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Upper Bounds for Some Ramsey Numbers

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ABSTRACT

The Ramsey number $M(p, q)$ is the greatest integer such that for each $n < M(p, q)$, it is possible to color the $\binom{n}{2}$ edges of the complete graph on n vertices with red and blue in such a way that no complete subgraph on p vertices has all its edges red, and no complete subgraph on q vertices has all its edges blue. Counting arguments are developed by means of which it is possible to prove the non-existence of such coloring schemes and thus to establish new upper bounds for many of the Ramsey numbers.

1. INTRODUCTION

The complete graph on n vertices will be called an n-clique. The $\binom{n}{2}$ edges of the n-clique are painted with red and blue. For $p, q \geq 2$, define a (p, q)-coloring to be one in which there is no red p-clique or blue q-clique. A theorem of Ramsey [1] implies the existence of a least integer $M(p, q)$ such that for $n \geq M$, no such coloring exists.

Apart from the trivial case $M(2, q) = q$, very few of these Ramsey numbers are known. Greenwood and Gleason [3] have evaluated $M(3, 3)$ $M(3, 4)$, $M(3, 5)$, and $M(4, 4)$, and they discuss colorings in more than two colors as well. In [5] and [6] it is proved that $M(3, 6) = 18$. All known non-trivial values of $M(p, q)$ are contained in Table 1. Note that by symmetry $M(p, q) = M(q, p)$.

Lower bounds have been obtained for some other Ramsey numbers. Abbott [1] has shown that $M(5, 5) \geq 38$. The remaining lower bounds

TABLE 1

RAMSEY NUMBERS
$M(p, q)$

$p \backslash q$	3	4	5	6
3	6	9	14	18
4	9	18		
5	14			
6	18			

given in Tables 2 and 3 were established in [4] and [5]. The best upper bounds previously available for these numbers are yielding by Lemma 2, originally given by Erdös and Szekeres [2]. Lemma 1 is also implicit in [2].

TABLE 2

BOUNDS FOR $M(3, q)$

q	7	8	9	10	11	12	13	14
New upper bound	24	30	37	45	54	63	73	84
Lower bound	23	27	36	39	46	49	58	63

TABLE 3

LOWER/UPPER BOUNDS FOR $M(p, q)$

$p \backslash q$	4	5	6
4	18	25/30	34/46
5	25/30	38/59	51/105
6	34/46	51/105	102/210

LEMMA 1: *In a (p, q)-coloring, there are at most $M(p-1, q) - 1$ red lines, and at most $M(p, q-1) - 1$ blue lines from any point.*

LEMMA 2: $M(p, q) \leq M(p-1, q) + M(p, q-1)$, *with strict inequality if both terms on the right are even.*

In this paper, the upper bounds given by Lemma 2 will be reduced by means of counting arguments applied to the number of red lines. Tables 2 and 3 list new upper bounds for several Ramsey numbers, together with lower bounds from [1], [4], and]5]. It becomes increasingly apparent that Lemma 2 yields upper bounds which are consistently too high, and which become worse as p and q increase.

2. New Upper Bounds for $M(3, q)$

Define $R(q, n)$ to be the minimum number of red lines possible in a $(3, q)$-coloring of the n-clique. [Take $R(q, n) = \infty$ if $n \geq M(3, q)$.] Table 4 gives a lower bound $r(q, n)$ for $R(q, n)$ for several values of n and q. For the first eight entries in the table, $R(q, n) = r(q, n)$. $R(3, 4)$, $R(3, 5)$, and $R(5, 13)$ follow from Lemma 1, and $R(4, 7)$, $R(4, 8)$, and $R(5, 12)$ are fairly easy to establish. Evaluation of $R(6, 16)$ and $R(6, 17)$ is more difficult (Chapter 4 in [5]).

