[1]:
```
using Markdown
```

[2]:
```
using Plots, LaTeXStrings, QuadGK, Roots,␣
 ↪DifferentialEquations
```

1 1D Bistable Example

Define separate production and degradation terms

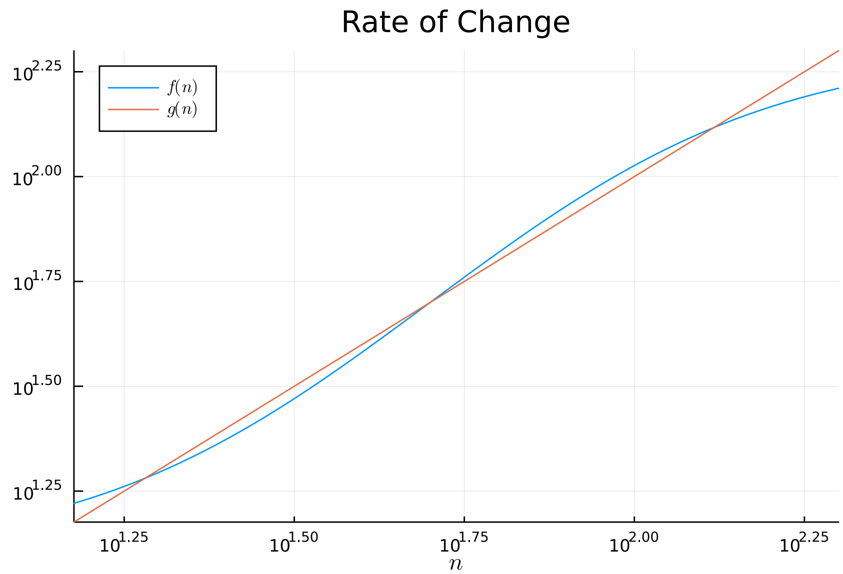
[3]:
```
begin
        n_max = 200.
        n_min = 1 # need to avoid 0 for log we'll evaluate␣
 ↪later
        n = n_min:0.1:n_max
          = 1
        ks = 1e-4
         0 = 12.5
         1 = 200
        production(n) = ( 0+  1 * ks*n^2)/(1+ks*n^2)
        degradation(n) =  *n
end
```

[3]: degradation (generic function with 1 method)

[4]:
```
begin

        plot(production, n, label=L"f(n)")
        p1 = plot!(degradation, n, label=L"g(n)",␣
 ↪xlabel=L"n", title="Rate of Change", scale=:log10,␣
 ↪xlimits=(15, n_max), ylimits=(15, 200), legend=:topleft)
# plot!(n->production(n)  - degradation(n), n,␣
 ↪label=L"\frac{dn}{dt}",
#                 ylims=(-20, 150), xlims=(0, 150), legend=:
 ↪topleft, title="Time evolution")
end
```

[4]:



*Compute the fixed points to split up the 2 attractors into **Basins of Attractions**"*

[5]:
```
fixed_points = find_zeros(n-> production(n) -␣
 ↪degradation(n), n_min, n_max)
```

[5]:
```
3-element Vector{Float64}:
  19.098300562505266
  50.000000000000014
 130.90169943749478
```

Integrate to get the quasi-potential

[6]:
```
begin
        f = production
        g = degradation

         (n) = quadgk(n -> -2*(f(n) -g(n))/(f(n) + g(n)),␣
 ↪n_min, n)[1]
end
```

[6]: (generic function with 1 method)

