An Introduction to the Happy Ending Problem and the Erdős–Szekeres Conjecture

Thesis

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Abstract

In 1932, Esther Klein made the observation that among any 5 points in general position in the plane, one can always find 4 points that form a convex quadrilateral. This problem came to be known as the Happy Ending Problem. She then posed the following generalization of her observation: how many points in the plane are needed to guarantee a convex n-gon, if such a number of points exists at all? Two of the mathematicians Klein was working with at the time, Paul Erdős and George Szekeres, published their progress on Klein's generalized problem in 1935. Their paper proved that for any natural number $n \geq 3$, there exists a minimal number $ES(n)$ such that any configuration of $ES(n)$ points in general position in the plane is guaranteed to contain a convex n-gon. They also attempted to find a formula for $ES(n)$, and while they did not actually prove that $ES(n)$ could be written as a function of n, they did conjecture that $ES(n) = 2^{n-2} + 1$. This has come to be known as the Erdős–Szekeres Conjecture. Despite the work of many mathematicians since, this conjecture still remains unproven. This paper details the history of the Happy Ending Problem and the Erdős–Szekeres Conjecture, provides proofs for values of $ES(n)$ for some specific n, and discusses the progress that has been made toward proving the Erdős–Szekeres Conjecture.

Acknowledgements

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Chapter 1

Introduction

Some of the most interesting problems in mathematics have a common structure: an easy to understand question that turns out to be extremely difficult to answer. The Happy Ending Problem and the related Erdős–Szekeres Conjecture are perfect examples of problems with this structure. While the initial Happy Ending Problem was born from a simple observation, its generalization to the Erdős–Szekeres Conjecture has remained unproven since 1935. In this chapter, we will take a look into the original Happy Ending Problem, discuss its origins and history, and examine the long standing conjecture to which it lead.

1.1 Original Problem

The original Happy Ending Problem was introduced by mathematician Esther Klein, who observed that given any set of 5 points in general position in the plane (that is, no 3 points lie on the same line), one can find a subset of the 5 points that form a convex quadrilateral [11]. We will prove this fact in the next chapter. After this observation, Klein proposed a more general related problem: how many points are needed to guarantee that any configuration of those points in general position will contain a subset that forms a convex n-gon?

At the time of Klein's proposed problem, she was working with a group of fellow mathematicians (who were students at the time), and two of her peers were Paul Erdős and George Szekeres [3]. Erdős and Szekeres ended up working quite a bit on Klein's general problem, which in turn ultimately lead to Klein and Szekeres marrying in 1937. This marriage led to Erdős naming the problem the Happy Ending Problem. Szekeres and Klein remained married until they died within one hour of each other in 2005.

1.2 Erdős–Szekeres Conjecture

Erdős and Szekeres considered Klein's problem, and they noted two important questions that were related to it. First, for any natural number n, can we always find a large enough set of points such that any configuration of the points in general position will be guaranteed to contain a convex n-gon? Or is it possible that for some n there exist arbitrarily large sets of points that do not contain a convex n-gon? Second, if we can always find a large enough set of points to guarantee a convex n-gon, is there a function of n that will give the minimal size of the set needed to do so [11].

As it turns out, the answer to their first question is "yes," a fact that Erdős and Szekeres proved in 1935 [4] and that we will prove in Chapter 3 of this paper. Their second question, however, was not as easy to answer. In fact, it remains unanswered as of this writing. Though Erdős and Szekeres were not able to definitively answer their second question, they were able to make some progress toward its answer and ultimately make a conjecture, which has come to be known as the Erdős–Szekeres Conjecture. If we let $ES(n)$ represent the minimum number of points in general position in the plane that are needed to guarantee that a subset of those points form a convex n-gon, the the conjecture states:

Erdős–Szekeres Conjecture. $ES(n) = 2^{n-2} + 1$ *for all* $n \geq 3$.

Throughout the rest of this paper, we will look at some of the progress that has been made toward proving this conjecture.

Chapter 2

Specific Results

Before discussing the progress that has been made toward proving the general Erdős–Szekeres conjecture, we will first take a look at some results concerning $ES(n)$ for specific values of n.

2.1 Notation, Terminology, and Definitions

We will need to discuss some notation, terminology, and definitions before we proceed. As at the end of the last chapter, we will continue to use $ES(n)$ to represent the minimum number of points in the plane that are needed to guarantee that any configuration of the points in general position will contain a subset of points that form a convex n-gon. We should also recall that a set of points are in "general position" if no three points of the set lie on the same line. We now note a few important definitions as they appear in [10].

Definition 2.1.1. *A set $C \subset \mathbb{R}^2$ is* convex *if for every two points $x, y \in C$ the whole segment xy is also contained in C. In other words, for every $t \in [0, 1]$, the point $tx + (1-t)y$ belongs to C.*

Definition 2.1.2. *Given a set $X \subset \mathbb{R}^2$, the* convex hull *of X, denoted* $\text{conv}(X)$, *is the intersection of all convex sets in $\mathbb{R}^2$ that contain X.*

We can now use this notion of convex hull to define what it means for a set of points to be in convex position.

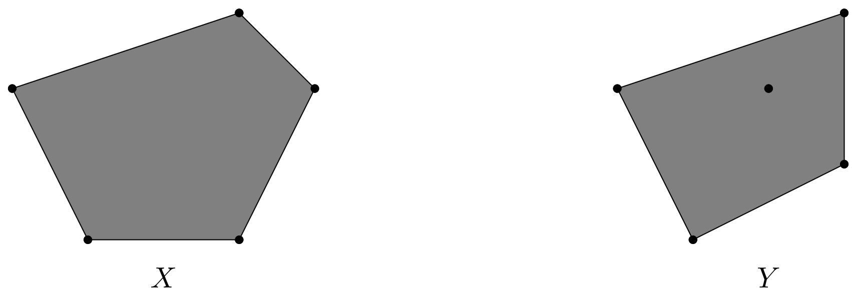


Figure 2.1: The set of points X is in convex position, while the set of points Y is not because one of its points is in the interior of $\operatorname{conv}(Y)$.

Definition 2.1.3. *A set of points $X \subset \mathbb{R}^2$ is in* convex position *if none of the points are interior points of conv(X). That is, a set X of points is in* convex position *if all of the points are vertices of conv(X).*

An example of a convex set of points X and a non-convex (or concave) set of points Y can be seen in Figure 2.1.

2.2 $ES(3) = 3$

We will now begin by taking a look at the most basic version of our problem, that is, the version of our problem with the smallest possible n. Because n is the number of vertices of a convex polygon, first note that it does not make any sense to consider $n = 1$ or $n = 2$, as we need at least three vertices to create a polygon in Euclidean space. Thus, we will start with the case of $n = 3$.

Theorem 2.2.1. $ES(3) = 3$. *In other words, any 3 points in general position in the plane is guaranteed to contain a convex triangle, and any fewer points will not be guaranteed to contain a convex triangle.*

Proof. First note that because we are trying to find the smallest number of points in general position that will give us a convex triangle, in particular we need at least 3 points. If we had fewer than 3 points, we would not be able to form a triangle (or any polygon for that matter), so we must have $ES(3) \geq 3$. Now consider 3 points in general position. Because the 3 points are in general position, they must not all lie on the same line. Thus, we can form a triangle with these three points as vertices. Note that any triangle in Euclidean space is convex. So we have $ES(3) \leq 3$, and conclude that we in fact have $ES(3) = 3$. ■

While this result may seem trivial, this is the most basic form of our problem and a good place to start when trying to find $ES(n)$ for specific n. Let us now consider the slightly more complex original problem.

2.3 $ES(4) = 5$

Now that we have taken care of the trivial case, we will consider the case of $n = 4$ that Erdős and Szekeres were first given by Klein. To do so, we first need a few lemmas.

Lemma 2.3.1. $ES(4) > 4$. *In other words, there exist configurations of 4 points in general position in the plane that do not contain a convex quadrilateral.*

Proof. In order to prove the statement of the lemma, it suffices to show one example of a 4-point set in general position that does not contain a convex quadrilateral. Consider the set of points $\{(0,0),(1,1),(1,2),(2,0)\}$ in the plane, which can be seen in Figure 2.2. The points of this set are in general position, and no convex quadrilateral can be formed using these points as vertices. So the lemma holds. ■

We now state and prove Klein's original observation about sets of 5 points.

Lemma 2.3.2. *All sets of 5 points in the plane in general position contain a subset of 4 points that are in convex position. That is, any 5-point set in general position contains a convex quadrilateral.*

Proof. Given a set X of 5 points in general position in the plane, we have three possibilities for $\text{conv}(X)$. The first possibility is that all 5 points of X are vertices of $\text{conv}(X)$, that is,

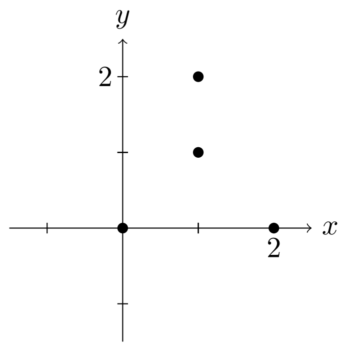


Figure 2.2: A 4-point set in general position that does not contain a convex quadrilateral.

conv(X) is a pentagon. In this case, we can choose any 4 of the 5 points of X to make a convex quadrilateral, as seen in Figure 2.3. The second possibility is that only 4 of the 5 points of X are vertices of conv(X). In this case, we can just take the four vertices of conv(X) to get our convex quadrilateral, as seen in Figure 2.4.

The last possibility is that only 3 points of X are vertices of conv(X) (note that conv(X) cannot have less than 3 vertices because conv(X) is a convex polygon). So in this case, conv(X) is a triangle and 2 points of X are in the interior of conv(X). Let us name the vertices of conv(X) A, B, and C and the interior points D and E. Note that the line through D and E must intersect two of the sides of the triangle $\triangle ABC$. Without loss of generality, we can assume that D and E are oriented in such a way that the line through them intersects segments $\overline{AB}$ and $\overline{AC}$, with ray $\overrightarrow{DE}$ intersecting segment $\overline{AB}$. Then we can now form convex quadrilateral $BCDE$, as seen in Figure 2.5.

We see that in all three possible cases for the shape of conv(X), we can always form a convex quadrilateral. Thus, the lemma holds. ■

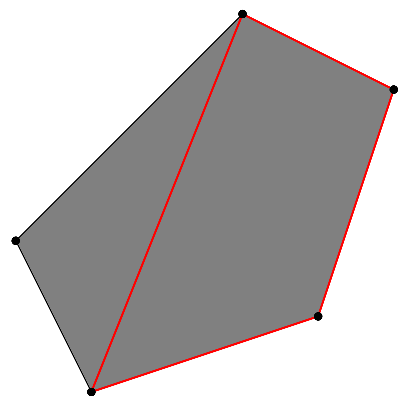


Figure 2.3: If X has 5 points and conv(X) is a pentagon, we can choose any 4 points of X to form a convex quadrilateral.

