4	0	0	0	80
13	4	1	1	0	0	109
14		2	0	1	0	137
15		3	0	0	1	125
16		4	0	0	0	109
17	5	1	1	0	0	130
18		2	0	1	0	165
19		3	0	0	1	128
20		4	0	0	0	96

We can use equation (15.13) to forecast quarterly sales for next year.

$$\text{Quarter 1: Sales} = 95.0 + 29.0(1) + 57.0(0) + 26.0(0) = 124$$
$$\text{Quarter 2: Sales} = 95.0 + 29.0(0) + 57.0(1) + 26.0(0) = 152$$
$$\text{Quarter 3: Sales} = 95.0 + 29.0(0) + 57.0(0) + 26.0(1) = 121$$
$$\text{Quarter 4: Sales} = 95.0 + 29.0(0) + 57.0(1) + 26.0(0) = 95$$

It is interesting to note that we could have obtained the quarterly forecasts for next year by simply computing the average number of umbrellas sold in each quarter, as shown in the following table:

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
1	125	153	106	88
2	118	161	133	102
3	138	144	113	80
4	109	137	125	109
5	130	165	128	96
Average	124	152	121	95

Nonetheless, for more complex types of problem situations, such as dealing with a time series that has both trend and seasonal effects, this simple averaging approach will not work.

TABLE 15.16 TELEVISION SET SALES TIME SERIES

Year	Quarter	Sales (1000s)
1	1	4.8
	2	4.1
	3	6.0
	4	6.5
2	1	5.8
	2	5.2
	3	6.8
	4	7.4
3	1	6.0
	2	5.6
	3	7.5
	4	7.8
4	1	6.3
	2	5.9
	3	8.0
	4	8.4

WEB file

TVSales

Seasonality and Trend

We now extend the curve-fitting approach to include situations where the time series contains both a seasonal effect and a linear trend, by showing how to forecast the quarterly television set sales time series introduced in Section 15.1. The data for the television set time series are shown in Table 15.16. The time series plot in Figure 15.14 indicates that sales are lowest in the second quarter of each year and increase in quarters 3 and 4. Thus, we conclude that a seasonal pattern exists for television set sales. But the time series also has an upward linear trend that will need to be accounted for in order to develop accurate forecasts of quarterly sales. This is easily done by combining the dummy variable approach for handling seasonality with the approach we discussed in Section 15.3 for handling linear trend.

The general form of the equation for modeling both the quarterly seasonal effects and the linear trend in the television set time series is

$$F_t = b_0 + b_1 \text{ Qtr1}_t + b_2 \text{ Qtr2}_t + b_3 \text{ Qtr3}_t + b_4 t$$

where

F_t = forecast of sales in period t

Qtr1_t = 1 if time period t corresponds to the first quarter of the year; 0 otherwise

Qtr2_t = 1 if time period t corresponds to the second quarter of the year; 0 otherwise

Qtr3_t = 1 if period t corresponds to the third quarter of the year; 0 otherwise

t = time period

WEB file

TVSales_Seas_Trend

Table 15.17 is the revised television set sales time series that includes the coded values of the dummy variables and the time period t. Using the data in Table 15.17 with the sum