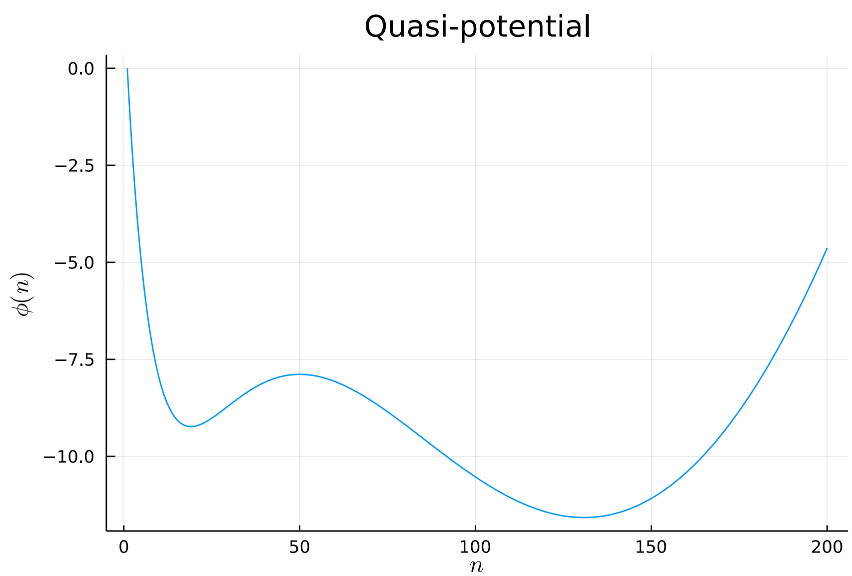
[7]:
```
_vec =  .(n)
```

[7]:
```
1991-element Vector{Float64}:
 -0.0
 -0.16905266634092922
 -0.33541586771004656
 -0.49913123834477463
 -0.660239686628211
 -0.818781410188954
 -0.9747959106256152
 -1.1283220078669904
 -1.2793978541784838
 -1.4280609478250232
 -1.574348146400372
 -1.7182956798324023
 -1.8599391630735853

 -4.858877155199968
 -4.838515233267167
 -4.818122088274848
 -4.797697722363832
 -4.777242137690655
 -4.756755336427483
 -4.736237320762113
 -4.715688092897896
 -4.695107655053741
 -4.674496009463997
 -4.653853158378498
 -4.633179104062437
```

[8]:
```
p2 = plot(n,  _vec, ylabel=L"\phi(n)", xlabel=L"n",␣
↪label="", title="Quasi-potential")
```

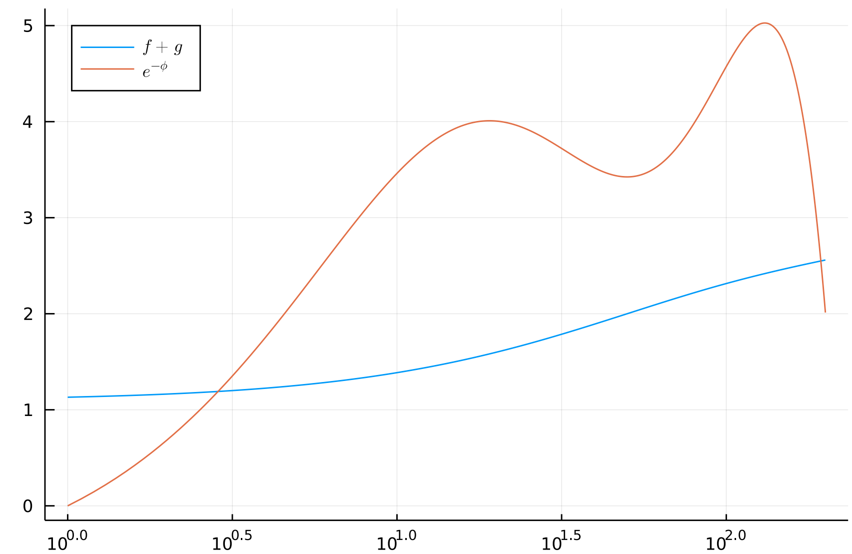
[8]:



Show the the probability due to the quasi-potential dominates compared to the other term in the equation

[9]:
```
begin
	plot(n->log10.(f(n) + g(n)), n, label=L"f + g")
	plot!(n, log10.( .^(- _vec)), label=L"e^{-\phi}")
	p3 = plot!(xaxis=:log10, legend=:topleft)
end
```

