AI1103 : Assignment 6

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Download Latex codes from

https://github.com/BokkaRajaRaviKiranReddy/AI1103/blob/main/Assignment6/Assignment6.tex

1 UGC NET 2013 JUNE Q.NO 101

Let X_1,X_2,...be the independent random variables each following exponential distribution with mean 1.then which of the following statements are correct?

1) $\Pr\left(X_n > \log n \ \ \forall n \geq 1\right)$=1
2) $\Pr\left(X_n > 2 \ \ \forall n \geq 1\right)$=1
3) $\Pr\left(X_n > \frac{1}{2} \ \ \forall n \geq 1\right)$=0
4) $\Pr\left(X_n > \log n, X_{n+1} > \log(n+1) \ \ \forall n \geq 1\right)$=0

2 SOLUTION

Lemma 2.1. *PDF of X_i is*

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean of $X_i = \frac{1}{\lambda_i}$

Proof.

$$E[X_i] = \int_{-\infty}^{\infty} x f_{X_i}(x) dx \tag{2.0.1}$$
$$= \int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} x \lambda_i e^{-\lambda_i x} \tag{2.0.2}$$
$$= \frac{1}{\lambda_i} \tag{2.0.3}$$

□

Lemma 2.2. *CDF of X_i is*

$$F_{X_i}(x) = \begin{cases} 1 - e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Proof.

$$F_{X_i}(x) = \int_{-\infty}^{x} f_{X_i}(x) dx \tag{2.0.4}$$

For $x < 0$

$$F_{X_i}(x) = \int_{-\infty}^{x} 0 dx \tag{2.0.5}$$
$$= 0 \tag{2.0.6}$$

For $x \geq 0$

$$F_{X_i}(x) = \int_{-\infty}^{x} f_{X_i}(x) dx \tag{2.0.7}$$
$$= \int_{-\infty}^{0} 0 dx + \int_{0}^{x} \lambda_i e^{-\lambda_i x} dx \tag{2.0.8}$$
$$= 1 - e^{-\lambda_i x} \tag{2.0.9}$$

so,

$$F_{X_i}(x) = \begin{cases} 1 - e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

□

From (2.0.3) and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \geq 1$ Now, for some constant $c \geq 0$

$$\Pr\left(X_n > c\right) = 1 - F_{X_n}(c) \tag{2.0.10}$$
$$= e^{-c} \tag{2.0.11}$$

We need Borel-Cantelli Lemmas

Lemma 2.3. *Let A_1,A_2,... be a sequence of events in some probability space. The Borel–Cantelli lemma states that, if the sum of the probabilities of the events A_n is finite*

$$\sum_{n=1}^{\infty} \Pr\left(A_n\right) < \infty \tag{2.0.12}$$

then the probability that infinitely many of them occur is 0

$$\Pr \lim_{n\to\infty} \sup A_n = 0 \tag{2.0.13}$$

Lemma 2.4. *If the events A_n are independent and the sum of the probabilities of the A_n diverges to* *infinity, then the probability that infinitely many of them occur is 1. If for independent events $A_1, A_2, ...$*

$$\sum_{n=1}^{\infty} \Pr(A_n) = \infty \tag{2.0.14}$$

Then

$$\Pr\left(\lim_{n\to\infty} \sup A_n\right) = 1 \tag{2.0.15}$$

1) OPTION 1: We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.
   from (2.0.11)
$$\sum_{n=1}^{\infty} \Pr(X_n > \log n) = \sum_{n=1}^{\infty} e^{-\log n} \tag{2.0.16}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} \tag{2.0.17}$$
$$= \infty \text{ (Cauchy's Criterion)} \tag{2.0.18}$$
   Now, from (2.4)
$$\Pr(X_n > \log n \ \ \forall n \geq 1) \tag{2.0.19}$$
$$= \Pr\left(\lim_{n\to\infty} \sup X_n > \log n\right) \tag{2.0.20}$$
$$= 1 \tag{2.0.21}$$
   $\therefore$ Option 1 is correct.
2) OPTION 2: We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as X_n are independent random variable.
   From (2.0.11)
$$\sum_{n=1}^{\infty} \Pr(X_n > 2) = \sum_{n=1}^{\infty} e^{-2} \tag{2.0.22}$$
$$= \infty \tag{2.0.23}$$
   Now, from (2.4)
$$\Pr(X_n > 2 \ \ \forall n \geq 1) \tag{2.0.24}$$
$$= \Pr\left(\lim_{n\to\infty} \sup X_n > 2\right) \tag{2.0.25}$$
$$= 1 \tag{2.0.26}$$
   $\therefore$ Option 2 is correct.
3) OPTION 3: We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as X_n are independent random variable.
   From (2.0.11)
$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}} \tag{2.0.27}$$
$$= \infty \tag{2.0.28}$$
   Now, from (2.4)
$$\Pr\left(X_n > \frac{1}{2} \ \ \forall n \geq 1\right) \tag{2.0.29}$$
$$= \Pr\left(\lim_{n\to\infty} \sup X_n > \frac{1}{2}\right) \tag{2.0.30}$$
$$= 1 \tag{2.0.31}$$
   $\therefore$ Option 3 is incorrect.
4) OPTION 4:We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.
   Let the event $X_n > \log n, X_{n+1} > \log(n+1)$ be represented by A_n'
   From (2.0.11)
$$\sum_{n=1}^{\infty} \Pr(A_n) \tag{2.0.32}$$
$$= \sum_{n=1}^{\infty} \Pr(X_n > \log n) \Pr(X_{n+1} > \log(n+1)) \tag{2.0.33}$$
$$= \sum_{n=1}^{\infty} e^{-\log n} e^{-\log(n+1)} \tag{2.0.34}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \tag{2.0.35}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \tag{2.0.36}$$
$$= 1 \tag{2.0.37}$$
   Now, from (2.3)
$$\Pr(X_n > \log n, X_{n+1} > \log(n+1) \ \ \forall n \geq 1) \tag{2.0.38}$$
$$= \Pr\left(\lim_{n\to\infty} \sup(X_n > \log n, X_{n+1} > \log(n+1))\right) \tag{2.0.39}$$
$$= 0 \tag{2.0.40}$$
   $\therefore$ Option 4 is correct.
   Correct options are **Option 1,2,4**