AI1103 : Assignment 6

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1 UGC NET 2013 JUNE Q.NO 101

Let $X_1, X_2,...$be the independent random variables each following exponential distribution with mean 1.then which of the following statements are correct?

1) $\Pr(X_n > \log n \ \forall n \geq 1) = 1$

2) $\Pr(X_n > 2 \ \forall n \geq 1) = 1$

3) $\Pr\left(X_n > \frac{1}{2} \ \forall n \geq 1\right) = 0$

4) $\Pr(X_n > \log n, X_{n+1} > \log(n+1) \ \forall n \geq 1) = 0$

2 SOLUTION

PDF of X_i is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean of X_i is expressed as

$$E[X_i] = \int_{-\infty}^{\infty} x f_{X_i}(x)dx \qquad (2.0.1)$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} x\lambda_i e^{-\lambda_i x} \qquad (2.0.2)$$

$$= \frac{1}{\lambda_i} \qquad (2.0.3)$$

From (2.0.3) and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \geq 1$
Now, for some constant $c \geq 0$

$$\Pr(X_n > c) = \int_{c}^{\infty} f_{X_n}(x)dx \qquad (2.0.4)$$

$$= \int_{c}^{\infty} e^{-x}dx \qquad (2.0.5)$$

$$= e^{-c} \qquad (2.0.6)$$

We need Borel-Cantelli Lemma 1 , Lemma 2

Lemma 2.1. *Let $A_1, A_2,...$ be a sequence of events in some probability space. The Borel–Cantelli lemma states that, if the sum of the probabilities of the events A_n is finite*

$$\sum_{n=1}^{\infty} \Pr(A_n) < \infty \qquad (2.0.7)$$

then the probability that infinitely many of them occur is 0

$$\Pr \limsup_{n \to \infty} A_n = 0 \qquad (2.0.8)$$

Lemma 2.2. *If the events A_n are independent and the sum of the probabilities of the A_n diverges to infinity, then the probability that infinitely many of them occur is 1. If for independent events $A_1, A_2, ...$*

$$\sum_{n=1}^{\infty} \Pr(A_n) = \infty \qquad (2.0.9)$$

Then

$$\Pr\left(\limsup_{n \to \infty} A_n\right) = 1 \qquad (2.0.10)$$

1) OPTION 1: We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.
   from (2.0.6)

$$\sum_{n=1}^{\infty} \Pr(X_n > \log n) = \sum_{n=1}^{\infty} e^{-\log n} \qquad (2.0.11)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \qquad (2.0.12)$$

$$= \infty \text{ (Cauchy's Criterion)} \qquad (2.0.13)$$

Now, from (2.2)

$$\Pr(X_n > \log n \ \forall n \geq 1) \qquad (2.0.14)$$

$$= \Pr\left(\limsup_{n \to \infty} X_n > \log n\right) \qquad (2.0.15)$$

$$= 1 \qquad (2.0.16)$$

$\therefore$ Option 1 is correct.

2) OPTION 2: We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as X_n are independent random variable.
From (2.0.6)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > 2\right) = \sum_{n=1}^{\infty} e^{-2} \qquad (2.0.17)$$

$$= \infty \qquad (2.0.18)$$

Now, from (2.2)

$$\Pr\left(X_n > 2 \ \forall n \geq 1\right) \qquad (2.0.19)$$

$$= \Pr\left(\limsup_{n\to\infty} X_n > 2\right) \qquad (2.0.20)$$

$$= 1 \qquad (2.0.21)$$

$\therefore$ Option 2 is correct.

3) OPTION 3: We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as X_n are independent random variable.
From (2.0.6)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}} \qquad (2.0.22)$$

$$= \infty \qquad (2.0.23)$$

Now, from (2.2)

$$\Pr\left(X_n > \frac{1}{2} \ \forall n \geq 1\right) \qquad (2.0.24)$$

$$= \Pr\left(\limsup_{n\to\infty} X_n > \frac{1}{2}\right) \qquad (2.0.25)$$

$$= 1 \qquad (2.0.26)$$

$\therefore$ Option 3 is incorrect.

4) OPTION 4: We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as X_n are independent random variable.
Let the event $X_n > \log n, X_{n+1} > \log(n+1)$ be represented by A_n'

From (2.0.6)

$$\sum_{n=1}^{\infty} \Pr\left(A_n\right) \qquad (2.0.27)$$

$$= \sum_{n=1}^{\infty} \Pr\left(X_n > \log n\right) \Pr\left(X_{n+1} > \log(n+1)\right) \qquad (2.0.28)$$

$$= \sum_{n=1}^{\infty} e^{-\log n} e^{-\log(n+1)} \qquad (2.0.29)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad (2.0.30)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \qquad (2.0.31)$$

$$= 1 \qquad (2.0.32)$$

Now, from (2.1)

$$\Pr\left(X_n > \log n, X_{n+1} > \log(n+1) \ \forall n \geq 1\right) \qquad (2.0.33)$$

$$= \Pr\left(\limsup_{n\to\infty}(X_n > \log n, X_{n+1} > \log(n+1))\right) \qquad (2.0.34)$$

$$= 0 \qquad (2.0.35)$$

$\therefore$ Option 4 is correct.
Correct options are **Option 1,2,4**