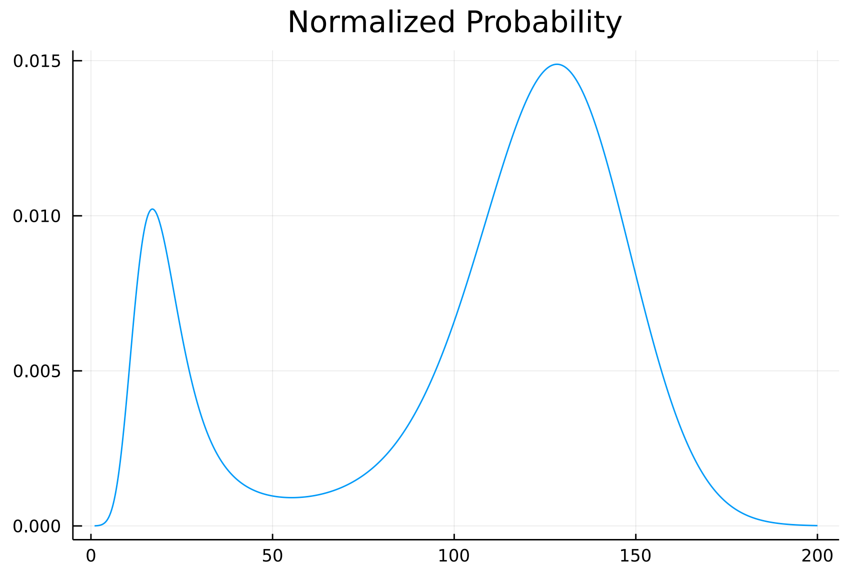
[9]:



Compute the normalized probability

[10]:
```
begin
        fn = f.(n)
        gn = g.(n)
        p_unnormal(n) =   ^(- (n)) / (f(n)+g(n))
        norm = quadgk(p_unnormal, n_min, n_max)[1]
        p_normal(n) = 1/norm * p_unnormal(n)
        p4 = plot(p_normal,n,  title = "Normalized␣
 ↪Probability", label="")
end
```

[10]:



Set up the jump process simulation

[11]:
```
random_count() = floor(Int, n_min + rand() * n_max)
```

[11]:
```
random_count (generic function with 1 method)
```

[12]:
```
begin
        affect1!(integrator) = integrator.u += 1
        affect2!(integrator) = integrator.u -= 1
        prodJump = ConstantRateJump((u, p, t) -> f(u),␣
 ↪affect1!)
        degJump = ConstantRateJump((u, p, t) -> g(u),␣
 ↪affect2!)
        u0 = random_count()
        t = (0.0, 5000.0)
        dprob = DiscreteProblem(u0, t)
        jprob = JumpProblem(dprob, Direct(), prodJump,␣
 ↪degJump)
end
```

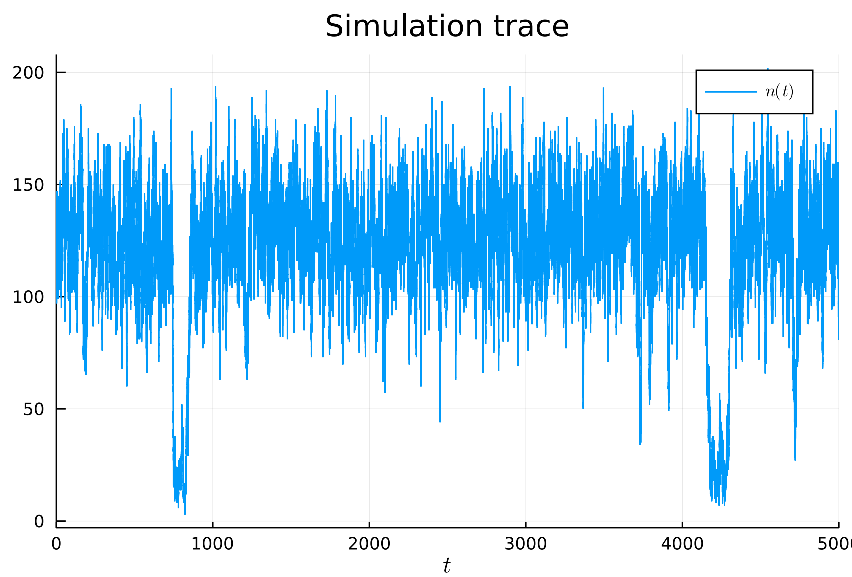
[12]:

```
Number of constant rate jumps: 2
Number of variable rate jumps: 0
```

[13]:

```
begin
        sim_res = solve(jprob, SSAStepper())
        p5 = plot(sim_res, xlabel=L"t", label=L"n(t)",␣
 ↪title="Simulation trace")
end
```

