Seminar Nr.7, Inequalities; Sequences of Random Variables

Theory Review

Hölder's Inequality: $E(|XY|) \leq (E(|X|^p))^{\frac{1}{p}} \cdot (E(|Y|^q))^{\frac{1}{q}}$, $\forall p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Markov's Inequality: $P(|X| \geq a) \leq \frac{1}{a} E(|X|)$, $\forall a > 0$.

Chebyshev's Inequality: $P(|X - E(X)| \geq \epsilon) \leq \frac{V(X)}{\epsilon^2}$, $\forall \epsilon > 0$.

Convergence:
1) **in probability** $X_n \xrightarrow{p} X$, if $\lim\limits_{n\to\infty} P\Big(|X_n - X| < \varepsilon\Big) = 1, \quad \forall \varepsilon > 0$;
2) **strongly** $X_n \xrightarrow{s} X$, if $\lim\limits_{n\to\infty} P\Big(\bigcap\limits_{k\geq n} \{|X_k - X| < \varepsilon\}\Big) = 1, \quad \forall \varepsilon > 0$;
3) **almost surely** $X_n \xrightarrow{a.s.} X$, if $P\Big(\lim\limits_{n\to\infty} X_n = X\Big) = 1$;
4) **in distribution** $X_n \xrightarrow{d} X$, if $\lim\limits_{n\to\infty} F_n(x) = F(x)$, $\forall x \in \mathbb{R}$ continuity point for F;
5) **in mean of order** r, $0 < r < \infty$ $X_n \xrightarrow{L^r} X$, if $\lim\limits_{n\to\infty} E(|X_n - X|^r) = 0$.

Properties
1. 2) <=> 3) => 1) => 4)
2. 5) => 1)

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1. (The 3σ Rule). For any random variable X, most of the values of X lie within 3 standard deviations away from the mean.

2. True or False: There is at least a 90% chance of the following happening: when flipping a coin 1000 times, the number of "heads" that appear is between 450 and 550.

3. Let X be a r. v. with pdf $f(x) = \frac{x^m e^{-x}}{m!}$, $x \geq 0$. Show that $P(0 < X < 2(m+1)) \geq \frac{m}{m+1}$.

4. Let $S = [0,1]$, $\mathcal{K} = \mathcal{B}[0,1]$ (the set of all open subsets of $[0,1]$) and let $P = \mu$ be the Lebesgue measure on $[0,1]$ (length, distance). In the probability space $(S, \mathcal{K}, P)$, consider the sequence of random variables given by

$$X_n(e) = \begin{cases} 2^n, & \text{if } 0 \leq e \leq \frac{1}{n} \\ 1, & \text{if } e > \frac{1}{n}. \end{cases}$$

Study the various types of convergence of X_n to $X = 1$.

5. Consider a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables such that each X_n is uniformly distributed on $[-n, n]$. Does $\{X_n\}_{n\in\mathbb{N}}$ converge in distribution?

6. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables and let X be a random variable such that $X_n \xrightarrow{L^r} X$.

Prove that $X_n \xrightarrow{L^s} X$ for each $s \leq r$.

7. Let $X_n \in N(\mu_n, \sigma_n)$ be a sequence of random variables such that $X_n \xrightarrow{L^r} 0$, $r \geq 2$. Show that $\lim\limits_{n\to\infty} \mu_n = \lim\limits_{n\to\infty} \sigma_n = 0$.