FIGURE 15.14 TELEVISION SET SALES TIME SERIES PLOT

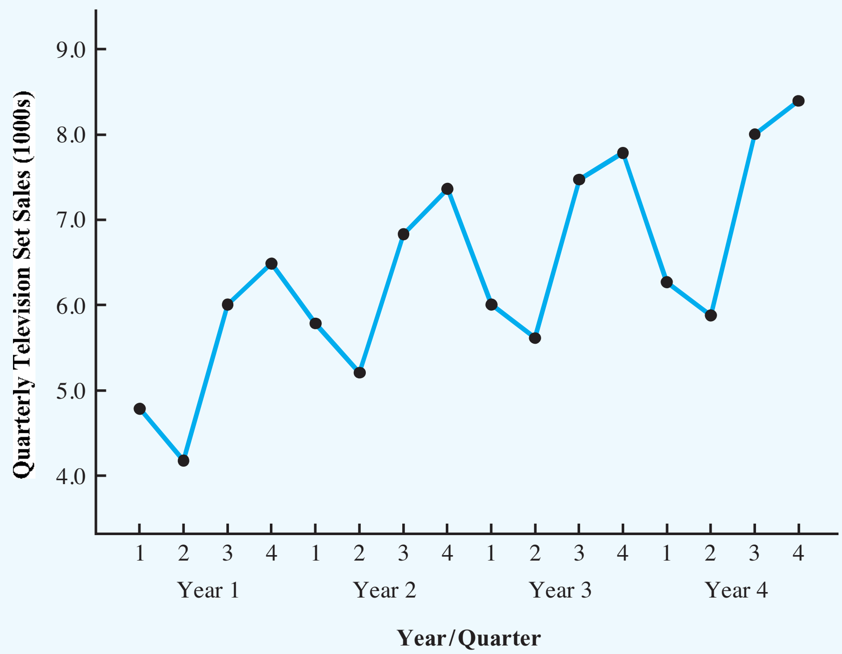


TABLE 15.17 TELEVISION SET SALES TIME SERIES WITH DUMMY VARIABLES AND TIME PERIOD

Period	Year	Quarter	Qtr1	Qtr2	Qtr3	Period	Sales (1000s)
1	1	1	1	0	0	1	4.8
2		2	0	1	0	2	4.1
3		3	0	0	1	3	6.0
4		4	0	0	0	4	6.5
5	2	1	1	0	0	5	5.8
6		2	0	1	0	6	5.2
7		3	0	0	1	7	6.8
8		4	0	0	0	8	7.4
9	3	1	1	0	0	9	6.0
10		2	0	1	0	10	5.6
11		3	0	0	1	11	7.5
12		4	0	0	0	12	7.8
13	4	1	1	0	0	13	6.3
14		2	0	1	0	14	5.9
15		3	0	0	1	15	8.0
16		4	0	0	0	16	8.4

of squared error minimization model with the seasonal and trend components, we obtain the following equation:

$$F_t = 6.07 - 1.36\ \text{Qtr1}_t - 2.03\ \text{Qtr2}_t - 0.304\ \text{Qtr3}_t + 0.146\ t \qquad (15.14)$$

We can now use equation (15.14) to forecast quarterly sales for next year. Next year is year 5 for the television set sales time series; that is, time periods 17, 18, 19, and 20.

Forecast for Time Period 17 (quarter 1 in year 5)

$$F_{17} = 6.07 - 1.36(1) - 2.03(0) - 0.304(0) + 0.146(17) = 7.19$$

Forecast for Time Period 18 (quarter 2 in year 5)

$$F_{18} = 6.07 - 1.36(0) - 2.03(1) - 0.304(0) + 0.146(18) = 6.67$$

Forecast for Time Period 19 (quarter 3 in year 5)

$$F_{19} = 6.07 - 1.36(0) - 2.03(0) - 0.304(1) + 0.146(19) = 8.54$$

Forecast for Time Period 20 (quarter 4 in year 5)

$$F_{20} = 6.07 - 1.36(0) - 2.03(0) - 0.304(0) + 0.146(20) = 8.99$$

Thus, accounting for the seasonal effects and the linear trend in television set sales, the estimates of quarterly sales in year 5 are 7190, 6670, 8540, and 8990.

The dummy variables in the equation actually provide four equations, one for each quarter. For instance, if time period t corresponds to quarter 1, the estimate of quarterly sales is

$$\text{Quarter 1: Sales} = 6.07 - 1.36(1) - 2.03(0) - 0.304(0) + 0.146t = 4.71 + 0.146t$$

Similarly, if time period t corresponds to quarters 2, 3, and 4, the estimates of quarterly sales are

$$\text{Quarter 2: Sales} = 6.07 - 1.36(0) - 2.03(1) - 0.304(0) + 0.146t = 4.04 + 0.146t$$

$$\text{Quarter 3: Sales} = 6.07 - 1.36(0) - 2.03(0) - 0.304(1) + 0.146t = 5.77 + 0.146t$$

$$\text{Quarter 4: Sales} = 6.07 - 1.36(0) - 2.03(0) - 0.304(0) + 0.146t = 6.07 + 0.146t$$

The slope of the trend line for each quarterly forecast equation is 0.146, indicating a growth in sales of about 146 sets per quarter. The only difference in the four equations is that they have different intercepts.

Models Based on Monthly Data

In the preceding television set sales example, we showed how dummy variables can be used to account for the quarterly seasonal effects in the time series. Because there were four levels for the categorical variable season, three dummy variables were required. However, many businesses use monthly rather than quarterly forecasts. For monthly data, season is a categorical variable with 12 levels and thus $12 - 1 = 11$ dummy variables are required. For example, the 11 dummy variables could be coded as follows:

$$\text{Month1} = \begin{cases} 1 \text{ if January} \\ 0 \text{ otherwise} \end{cases}$$

$$\text{Month2} = \begin{cases} 1 \text{ if February} \\ 0 \text{ otherwise} \end{cases}$$

$$\vdots$$

$$\text{Month11} = \begin{cases} 1 \text{ if November} \\ 0 \text{ otherwise} \end{cases}$$

Whenever a categorical variable such as season has k levels, k – 1 dummy variables are required.

Other than this change, the approach for handling seasonality remains the same.

SUMMARY

This chapter provided an introduction to the basic methods of time series analysis and forecasting. First, we showed that the underlying pattern in the time series can often be identified by constructing a time series plot. Several types of data patterns can be distinguished, including a horizontal pattern, a trend pattern, and a seasonal pattern. The forecasting methods we have discussed are based on which of these patterns are present in the time series.