Let 1 be any point in a $(3, q)$-coloring of the N-clique. If there are x red lines and y blue lines incident with 1, 1 will be called x-valent in red, and y-valent in blue. Obviously, $x + y + 1 = N$, and, by Lemma 1, $x \leq q - 1$, $y \leq M(3, q) - 1$. X shall denote the subgraph formed by the x points joined to 1 by red lines, and Y the subgraph of y points joined to 1 by blue lines. A $(3, q)$-coloring of the N-clique contains no set of three vertices interjoined only by red lines (that is, no red triangle), and no set of q vertices interjoined only by blue lines (no blue q-clique). Since there are no red triangles, X contains no red lines, and there are just three types of red lines: those from 1 to X, those from X to Y, and those within Y. An upper bound may be established on the number of red lines lying in Y. Since there are no red triangles or blue q-cliques in the graph, Y contains no red triangle and no blue $(q-1)$-clique (which would be joined to 1 entirely by blue lines). Thus Y is $(3, q-1)$-colored, and contains at least $R(3, q-1)$ red lines. If this lower bound on the number of red lines in Y exceeds the upper bound obtained, a

TABLE 4

LOWER BOUND $r(q, n)$ FOR THE NUMBER OF RED LINES IN A $(3, q)$-COLORING OF THE n-CLIQUE

n	q	$r(q, n)$	n	q	$r(q, n)$
4	3	2	40	10	139
5	3	5	41	10	151
7	4	6	42	10	164
8	4	10	43	10	175
12	5	20	44	10	192
13	5	26	49	11	200
16	6	32	50	11	215
17	6	40	51	11	229
21	7	49	52	11	243
22	7	59	53	11	262
23	7	69	59	12	279
27	8	75	60	12	296
28	8	85	61	12	310
29	8	97	62	12	329
33	9	99	70	13	379
34	9	112	71	13	396
35	9	123	72	13	415
36	9	138			

contradiction results, and the non-existence of a $(3, q)$-coloring of the N-clique is established.

The above argument, with some refinements, may be used to obtain the upper bounds in Table 2 from the bounds $r(q, n)$ in Table 4. Here, the upper bounds for $M(3, 7)$ and $M(3, 8)$ will be established; also $r(7, 22)$ and $r(7, 23)$ will be obtained to illustrate the way in which Table 4 is set up. Further details will be found in [5, Chapter 5].

THEOREM 1. $M(3, 7) \leq 24$.

PROOF: By Lemma 2, $M(3, 7) \leq M(2, 7) + M(3, 6) = 25$. By Lemma 1, in a $(3, 7)$-coloring each point is at most 6-valent in red and at most 17-valent in blue. There are 23 lines incident with each point in the 24-clique. Thus in a $(3, 7)$-coloring of the 24-clique all points are 6-valent in red and 17-valent in blue.

Let 1 be any point in such a coloring. It is joined by red lines to 6 points (subgraph X) and by blue lines to 17 points (subgraph Y). Since there are no red triangles, all edges in X are blue. But every vertex is 6-valent in red. Thus there are 5 red lines from each point of X to some 5 points in Y, and in all there are $5 \cdot 6 = 30$ red lines from X to Y. Also, every point of Y is 6-valent in red, and in all there are $6 \cdot 17 = 102$ red incidences with points of Y. Since 30 of these incidences are with red lines from X, there are $\frac{1}{2}(102 - 30) = 36$ red lines within Y.

But the 17-clique Y must be (3, 6)-colored; for the graph contains no red triangles, and a blue 6-clique in Y joined by blue lines to 1 would give a blue 7-clique. Since $r(6, 17) = 40$, Y must contain at least 40 red lines—a contradiction. There can exist no (3, 7)-coloring of the 24-clique, and Theorem 1 follows.

In the next two theorems, the following fact is required: in a (3, 6)-coloring of the 17-clique, all points are 4- or 5-valent in red, and at most 5 are 4-valent. A proof of this is given in [5].

