

AI1103 : Assignment 6

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<https://github.com/BokkaRajaRaviKiranReddy/AI1103/blob/main/Assignment6/Assignment6.tex>

1 UGC NET 2013 JUNE Q.NO 101

Let A_1, A_2, \dots be the independent random variables each following exponential distribution with mean 1. then which of the following statements are correct?

- 1) $\Pr(X_n > \log n \text{ for infinitely many } n \geq 1) = 1$
- 2) $\Pr(X_n > 2 \text{ for infinitely many } n \geq 1) = 1$
- 3) $\Pr(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1) = 0$
- 4) $\Pr(X_n > \log n, X_{n+1} > \log(n+1) \text{ for infinitely many } n \geq 1) = 0$

2 SOLUTION

PDF of X_i is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean of X_i is expressed as

$$E[X_i] = \int_{-\infty}^{\infty} x f_{X_i}(x) dx \quad (2.0.1)$$

$$= \int_{-\infty}^0 0 dx + \int_0^{\infty} x \lambda_i e^{-\lambda_i x} dx \quad (2.0.2)$$

$$= \frac{1}{\lambda_i} \quad (2.0.3)$$

From (2.0.3) and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \geq 1$
Now, for some constant $c \geq 0$

$$\Pr(X_n > c) = \int_c^{\infty} f_{X_n}(x) dx \quad (2.0.4)$$

$$= \int_c^{\infty} e^{-x} dx \quad (2.0.5)$$

$$= e^{-c} \quad (2.0.6)$$

We need Borel-Cantelli Lemma 1, Lemma 2

Lemma 2.1. Let A_1, A_2, \dots be a sequence of events in some probability space. The Borel-Cantelli lemma states that, if the sum of the probabilities of the events A_n is finite

$$\sum_{n=1}^{\infty} \Pr(A_n) < \infty \quad (2.0.7)$$

then the probability that infinitely many of them occur is 0

$$\Pr \limsup_{n \rightarrow \infty} A_n = 0 \quad (2.0.8)$$

Lemma 2.2. If the events A_n are independent and the sum of the probabilities of the A_n diverges to infinity, then the probability that infinitely many of them occur is 1. If for independent events A_1, A_2, \dots

$$\sum_{n=1}^{\infty} \Pr(A_n) = \infty \quad (2.0.9)$$

Then

$$\Pr \left(\limsup_{n \rightarrow \infty} A_n \right) = 1 \quad (2.0.10)$$

- 1) OPTION 1: We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.
from (2.0.6)

$$\sum_{n=1}^{\infty} \Pr(X_n > \log n) = \sum_{n=1}^{\infty} e^{-\log n} \quad (2.0.11)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \quad (2.0.12)$$

$$= \infty \quad (2.0.13)$$

Now, from (2.2)

$$\Pr(X_n > \log n \text{ for infinitely many } n \geq 1) \quad (2.0.14)$$

$$= \Pr \left(\limsup_{n \rightarrow \infty} X_n > \log n \right) \quad (2.0.15)$$

$$= 1 \quad (2.0.16)$$

\therefore Option 1 is correct.

- 2) OPTION 2: We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (2.0.6)

$$\sum_{n=1}^{\infty} \Pr(X_n > 2) = \sum_{n=1}^{\infty} e^{-2} \quad (2.0.17)$$

$$= \infty \quad (2.0.18)$$

Now, from (2.2)

$$\Pr(X_n > 2 \text{ for infinitely many } n \geq 1) \quad (2.0.19)$$

$$= \Pr\left(\limsup_{n \rightarrow \infty} X_n > 2\right) \quad (2.0.20)$$

$$= 1 \quad (2.0.21)$$

\therefore Option 2 is correct.

- 3) OPTION 3: We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as X_n are independent random variable.

From (2.0.6)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}} \quad (2.0.22)$$

$$= \infty \quad (2.0.23)$$

Now, from (2.2)

$$\Pr\left(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1\right) \quad (2.0.24)$$

$$= \Pr\left(\limsup_{n \rightarrow \infty} X_n > \frac{1}{2}\right) \quad (2.0.25)$$

$$= 1 \quad (2.0.26)$$

\therefore Option 3 is incorrect.

- 4) OPTION 4: We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.

Let the event $X_n > \log n, X_{n+1} > \log(n+1)$ be represented by A_n ,

From (2.0.6)

$$\sum_{n=1}^{\infty} \Pr(A_n) \quad (2.0.27)$$

$$= \sum_{n=1}^{\infty} \Pr(X_n > \log n) \Pr(X_{n+1} > \log(n+1)) \quad (2.0.28)$$

$$= \sum_{n=1}^{\infty} e^{-\log n} e^{-\log(n+1)} \quad (2.0.29)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad (2.0.30)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \quad (2.0.31)$$

$$= 1 \quad (2.0.32)$$

Now, from (2.1)

$$\Pr(A_n \text{ for infinitely many } n \geq 1) \quad (2.0.33)$$

$$= \Pr\left(\limsup_{n \rightarrow \infty} (X_n > \log n, X_{n+1} > \log(n+1))\right) \quad (2.0.34)$$

$$= 0 \quad (2.0.35)$$

\therefore Option 4 is correct.

SO, option 1,2,4 are correct.