Example 7: Using the theorem of deduction prove that the formula $(\forall x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\forall x)B(x))$ is a theorem.

We apply the reverse of the theorem of deduction:

$\vdash (\forall x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\forall x)B(x))$ ==>

$(\forall x)(A(x) \rightarrow B(x)) \vdash ((\forall x)A(x) \rightarrow (\forall x)B(x))$ ==>

$(\forall x)(A(x) \rightarrow B(x)), (\forall x)A(x) \vdash (\forall x)B(x)$

We prove that $(\forall x)(A(x) \rightarrow B(x)), (\forall x)A(x) \vdash (\forall x)B(x)$ using the definition of a deduction building the sequence (f1, f2,…, f8) of predicate formulas:

f1: $(\forall x)(A(x) \rightarrow B(x))$

f2: $(\forall x)(A(x) \rightarrow B(x)) \rightarrow (A(y) \rightarrow B(y))$ ---- axiom A4, x is instantiated with y-term

f1, f2 $|-_{mp}$ f3 = $A(y) \rightarrow B(y)$

f4: $(\forall x)A(x)$

f5: $(\forall x)A(x) \rightarrow A(y)$ ---- axiom A4, x is instantiated with y-term

f4, f5 $|-_{mp}$ f6 = $A(y)$

f3, f6 $|-_{mp}$ f7 = $B(y)$

f7 $|-_{gen}$ f8 = $(\forall x)\, B(x)$---- axiom A5

(f1,…, f8) is the deduction of $(\forall x)\, B(x)$ from the hypothesis $(\forall x)(A(x) \rightarrow B(x))$ and $(\forall x)A(x)$.

Using this deduction and applying twice the theorem of deduction we obtain:

$(\forall x)(A(x) \rightarrow B(x)), (\forall x)A(x) \vdash (\forall x)B(x)$ ==>

$(\forall x)(A(x) \rightarrow B(x)) \vdash (\forall x)A(x) \rightarrow (\forall x)B(x)$ ==>

$\vdash (\forall x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\forall x)B(x))$

Thus we have proved that the initial formula is a theorem.

The Skolem normal forms without quantifiers are obtained by eliminating the universal quantifiers.

$A_1^{Sq} = (\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(a) = A_1^C$ **(clausal form)**

$A_2^{Sq} = (\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(f(x)) = A_2^C$

$A_3^{Sq} = (\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(g(x,y)) = A_3^C$

Example 8: Transform into prenex normal form, Skolem normal form and clausal normal form the formula: $U = (\exists x)(\forall y)P(x,y) \vee (\exists z)(\neg Q(z) \vee (\forall u)(\exists t)R(z,u,t))$

$U^P = (\exists x)(\forall y)(\exists z)(\forall u)(\exists t)(P(x,y) \vee \neg Q(z) \vee R(z,u,t))$

$U^S = (\forall y)(\forall u)(P(a,y) \vee \neg Q(f(y)) \vee R(f(y),u,g(y,u)))$, where:

$[x \leftarrow a],[z \leftarrow f(y)],[t \leftarrow g(y,u)]$, a = Skolem constant, f,g = Skolem functions

$U^{Sq} = P(a,y) \vee \neg Q(f(y)) \vee R(f(y),u,g(y,u)) = U^C$

Theorem 3 (Church 1936):

The problem of validity of a first-order formula is ***undecidable***, but is ***semi-decidable***. If a procedure P is used to check the validity of a formula we have the following situations:

- if a formula A is valid, then P ends with the corresponding answer.
- if the formula A is not valid, then P ends with the corresponding answer or P may never stop.

Theorem of deduction: if $X \cup \{A\} \vdash B$ then $X \vdash (A \rightarrow B)$, is used in theorem proving.
Refutation theorem: if $X \cup \{\neg A\}$ is inconsistent then $X \vdash A$.

This theorem is used in proof methods such as: resolution, semantic tableaux method, called *refutation proof methods*.

Theorem of soundness and completness states the equivalence between the "logical consequence" concept and "syntactic consequence" concept.
Let Γ be a set of first-order formulas and γ a first-order formula.

- *completness*: if $\Gamma \models \gamma$ then $\Gamma \vdash \gamma$.
- *soundness*: if $\Gamma \vdash \gamma$ then $\Gamma \models \gamma$.

A particular case of this theorem is the following result:
"A formula is a tautology if and only if it is a theorem in first-order logic."