THEOREM 2. $r(7, 23) = 69$; *in a* (3, 7)*-coloring of the* 23*-clique, all points are* 6*-valent in red, and there are* 69 *red lines.*

PROOF: By Lemma 1, points in a (3, 7)-coloring of the 23-clique may be of two types only: 5-valent in red and 17-valent in blue (type A), 6-valent in red and 16-valent in blue (type B). Suppose there is a coloring containing a point 1 of type A. A contradiction will be obtained, and the theorem will follow.

1 is joined by red lines to 5 points (subgraph X) and by blue lines to 17 points (subgraph Y). Let there be y_i points in Y joined by red lines to i points in X ($i = 0, 1, 2, \ldots$). Y must be (3, 6)-colored. Thus

(i) $y_i = 0$ for $i > 2$, since points of Y must be at least 4-valent in red within Y in a (3, 6)-coloring of Y.

(ii) $y_2 \leq 5$, since at most 5 points in Y are 4-valent in red within Y in a (3, 6)-coloring of Y.

Also, there are no red triangles or blue 7-cliques in the overall graph. Thus

(iii) $y_1 \leq 10$. For if there were 3 points of Y each joined by a red line to the same point in X but to no others, these 3 points, together with the remaining 4 in X, would give a blue 7-clique.

(iv) $y_0 \leq 2$, for 3 points in Y joined by blue lines to all points in X would contain a blue line (they cannot form a red triangle). The 2 ends of the blue line and the 5 points of X would form a blue 7-clique.

However, $\Sigma y_i = 17$, and so $y_0 = 2$, $y_1 = 10$, $y_2 = 5$. The number of red lines from points of Y to points of X is

$$0.2 + 1.10 + 2.5 = 20$$

Therefore, all points of X are 5-valent in red (type A).

But 1 was any point of type A, and therefore every point of type A is joined by red lines only to points of type A. Since $y_2 = 5$, there is a point P of type B. Since no A is joined to a B, no B is joined to an A by red, and P is joined by red lines to 6 points also of type B. These 6 points form a blue 6-clique which is joined entirely by blue lines to every point of type A. Therefore, there exists no (3, 7)-coloring of the 23-clique containing a point of type A.

THEOREM 3. $r(7, 22) = 59$.

PROOF: By Lemma 1, points in a (3, 7)-coloring of the 22-clique may be of three types: A (incident with 4 red lines and 17 blue lines), B (5-valent in red), and C (6-valent in red). If there are a A's, b B's, and c C's, the number of red lines is $\frac{1}{2}$ (4a + 5b + 6c).

First, suppose $a > 0$, and let 1 be any point of type A. Denote by X and Y the subgraphs formed by points joined to 1 by red lines and blue lines, respectively. If X contains 2 points P, Q of type A, then PQ is blue, and Q is at most 3-valent in red in the 17-clique joined to P by blue. But this 17-clique must be (3, 6)-colored—impossible. Therefore, X contains at most one A, and there are at least $3 + 3.4 = 15$ red lines from X to Y. Since Y is (3,6)-colored, it contains at least $r(6, 17) = 40$ red lines. Including the 4 red lines from 1 to X, there are at least $4 + 15 + 40 = 59$ red lines.

Therefore, assume $a = 0$. Then $b + c = 22$, and the number of red lines is $\frac{1}{2}$ (5b + 6c) = 66 − (b/2). A coloring with fewer than 59 red lines contains at least 16 B's. Let 1 be any B and define X, Y in the usual manner. There are at least $4 \cdot 5 = 20$ red lines from X to Y ($a = 0$), and Y, being (3, 6)-colored, contains at least $r(6, 16) = 32$ red lines. In all, there are at least $5 + 20 + 32 = 57$ red lines.

In a coloring with 57 red lines, $b = 18$ and $c = 4$. Since there are just 4 points 6-valent in red, there exists a point of type B which is joined by a red line to at least one C. Choosing this point to be 1, there are at least $4.4 + 1.5 = 21$ red lines from X to Y, and at least 58 lines in all —contradiction.

