Sankt Petersburg Paradox

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Example 1 (St. Petersburg paradox) *This paradox was noticed by a Swiss mathematician Daniel Bernoulli (1700–1782), a nephew of Jacob. It describes a gambling strategy that enables one to win any desired amount of money with probability one.*

Isn't it a very attractive strategy? It's real, there is no cheating!

Consider a game that can be played any number of times. Rounds are independent, and each time your winning probability is p. The game does not have to be favorable to you or even fair; this p can be any positive probability. For each round, you bet some amount x. In case of a success, you win x. If you lose a round, you lose x.

The strategy is simple. Your initial bet is the amount that you desire to win eventually. Then, if you win a round, stop. If you lose a round, double your bet and continue.

Let the desired profit be \$100. The game will progress as follows.

		Balance...	
Round	Bet	...if lose	...if win
1	100	-100	+100 and stop
2	200	-300	+100 and stop
3	400	-700	+100 and stop
...	...	...	...

Sooner or later, the game will stop, and at this moment, your balance will be \$100. Guaranteed! However, this is not what D. Bernoulli called a paradox.

How many rounds should be played? Since each round is a Bernoulli trial, the number of them, X, until the first win is a Geometric random variable with parameter p.

Is the game endless? No, on the average, it will last $E(X) = 1/p$ rounds. In a fair game with p = 1/2, one will need 2 rounds, on the average, to win the desired amount. In an "unfair" game, with $p < 1/2$, it will take longer to win, but still a finite number of rounds. For example, if $p = 0.2$, i.e., one win in five rounds, then on the average, one stops after $1/p = 5$ rounds. This is not a paradox yet.

Finally, how much money does one need to have in order to be able to follow this strategy? Let Y be the amount of the last bet. According to the strategy,

$Y = 100 \cdot 2^{X-1}$. It is a discrete random variable whose expectation equals

$$E(Y) = \sum_x 100 \cdot 2^{x-1} P(x) = 100 \sum_{x=1}^{\infty} 2^{x-1}(1-p)^{x-1} p$$
$$= 100p \sum_{x=1}^{\infty} [2(1-p)]^{x-1} = \begin{cases} \frac{100p}{2(1-p)} & \text{if } p > 1/2 \\ \infty & \text{if } p \le 1/2 \end{cases}$$

This is the St. Petersburg Paradox! A random variable that is always finite has an infinite expectation! Even when the game is fair offering a 50-50 chance to win, one has to be (on the average!) infinitely wealthy to follow this strategy.

To the best of our knowledge, every casino has a limit on the maximum bet, making sure gamblers cannot fully apply this St. Petersburg strategy. When such a limit is enforced, it can be proved that a winning strategy does not exist.