For a time series with a horizontal pattern, we showed how moving averages and exponential smoothing can be used to develop a forecast. The moving averages method consists of computing an average of past data values and then using that average as the forecast for the next period. In the exponential smoothing method, a weighted average of past time series values is used to compute a forecast. These methods also adapt well when a horizontal pattern shifts to a different level but maintains a horizontal pattern at the new level.

An important factor in determining what forecasting method to use involves the accuracy of the method. We discussed three measures of forecast accuracy: mean absolute error (MAE), mean squared error (MSE), and mean absolute percentage error (MAPE). Each of these measures is designed to determine how well a particular forecasting method is able to reproduce the time series data that are already available. By selecting a method that has the best accuracy for the data already known, we hope to increase the likelihood that we will obtain better forecasts for future time periods.

For time series that have only a long-term linear trend, we showed how curve fitting can be used to make trend projections. For a time series with a curvilinear or nonlinear trend, we showed how curve-fitting optimization can be used to fit a quadratic trend equation or an exponential trend equation to the data.

For a time series with a seasonal trend, we showed how the use of dummy variables can be used to develop an equation with seasonal effects. We then extended the approach to include situations where the time series contains both a seasonal and a linear trend effect by showing how to combine the dummy variable approach for handling seasonality with the approach for handling linear trend.

GLOSSARY

Time series A sequence of observations on a variable measured at successive points in time or over successive periods of time.

Time series plot A graphical presentation of the relationship between time and the time series variable. Time is shown on the horizontal axis and the time series values are shown on the verical axis.

Stationary time series A time series whose statistical properties are independent of time. For a stationary time series the process generating the data has a constant mean and the variability of the time series is constant over time.

Trend pattern A trend pattern exists if the time series plot shows gradual shifts or movements to relatively higher or lower values over a longer period of time.

Seasonal pattern A seasonal pattern exists if the time series plot exhibits a repeating pattern over successive periods. The successive periods are often one-year intervals, which is where the name seasonal pattern comes from.

Cyclical pattern A cyclical pattern exists if the time series plot shows an alternating sequence of points below and above the trend line lasting more than one year.

Forecast error The difference between the actual time series value and the forecast.

Mean absolute error (MAE) The average of the absolute values of the forecast errors.

Mean squared error (MSE) The average of the sum of squared forecast errors.

Mean absolute percentage error (MAPE) The average of the absolute values of the percentage forecast errors.

Moving averages A forecasting method that uses the average of the most recent *k* data values in the time series as the forecast for the next period.

Weighted moving averages A forecasting method that involves selecting a different weight for the most recent *k* data values values in the time series and then computing a weighted average of the values. The sum of the weights must equal 1.

Exponential smoothing A forecasting method that uses a weighted average of past time series values as the forecast; it is a special case of the weighted moving averages method in which we select only one weight—the weight for the most recent observation.

Smoothing constant A parameter of the exponential smoothing model that provides the weight given to the most recent time series value in the calculation of the forecast value.

PROBLEMS

SELF test

1. Consider the following time series data:

Week	1	2	3	4	5	6
Value	18	13	16	11	17	14

Using the naïve method (most recent value) as the forecast for the next week, compute the following measures of forecast accuracy:

a. Mean absolute error
b. Mean squared error

c. Mean absolute percentage error
d. What is the forecast for week 7?

2. Refer to the time series data in Problem 1. Using the average of all the historical data as a forecast for the next period, compute the following measures of forecast accuracy:
   a. Mean absolute error
   b. Mean squared error
   c. Mean absolute percentage error
   d. What is the forecast for week 7?

SELF test

3. Problems 1 and 2 used different forecasting methods. Which method appears to provide the more accurate forecasts for the historical data? Explain.

4. Consider the following time series data:

Month	1	2	3	4	5	6	7
Value	24	13	20	12	19	23	15

   a. Compute MSE using the most recent value as the forecast for the next period. What is the forecast for month 8?
   b. Compute MSE using the average of all the data available as the forecast for the next period. What is the forecast for month 8?
   c. Which method appears to provide the better forecast?

SELF test

5. Consider the following time series data:

Week	1	2	3	4	5	6
Value	18	13	16	11	17	14

   a. Construct a time series plot. What type of pattern exists in the data?
   b. Develop a three-week moving average for this time series. Compute MSE and a forecast for week 7.
   c. Use $\alpha = 0.2$ to compute the exponential smoothing values for the time series. Compute MSE and a forecast for week 7.
   d. Compare the three-week moving average forecast with the exponential smoothing forecast using $\alpha = 0.2$. Which appears to provide the better forecast based on MSE? Explain.
   e. Use Excel Solver or LINGO to find the value of α that minimizes MSE. (*Hint:* Minimize the sum of squared error.)

6. Consider the following time series data:

Month	1	2	3	4	5	6	7
Value	24	13	20	12	19	23	15

   a. Construct a time series plot. What type of pattern exists in the data?
   b. Develop a three-week moving average for this time series. Compute MSE and a forecast for week 8.
   c. Use $\alpha = 0.2$ to compute the exponential smoothing values for the time series. Compute MSE and a forecast for week 8.

d. Compare the three-week moving average forecast with the exponential smoothing forecast using $\alpha = 0.2$. Which appears to provide the better forecast based on MSE?

e. Use Excel Solver or LINGO to find the value of α that minimizes MSE. (*Hint:* Minimize the sum of squared error.)

