SEMINAR 8

11. IV. 2011

(1) $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b}) \cdot \bar{c} = \begin{vmatrix} \bar{b} & \bar{c} \\ \bar{a} \cdot \bar{b} & \bar{a} \cdot \bar{c} \end{vmatrix}$

$$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{b} \cdot \bar{c})\bar{a} = \begin{vmatrix} \bar{b} & \bar{a} \\ \bar{b} \cdot \bar{c} & \bar{a} \cdot \bar{c} \end{vmatrix}$$

<u>Laplace identity</u>

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{b} \cdot \bar{c})(\bar{a} \cdot \bar{d}) =$$

$$= \begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{a} \cdot d \\ \bar{b} \cdot \bar{c} & \bar{b} \cdot \bar{d} \end{vmatrix}$$

$$\begin{aligned}
(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) &= (\bar{a} \times \bar{b}, \bar{c}, \bar{d}) = \\
&= (\bar{d}, \bar{a} \times \bar{b}, \bar{c}) \\
&= \bar{d}\,((\bar{a} \times \bar{b}) \times \bar{c}) \\
&= \bar{d}\,((\bar{a} \cdot \bar{c})\bar{b} - (\bar{b} \cdot \bar{c})\bar{a}) \\
&= (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{b} \cdot \bar{c})(\bar{a} \cdot \bar{d}) \checkmark
\end{aligned}$$

<u>Jacobi identity</u>

$$(\bar{a} \times \bar{b}) \times \bar{c} + (\bar{b} \times \bar{c}) \times \bar{a} + (\bar{c} \times \bar{a}) \times \bar{b} = \bar{0}$$

$$S = \cancel{(\bar{a} \cdot \bar{c})\bar{b}} - (\bar{b} \cdot \bar{c})\bar{a} + \cancel{(\bar{b} \cdot \bar{a})\bar{c}} - \cancel{(\bar{c} \cdot \bar{a})\bar{b}} +$$
$$+ (\bar{c} \cdot \bar{b})\bar{a} - \cancel{(\bar{a} \cdot \bar{b})\bar{c}}$$

② Let d_1, d_2, d_3, d_4 be pairwise skew lines.
Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$
s.t. $d_{14} \perp d_{23}$, where d_{ik} is common perpendicular on d_i and d_k

$$\begin{cases} d_{12} \perp d_1 \\ d_{12} \perp d_2 \end{cases}$$

$$\begin{cases} d_{13} \perp d_1 \\ d_{13} \perp d_3 \end{cases}$$

<u>$d_{12} \perp d_{34}$</u> $\Rightarrow \vec{v}_{12} \perp \vec{v}_{34}$

$\vec{v}_{12} = \vec{v}_1 \times \vec{v}_2$

$\vec{v}_{34} = \vec{v}_3 \times \vec{v}_4$

$\vec{v}_1 \times \vec{v}_2 \perp \vec{v}_3 \times \vec{v}_4 \Rightarrow (\vec{v}_1 \times \vec{v}_2)(\vec{v}_3 \times \vec{v}_4) = 0$

$\Rightarrow (\vec{v}_1 \vec{v}_3)(\vec{v}_2 \cdot v_4) - (\bar{v}_2 \bar{v}_3)(\bar{v}_1 \bar{v}_4) = 0 \quad (1)$

<u>$d_{13} \perp d_{24}$</u> $\Rightarrow \bar{v}_{13} \perp \bar{v}_{24} \Rightarrow \vec{v}_1 \times \vec{v}_3 \perp \vec{v}_2 \times \vec{v}_4 \Rightarrow$
$(\vec{v}_1 \times \vec{v}_3)(\vec{v}_2 \times \vec{v}_4) = 0$

$\Rightarrow (\vec{v}_1 \vec{v}_2)(\vec{v}_3 \vec{v}_4) - (\vec{v}_3 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{v}_4) = 0 \quad (2)$

<u>$d_{14} \perp d_{23}$</u> $\Rightarrow (\vec{v}_1 \times \vec{v}_4)(\vec{v}_2 \times \vec{v}_3) = 0$

$\Rightarrow (\vec{v}_1 \vec{v}_2)(\vec{v}_4 \vec{v}_3) - (\vec{v}_4 \vec{v}_2)(\vec{v}_1 \vec{v}_3) = 0$

$\underline{\overset{(1)\,(2)}{\Longrightarrow}}$ $(\vec{v}_3 \vec{v}_2)(\vec{v}_1 \vec{v}_4) - (\vec{v}_2 \vec{v}_3)(\vec{v}_1 \vec{v}_4) = 0$ true

$\Rightarrow d_{13} \perp d_{24}$

③ The vertices of ABCD are $A(4,3)$, $B(5,-4)$, $C(-1,-3)$, $D(-3,-1)$

a) Find coord of $\{E\} = AB \cap CD$

$\{F\} = AD \cap BC$

b) Prove that the midpoints of $[AC]$, $[BD]$, $[EF]$ are collinear.