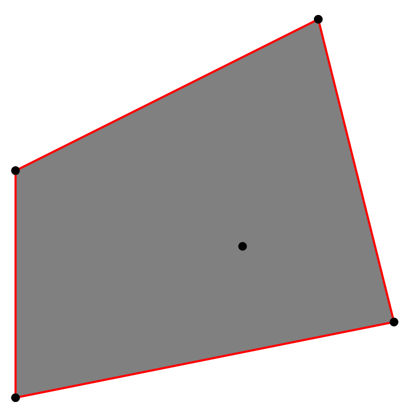


Figure 2.4: If X has 5 points and conv(X) has 4 vertices, we can choose the 4 vertices of conv(X) to form a convex quadrilateral.

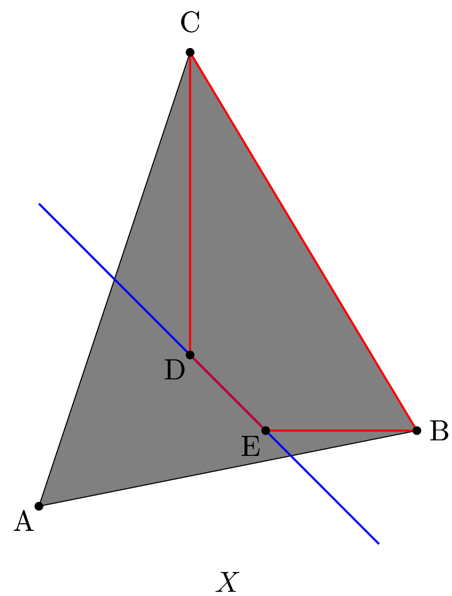


Figure 2.5: If X has 5 points and conv(X) has 3 vertices, we can always form a convex quadrilateral with 2 of the vertices and the 2 interior points.

Now we can proceed with the main theorem of this section.

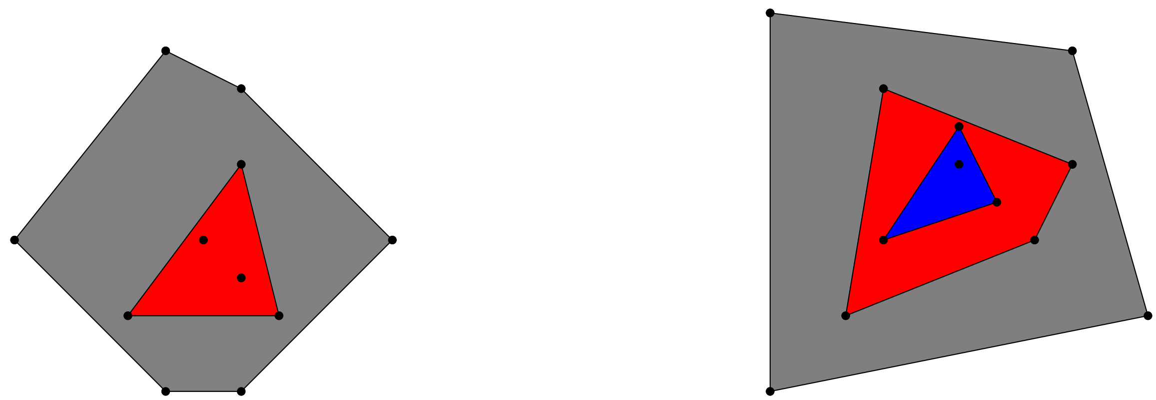
Theorem 2.3.3. $ES(4) = 5$. *In other words, any set of 5 points in general position in the plane is guaranteed to contain a convex quadrilateral, and any fewer points will not guarantee a convex quadrilateral.*

Proof. By Lemma 2.3.1, $ES(4) > 4$. So in particular, $ES(4) \geq 5$. By Lemma 2.3.2, $ES(4) \leq 5$. Thus, we can conclude $ES(4) = 5$. ∎

We see that the proof of $ES(4) = 5$ was a little more involved that the proof of $ES(3) = 3$. This trend will continue as we take a look at $ES(5)$ in the next section.

2.4 $ES(5) = 9$

As stated in [11], Erdős and Szekeres note in [4] that Makai had proven $ES(5) = 9$ and later credited Makai and Turán in [5] with first proving the result, but a proof did not appear



X Y

Figure 2.6: An example of a $(6, 3, 2)$ set of points X and a $(4, 4, 3, 1)$ set of points Y.

in literature until 1970 in a paper by Kalbfleisch, Kalbfleisch, and Stanton [7]. We will be following the proof of Bonnice [1] as it is outlined in [11].

We will first need to discuss some more notation. Given a finite set of points X in the plane, we will say that X is $(k_1, k_2, ..., k_i)$ if $|X| = k_1 + k_2 + ... + k_i$ and $\text{conv}(X)$ is a k_1-gon, the set $\text{conv}(X \setminus \{\text{The vertices of conv}(X)\})$ is a k_2-gon, etc. In other words, the sum $k_1 + k_2 + ... + k_i$ gives the total number of points of X, the convex hull of X has k_1 vertices, the convex hull of the points that would remain when removing the vertices of $\text{conv}(X)$ would be a k_2-gon, and so on. An example of a set X that is $(6, 3, 2)$ and a set Y that is $(4, 4, 3, 1)$ can be seen in Figure 2.6. In addition, if A, B, and C are not collinear, beam $A : BC$ will denote the infinite section of the plane obtained by removing $\text{conv}(\{A, B, C\})$ from the convex cone which has vertex A and is bounded by rays $\overrightarrow{AB}$. and $\overrightarrow{AC}$. Also, if $PQRS$ is a quadrilateral with vertices ordered clockwise, beam $PQ : RS$ will denote the part of the plane obtained by removing $\text{conv}(\{P, Q, R, S\})$ from the convex section of the plane bounded by segment $\overline{PQ}$ and rays $\overrightarrow{PS}$ and $\overrightarrow{QR}$. Examples of these can be seen in Figure 2.7 and Figure 2.8.

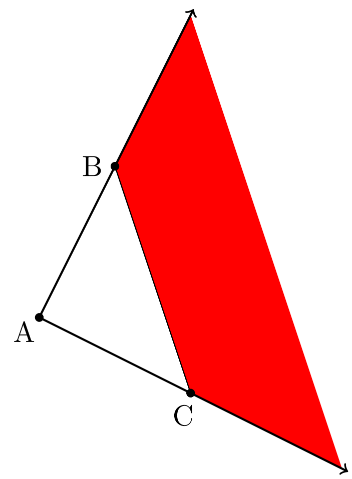


Figure 2.7: The red portion of this picture is beam $A : BC$.

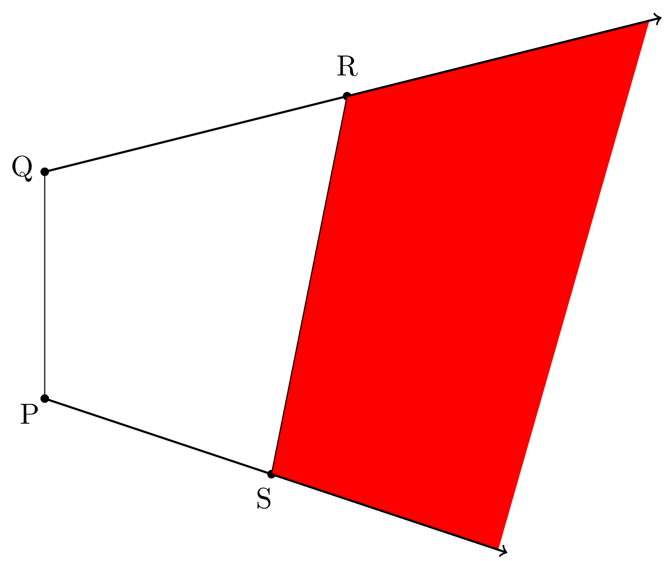


Figure 2.8: The red portion of this picture is beam $PQ : RS$.

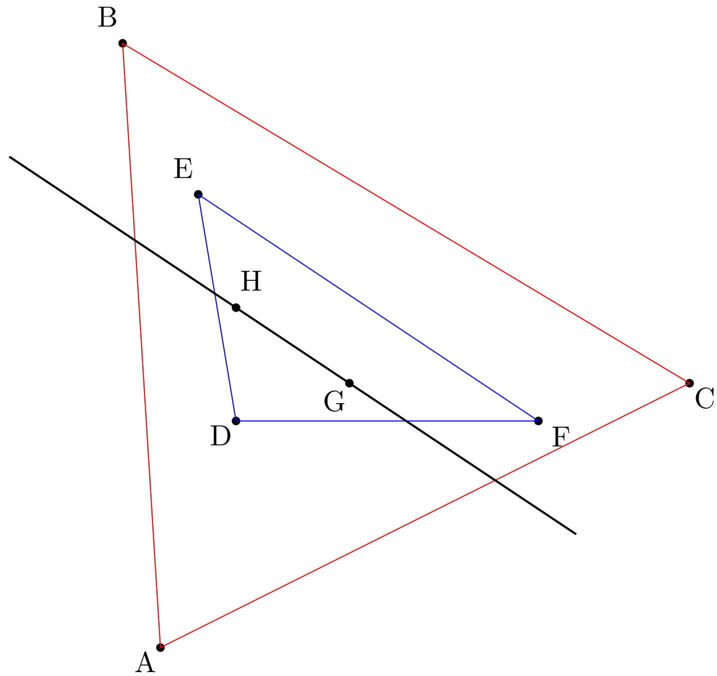


Figure 2.9: Our chosen orientation of D, E, F, G, and H and possible locations for A, B, and C in the $(3, 3, 2)$ case of the proof of Lemma 2.4.1.

We also need a lemma to help prove our theorem about $ES(5)$, which we will state and prove now.

Lemma 2.4.1. *If a planar set X in general position is $(3, 3, 2), (4, 3, 1)$, or $(3, 4, 2)$, then X contains a convex pentagon.*

Proof. We have three configurations to check, so first assume that X is $(3, 3, 2)$. We can call the vertices of $\text{conv}(X)$ A, B, and C, the vertices of the inner triangle D, E, and F (that is, D, E, and F are the vertices of $\text{conv}(X \setminus \{A, B, C\})$), and G and H the interior points of triangle $\triangle DEF$. Without loss of generality, we can assume that G and H are oriented in such a way the line $\overleftrightarrow{GH}$ intersects segments $\overline{DE}$ and $\overline{DF}$ and ray $\overrightarrow{GH}$ intersects ray $\overrightarrow{DE}$. Such an orientation can be seen in Figure 2.9.