WEB file
Gasoline

7. Refer to the gasoline sales time series data in Table 15.1.

a. Compute four-week and five-week moving averages for the time series.

b. Compute the MSE for the four-week and five-week moving average forecasts.

c. What appears to be the best number of weeks of past data (three, four, or five) to use in the moving average computation? Recall that MSE for the three-week moving average is 10.22.

WEB file
Gasoline

8. Refer again to the gasoline sales time series data in Table 15.1.

a. Using a weight of ½ for the most recent observation, ⅓ for the second most recent, and ⅙ for third most recent, compute a three-week weighted moving average for the time series.

b. Compute the MSE for the weighted moving average in part (a). Do you prefer this weighted moving average to the unweighted moving average? Remember that the MSE for the unweighted moving average is 10.22.

c. Suppose you are allowed to choose any weights as long as they sum to 1. Could you always find a set of weights that would make the MSE at least as small as for a weighted moving average than for an unweighted moving average? Why or why not?

WEB file
Gasoline

9. With the gasoline time series data from Table 15.1, show the exponential smoothing forecasts using $\alpha = 0.1$.

a. Applying the MSE measure of forecast accuracy, would you prefer a smoothing constant of $\alpha = 0.1$ or $\alpha = 0.2$ for the gasoline sales time series?

b. Are the results the same if you apply MAE as the measure of accuracy?

c. What are the results if MAPE is used?

10. With a smoothing constant of $\alpha = 0.2$, equation (15.5) shows that the forecast for week 13 of the gasoline sales data from Table 15.1 is given by $F_{13} = 0.2Y_{12} + 0.8F_{12}$. However, the forecast for week 12 is given by $F_{12} = 0.2Y_{11} + 0.8F_{11}$. Thus, we could combine these two results to show that the forecast for week 13 can be written

$$F_{13} = 0.2Y_{12} + 0.8(0.2Y_{11} + 0.8F_{11}) = 0.2Y_{12} + 0.16Y_{11} + 0.64F_{11}$$

a. Making use of the fact that $F_{11} = 0.2Y_{10} + 0.8F_{10}$ (and similarly for F_{10} and F_9), continue to expand the expression for F_{13} until it is written in terms of the past data values Y_{12}, Y_{11}, Y_{10}, Y_9, Y_8, and the forecast for period 8.

b. Refer to the coefficients or weights for the past values Y_{12}, Y_{11}, Y_{10}, Y_9, Y_8; what observation can you make about how exponential smoothing weights past data values in arriving at new forecasts? Compare this weighting pattern with the weighting pattern of the moving averages method.

11. For the Hawkins Company, the monthly percentages of all shipments received on time over the past 12 months are 80, 82, 84, 83, 83, 84, 85, 84, 82, 83, 84, and 83.

a. Construct a time series plot. What type of pattern exists in the data?

b. Compare a three-month moving average forecast with an exponential smoothing forecast for $\alpha = 0.2$. Which provides the better forecasts using MSE as the measure of model accuracy?

c. What is the forecast for next month?

12. Corporate triple A bond interest rates for 12 consecutive months follow:

9.5	9.3	9.4	9.6	9.8	9.7	9.8	10.5	9.9	9.7	9.6	9.6

a. Construct a time series plot. What type of pattern exists in the data?
b. Develop three-month and four-month moving averages for this time series. Does the three-month or four-month moving average provide the better forecasts based on MSE? Explain.
c. What is the moving average forecast for the next month?

SELF test

13. The values of Alabama building contracts (in millions of dollars) for a 12-month period follow:

240 350 230 260 280 320 220 310 240 310 240 230

a. Construct a time series plot. What type of pattern exists in the data?
b. Compare a three-month moving average forecast with an exponential smoothing forecast. Use $\alpha = 0.2$. Which provides the better forecasts based on MSE?
c. What is the forecast for the next month?

14. The following time series shows the sales of a particular product over the past 12 months:

Month	Sales	Month	Sales
1	105	7	145
2	135	8	140
3	120	9	100
4	105	10	80
5	90	11	100
6	120	12	110

a. Construct a time series plot. What type of pattern exists in the data?
b. Use $\alpha = 0.3$ to compute the exponential smoothing values for the time series.
c. Use Excel Solver or LINGO to find the value of α that minimizes MSE. (*Hint*: Minimize the sum of squared error.)

15. Ten weeks of data on the Commodity Futures Index are 7.35, 7.40, 7.55, 7.56, 7.60, 7.52, 7.52, 7.70, 7.62, and 7.55.
a. Construct a time series plot. What type of pattern exists in the data?
b. Use Excel Solver or LINGO to find the value of α that minimizes MSE. (*Hint:* Minimize the sum of squared error.)