In a coloring with 58 red lines, $b = 16$ and $c = 6$. The subgraph formed by the 6 points of type C must be colored in such a way that it contains no red triangles, and it is easy to see that it can contain at most 9 red lines. Therefore, there are at least $6.6 - 2.9 = 18$ red lines from the 6 C's to the 16 B's. There exists a B joined by red lines to at least 2 C's. Choosing this point to be 1, there are at least $3.4 + 2.5 = 22$ red lines from X to Y, and at least 59 red lines in all—contradiction.

Therefore, there exists no (3, 7)-coloring of the 22-clique with fewer than 59 red lines.

The value of $M(3, 7)$ is not known. However, in [5] a (3, 7)-coloring of the 22-clique containing 64 red lines is constructed, so that $M(3, 7)$ is 23 or 24. Knowledge of $M(3, 7)$ would be of little or no help in evaluating $M(3, 8)$, since it can be shown that no (3, 8)-coloring of the 28-clique or 29-clique can contain a point 23-valent in blue. Evaluation of each successive $M(3, q)$ seems to be a new and difficult problem in itself.

THEOREM 4. $M(3, 8) \leq 30$.

PROOF: With $M(3, 7) \leq 24$, the strict inequality in Lemma 2 applies, and $M(3, 8) \leq 31$. By Lemma 1, in a (3, 8)-coloring of the 30-clique, points are of two types only: A (incident with 6 red lines and 23 blue lines), and B (incident with 7 red lines and 22 blue lines).

Suppose there exists a point 1 of type A, and define X, Y in the usual manner. There are at least $5 \cdot 6 = 30$ red lines from X to Y, and Y contains at most $\frac{1}{2}(7.23 - 30)$, that is, at most 65 red lines. But the 23-clique Y must be (3, 7)-colored—impossible by Theorem 2.

Therefore, all points are of type B. Take any point 1 and define X and Y as usual. There are $6.7 = 42$ red lines from X to Y, and Y contains $\frac{1}{2}(7.22 - 42) = 56$ red lines. But the 22-clique Y must be (3, 7)-colored —impossible by Theorem 3. There exists no (3, 8)-coloring of the 30-clique, and $M(3, 8) \leq 30$.

3. FURTHER RESULTS

It should be possible to improve on several of the lower bounds in Table 4, and perhaps to reduce further some of the upper bounds in Table 2. It appears that a result of the form $R(q, n) \geq R(q, n - 1) + 2q - 4$ might hold under suitable restrictions on n, q. Such a result would be quite helpful.

The arguments of the last section may be used to reduce the upper bound provided by Lemma 2 for further values of $M(3, q)$. It appears that

$$M(3, q) \leq M(3, q-1) + q - \varepsilon(q)$$

where ε increases as q increases.

New upper bounds for $M(4, 5)$ and $M(4, 6)$ may also be obtained by these methods. Lemma 1 implies that in a (4, 4)-coloring of the 17-clique all points are 8-valent in red, and in a (4, 4)-coloring of the 16-clique all points are 7- or 8-valent in red. These facts, together with $R(5, 12) = 20$ and $R(5, 13) = 26$, may be used to prove that there exists no (4, 5)-coloring of the 30-clique.

Lemma 2 now implies $M(5, 5) \leq 59$ and $M(4, 6) \leq 47$. It can be proved that there are at least 148 red lines in a (4, 5)-coloring of the 28-clique, and at least 167 red lines in a (4, 5)-coloring of the 29-clique. These facts, together with $R(6, 16) = 32$ and $R(6, 17) = 40$, imply the non-existence of a (4, 6)-coloring of the 46-clique. The remaining upper bounds in Table 3 follow from Lemma 2, and can quite likely be further reduced by the above methods.

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