Now note that because the union of the regions beam $GH : EF$, beam $G : DF$, and beam $H : DE$ covers the entire plane outside of $\triangle DEF$ as seen in Figure 2.10, we must

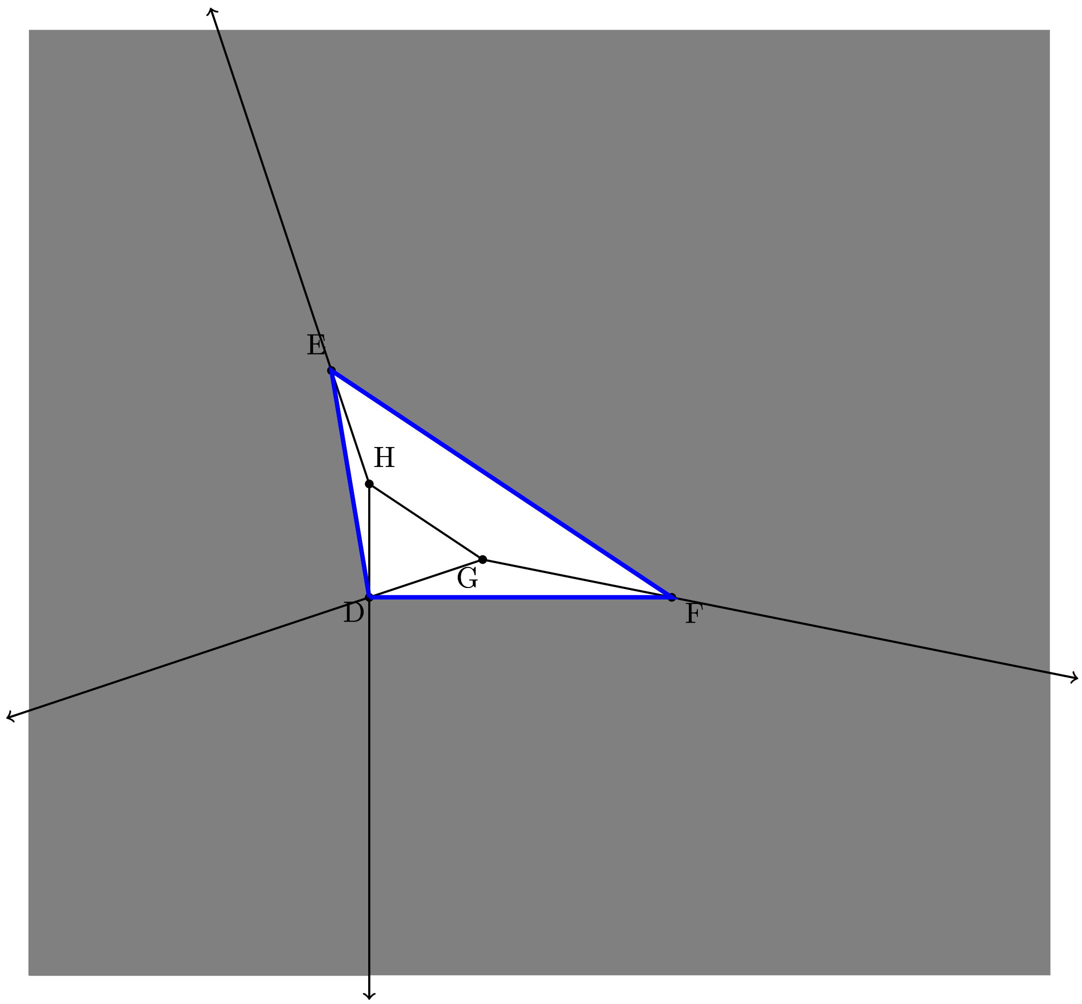


Figure 2.10: The points A, B, and C must be in the union of beams $GH : EF$, $G : DF$, and $H : DE$, shown in gray above.

have that points A, B, and C are contained in the union of these beams.

If one of these points, say A, is contained in beam $GH : EF$, we have that $GHEAF$ is a convex pentagon, as seen in Figure 2.11.

If none of A, B, and C are contained in beam $GH : EF$, then by the pigeonhole principle we must have at least two of the points A, B, and C contained in either beam $G : DF$ or beam $H : DE$. Without loss of generality, let us assume the two points are A and B and they are contained in beam $G : DF$. Because D, E, and F are contained in $\text{conv}(\{A, B, C\})$, we

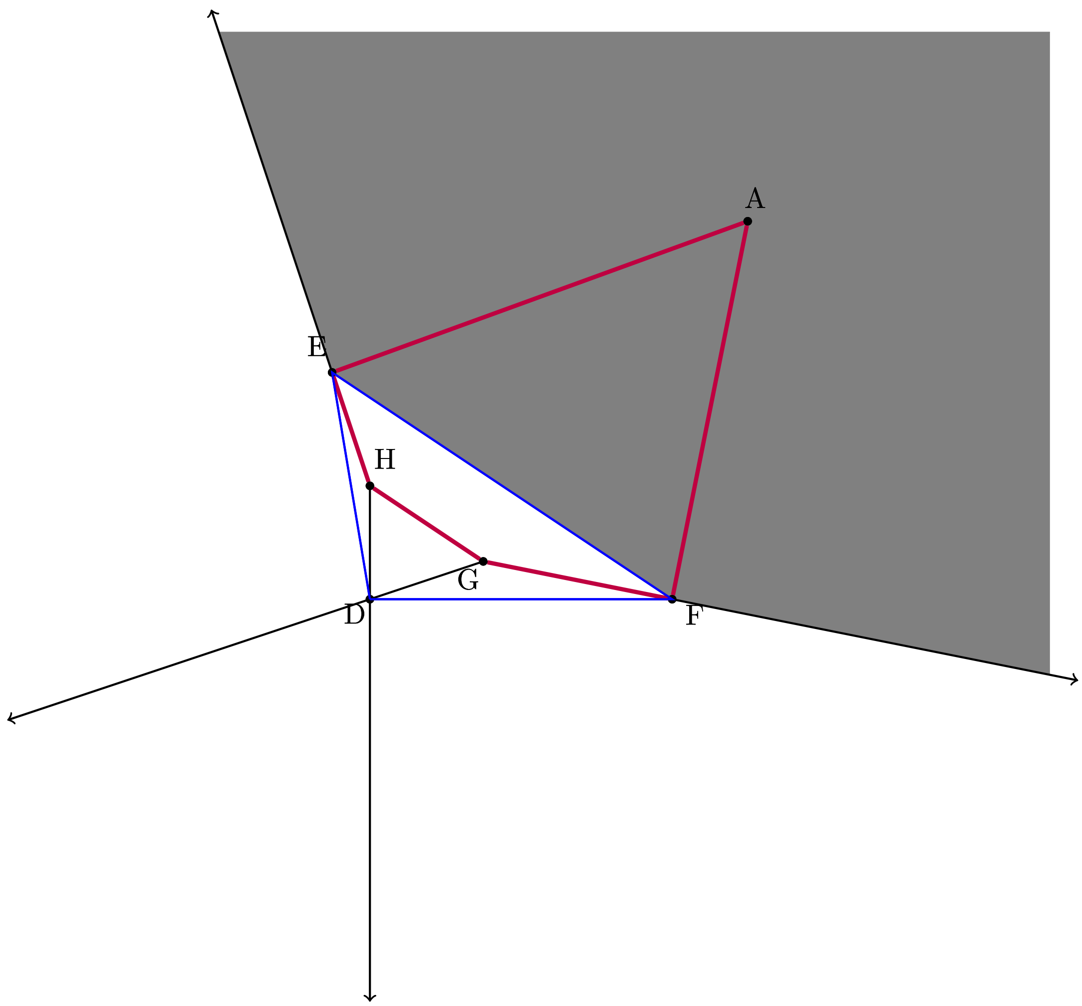


Figure 2.11: If A is in beam $GH : EF$, then $GHEAF$ is a convex pentagon.

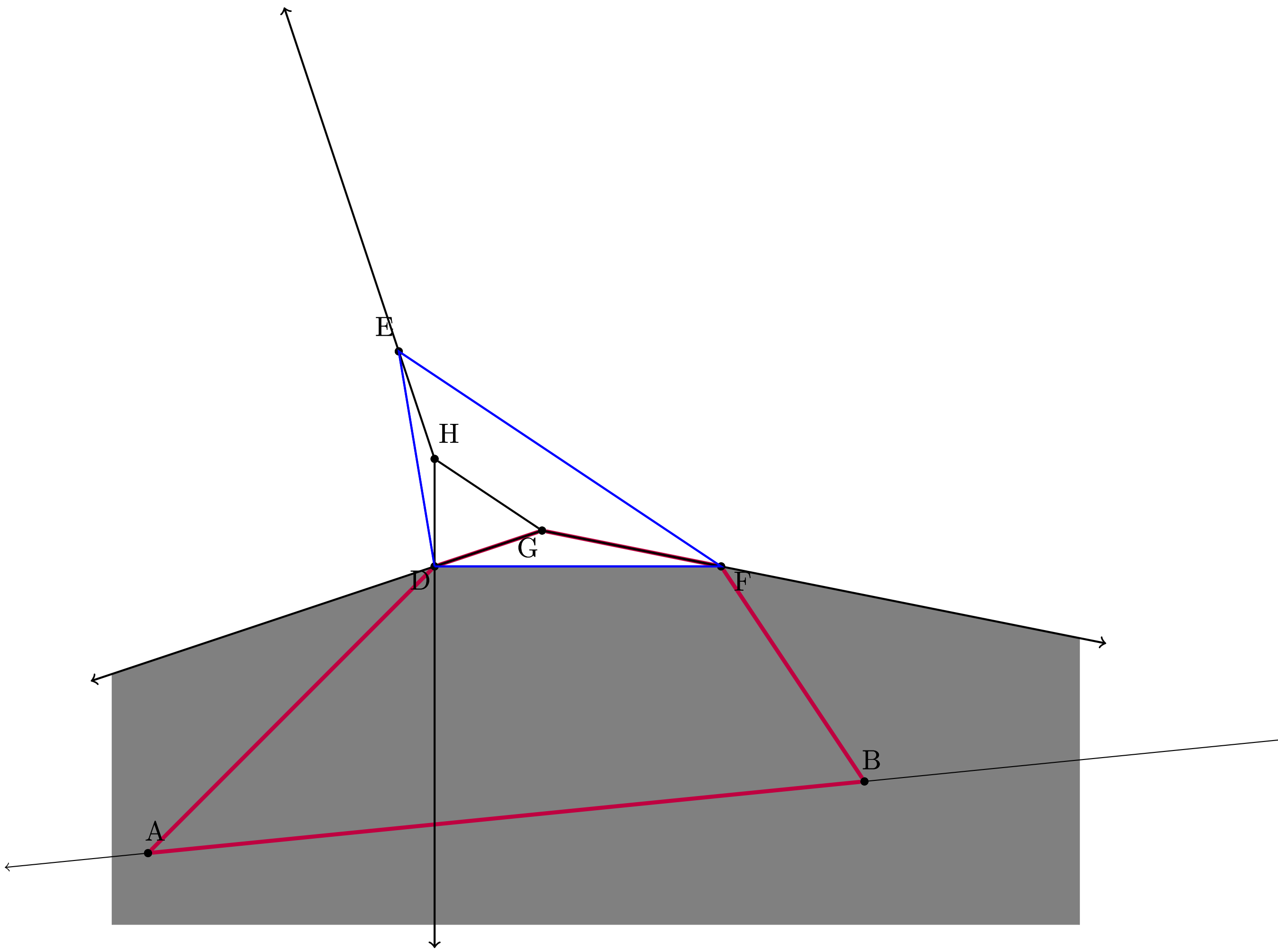


Figure 2.12: If A and B are in beam $G : DF$, then A, B, D, F, and G will determine a convex pentagon.

must have that triangle $\triangle DEF$ lies completely on one side of the line $\overleftrightarrow{AB}$. So in particular, $\overleftrightarrow{AB}$ does not intersect $\text{conv}(\{D, E, F\})$, and we can form a convex pentagon from the points A, B, D, F, and G as seen in Figure 2.12 (note that the orientation of A and B will determine the order in which the vertices of the pentagon can be written). If these 5 points did not form a convex pentagon, we would be forced to have $\overleftrightarrow{AB}$ intersect $\text{conv}(\{D, E, F\})$, which would contradict what we already know about the orientation of the points. So we have shown that any set X that is $(3, 3, 2)$ must indeed contain a convex pentagon.