WEB file
Masters

16. The Nielsen ratings (percentage of U.S. households that tuned in) for the Masters golf tournament from 1997 through 2008 follow (*Golf Magazine*, January 2009):

Year	Rating
1997	11.2
1998	8.6
1999	7.9
2000	7.6
2001	10.7
2002	8.1
2003	6.9
2004	6.7
2005	8.0
2006	6.9
2007	7.6
2008	7.3

The rating of 11.2 in 1997 indicates that 11.2% of U.S. households tuned in to watch Tiger Woods win his first major golf tournament and become the first African-American to win the Masters. Tiger Woods also won the Masters in 2001 and 2005.

a. Construct a time series plot. What type of pattern exists in the data? Discuss some of the factors that may have resulted in the pattern exhibited in the time series plot for this time series.

b. Given the pattern of the time series plot developed in part (a), do you think the forecasting methods discussed in this section are appropriate to develop forecasts for this time series? Explain.

c. Would you recommend using only the Nielsen ratings for 2002–2008 to forecast the rating for 2009, or should the entire time series from 1997–2008 be used? Explain.

SELF test

17. Consider the following time series:

t	1	2	3	4	5
Y_t	6	11	9	14	15

a. Construct a time series plot. What type of pattern exists in the data?

b. Use Excel Solver or LINGO to find the parameters for the line that minimizes MSE this time series.

c. What is the forecast for $t = 6$?

18. The following table reports the percentage of stocks in a portfolio for nine quarters from 2007 to 2009:

Quarter	Stock %
1st—2007	29.8
2nd—2007	31.0
3rd—2007	29.9
4th—2007	30.1
1st—2008	32.2
2nd—2008	31.5
3rd—2008	32.0
4th—2008	31.9
1st—2009	30.0

a. Construct a time series plot. What type of pattern exists in the data?

b. Use exponential smoothing to forecast this time series. Using Excel Solver or LINGO find the value of α that minimizes the sum of squared error.

c. What is the forecast of the percentage of stocks in a typical portfolio for the second quarter of 2009?

19. Consider the following time series:

t	1	2	3	4	5	6	7
Y_t	120	110	100	96	94	92	88

a. Construct a time series plot. What type of pattern exists in the data?

b. Use Excel Solver or LINGO to find the parameters for the line that minimizes MSE this time series.

c. What is the forecast for $t = 8$?

20. Consider the following time series:

t	1	2	3	4	5	6	7
Y_t	82	60	44	35	30	29	35

a. Construct a time series plot. What type of pattern exists in the data?
b. Using LINGO or EXCEL Solver, develop the quadratic trend equation for the time series.
c. What is the forecast for $t = 8$?

SELF test

21. Because of high tuition costs at state and private universities, enrollments at community colleges have increased dramatically in recent years. The following data show the enrollment (in thousands) for Jefferson Community College from 2001–2009:

Year	Period (t)	Enrollment (1000s)
2001	1	6.5
2002	2	8.1
2003	3	8.4
2004	4	10.2
2005	5	12.5
2006	6	13.3
2007	7	13.7
2008	8	17.2
2009	9	18.1

a. Construct a time series plot. What type of pattern exists in the data?
b. Use Excel Solver or LINGO to find the parameters for the line that minimizes MSE this time series.
c. What is the forecast for 2010?

22. The Seneca Children's Fund (SCC) is a local charity that runs a summer camp for disadvantaged children. The fund's board of directors has been working very hard over recent years to decrease the amount of overhead expenses, a major factor in how charities are rated by independent agencies. The following data show the percentage of the money SCC has raised that were spent on administrative and fund-raising expenses for 2003–2009:

Year	Period (t)	Expense (%)
2003	1	13.9
2004	2	12.2
2005	3	10.5
2006	4	10.4
2007	5	11.5
2008	6	10.0
2009	7	8.5

a. Construct a time series plot. What type of pattern exists in the data?
b. Use Excel Solver or LINGO to find the parameters for the line that minimizes MSE this time series.
c. Forecast the percentage of administrative expenses for 2010.
d. If SCC can maintain their current trend in reducing administrative expenses, how long will it take them to achieve a level of 5% or less?

23. The president of a small manufacturing firm is concerned about the continual increase in manufacturing costs over the past several years. The following figures provide a time series of the cost per unit for the firm's leading product over the past eight years:

Year	Cost/Unit ($)	Year	Cost/Unit ($)
1	20.00	5	26.60
2	24.50	6	30.00
3	28.20	7	31.00
4	27.50	8	36.00

a. Construct a time series plot. What type of pattern exists in the data?
b. Use Excel Solver or LINGO to find the parameters for the line that minimizes MSE this time series.
c. What is the average cost increase that the firm has been realizing per year?
d. Compute an estimate of the cost/unit for next year.