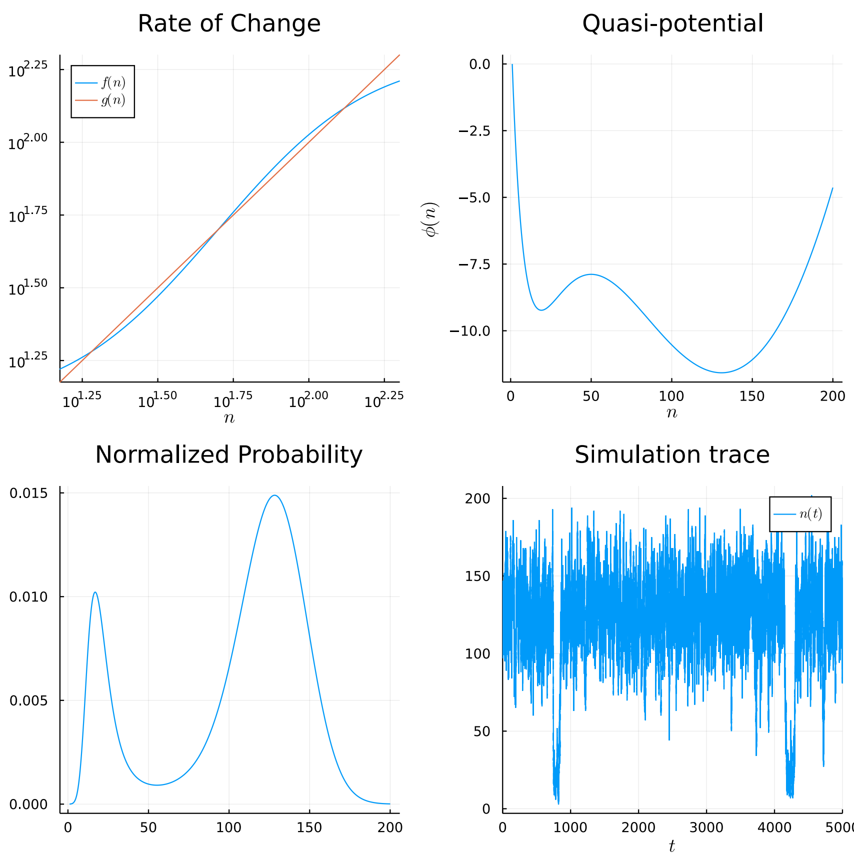
[13]:



[14]:

```
begin
        pf = plot(p1, p2, p4, p5, layout = grid(2,2),␣
 ↪size=(750, 750))
        # savefig(pf, "ch3_bistable_prob.pdf")
        pf
end
```

[14]:



Setup functions to do a longer time simulation and compare to integrated probability distribution

[15]:
```
function calculate_time_fraction(sim_result, split)
        dts = diff(sim_result.t)
        dt_vals = collect(zip(dts, map(last, sim_result.u[1:
 ↪end])))
        high = sum(map(first, filter(dtu -> last(dtu) >␣
 ↪split, dt_vals)))
        low = sum(map(first, filter(dtu -> last(dtu) <␣
 ↪split, dt_vals)))
        high, low
```

```
end
```

[15]: calculate_time_fraction (generic function with 1 method)

[16]:
```
function big_sim(num_sim = 25)
        function prob_func(prob, i, repeat)
        remake(prob, u0=random_count())
    end
        num_steps = 0
    ensemble_prob = EnsembleProblem(jprob,␣
↪prob_func=prob_func)
        sim = solve(ensemble_prob, SSAStepper(),␣
↪trajectories=num_sim)
        totals = [0., 0.]
        for j in 1:num_sim
                totals.+= calculate_time_fraction(sim[j],␣
↪fixed_points[2])
                num_steps += length(sim[j])
        end
        totals, num_steps
end
```

[16]: big_sim (generic function with 2 methods)

[17]:
```
totals, num_steps = big_sim()
```

[17]: ([115699.83688188824, 9300.163118111735], 29192028)

[18]:
```
begin
        sim_frac = totals./sum(totals)
end
```

[18]: 2-element Vector{Float64}:
 0.9255986950551061
 0.07440130494489389

Time fractions from simulations is $(sim_frac)

[19]:
```
begin
        p_low = quadgk(p_normal, n_min, fixed_points[2])[1]
```

```
        p_high = quadgk(p_normal, fixed_points[2], n_max)[1]
        comp_frac = p_high, p_low
end
```

[19]: (0.8055671593777174, 0.19443284062228242)

Time fractions from integration of probability is $(comp_frac)