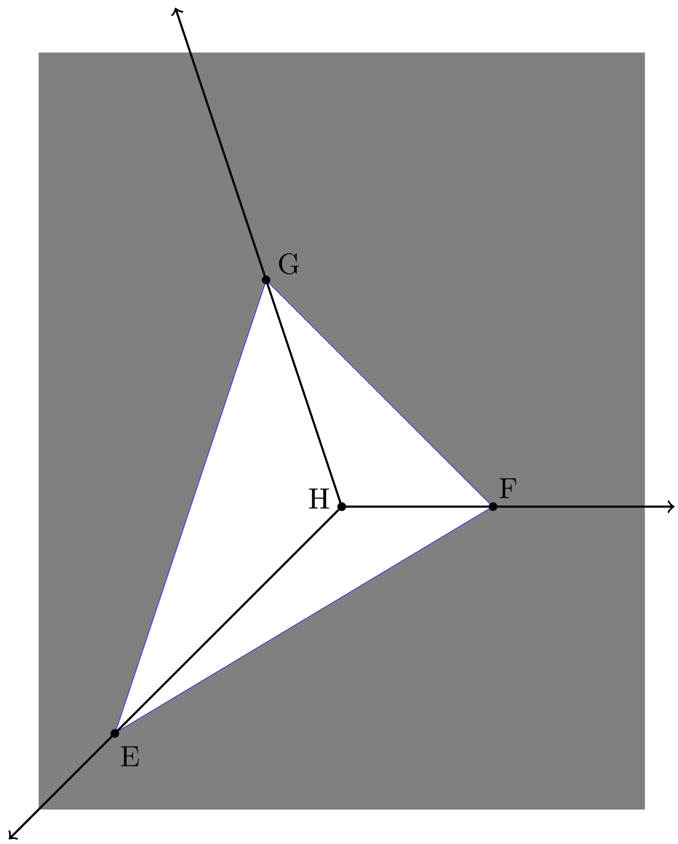


Figure 2.13: In the $(4, 3, 1)$ case of the proof of Lemma 2.4.1, the 3 beams $H : EF$, $H : EG$, and $H : FG$ partition the part of the plane outside of $\triangle EFG$.

We now move onto the $(4, 3, 1)$ orientation. So assume that X is $(4, 3, 1)$, and let us name the vertices of $\text{conv}(X)$ A, B, C and D, the vertices of the next inner triangle E, F, and G, and the last point H. We can use beams $H : EF$, $H : EG$, and $H : FG$ to partition the portion of the plane outside of triangle $\triangle EFG$, as seen in Figure 2.13.

Because A, B, C and D must lie in these beams, we must have that at least two of these points lie in the same beam. Without loss of generality, assume that the points are A and B and that they lie in beam $H : EG$. Then we must have that A, B, G, H, and E determine a convex pentagon as seen in Figure 2.14 (note that the orientation of A and B will determine the order in which the vertices of the pentagon can be written). So we have shown that any set X that is $(4, 3, 1)$ must indeed contain a convex pentagon.

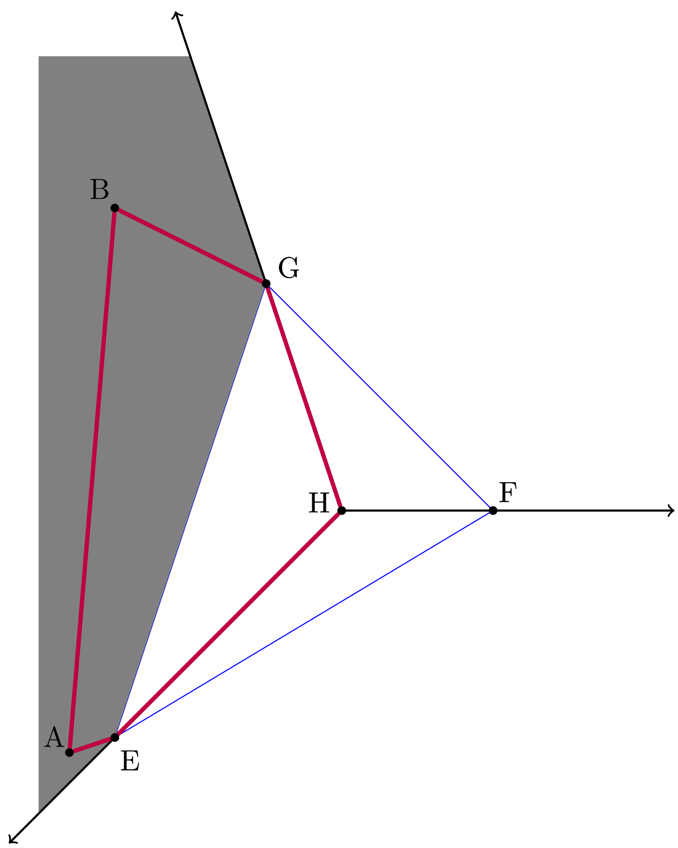


Figure 2.14: If A and B are contained in beam $H : EG$, then A, B, G, H, and E determine a convex pentagon.

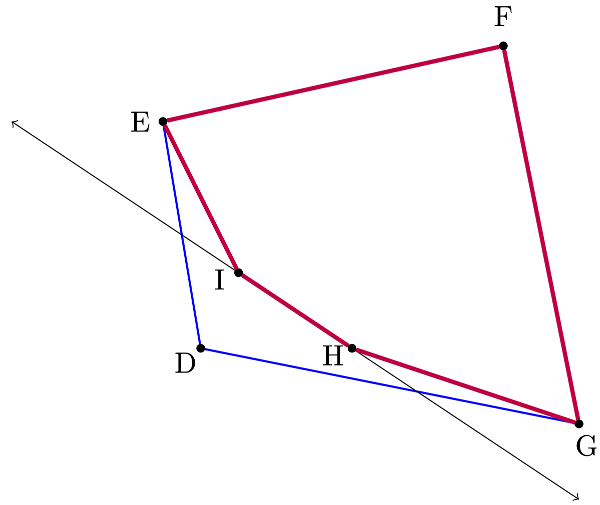


Figure 2.15: If the line through H and I intersects adjacent sides of quadrilateral $DEFG$, then we can form a convex pentagon using H, I, and three of the vertices of $DEFG$.

Lastly, we need to consider the $(3,4,2)$ orientation, so assume that X is $(3,4,2)$ and let A, B, and C be the vertices of $\text{conv}(X)$, let D, E, F, and G be the vertices of $\text{conv}(X \setminus \{A, B, C\})$ oriented clockwise, and let H and I be the interior points of $\text{conv}(\{D, E, F, G\})$. If the line through H and I intersects adjacent sides of quadrilateral $DEFG$, then one vertex of of $DEFG$ will be on one side of line $\overleftrightarrow{HI}$ and the other three vertices of $DEFG$ will be on the other side of $\overleftrightarrow{HI}$. Without loss of generality, assume that $\overleftrightarrow{HI}$ intersects segments $\overline{DE}$ and $\overline{DG}$. Then we would have D on one side of $\overleftrightarrow{HI}$ and E, F, and G on the other side of $\overleftrightarrow{HI}$, and we can form a convex quadrilateral using points E, F, G, H, and I as seen in Figure 2.15 (note that the orientation of H and I will determine the order in which we can write the vertices of the convex pentagon).

If $\overleftrightarrow{HI}$ does not intersect adjacent sides of $DEFG$, it must intersect opposite sides of $DEFG$. Without loss of generality, assume that $\overleftrightarrow{HI}$ intersects sides $\overline{DG}$ and $\overline{EF}$ such that ray $\overrightarrow{HI}$ intersects side $\overline{EF}$. Note that A, B, and C must be contained in the beams $HI : FG$, $HI : ED$, $I : EF$, and $H : DG$. If one of A, B, or C is contained in either beam $HI : FG$ or beam $HI : ED$, we can form a convex pentagon. To see this, without loss of generality assume that A is contained in beam $HI : FG$. Then we can form convex

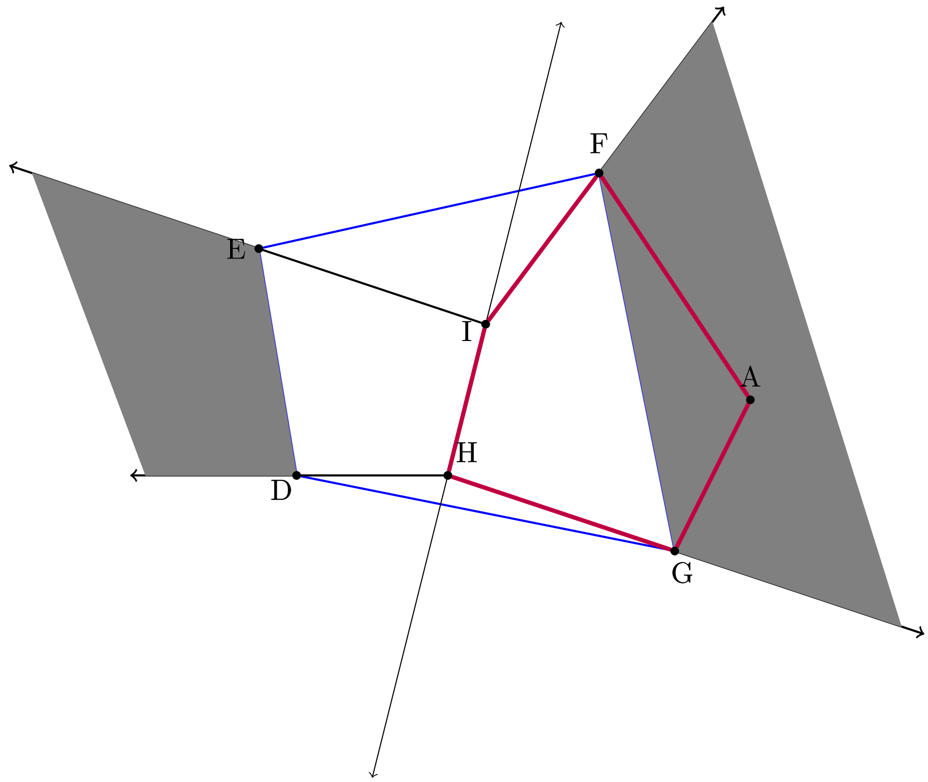


Figure 2.16: If one of the points A, B, or C is contained in beam $HI : FG$ or beam $HI : ED$, we can form a convex pentagon.

pentagon $AFIHG$ as seen in Figure 2.16.