WEB file
Exchange Rate

24. FRED® (Federal Reserve Economic Data), a database of more than 3000 U.S. economic time series, contains historical data on foreign exchange rates. The following data show the foreign exchange rate for the United States and China (http://research.stlouisfed.org/fred2/). The units for Rate are the number of Chinese yuan renmimbis to one U.S. dollar.

Year	Month	Rate
2007	October	7.5019
2007	November	7.4210
2007	December	7.3682
2008	January	7.2405
2008	February	7.1644
2008	March	7.0722
2008	April	6.9997
2008	May	6.9725
2008	June	6.8993
2008	July	6.8355

a. Construct a time series plot. Does a linear trend appear to be present?
b. Use Excel Solver or LINGO to find the parameters for the line that minimizes MSE this time series.
c. Use the trend equation to forecast the exchange rate for August 2008.
d. Would you feel comfortable using the trend equation to forecast the exchange rate for December 2008?

25. Automobile unit sales at B. J. Scott Motors, Inc., provided the following 10-year time series:

Year	Sales	Year	Sales
1	400	6	260
2	390	7	300
3	320	8	320
4	340	9	340
5	270	10	370

a. Construct a time series plot. Comment on the appropriateness of a linear trend.
b. Using Excel Solver or LINGO, develop a quadratic trend equation that can be used to forecast sales.

c. Using the trend equation developed in part (b), forecast sales in year 11.
d. Suggest an alternative to using a quadratic trend equation to forecast sales. Explain.

WEB file
Pasta

26. Giovanni Food Products produces and sells frozen pizzas to public schools throughout the eastern United States. Using a very aggressive marketing strategy they have been able to increase their annual revenue by approximately $10 million over the past 10 years. But, increased competition has slowed their growth rate in the past few years. The annual revenue, in millions of dollars, for the previous 10 years is shown below.

Year	Revenue
1	8.53
2	10.84
3	12.98
4	14.11
5	16.31
6	17.21
7	18.37
8	18.45
9	18.40
10	18.43

a. Construct a time series plot. Comment on the appropriateness of a linear trend.
b. Using Excel Solver or LINGO, develop a quadratic trend equation that can be used to forecast revenue.
c. Using the trend equation developed in part (b), forecast revenue in year 11.

WEB file
NFL Value

27. *Forbes* magazine (www.Forbes.com) ranks NFL teams by value each year. The data below are the value of the Indianapolis Colts from 1998 to 2008.

Year	Period	Value ($ million)
1998	1	227
1999	2	305
2000	3	332
2001	4	367
2002	5	419
2003	6	547
2004	7	609
2005	8	715
2006	9	837
2007	10	911
2008	11	1076

a. Construct a time series plot. What type of pattern exists in the data?
b. Using Excel Solver or LINGO, develop the quadratic trend equation that can be used to forecast the team's value.
c. Using Excel Solver or LINGO, develop the exponential trend equation that can be used to forecast the team's value.
d. Using Excel Solver or LINGO, develop the linear trend equation that can be used to forecast the team's value.
e. Which equation would you recommend using to estimate the team's value in 2009?
f. Use the model you recommended in part (e) to forecast the value of the Colts in 2009.

SELF test

28. Consider the following time series:

Quarter	Year 1	Year 2	Year 3
1	71	68	62
2	49	41	51
3	58	60	53
4	78	81	72

a. Construct a time series plot. What type of pattern exists in the data?
b. Use an Excel or LINGO model with dummy variables as follows to develop an equation to account for seasonal effects in the data. Qtr1 $=$ 1 if Quarter 1, 0 otherwise; Qtr2 $=$ 1 if Quarter 2, 0 otherwise; Qtr3 $=$ 1 if Quarter 3, 0 otherwise.
c. Compute the quarterly forecasts for next year.

29. Consider the following time series data:

Quarter	Year 1	Year 2	Year 3
1	4	6	7
2	2	3	6
3	3	5	6
4	5	7	8

a. Construct a time series plot. What type of pattern exists in the data?
b. Use an Excel or LINGO model with dummy variables as follows to develop an equation to account for seasonal effects in the data. Qtr1 $=$ 1 if Quarter 1, 0 otherwise; Qtr2 $=$ 1 if Quarter 2, 0 otherwise; Qtr3 $=$ 1 if Quarter 3, 0 otherwise.
c. Compute the quarterly forecasts for next year.

30. The quarterly sales data (number of copies sold) for a college textbook over the past three years follow:

Quarter	Year 1	Year 2	Year 3
1	1690	1800	1850
2	940	900	1100
3	2625	2900	2930
4	2500	2360	2615

a. Construct a time series plot. What type of pattern exists in the data?
b. Use an Excel or LINGO model with dummy variables as follows to develop an equation to account for seasonal effects in the data. Qtr1 $=$ 1 if Quarter 1, 0 otherwise; Qtr2 $=$ 1 if Quarter 2, 0 otherwise; Qtr3 $=$ 1 if Quarter 3, 0 otherwise.
c. Compute the quarterly forecasts for next year.
d. Let $t = 1$ to refer to the observation in quarter 1 of year 1; $t = 2$ to refer to the observation in quarter 2 of year 1; . . . and $t = 12$ to refer to the observation in quarter 4 of year 3. Using the dummy variables defined in part (b) and t, develop an equation to account for seasonal effects and any linear trend in the time series. Based upon the seasonal effects in the data and linear trend, compute the quarterly forecasts for next year.

