CURS 8

13. IV. 2011

The distance from a point to a line

$$\delta(P,d) = PP' = \frac{\text{area of paral. constructed on } \overline{A_0P} \text{ and } \bar{v}}{|\bar{v}|}$$

$$= \frac{|\overrightarrow{A_0P} \times \vec{v}|}{|\vec{v}|} = \frac{\begin{vmatrix} i & j & k \\ x_p - x_0 & y_p - y_0 & z_p - z_0 \\ p & q & r \end{vmatrix}}{\sqrt{p^2+q^2+r^2}}$$

$$= \frac{\sqrt{\begin{vmatrix} y_p - y_0 & z_p - z_0 \\ q & r \end{vmatrix}^2 + \begin{vmatrix} x_p - x_0 & z_p - z_0 \\ p & r \end{vmatrix}^2 + \begin{vmatrix} x_p - x_0 & y_p - y_0 \\ p & q \end{vmatrix}^2}}{\sqrt{p^2+q^2+r^2}}$$

Second method - find the coord of P'

$$\pi: \ p(x - x_p) + q(y - y_p) + r(z - z_p) = 0$$

$$d: \begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}$$

$$\{P'\} = d \cap \pi$$

$$p(x_0 + pt - x_p) + q(y_0 + qt - y_p) + r(z_0 + rt - z_p) = 0$$

$$p(x_0 - x_p) + q(y_0 - y_p) + r(z_0 - z_p) + (p^2 + q^2 + r^2)t = 0$$

$$\bar{v} \cdot \overline{PA_0} + |\vec{v}|t = 0 \iff t = \frac{\overline{A_0P} \cdot \bar{v}}{|\vec{v}|^2}$$

$$P'\left(x_0 + p\frac{\overline{A_0P} \cdot \bar{v}}{|\vec{v}|^2}, \ y_0 + q\frac{\overline{A_0P} \cdot \bar{v}}{|\vec{v}|^2}, \ z_0 + r\frac{\overline{A_0P} \cdot \bar{v}}{|\vec{v}|^2}\right)$$

The distance between 2 parallel planes

$$d(\pi_1, \pi_2) = d(P_1, \pi_2) \text{ where } P_1 \in \pi_1$$

The distance ~~fro~~ between 2 lines d_1, d_2

I d_1, d_2 - coplanar

① d_1, d_2 - concurrent $\Rightarrow d(d_1, d_2) = 0$

$d_1 \parallel d_2 \Rightarrow d(d_1, d_2) = d(P_1, d_2)$ where $P_1 \in d_1$

II d_1, d_2 - skew $\Rightarrow d_1 \cap d_2 = \emptyset$ and $d_1 \nparallel d_2$

Remark: There exist a unique pair of planes π_1, π_2 s.t. $d_1 \subset \pi_1$, $d_2 \subset \pi_2$ and $\pi_1 \parallel \pi_2$

$$\pi_1 : \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0$$

$$\pi_2 : \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0$$

$$\Leftrightarrow A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

where $A = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}$ $B = \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}$ $C = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$

$$A(x - x_2) + B(y - y_2) + C(z - z_2) = 0$$

$$\delta(d_1, d_2) = \delta(\bar{u}_1, \bar{u}_2) = \delta(A, \bar{u}_2) =$$
$$= \frac{|A(x_1 - x_2) + B(y_1 - y_2) + C(z_1 - z_2)|}{\sqrt{A^2 + B^2 + C^2}}$$

CONICS

① Conics defined through canonical equations

• The circle

Def. A circle is a closed curve defined as the locus of point in the plane at a given distance R from a fixed point I.

$\mathcal{C}(I, R)$

$M(x, y) \in \mathcal{C}(I, R) \iff |\overline{MI}| = R$

$\iff |\overline{MI}| = R^2 \iff (x-a)^2 + (y-b)^2 = R^2$

$x^2 + y^2 - 2ax - 2by + c = 0$ where $c = a^2 + b^2 - R^2$

Remark The eq. $x^2 + y^2 - 2ax + 2by + c = 0$ (*) represents either a circle, or a point, or the empty set.

- If $a^2 + b^2 - c > 0$, then (*) = general eq. of the circle represents the eq of the circle $\mathcal{C}(I, R)$, $I(a, b)$ $R = \sqrt{a^2 + b^2 - c}$
- If $a^2 + b^2 - c = 0$, then (*) repr. the point $I(a, b)$
- If $a^2 + b^2 - c < 0$ then (*), repr. the empty

• the circle determined by 3 non-collinear points

$M_1(x_1, y_1)$ $M_2(x_2, y_2)$ $M_3(x_3, y_3)$ - non collinear

$M(x,y) \in \mathcal{C}(M_1, M_2, M_3): x^2+y^2-2ax-2by+c=0$

$$\Leftrightarrow \begin{cases} x^2+y^2-2ax-2by+c=0 \\ x_1^2+y_1^2-2ax_1-2by_1+c=0 \\ x_2^2+y_2^2-2ax_2-2by_2+c=0 \\ x_3^2+y_3^2-2ax_3-2by_3+c=0 \end{cases}$$

$\mathcal{C}(M_1, M_2, M_3)$ - the circle det. by M_1, M_2, M_3
= circumcircle of $\Delta M_1M_2M_3$

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Remark: the points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$, $M_4(x_4, y_4)$ belong to the same circle

iff
$$\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_1^2 & x_2 & y_2 & 1 \\ x_3^2+y_2^2 & x_3 & y_3 & 1 \\ x_4^2+y_3^2 & x_4 & y_4 & 1 \end{vmatrix} = 0$$

The intersection of a circle and a line

Let $\mathcal{C} \subseteq \mathcal{E}_2$ be a circle and $d \subseteq \mathcal{E}_2$ be a line. Choose the reference system $R = (O, \vec{i}, \vec{j})$ of $\mathcal{E}_2$ in such a way that O is the center of $\mathcal{C}$.

$\mathcal{C}: x^2 + y^2 = R^2$

$d: y = mx + n$

$$d \cap \mathcal{C} : \begin{cases} x^2 + y^2 = R^2 \\ y = mx + n \end{cases} \Rightarrow x^2 + (mx+n)^2 = R^2$$

$$x^2 + m^2x^2 + 2mnx + n^2 - R^2 = 0$$

$$(1+m^2)x^2 + 2mnx + n^2 - R^2 = 0$$

$$\Delta = 4m^2n^2 - 4(1+m^2)(n^2-R^2) = \cancel{4m^2n^2} -$$

$$- 4n^2 + 4R^2 - \cancel{4m^2n^2} + 4m^2R^2 =$$

$$= 4(m^2R^2 - n^2 + R^2)$$

- If $R^2 + m^2R^2 - n^2 < 0$ then $d \cap \mathcal{C} = \emptyset$
- If $R^2 + m^2R^2 - n^2 = 0$, then there is a double point (tangency point) between d and $\mathcal{C}$. The line is tangent to the circle
- If $R^2 + m^2R^2 - n^2 > 0$, then there are 2 intersection points between d and $\mathcal{C}$ - d is secant to the circle

The tangents to a circle having a given slope

m - slope

$y = mx + n$ - tangent to $\mathcal{C}$ $\Leftrightarrow$

$R^2 + m^2R^2 - n^2 = 0$

$n = \pm R\sqrt{m^2+1}$

$y = mx \pm R\sqrt{m^2+1}$