Now if none of A, B, and C are contained in the beams $HI : FG$ and $HI : ED$, then we must have that all three of A, B, and C are contained in the union of the beams $I : EF$ and $H : DG$. So in particular, we must have that two of A, B, and C both lie in either beam $I : EF$ or beam $H : DG$. Without loss of generality, assume that A and B are contained in beam $I : EF$. As before in the $(3, 3, 2)$ case, because all other points of X are contained in $\text{conv}(\{A, B, C\})$, all other points of X must lie on one side of the line $\overleftrightarrow{AB}$. So in particular, $\overleftrightarrow{AB}$ cannot intersect $\text{conv}(\{D, E, F, G\})$, which in turn makes it so that A, B, F, I, and E determine a convex pentagon, as seen in Figure 2.17 (again, the orientation of A and B determine the order in which we can write the vertices of the pentagon).

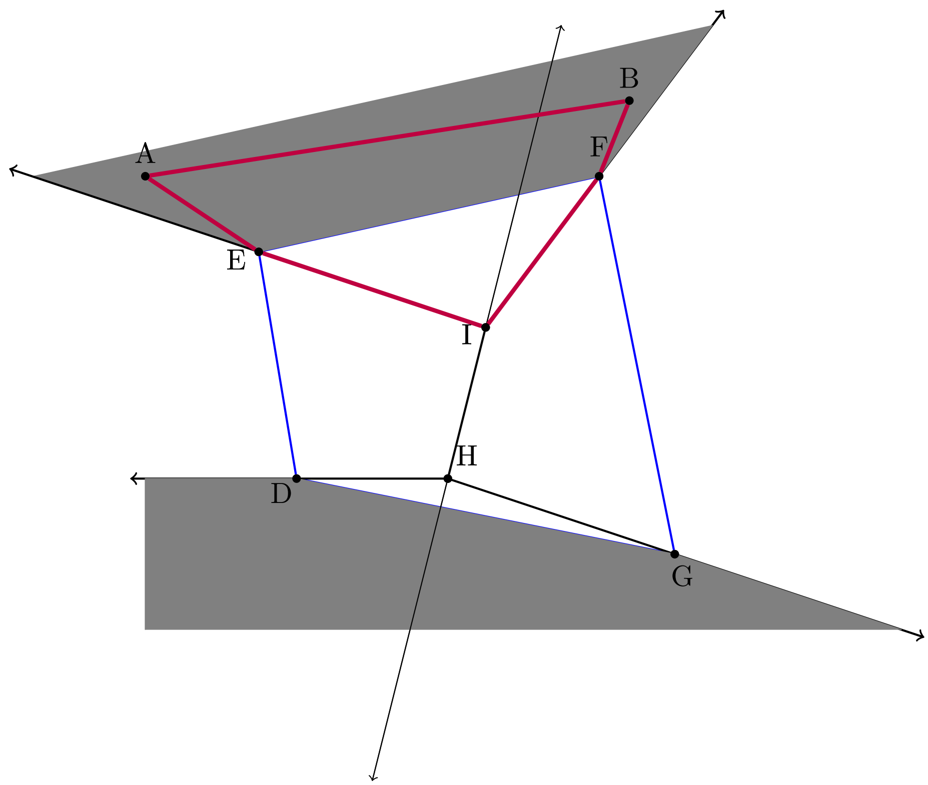


Figure 2.17: If A and B are contained in beam $I : EF$, we can form a convex pentagon.

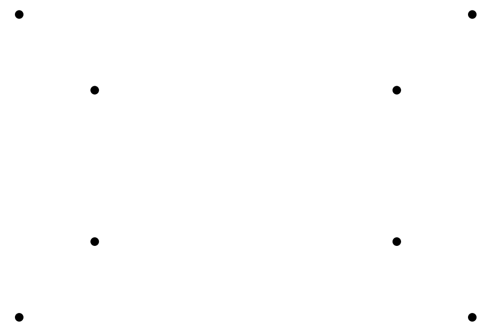


Figure 2.18: An example of an 8-point set in general position that contains no convex pentagon.

So in all cases, X contains a convex pentagon when it is $(3, 4, 2)$. So our lemma has been proven.

■

Now that we have proven this lemma, we can prove the main theorem of this section.

Theorem 2.4.2. $ES(5) = 9$.

Proof. We will first show that $ES(5) > 8$. To do so, it suffices to give one example of an 8-point set in general position that does not contain a convex pentagon. Such a set can be seen in Figure 2.18. So in particular, $ES(5) \geq 9$.

Now we must show that $ES(5) \leq 9$. If we assume that there exist orientations of a 9-point set X in general position without a convex pentagon, the set must be $(4, 4, 1)$, $(4, 3, 2)$, $(3, 4, 2)$, or $(3, 3, 3)$. Note that these are all of the possible orientations because if any of the entries of these orientations was 5 or greater, that would imply that we have a convex pentagon. Also note that the only entry of these orientations that can be less than 3 is the last entry, as all other previous entries must be the number of vertices of a convex hull, and thus the number of vertices of a convex polygon.

Now note that if X is $(4, 4, 1)$, it in particular must contain a subset that is $(4, 3, 1)$. This is because if we draw a diagonal of the inner quadrilateral, the interior point of that quadrilateral must lie on one side of that diagonal. We can then take the two vertices from

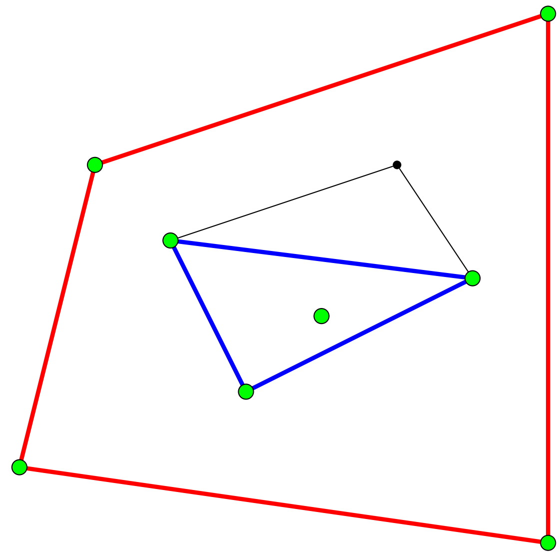


Figure 2.19: A set that is $(4, 4, 1)$ will always contain a set that is $(4, 3, 1)$, whose points are shown in green.

which we drew the diagonal and the vertex that lies on the same side of the diagonal as the interior point to form a triangle, and in turn take the outer quadrilateral, this newly formed triangle, and the interior point to get a $(4, 3, 1)$ subset, as seen in Figure 2.19. So by Lemma 2.4.1, X must contain a convex pentagon.

If X is $(4, 3, 2)$, it must also contain a subset that is $(4, 3, 1)$. This can be done by keeping the outer quadrilateral and inner triangle of X and then choosing either of the 2 interior points of the inner triangle. This can be seen in Figure 2.20. So again by Lemma 2.4.1, X must contain an convex pentagon.

Now if X is $(3, 4, 2)$, this is an orientation that we have already shown must contain a convex pentagon by Lemma 2.4.1.

Lastly, if X is $(3, 3, 3)$, it must contain a subset that is $(3, 3, 2)$. This can be done by keeping the outer and middle triangles and then choosing any 2 of the vertices of the inner triangle. This can be seen in Figure 2.21. Thus, again by Lemma 2.4.1, X must contain a convex pentagon.

So we have shown that all possible orientations of a 9-point set that had the potential

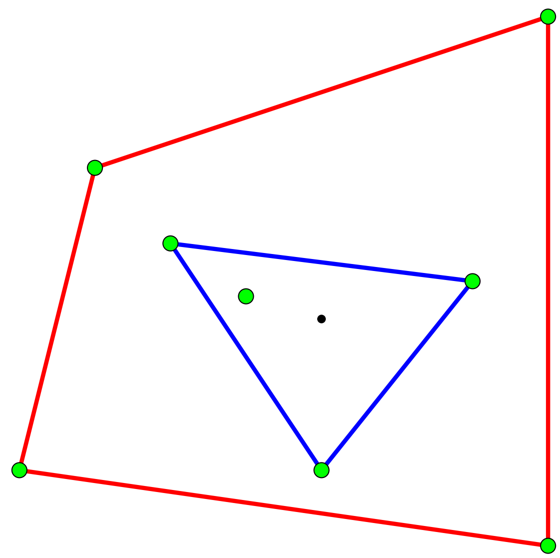


Figure 2.20: A set that is $(4, 3, 2)$ will always contain a set that is $(4, 3, 1)$, whose points are shown in green.

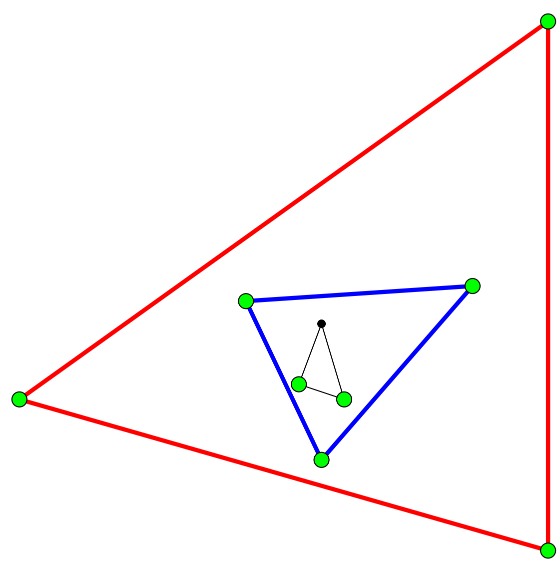


Figure 2.21: A set that is $(3, 3, 3)$ will always contain a set that is $(3, 3, 2)$, whose points are shown in green.

to not contain a convex pentagon actually must contain a convex pentagon. Thus, we have $ES(5) \leq 9$. Therefore, we can conclude that in fact $ES(5) = 9$.

$\blacksquare$

Now that we have proven the main result of this section, let us consider the work we have done so far. If we recall, the proof that $ES(3) = 3$ was trivial. The proof of $ES(4) = 5$ was a little more involved, but ultimately came down to exhibiting a 4-point set without a convex quadrilateral and then checking the three possible orientations of 5-point sets. So while $ES(4) = 5$ was not trivial, one could argue that it was still straightforward and not overly complex. When we jump to proving $ES(5) = 9$, however, we see that the complexity increases dramatically. We needed to check many possible configurations of 9-point sets and argue, with the help of a non-obvious lemma, that they contained subsets that contained a convex pentagon.

2.5 $ES(6) = 17$ and Further Specific Results

While the early results of $ES(3) = 3$ and $ES(4) = 5$ were known in the 1930s and the proof that $ES(5) = 9$ first appeared in literature in 1970 (though Erdős and Szekeres claimed that it had been proven earlier), the value of $ES(6)$ is a much more recent discovery. While the conjectured value for $ES(6)$ was 17, this was actually not proven to be true until 2006 by George Szekeres himself and Lindsay Peters, a student of Szekeres [13].