WEB file

Pollution

31. Air pollution control specialists in southern California monitor the amount of ozone, carbon dioxide, and nitrogen dioxide in the air on an hourly basis. The hourly time series data exhibit seasonality, with the levels of pollutants showing patterns that vary over the hours

in the day. On July 15, 16, and 17, the following levels of nitrogen dioxide were observed for the 12 hours from 6:00 A.M. to 6:00 P.M.

July 15:	25	28	35	50	60	60	40	35	30	25	25	20
July 16:	28	30	35	48	60	65	50	40	35	25	20	20
July 17:	35	42	45	70	72	75	60	45	40	25	25	25

a. Construct a time series plot. What type of pattern exists in the data?

b. Use an Excel or LINGO model with dummy variables as follows to develop an equation to account for seasonal effects in the data:

Hour1 = 1 if the reading was made between 6:00 A.M. and 7:00 A.M.; 0 otherwise
Hour2 = 1 if the reading was made between 7:00 A.M. and 8:00 A.M.; 0 otherwise
.
.
.
Hour11 = 1 if the reading was made between 4:00 P.M. and 5:00 P.M., 0 otherwise.
Note that when the values of the 11 dummy variables are equal to 0, the observation corresponds to the 5:00 P.M. to 6:00 P.M. hour.

c. Using the equation developed in part (b), compute estimates of the levels of nitrogen dioxide for July 18.

d. Let $t = 1$ refer to the observation in hour 1 on July 15; $t = 2$ to refer to the observation in hour 2 of July 15; . . . and $t = 36$ to refer to the observation in hour 12 of July 17. Using the dummy variables defined in part (b) and ts, develop an equation to account for seasonal effects and any linear trend in the time series. Based upon the seasonal effects in the data and the linear trend, compute estimates of the levels of nitrogen dioxide for July 18.

WEB file
SouthShore

32. South Shore Construction builds permanent docks and seawalls along the southern shore of Long Island, New York. Although the firm has been in business only five years, revenue has increased from $308,000 in the first year of operation to $1,084,000 in the most recent year. The following data show the quarterly sales revenue in thousands of dollars:

Quarter	Year 1	Year 2	Year 3	Year 4	Year 5
1	20	37	75	92	176
2	100	136	155	202	282
3	175	245	326	384	445
4	13	26	48	82	181

a. Construct a time series plot. What type of pattern exists in the data?

b. Use an Excel or LINGO model with dummy variables as follows to develop an equation to account for seasonal effects in the data. Qtr1 = 1 if Quarter 1, 0 otherwise; Qtr2 = 1 if Quarter 2, 0 otherwise; Qtr3 = 1 if Quarter 3, 0 otherwise.

c. Let Period = 1 to refer to the observation in quarter 1 of year 1; Period = 2 to refer to the observation in quarter 2 of year 1; . . . and Period = 20 refer to the observation in quarter 4 of year 5. Using the dummy variables defined in part (b) and Period, develop an equation to account for seasonal effects and any linear trend in the time series. Based upon the seasonal effects in the data and linear trend, compute estimates of quarterly sales for year 6.

TABLE 15.18 FOOD AND BEVERAGE SALES FOR THE VINTAGE RESTAURANT ($1000s)

WEB file
Vintage

Month	First Year	Second Year	Third Year
January	242	263	282
February	235	238	255
March	232	247	265
April	178	193	205
May	184	193	210
June	140	149	160
July	145	157	166
August	152	161	174
September	110	122	126
October	130	130	148
November	152	167	173
December	206	230	235

Case Problem 1 FORECASTING FOOD AND BEVERAGE SALES

The Vintage Restaurant, on Captiva Island near Fort Myers, Florida, is owned and operated by Karen Payne. The restaurant just completed its third year of operation. During that time, Karen sought to establish a reputation for the restaurant as a high-quality dining establishment that specializes in fresh seafood. Through the efforts of Karen and her staff, her restaurant has become one of the best and fastest-growing restaurants on the island.

To better plan for future growth of the restaurant, Karen needs to develop a system that will enable her to forecast food and beverage sales by month for up to one year in advance. Table 15.18 shows the value of food and beverage sales ($1000s) for the first three years of operation.

Managerial Report

Perform an analysis of the sales data for the Vintage Restaurant. Prepare a report for Karen that summarizes your findings, forecasts, and recommendations. Include the following:

1. A time series plot. Comment on the underlying pattern in the time series.
2. Using the dummy variable approach, forecast sales for January through December of the fourth year.