$AB: \quad \dfrac{x - x_A}{x_B - x_A} = \dfrac{y - y_A}{y_B - y_A}$

$$\frac{x-4}{5-4} = \frac{y-3}{-4-3}$$

$$y-3 = -7(x-4)$$

$AB: \quad 7x + y - 31 = 0$

$CD: \quad \dfrac{x+1}{-3+1} = \dfrac{y+3}{-1+3}$

$CD: \quad x + y + 4 = 0$

$$\left\{\begin{array}{l} 7x + y - 31 = 0 \\ x + y + 4 = 0 \end{array}\right. \quad \Big| -$$

$$6x - 35 = 0$$

$$x = \frac{35}{6} \qquad y = -\frac{59}{6}$$

$$E\left(\frac{35}{6}; -\frac{59}{6}\right)$$

$$AD: \frac{x+3}{4+3} = \frac{y+1}{3+1}$$

$$4(x+3) = 7(y+1)$$

$$AD: \ 4x - 7y + 5 = 0$$

$$BC: \frac{x+1}{5+1} = \frac{y+3}{-4+3}$$

$$BC: \ x + 6y + 19 = 0$$

$$\begin{cases} 4x - 7y + 5 = 0 \\ x + 6y + 19 = 0 \end{cases}$$

$$\Rightarrow F\left(-\frac{163}{31}; -\frac{71}{31}\right)$$

$$E\left(\frac{35}{6}; -\frac{59}{6}\right)$$

$$x = -\frac{163}{31}$$

$$y = -\frac{71}{31}$$

$$\frac{35}{6} - \frac{163}{31} = \frac{1085-978}{186} = \frac{107}{186}$$

$$-\frac{59}{6} + \frac{71}{31} = \frac{-1829-426}{186} = \frac{-2255}{186}$$

M - midpoint of AC

$$M\left(\frac{3}{2}; 0\right)$$

N - midpoint of BD

$$N\left(1; -\frac{5}{2}\right)$$

P - midpoint of EF

$$P\left(\frac{107}{372}; \frac{-2255}{372}\right)$$

$$\begin{vmatrix} \frac{3}{2} & 0 & 1 \\ 1 & -\frac{5}{2} & 1 \end{vmatrix} = 0$$

④ $A(6,0)$, $B(1,5)$, $C(0,4)$

a) Compute the length of sides of ΔABC

b) S.t. $OABC$ is a cyclic quadrilateral

c) Let $OA' \perp BC$, $A' \in BC$, $OC' \perp AB$, $C' \in AB$, $OB' \perp AC$, $B' \in AC$. S.t. A', B', C' are collinear.

a) $AB = \sqrt{(1-6)^2 + (5-0)^2} = \sqrt{25+25} = \cancel{2\sqrt{3}}\ 5\sqrt{2} = \sqrt{50}$

$BC = \sqrt{1^2 + 1^2} = \sqrt{2}$

$AC = \sqrt{(-6)^2 + 4^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

b) $OABC$ cyclic $\Leftrightarrow m(\hat{B}) + m(\hat{O}) = 180°$

$m(\hat{C}) + m(\hat{A}) = 180°$

$AC^2 = AB^2 + BC^2$

$52 = 50 + 2$

$\Rightarrow ABC$ right Δ $\Rightarrow$

$m(\hat{B}) + m(\hat{O}) = 180°$

$BC: \quad \dfrac{x-1}{-1} = \dfrac{y-5}{-1}$

$-x + y - 4 = 0$

$m_{BC} = 1 \Rightarrow OA' \perp BC \Rightarrow m_{OA'} = -1$

$\Rightarrow OA': \quad y - y_0 = m(x - x_0)$

$y = -x$

$OA: \quad y + x = 0$

$\{A'\} = BC \cap OA' \quad \begin{cases} -x + y - 4 = 0 \\ x + y = 0 \end{cases}$

$OC' \perp BA$

$BA: \frac{x-1}{5} = \frac{y-5}{-5}$

$BA: x - y + 6 = 0$

$m_{BA} = -1 \Rightarrow m_{OC'} = 1$

$OC': y - x = 0$

$\begin{cases} x - y = 0 \\ x + y = 6 \end{cases} \Rightarrow C'(3, 3)$

$OB' \perp AC$

$AC: \frac{x-6}{-6} = \frac{y}{4}$

$AC: 2x + 3y - 12 = 0$

$m_{AC} = -\frac{2}{3} \Rightarrow m_{OB'} = \frac{3}{2}$

$OB': \frac{3}{2}x - y = 0$

$\begin{cases} \frac{3}{2}x - y = 0 \\ 2x + 3y - 12 = 0 \end{cases} \Rightarrow B'\left(\frac{24}{13} ; \frac{36}{13}\right)$

$\begin{vmatrix} 2 & 2 & 1 \\ 3 & 3 & 1 \\ \frac{24}{13} & \frac{36}{13} & 1 \end{vmatrix} = 0 \Rightarrow A', B', C'$ - coliniari

(5) Find the angle det by plane (xoy) with line M_1M_2, $M_1(1,2,3)$, $M_2(-2,1,4)$

$\bar{v}_{M_1M_2}(-3,-1,1)$

$M_1M_2: \frac{x-1}{-3} = \frac{y-2}{-1} = \frac{z-3}{1} = t$

$\bar{n}_{xoy} = \bar{k}$

$\bar{k}(0,0,1)$

$\bar{k} \cdot \bar{v}_{M_1M_2} = 1 > 0 \Rightarrow m(xoy, M_1M_2) = \cancel{\frac{\pi}{2} \angle \cos(\bar{k}, \bar{v}_{M_1M_2})}$

$= \frac{\pi}{2} - m(\bar{k}, \bar{v}_{M_1M_2}) =$

$= \frac{\pi}{2} - \arccos \frac{1}{\sqrt{11}}$