If we recall our proof of $ES(5) = 9$, we should note that this approach would be very difficult to extend to $ES(6)$ due to how many orientations we would have to check. With only 9-point sets, we had to check quite a few configurations, but as noted in [11], we would have to check 70 different $(k_1, k_2, ..., k_i)$ orientations of 17-point sets that could possibly not contain a convex hexagon. Instead of extending a proof similar to the one we used for $ES(5) = 9$, Szekeres and Lindsay used another combinatorial model of possible orientations of 17-point sets in conjunction with a computer proof to show that $ES(6) = 17$. They claim that the computer proof is reproducible, as three different implementations of the proof have been independently developed.

As of this writing, $ES(6) = 17$ is the last discovered value for $ES(n)$ for a specific n. While the values of $ES(7), ES(8)$, etc., have been conjectured, their actual values remain unverified. While one could try to find these specific results, the increased complexity required for the proof for each successive n potentially makes it easier (and certainly more powerful) to try and prove the general conjecture that $ES(n) = 2^{n-2} + 1$ for all $n \geq 3$.

Chapter 3

General Results and Progress Toward the Lower and Upper Bounds

As we saw in the previous chapter, the level of complexity needed to prove results about $ES(n)$ for specific values of n increased drastically as we increased n. Combine this with the fact that we cannot possibly prove the conjecture for each value of n individually, as we would have infinitely many things to prove, and we see that we need to move to more general arguments to fully prove the conjecture. These more general arguments concern the upper and lower bounds of the conjecture, with the ultimate goal being to make the upper and lower bounds the same as the conjectured value of $ES(n)$.

In this chapter, we will take a look at an area of mathematics called Ramsey Theory, and then use the ideas of Ramsey Theory to prove the existence of upper bounds on $ES(n)$. We will also discuss the progress that has been made toward improving the lower and upper bounds of $ES(n)$.

3.1 An introduction to Ramsey Theory

While we have found exact values of $ES(n)$ for some specific values of n, we were working under the assumption that such a minimal number of points in general position actually exists to guarantee a convex n-gon. But for larger n, why should we expect that we can always find this minimal number of points? Is it possible that for some integer n, there is actually no limit to the number of points we can have and still not have a convex n-gon?

As it turns out, this is not the case. For every natural number $n \geq 3$, there does in fact exist a smallest natural number of points in general position that will guarantee a convex n-gon. To actually prove this, we will need discuss a branch of mathematics called Ramsey Theory. This area of mathematics deals with problems about finding the conditions of a structure that are necessary to guarantee that the structure has certain properties. You may note that our problem is very closely related to Ramsey Theory, and we will actually be directly using results from Ramsey Theory to prove some results about our problem.

Before we can prove a cornerstone idea of Ramsey Theory known as Ramsey's Theorem, we need to discuss some more definitions and notation. These definitions, as well as some of the ideas for the upcoming proofs, have been taken from [15].

Definition 3.1.1. *A* graph *G consists of a set $V(G)$ of* vertices*, a set $E(G)$ of* edges*, and a mapping associating each edge $e \in E(G)$ and unordered pair x, y of vertices called the* endpoints *of e. We say that an edge is* incident *with its endpoints, and that it* joins *the endpoints.*

Definition 3.1.2. *An edge of a graph is called a* loop *if it joins a vertex to itself. A graph is said to have* multiple edges *if there exist more than one edge that joins the same two vertices. A graph that does not contain any loops or multiple edges is called* simple*.*

An example of a non-simple graph G (both because it has a loop and because it has multiple edges) with 4 vertices and 5 edges can be seen in Figure 3.1, and an example of a simple graph with 6 vertices and 8 edges can be seen in Figure 3.2.

We will be concerned with a special type of simple graph called a complete graph, which we define now.

Definition 3.1.3. *The* complete graph *on n vertices, denoted K_n, is a simple graph that has all $\binom{n}{2}$ possible edges.*

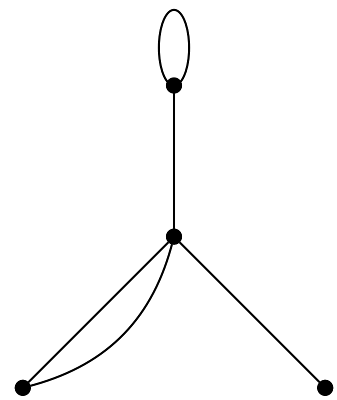


Figure 3.1: A non-simple graph with 4 vertices and 5 edges. It fails to be simple for two reasons: it has a loop and it has multiple edges joining the same two vertices.

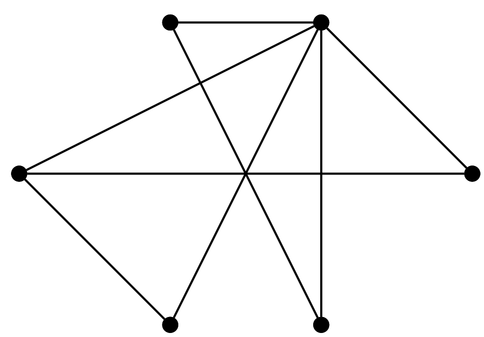


Figure 3.2: A simple graph with 6 vertices and 8 edges. It is simple because it contains no loops and no multiple edges.

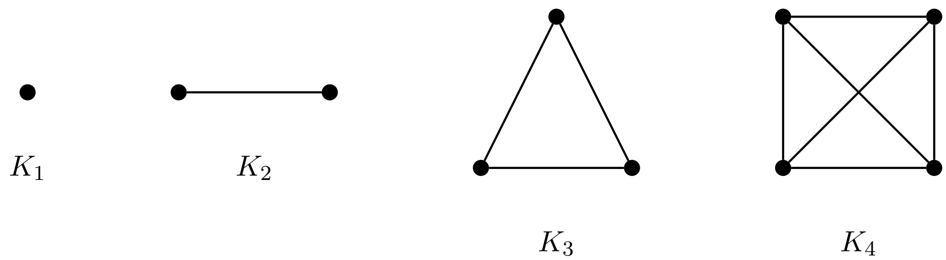


Figure 3.3: Complete graphs on 1, 2, 3, and 4 vertices, respectively.

In other words, a complete graph is a simple graph where every vertex is connected to every other vertex. Examples of the complete graphs on 1, 2, 3, and 4 vertices are shown in Figure 3.3.

Definition 3.1.4. *An* edge coloring *of a graph G is a function from the edges of G to a set $C = \{c_1, c_2, ..., c_i\}$ of colors such that every edge of G is assigned a color of C. If $|C| = i$ and the function is surjective, we say that it is an i-coloring.*

Note that our use of the term "edge coloring" may more often be referred to as an "edge labelling." Also note that the term "edge coloring" may be used to describe an assignment of colors in a way such that no two adjacent edges have the same color. Our definition of "edge coloring" does not have this restriction, and in fact we will actually want to have many adjacent edges with the same assigned color.

We are now ready to state and prove the usual basic case of Ramsey's Theorem, which involves coloring a complete graph with 2 colors, say red and blue.

Theorem 3.1.1. ***Ramsey's Theorem:*** *For any natural numbers r and b, there exists a smallest natural number $R(r, b)$ such that any edge coloring using 2 colors, red and blue, of the edges of a complete graph with at least $R(r, b)$ points contains either a complete subgraph on r vertices with all edges colored red or a complete subgraph on b vertices with all edges colored blue. In other words, any complete graph on $R(r, b)$ contains a red K_r or a blue K_b.*

Proof. We proceed by induction on $r + b$.

Base Case: Note that if $r = 1$ and b is any natural number, $R(r, b) = R(1, b) = 1$. This is because a complete graph with only one vertex has no edges, so we can take the the entire graph (that is, the one vertex) as our subgraph on one vertex with all edges vacuously colored red. Similarly, if $b = 1$ and r is any natural number, we have $R(r, b) = R(r, 1) = 1$.

Inductive Step: Assume that $R(r-1, b)$ and $R(r, b-1)$ exist for $r, b \in \mathbb{N}$. We now prove $R(r, b)$ exists by finding an upper bound on $R(r, b)$. We claim that $R(r, b) \leq R(r-1, b) + R(r, b-1)$. Let G be a graph with $R(r-1, b) + R(r, b-1)$ vertices, and choose one of the vertices and call it x. We can partition the rest of the vertices into two sets A and B defined as $A = \{y \in G | \{x, y\}$ is colored red$\}$ and $B = \{y \in G | \{x, y\}$ is colored blue$\}$, where $\{x, y\}$ denotes the edge that joins vertices x and y. In other words, A is the set of all vertices whose edge connected to x is red, and B is the set of all vertices whose edge connected to x is blue. Then note we have $|G| = R(r-1, b) + R(r, b-1)$ and $|G| = |A| + |B| + 1$ (the $+1$ coming from x itself), so in particular we have $R(r-1, b) + R(r, b-1) = |A| + |B| + 1$.

We claim that we either have $R(r-1, b) \leq |A|$ or $R(r, b-1) \leq |B|$. To see this, note that if $R(r-1, b) \leq |A|$ then we are done. So assume that $R(r-1, b) > |A|$. Because $R(r-1, b) > |A|$, we have $R(r-1, b) - |A| = k$ for some integer $k \geq 1$. Thus, by subtracting $|A|$ from both sides of $R(r-1, b) + R(r, b-1) = |A| + |B| + 1$, we get $k + R(r, b-1) = |B| + 1$, which in turn implies $R(r, b-1) = |B| + 1 - k$. Note that $1 - k \leq 0$, so in particular $|B| + 1 - k \leq |B|$. We conclude that $R(r, b-1) \leq |B|$. So our claim holds.

Now if $R(r-1, b) \leq |A|$, then either A contains a blue K_b or a red K_{r-1}. If A contains a blue K_b, then that blue K_b is contained in G and we are done. If A does not contain a blue K_b, then it must contain a red K_{r-1}, and by our construction we then have $A \cup \{x\}$ as a red K_r in G. Similarly, if $R(r, b-1) \leq |B|$, then either B contains a red K_r or a blue K_{b-1}. If B contains a red K_r, then that red K_r is contained in G and we are done. If B does not contain a red K_r, then it must contain a blue K_{b-1}, and by our construction we then have $B \cup \{x\}$ as a blue K_b in G. So in all cases, G contains either a red K_r or a blue K_b, so our claim that $R(r, b) \leq R(r-1, b) + R(r, b-1)$ holds.

This completes our proof by induction. ■

We call these $R(r, b)$ *Ramsey numbers*. As we noted earlier, this is the usual basic form of Ramsey's Theorem. However, the theorem can actually be extended in one of two ways (or even in both ways at once). The first extension involves using more than just two colors, while the second extension involves looking at hypergraphs. For our purposes with the Erdős–Szekeres Conjecture, we actually only need the later extension. That being said, we include the statements and proofs of both extensions on their own, as they are interesting results in their own right.

Theorem 3.1.2. ***Ramsey's Theorem for More Than 2 Colors:*** *For any natural numbers $c_1, c_2, ..., c_k$, there exists a smallest natural number $R(c_1, c_2, ..., c_k)$ such that any edge coloring using k colors, $COLOR_1, COLOR_2, ..., COLOR_k$, of the edges of a complete graph with at least $R(c_1, c_2, ..., c_k)$ points contains a complete subgraph on c_i vertices with all edges colored $COLOR_i$ for some $1 \leq i \leq k$. In other words, any complete graph on $R(c_1, c_2, ..., c_k)$ contains a K_{c_i} colored $COLOR_i$ for some $1 \leq i \leq k$.*

Proof. We proceed by induction on k.

