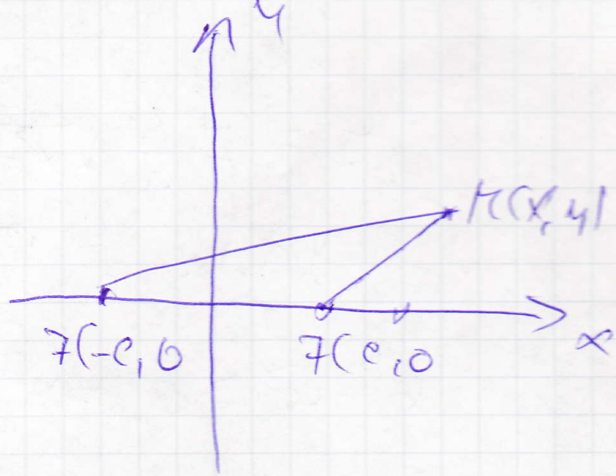
CURS 10 — 4. V. 2011

The hyperbola is a plane curve defined as the locus of points whose absolute value difference to two fixed points is constant. The 2 fixed points are called the foci and the distance between them is called the focal distance

F, F' - foci

$|FF'| = 2c$



$M(x, y)$

$||MF| - |MF'|| = 2a$

Remark. In $\Delta MFF'$ we have $|MF| < |MF'| + 2c$

$|MF'| < |MF| + 2c$

$\Leftrightarrow ||MF| - |MF'|| < 2c \Leftrightarrow a < c$

$|MF| - |MF'| = \pm 2a \Leftrightarrow$

$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a$

$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}$

$\Rightarrow \cancel{x^2} - 2cx + \cancel{c^2} + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + \cancel{x^2} +$
$+ 2cx + \cancel{c^2} + y^2$

$\pm a\sqrt{(x+c)^2 + y^2} = a^2 + cx \Rightarrow$

$\Rightarrow a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$

$\Leftrightarrow a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$

$$\Leftrightarrow \underbrace{(c^2-a^2)}_{b^2} x^2 - a^2y^2 = a^2\underbrace{(c^2-a^2)}_{>0} \quad ; \quad b^2 = c^2 - a^2$$

$$\Leftrightarrow b^2x^2 - a^2y^2 = a^2b^2 \quad | : a^2b^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Leftrightarrow b^2x^2 - a^2y^2 = a^2b^2$$

$$a^2y^2 = b^2(x^2-a^2)$$

$$y = \pm \frac{b}{a}\sqrt{x^2-a^2}$$

$$f(x) = \frac{b}{a}\sqrt{x^2-a^2}$$

$$x^2 - a^2 \geq 0 \Leftrightarrow |x| \geq a \Leftrightarrow x \in (-\infty; -a] \cup [a; \infty)$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{b}{a} \frac{|x|\sqrt{1-\frac{a^2}{x^2}}}{x} = \frac{b}{a}\lim_{x\to\pm\infty}\frac{|x|}{x}$$

$$= \pm\frac{b}{a}$$

$$y = \pm\frac{b}{a}x$$

$$\lim_{x\to\pm\infty}(f(x) - m_{\pm}x) = \lim_{x\to\pm\infty}\left(\frac{b}{a}\sqrt{x^2-a^2} \mp \frac{b}{a}x\right)$$

$$= \frac{b}{a}\lim_{x\to\pm\infty}\left(\sqrt{x^2-a^2} \mp x\right) = 0$$

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2-a^2}}$$

x	$-\infty$		$-a$		a		$+\infty$
$f'(x)$		$-$	|		|	$+$	$+$
$f(x)$	$+\infty$	$\searrow$	0		0	$\nearrow$	$+\infty$
$f''(x)$		$-$ $\frown$	|		|	$-$ $\frown$	

$$\lim_{x \to \pm\infty} f(x) = +\infty$$

$$b^2 = c^2 - a^2$$

$e = \frac{c}{a}$ - the eccentricity of the hyperbola

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 + b^2}{a^2} = 1 + \left(\frac{b}{a}\right)^2$$

The intersection between a hyperbola and a line

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ y = mx + n \end{cases} \Rightarrow b^2 x^2 - a^2 (mx + n)^2 = a^2 b^2$$

$$\Leftrightarrow b^2 x^2 - a^2 m^2 x^2 - 2a^2 mn x - a^2 n^2 = a^2 b^2$$

$$\Rightarrow (b^2 - a^2 m^2) x^2 - 2a^2 mn x - a^2 (n^2 + b^2) = 0$$

- If $b^2 - a^2 m^2 = 0 \Leftrightarrow m = \pm \frac{b}{a}$ the eq becomes $2a^2 mn x + a^2 (n^2 + b^2) = 0$
  $\Rightarrow 2mn x + n^2 + b^2 = 0$

- If $n = 0 \Rightarrow b^2 = 0$ incompatible $H \cap d = \emptyset$

- If $n \neq 0$, then $H \cap d$ = a unique point
- If $b^2 - a^2m^2 \neq 0$

$$\Delta = 4[a^4m^2n^2 + a^2(n^2+b^2)(b^2-a^2m^2)]$$

- If $\Delta < 0 \Rightarrow H \cap d = \emptyset$
- If $\Delta = 0 \Rightarrow H \cap d$ = a unique point
- If $\Delta > 0 \Rightarrow H \cap d$ = two different points

The tangents to a hyperbola

The tangent for a given direction

$$\Delta = 0 \Leftrightarrow a^4m^2n^2 + a^2(n^2b^2 - a^2m^2n^2 + b^4 - a^2b^2m^2) = 0$$

$$\Leftrightarrow a^4m^2n^2 + a^2b^2n^2 - a^4m^2n^2 + a^2b^4 - a^4b^2m^2 = 0$$

$$\Rightarrow a^2b^2(n^2 + b^2 - a^2m^2) = 0$$

$$\Rightarrow a^2m^2 - n^2 - b^2 = 0$$

$$\Rightarrow n = \pm\sqrt{a^2m^2 - b^2}$$

$$\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$$

the tangent line to a hyperbola at a given point

$$\text{If } \ldots \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$M(x_0, y_0) \in H$

$H = F^{-1}(0)$ where $F(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$

$$\nabla F(x_0, y_0) = \left(\frac{\partial F}{\partial x}(x_0, y_0), \frac{\partial F}{\partial y}(x_0, y_0)\right)$$

$\downarrow$

The normal vector of $T_M(H)$

$$\nabla F(x_0, y_0) = \left(\frac{2x_0}{a^2} ; \frac{2y_0}{b^2}\right)$$

$$T_M(H) \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0) = 0$$

$$T_M(H) \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) = 0$$

$$T_M(H) \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \underbrace{\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}}_{1}$$

$$T_M(H) \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

<u>The parabola</u>: it is the plane curve defined as the locus of points whose distance to a fixed line d equals the distance to a fixed point F

d - direction line

F - focus

$M(x;y) \in$ Parabolei

$$\Leftrightarrow |MF| = \frac{d(M,a)}{p^2}$$

$$\Leftrightarrow \sqrt{\left(x-\frac{p}{2}\right)^2+y^2} = \left|x+\frac{p}{2}\right|$$

$$x^2 - px + \frac{p^2}{4} = x^2 + px + \frac{p^2}{4}$$

$$\Leftrightarrow \boxed{y^2 = 2px}$$