Seminar Nr.5, Continuous Random Variables; Continuous Random Vectors; Functions of Continuous Random Variables

Theory Review

$X : S \to \mathbb{R}$ continuous random variable with pdf $f : \mathbb{R} \to \mathbb{R}$ and cdf $F : \mathbb{R} \to \mathbb{R}$. Properties:

1. F is absolutely continuous and $F(x) = P(X < x) = \int\limits_{-\infty}^{x} f(t)dt$

2. $f(x) \geq 0$, $\forall x \in \mathbb{R}$, $\int\limits_{\mathbb{R}} f(x) = 1$

3. $P(X = x) = 0$, $\forall x \in \mathbb{R}$, $P(a < X < b) = \int\limits_{a}^{b} f(t)dt$

4. F is left continuous and monotonely increasing
5. $F(-\infty) = 0$, $F(\infty) = 1$

$(X, Y) : S \to \mathbb{R}^2$ continuous random vector with pdf $f = f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$ and

cdf $F = F_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = P(X < x, Y < y) = \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} f(u, v)\ dv\ du$, $\forall (x, y) \in \mathbb{R}^2$. Properties:

1. $P(a_1 \leq X < b_1, a_2 \leq Y < b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$
2. F is left continuous and monotonely increasing in each variable
3. $F(\infty, \infty) = 1$, $F(-\infty, y) = F(x, -\infty) = 0$, $\forall x, y \in \mathbb{R}$
4. X and Y are independent $<=>$ $F(x, y) = F_X(x)F_Y(y)$, $\forall (x, y) \in \mathbb{R}^2$
5. $F_X(x) = F(x, \infty)$, $F_Y(y) = F(\infty, y)$, $\forall x, y \in \mathbb{R}$ (marginal cdf's)
6. $P((X, Y) \in D) = \iint\limits_{D} f(x, y)\ dy\ dx$
7. $f_X(x) = \int\limits_{\mathbb{R}} f(x, y)dy$, $\forall x \in \mathbb{R}$, $f_Y(y) = \int\limits_{\mathbb{R}} f(x, y)dx$, $\forall y \in \mathbb{R}$ (marginal densities)
8. X and Y are independent $<=>$ $f_{(X,Y)}(x, y) = f_X(x)f_Y(y)$, $\forall (x, y) \in \mathbb{R}^2$.

Function $Y = g(X)$: X r.v., $g : \mathbb{R} \to \mathbb{R}$ differentiable with $g' \neq 0$, strictly monotone

$f_Y(y) = \dfrac{f_X\left(g^{-1}(y)\right)}{\left|g'\left(g^{-1}(y)\right)\right|}$, $y \in g(\mathbb{R})$

(X, Y) continuous random vector with joint pdf $f_{(X,Y)}$

Sum: $f_{X+Y}(z) = \int\limits_{\mathbb{R}} f_{(X,Y)}(u, z - u)du \overset{X,Y ind}{=} \int\limits_{\mathbb{R}} f_X(u)f_Y(z - u)du$

Product: $f_{XY}(z) = \int\limits_{\mathbb{R}} f_{(X,Y)}\left(u, \dfrac{z}{u}\right)\dfrac{1}{|u|}du \overset{X,Y ind}{=} \int\limits_{\mathbb{R}} f_X(u)f_Y\left(\dfrac{z}{u}\right)\dfrac{1}{|u|}du$

Quotient: $f_{X/Y}(z) = \int\limits_{\mathbb{R}} f_{(X,Y)}(uz, u)\,|u|du \overset{X,Y ind}{=} \int\limits_{\mathbb{R}} f_X(uz)f_Y(u)\,|u|du$

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1. Let $f(x) = kx$, $2 \leq x \leq 4$. Find
a) the constant k that makes f a density function (of some variable X);
b) the corresponding cdf F;
c) $P(2.5 \leq X \leq 3)$, $P(X = 2.5)$, $P(2.5 < X < 3)$.

2. Let $F(x) = a + b\ \mathrm{arctg}x$, $\forall x \in \mathbb{R}$. Find
a) the constants a, b so that F is the cdf of a random variable X;
b) $P(-1 \leq X < \sqrt{3})$;
c) the corresponding pdf, f.

3. The joint density for (X, Y) is $f_{(X,Y)}(x, y) = \frac{1}{16}x^3y^3$, x, $y \in [0, 2]$.
a) Find the marginal densities f_X, f_Y.
b) Are X and Y independent?
c) Find $P(X \leq 1)$.

4. Let X be a random variable with density $f_X(x) = \frac{1}{4}xe^{-\frac{x}{2}}$, $x \geq 0$ and let $Y = \frac{1}{2}X + 2$. Find f_Y.

5. Let $X \in N(0, 1)$. Find the probability density function of $Y = |X|$, $Z = e^{X^2}$, $T = X^2 - 1$.

6. Let X and Y be independent uniformly distributed variables over $(0, a)$ and $(0, b)$, respectively $(0 < a < b)$. Find the probability density function of $Z = XY$.

Bonus Problems:

7. The joint density for (X, Y) is $f_{(X,Y)}(x, y) = kxye^{-x}e^{-y}$, x, $y > 0$.
a) Find the constant k that makes this a density.
b) Find $P(X < Y)$, $P(X > 1)$.
c) Are X and Y independent?

8. Let X, $Y \in N(0, 1)$ be independent random variables. Let D_r be the disk centered at the origin with radius r. Find r such that $P((X, Y) \in D_r) = \alpha$, $0 < \alpha < 1$, given.