Base Case: Note that the case of $k = 1$ would mean that we we are coloring all of the edges of an entire graph one color. So in particular, for any $c_1 \in \mathbb{N}$, $R(c_1) = c_1$. That is, if we have c_1 points and color all edges $COLOR_1$, we can take the entire graph as our K_{c_1}. So our base case holds. (Note that because we are ultimately interested in more than two colors, we could have also taken our base case as $k = 2$, which we proved to hold in Theorem 3.1.1).

Inductive Step: Now assume the theorem holds for $k = n$, that is, $R(c_1, c_2, ..., c_n)$ exists. we will show that $R(c_1, c_2, ..., c_n, c_{n+1})$ exists. We claim that

$$R(c_1, c_2, ..., c_n, c_{n+1}) \leq R(c_1, c_2, ..., c_{n-1}, R(c_n, c_{n+1}))$$

Note that the right hand side of this inequality is a Ramsey number on n colors whose last entry is a Ramsey number on 2 colors, so in particular the right hand side of this inequality exists by our inductive hypothesis. Let $R(c_1, c_2, ..., c_{n-1}, R(c_n, c_{n+1})) = a$. Let us consider a complete graph on a points and color it with $n+1$ colors. We can then momentarily treat $COLOR_n$ and $COLOR_{n+1}$ as the same color and think of the graph as now having only n colors. By the inductive hypothesis, there either exists K_{c_i} all colored $COLOR_i$ for some $1 \leq i \leq n-1$, in which case we are done, or there exists a $K_{R(c_n,c_{n+1})}$ colored a mix of $COLOR_n$ and $COLOR_{n+1}$. In the later case, note that we have a complete graph on $R(c_n, c_{n+1})$ points, so in particular it must contain a K_{c_n} all colored $COLOR_n$ or a $K_{c_{n+1}}$ all colored $COLOR_{n+1}$. So in all cases, we have a K_i colored $COLOR_i$ for some $0 \leq i \leq n+1$, and our theorem holds by induction. ■

This was our first extension of Ramsey's Theorem. Our second extension involves hypergraphs. Note that in our versions of Ramsey's Theorem that we have discussed so far, edges have been made up of 2 vertices. However, in the hypergraph case, our "edges" will be made up of k vertices for some natural number k. Before stating and proving Ramsey's Theorem for hypergraphs, let us formally define some needed ideas.

Definition 3.1.5. *A* k-hypergraph *H consists of a set $V(H)$ of* vertices*, a set $E(H)$ of* hyperedges*, and a mapping associating each hyperedge $e \in E(H)$ and unordered set $x_1, x_2, ..., x_k$ of vertices. We say that a hyperedge contains the vertices with which it is associated, and if a hyperedge contains k vertices we call it a k-edge.*

Definition 3.1.6. *A* complete k-hypergraph *on n vertices, denoted K_n^k, is the k-hypergraph on n vertices in which each of the possible $\binom{n}{k}$ k-edges exists and there are no other edges.*

Note that K_n^2 would denote the regular complete graphs with which we have been working.

Definition 3.1.7. *A* k-edge coloring *of a hypergraph* H *is a function from the* k*-edges of* H *to a set* $C = \{c_1, c_2, ..., c_i\}$ *of colors such that every* k*-edge of* H *is assigned a color of* C*. If* $|C| = i$ *and the function is surjective, we say that it is an* i*-coloring.*

We can now formulate and prove the version of Ramsey's Theorem that we actually need to continue our discussion the Erdős–Szekeres Conjecture.

Theorem 3.1.3. ***Ramsey's Theorem for 2-Colored Hypergraphs:*** *For any natural numbers* r *and* b*, there exists a smallest natural number* $R_k(r, b)$*, called a hypergraph Ramsey number, such that any* k*-edge coloring using 2 colors, red and blue, of the* k*-edges of a complete* k*-hypergraph with at least* $R_k(r, b)$ *points contains either a complete* k*-subhypergraph on* r *vertices with all* k*-edges colored red or a complete* k*-subhypergraph on* b *vertices with all* k*-edges colored blue. In other words, any 2-colored complete hypergraph on* $R_k(r, b)$ *contains a red* K_r^k *or a blue* K_b^k*.*

Proof. We proceed by induction on k, and by induction on $r + b$ for a fixed k.

Base Case: Let $k = 1$. Note that this coincides with coloring each point (or "1-edge") of a graph. We claim that if we have $r + b - 1$ points in our graph and color it using red and blue, then we are guaranteed to have either a red K_r^1 or a blue K_b^1. In other words, we are guaranteed to have either r red points or b blue points. To see this, note that if we have r red points, we are done, so assume that we have less than r red points. Then we at most have $r-1$ red points, which in turn implies that we have at least $(r+b-1)-(r-1) = b$ blue points. Thus, we always have the desired result. Note that this proves that $R_1(r, b) \leq r + b - 1$ (but in fact $R_1(r, b) = r + b - 1$). (Note that because we are ultimately interested in the case where $k > 2$, we could have also taken our base case as $k = 2$, which we proved to hold in Theorem 3.1.1).

Inductive Step: Now assume the theorem holds $k = n$ for some $n \in \mathbb{N}$. Note that for any $m \in \mathbb{N}$, we have $R_m(1, b) = b$. This is because if all b m-edges are colored blue then our entire hypergraph is K_b^m, but if just one m-edge is colored red we can take that m-edge as our red K_1^m. Similarly, if $b = 1$, we have $R_m(r, 1) = r$. So the cases of $r = 1$ and $b = 1$ exist, so we may assume $r > 1$ and $b > 1$ and induct on $r + b$ by assuming $R_m(r - 1, b)$ and $R_m(r, b - 1)$ exist for some $r, b \in \mathbb{N}$ and any $m \in \mathbb{N}$. We will show that $R_{n+1}(r, b)$ is bounded above.

Now let $N = R_n(R_{n+1}(r-1, b), R_{n+1}(r, b-1))+1$. Note that by our inductive hypotheses (both that our theorem holds for $k = n$ and that $R_m(r - 1, b)$ and $R_m(r, b - 1)$ exist for any natural number m), all of these hypergraph Ramsey numbers exist and are finite.

Now consider a complete $(n + 1)$-hypergraph H on N vertices whose $(n + 1)$-edges are colored arbitrarily red or blue. Choose a point of H and call it x. Now consider the n-edge colored hypergraph G obtained by removing x from H but where all n-edges have kept the same color they had when they included x and were $(n + 1)$-edges of H. For example, if $\{v_1, v_2, ..., v_n, x\}$ was a red $(n + 1)$-edge of H, then $\{v_1, v_2, ..., v_n\}$ will be a red n-edge of G. Note that $(n + 1)$-edges of H that did not contain x need not be considered when constructing G because we are only interested in n-edges of G, all of which would be obtained by removing x from the $(n + 1)$-edges of H that contained x.

Now G is a complete n-hypergraph on $N - 1 = R_n(R_{n+1}(r - 1, b), R_{n+1}(r, b - 1))$ vertices. So in particular, there must either exist a complete n-subhypergraph on $R_{n+1}(r - 1, b)$ vertices that is completely colored red or a complete n-subhypergraph on $R_{n+1}(r, b - 1)$ vertices that is completely colored blue. In other words, we must have either a red $K^n_{R_{n+1}(r-1,b)}$, which we will call A, or a blue $K^n_{R_{n+1}(r,b-1)}$, which we will call B.

If A is realized, note that A is made up of $R_{n+1}(r - 1, b)$ points. So in particular, we have two possibilities within A. The first possibility is that A contains a complete $(n + 1)$-subhypergraph on b vertices that is completely colored blue, in which case we would be done. The second possibility is that A contains a complete $(n + 1)$-subhypergraph on $r - 1$ vertices that is completely colored red. Now recall that any n-edge of A, say $\{a_1, a_2, ..., a_n\}$, is red. So by our construction, any $(n+1)$-edge $\{a_1, a_2, ..., a_n, x\}$ of $A \cup \{x\}$ is also red. This means that we can take any of the $r - 1$ red $(n + 1)$-edges of A, delete one of the vertices and replace it with x, and the new $(n+1)$-edge would still be red. So in particular, because A contains a complete $(n+1)$-subhypergraph on $r-1$ points that is all colored red, we must have that $A \cup \{x\}$ contains a complete $(n+1)$-subhypergraph on r vertices that is all colored red. Again in this case, we are done.

Similarly, if B is realized, we must either have that B contains a complete $(n+1)$-subhypergraph on r vertices that is completely colored red or that $B \cup \{x\}$ contains a complete $(n+1)$-subhypergraph on b vertices that is all colored blue. So in all cases, we either have a red complete $(n+1)$-subhypergraph on r vertices or a blue complete $(n+1)$-subhypergraph on b. Thus, we can conclude

$$R_{n+1}(r,b) \leq R_n(R_{n+1}(r-1,b), R_{n+1}(r,b-1)) + 1$$

.

This completes our proof by induction, and we have that $R_k(r,b)$ exists for any natural numbers k, r, and b.

■

Szekeres actually independently proved this 2-color hypergraph version of Ramsey's theorem in [4]. With this theorem in particular, we now have enough tools from Ramsey Theory to continue our exploration of the Erdős–Szekeres Conjecture.

3.2 The Existence of an Upper Bound on $ES(n)$

Now that we have discussed Ramsey's Theorem for 2-colored hypergraphs, we will use it to prove that $ES(n)$ exists for all $n \geq 3$ by finding explicit upper bounds on $ES(n)$. As in [11], we will provide three proofs for upper bounds. The first theorem and proof was what Erdős and Szekeres used in [4]. But before we actually state and prove their theorem about an upper bound on $ES(n)$, we need a lemma.

Lemma 3.2.1. *A set of at least 3 points X in general position is in convex position if and only if all 4-point subsets of X are in convex position.*

Proof. Note that this is vacuously true if $|X| = 3$, so we may assume $|X| \geq 4$. We proceed by contrapositive in both directions.

First assume that there exists a 4-point subset $A \subseteq X$ such that A is not in convex position. Then in particular, a point $a \in A$ is an interior point of $\text{conv}(A)$. Because $\text{conv}(A) \subseteq \text{conv}(X)$, we must have that a is an interior point of $\text{conv}(X)$. Thus, X is not in convex position.

Now assume that X is not in convex position. Then there exists a point $x \in X$ that is an interior point of $\text{conv}(X)$. Note that because $\text{conv}(X)$ is a convex polygon, we may partition $\text{conv}(X)$ into triangles in the following way. Name the vertices of $\text{conv}(X)$ $v_1, v_2, ..., v_k$ clockwise around $\text{conv}(X)$. We can make triangles $v_1v_2v_3, v_1v_3v_4, ..., v_1v_{k-1}v_k$ (note that all of these triangles have v_1 as a vertex). Because this partitions the portion of $\text{conv}(X)$ where our interior point can exist, we must have that x is contained in the interior of one of our constructed triangles, say $v_1v_iv_{i+1}$. Thus, we can take the set $\{x, v_1, v_i, v_{i+1}\}$ as a 4-point set that is not in convex position.