Assume that January sales for the fourth year turn out to be $295,000. What was your forecast error? If this error is large, Karen may be puzzled about the difference between your forecast and the actual sales value. What can you do to resolve her uncertainty in the forecasting procedure?

Case Problem 2 FORECASTING LOST SALES

The Carlson Department Store suffered heavy damage when a hurricane struck on August 31. The store was closed for four months (September through December), and Carlson is now involved in a dispute with its insurance company about the amount of lost sales during the time the store was closed. Two key issues must be resolved: (1) the

TABLE 15.19 SALES FOR CARLSON DEPARTMENT STORE ($ MILLIONS)

WEB file
CarlsonSales

Month	Year 1	Year 2	Year 3	Year 4	Year 5
January		1.45	2.31	2.31	2.56
February		1.80	1.89	1.99	2.28
March		2.03	2.02	2.42	2.69
April		1.99	2.23	2.45	2.48
May		2.32	2.39	2.57	2.73
June		2.20	2.14	2.42	2.37
July		2.13	2.27	2.40	2.31
August		2.43	2.21	2.50	2.23
September	1.71	1.90	1.89	2.09	
October	1.90	2.13	2.29	2.54	
November	2.74	2.56	2.83	2.97	
December	4.20	4.16	4.04	4.35	

amount of sales Carlson would have made if the hurricane had not struck, and (2) whether Carlson is entitled to any compensation for excess sales due to increased business activity after the storm. More than $8 billion in federal disaster relief and insurance money came into the county, resulting in increased sales at department stores and numerous other businesses.

Table 15.19 gives Carlson's sales data for the 48 months preceding the storm. Table 15.20 reports total sales for the 48 months preceding the storm for all department stores in the county, as well as the total sales in the county for the four months the Carlson Department Store was closed. Carlson's managers asked you to analyze these data and develop estimates of the lost sales at the Carlson Department Store for the months of September through December. They also asked you to determine whether a case can be made for excess storm-related sales during the same period. If such a case can be made, Carlson is entitled to compensation for excess sales it would have earned in addition to ordinary sales.

TABLE 15.20 DEPARTMENT STORE SALES FOR THE COUNTY ($ MILLIONS)

WEB file
CountySales

Month	Year 1	Year 2	Year 3	Year 4	Year 5
January		46.80	46.80	43.80	48.00
February		48.00	48.60	45.60	51.60
March		60.00	59.40	57.60	57.60
April		57.60	58.20	53.40	58.20
May		61.80	60.60	56.40	60.00
June		58.20	55.20	52.80	57.00
July		56.40	51.00	54.00	57.60
August		63.00	58.80	60.60	61.80
September	55.80	57.60	49.80	47.40	69.00
October	56.40	53.40	54.60	54.60	75.00
November	71.40	71.40	65.40	67.80	85.20
December	117.60	114.00	102.00	100.20	121.80

Managerial Report

Prepare a report for the managers of the Carlson Department Store that summarizes your findings, forecasts, and recommendations. Include the following:

1. An estimate of sales for Carlson Department Store had there been no hurricane
2. An estimate of countywide department store sales had there been no hurricane
3. An estimate of lost sales for the Carlson Department Store for September through December

In addition, use the countywide actual department stores sales for September through December and the estimate in part (2) to make a case for or against excess storm-related sales.

Appendix 15.1 FORECASTING WITH EXCEL DATA ANALYSIS TOOLS

In this appendix we show how Excel can be used to develop forecasts using three forecasting methods: moving averages, exponential smoothing, and trend projection. We also show how to use Excel Solver for least-squares fitting of models to data.

Moving Averages

WEB file
Gasoline

To show how Excel can be used to develop forecasts using the moving averages method, we will develop a forecast for the gasoline sales time series in Table 15.1 and Figure 15.1. The sales data for the 12 weeks are entered into worksheet rows 2 through 13 of column B. The following steps can be used to produce a three-week moving average.

Step 1. Click the **Data** tab on the Ribbon
Step 2. In the **Analysis** group, click **Data Analysis**
Step 3. Choose **Moving Average** from the list of Analysis Tools
Click **OK**
Step 4. When the Moving Average dialog box appears:
Enter B2:B13 in the **Input Range** box
Enter 3 in the **Interval** box
Enter C2 in the **Output Range** box
Click OK

The three-week moving averages will appear in column C of the worksheet. The forecast for week 4 appears next to the sales value for week 3, and so on. Forecasts for periods of other length can be computed easily by entering a different value in the Interval box.

Exponential Smoothing

To show how Excel can be used for exponential smoothing, we again develop a forecast for the gasoline sales time series in Table 15.1 and Figure 15.1. The sales data for the 12 weeks are entered into worksheet rows 2 through 13 of column B. The following steps can be used to produce a forecast using a smoothing constant of $\alpha = .2$.

Step 1. Click the **Data** tab on the Ribbon
Step 2. In the **Analysis** group, click **Data Analysis**
Step 3. Choose **Exponential Smoothing** from the list of Analysis Tools
Click **OK**