This completes our proof by contrapositive.

■

We now proceed to the Erdős and Szekeres proof of an upper bound on $ES(n)$.

Theorem 3.2.2. *$ES(n) \leq R_4(n, 5)$.*

Proof. Let X be any set with at least $R_4(n, 5)$ vertices in general position (note that we know $R_4(n, 5)$ exists and is finite by Theorem 3.1.3). We will now color all 4-point subsets of X that are in convex position red and all other 4-point subsets of X blue. Note by Lemma 2.3.2, it is impossible for all 4-point subsets of a 5-point subset of X to be colored blue. Thus, because X has at least $R_4(n, 5)$ points, there must exist an n-point subset $A \subseteq X$ with each 4-point subset of A colored red. By our construction, that means that all 4-point subsets of A are in convex position. Now by Lemma 3.2.1, we must have that A is in convex position. Thus, we have that the vertices of A form a convex n-gon. So $ES(n) \leq R_4(n, 5)$. ■

We now also prove a different upper bound on $ES(n)$ in two different ways. The first appears in [9], in which Lewin attributes the solution to Tarsy, an undergraduate student that discovered the proof during an exam. The second is attributed to Johnson and is found in [6].

Theorem 3.2.3. *$ES(n) \leq R_3(n, n)$.*

Proof.

Proof 1. Let $X = \{x_1, x_2, ..., x_k\}$ be a set of points in general position in the plane with $k \geq R_3(n, n)$. Color a 3-point set $\{x_a, x_b, x_c\} \subseteq X$ for $a < b < c$ red if we find the points in the order (x_a, x_b, x_c) when traveling clockwise around $\text{conv}(\{x_a, x_b, x_c\})$, and color the set blue otherwise.

We claim that if all of the 3-point subsets of a 4-point set A are colored the same, then A is in convex position. To prove this claim, we consider the contrapositive, that being that if a 4-point set A is not in convex position, then there must be at least one red and one blue 3-point subset of A. To prove this, we can consider all possible cases. Let us call the points of A v_1, v_2, v_3, and v_4.

Note there are $4! = 24$ ways that we can orient these points so that they are not in convex position (as we need to pick 3 vertices of $\text{conv}(A)$ and 1 interior point). However, we can eliminate many of these configurations by rotational symmetry. We can also eliminate cases that are "mirrors" of each other (for example, if one can encounter the points v_4, v_3, and v_2 in that order running clockwise around $\text{conv}(A)$ and v_1 is an interior point, then the 3-point subsets of A will have the exact opposite coloring as if we encounter the points v_1, v_2, and v_3 in that order running clockwise around $\text{conv}(A)$ and v_4 is an interior point). Through all of this reduction, we actually only have to check four cases.

Case 1 would be if the vertices of conv(A) force us to color the edges of conv(A) red and the interior point is an "extreme point", that being either v_1 or v_4. Case 2 would be if the vertices of conv(A) force us to color the edges of conv(A) blue and the interior point is an "extreme point". Case 3 would be if the vertices of conv(A) force us to color the edges of conv(A) red and the interior point is an "intermediate point", that being either v_2 or v_3. Case 4 would be if the vertices of conv(A) force us to color the edges of conv(A) blue and the interior point is an "intermediate point". One can check that all four of these cases will indeed have two colors. So our claim holds.

Now our claim in conjunction with Lemma 3.2.1 implies that if all 3-point subsets of an n-point subset of X are the same color, then that n-point subset is in convex position. Because X has at least $R_3(n,n)$ points (and $R_3(n,n)$ exists by Theorem 3.1.3), we must either have an n-point subset of X whose 3-point subsets are all colored red or an n-point subset of X whose 3-point subsets are all colored blue. In either case, we must then have an n-point subset of X that is in convex position, and thus a convex n-gon contained in X. Thus, $ES(n) \leq R_3(n,n)$.

Proof 2. Let X be a set with at least $R_3(n,n)$ points. We will color a 3-point subset $Y \subseteq X$ red if there are an even number of interior points in conv(Y) and blue otherwise. We claim that if all of the 3-point subsets of a 4-point set A are colored the same, then A is in convex position. Again, we proceed by proving the contrapositive, that being that if A is not in convex position, then there exists at least one red and one blue 3-point subset of A.

Note that if A is a 4-point set that is not in convex position, then in particular we have that conv(A) is a triangle, say with vertices v_1, v_2, and v_3, with one interior point that is also a member of A. Call this interior point a. Then we have 4 triangles to consider: the outer triangle $v_1v_2v_3$ and the inner triangles v_1v_2a, v_1v_3a, and v_2v_3a. Note that if one of the inner triangles has an even number of interior points from X and another of the interior triangles has an odd number of interior points from X, then we will have two different colored triangles and be done. So assume that all of the interior triangles all have an even number of interior points from X (and thus are colored red). Then all of the interior points of these three triangles will also be interior points of $v_1v_2v_3$, but so will a. Thus, $v_1v_2v_3$ will have an EVEN + EVEN + EVEN +1 = ODD number of points and thus be colored blue. Similarly, if all of the interior triangles have an odd number of interior points from X (and thus are colored blue), then all of the interior points of these three triangles will also be interior points of $v_1v_2v_3$. But a will also be an interior point of $v_1v_2v_3$, so $v_1v_2v_3$ will have an ODD + ODD + ODD +1 = EVEN number of points and thus be colored red. In all cases, we have 3-point subsets of A that are different colors, so our claim holds.

Now our claim in conjunction with Lemma 3.2.1 implies that if all 3-point subsets of an n-point subset of X are the same color, then the n-point subset is in convex position. Because X has at least $R_3(n,n)$ points (and $R_3(n,n)$ exists by Theorem 3.1.3), we must either have an n-point subset of X whose 3-point subsets are all colored red or an n-point subset of X whose 3-point subsets are all colored blue. In either case, we must then have an n-point subset of X that is in convex position, and thus a convex n-gon contained in X. Thus, $ES(n) \leq R_3(n,n)$.

■

By combining the results of Theorem 3.2.2 and Theorem 3.2.3, we also have the immediate corollary.

Corollary 3.2.1. $ES(n) \leq min\{R_4(n,5), R_3(n,n)\}$.

With the theorems of this section, we now know that $ES(n)$ exists for all $n \geq 3$.

3.3 Best Known Lower and Upper Bounds

Now that we have established that $ES(n)$ exists for all $n \geq 3$, our next goal is to improve our bounds on $ES(n)$ as much as possible in the hopes of either proving or disproving the

Erdős–Szekeres Conjecture. Note that the Ramsey numbers $R_4(n, 5)$ and $R_3(n, n)$ that we found as upper bounds in the previous section are generally much larger than the conjectured value of $ES(n)$. To try to improve these upper bounds and start discussing the lower bound of $ES(n)$, we will need to discuss a few more definitions that we take from [11].

Definition 3.3.1. *Assume that we are using the standard Cartesian coordinate system on the plane. Let X be a set of points $\{(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)\}$ in general position, with $x_i \neq x_j$ for $i \neq j$. A subset of points $\{(x_{i_1} y_{i_1}), (x_{i_2}, y_{i_2}), ..., (x_{i_r}, y_{i_r})\}$ is called an* r-cup *if $x_{i_1} < x_{i_2} < ... < x_{i_r}$ and*

$$\frac{y_{i_1} - y_{i_2}}{x_{i_1} - x_{i_2}} < \frac{y_{i_2} - y_{i_3}}{x_{i_2} - x_{i_3}} < ... < \frac{y_{i_{r-1}} - y_{i_r}}{x_{i_{r-1}} - x_{i_r}}$$

Similarly, the subset is called an r-cap *if $x_{i_1} < x_{i_2} < ... < x_{i_r}$ and*

$$\frac{y_{i_1} - y_{i_2}}{x_{i_1} - x_{i_2}} > \frac{y_{i_2} - y_{i_3}}{x_{i_2} - x_{i_3}} > ... > \frac{y_{i_{r-1}} - y_{i_r}}{x_{i_{r-1}} - x_{i_r}}$$

In other words, a set of points in an r-cup if, from left to right, the slopes of the segments connecting each successive pair of points are monotonically increasing, and a set of points in an r-cap if, from left to right, the slopes of the segments connecting each successive pair of points are monotonically decreasing.

With this definition, we now state a lemma proved by Erdős and Szekeres [4].

Lemma 3.3.1. *Let $f(k, l)$ be the smallest positive integer such that if a set of points X in general position has at least $f(k, l)$ points, then it must contain either a k-cup or an l-cap. Then $f(k, l) = \binom{k+l-4}{k-2} + 1$*

Erdős and Szekeres used this to construct sets of 2^{n-2} points that proved the following theorem about the lower bound of $ES(n)$ [5].

Theorem 3.3.2. $ES(n) \geq 2^{n-2} + 1$.

Note that this is the conjectured lower bound for $ES(n)$. If we could similarly prove that the conjectured upper bound for $ES(n)$ also holds, we would prove the Erdős–Szekeres Conjecture. Unfortunately, that feat has remained unaccomplished since Erdős and Szekeres first made their conjecture. One should note that proving the conjectured lower bound is in a sense "easier" than proving the conjectured upper bound. This is because to prove the lower bound, we need to exhibit only ONE example of a set of 2^{n-2} vertices in general position that does not contain a convex n-gon. However, to prove the upper bound, we would need to show that ALL sets of $2^{n-2}+1$ points contain a convex n-gon.

While we have shown that upper bounds on $ES(n)$ exist in Theorem 3.2.2 and Theorem 3.2.3, better upper bounds have been proven to exist. What follows are several results that improve our upper bound on $ES(n)$, starting with Erdős and Szekeres [4] and ending with the most recent best upper bound, attributed to Andrew Suk [12]. Erdős and Szekeres found their original best upper bound with the following theorem.

Theorem 3.3.3. [4] *$ES(n) \leq \binom{2n-4}{n-2} + 1$.*

Note that $\binom{2n-4}{n-2} + 1 = 4^{n-o(n)}$. This stood as the best upper bound on $ES(n)$ for 63 years until Chung and Graham were able to improve it by 1 with the following theorem [2].

Theorem 3.3.4. *$ES(n) \leq \binom{2n-4}{n-2}$.*

The upper bound was again improved by Kleitman and Pachter with the following theorem [8].

Theorem 3.3.5. *$ES(n) \leq \binom{2n-4}{n-2} + 7 - 2n$.*

A further improvement was made by Tòth and Valtr [14].

Theorem 3.3.6. *$ES(n) \leq \binom{2n-5}{n-3} + 2$.*

Note that while these improvements do bring our upper bound close to the conjectured value, they are still quite a ways off from that value. Also note that these successive theorems only marginally improve the upper bound set by Erdős and Szekeres. However, a recent result by Suk has significantly improved this upper bound [12].

Theorem 3.3.7. $ES(n) \leq 2^{n+o(n)}$.

While this is a monumental improvement and remains the best upper bound to date on $ES(n)$, note that it is still not the conjectured upper bound. This ultimately leaves the Erdős–Szekeres Conjecture standing as an open problem in mathematics.

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