Course 11

The intersection of parabola with a line

$$\begin{cases} \mathcal{P}: y^2 = 2px \\ d: y = mx+n \end{cases} \Rightarrow (mx+n)^2 = 2px \Leftrightarrow m^2x^2 + 2mnx + n^2 = 2px \Leftrightarrow m^2x^2 + 2(mn-p)x + n^2 = 0$$

$$\Delta = 4[(mn-p)^2 - m^2n^2] = 4(m^2n^2 - 2mnp + p^2 - m^2n^2) = 4p(p-2mn)$$

$\Delta < 0 \Rightarrow \mathcal{P} \cap d = \phi$

$\Delta > 0 \Rightarrow \mathcal{P} \cap d$ consists in exactely two points

~~$x_{1,2} = \sqrt{\ }$~~

$$x_{1,2} = \frac{p - mn \pm \sqrt{p(p-2mn)}}{m^2}, \quad y_{1,2} = mx_1 + n$$

$(x_1, y_1), (x_2, y_2)$

$\Delta = 0 \Rightarrow \mathcal{P} \cap d =$ one point

The tangent line to a parabola

- The tangent as a given direction

$y = mx + n$

$\Delta = 0 \Leftrightarrow 4p(p-2mn) = 0 \Rightarrow n = \frac{p}{2m}$

$$\boxed{y = mx + \frac{p}{2m}}$$

- The tangent of a parabola at a given point

$\mathcal{P}$: ~~x~~ $y^2 = 2px$, $M_0(x_0, y_0) \in \mathcal{P} \Leftrightarrow y_0^2 = 2px_0$

$\mathcal{P}$: $f^{-1}(0)$, where $f(x,y) = y^2 - 2px$

$$(\nabla f)(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right) = (-2p, 2y_0)$$

$(p, -y_0)$ - a normal vector of

$\parallel$

$-\frac{1}{2}(\nabla f)(x_0, y_0)$

$$T_{M_0}(\mathcal{P}): \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = 0$$

$p(x - x_0) - y_0(y - y_0) = 0$

$px - px_0 - y_0 y + y_0^2 = 0$

$T_{M_0}(P) \quad y_0 y = p(x+x_0)$

QUADRIC SURFACES

$V_3 \simeq \mathbb{R}^3$

$F \in \mathbb{R}[x,y,z]$, $\deg(F) = 2$

$F = a_{00} + 2a_{01}x + 2a_{02}y + 2a_{03}z + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + a_{11}x^2 + a_{22}y^2 + a_3$

<u>Def</u>: A function $f: \mathbb{R}^3 \to \mathbb{R}$ is said to be <u>polynomial</u> if $f(x,y,z) = F(x,y,z)$ for some polynomial $F \in \mathbb{R}[x,y,z]$

<u>Def</u>: A <u>quadric surface</u> is an algebraic surface of degree two; i.e. the collection of zeros of a polynomial function

- <u>Quadric surfaces given by their reduced equations</u>
  - The <u>ellipsoid</u> is the quadric surface

$$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a,b,c>0$$

$$M(x,y,z) \in \mathcal{E} \Rightarrow N(-x,y,z), P(x,-y,z), Q(x,y,-z) \in \mathcal{E}$$
$$R(-x,-y,z), S(-x,y,-z), T(x,-y,-z) \in \mathcal{E}$$
$$U(-x,-y,-z) \in \mathcal{E}$$

Thus: • the coordinate planes are planes of symmetry of $\mathcal{E}$

• – // – axes – // – axes – // –

• The origine $O(0,0,0)$ is center of symmetry for $\mathcal{E}$

$$(z=\lambda) \cap \mathcal{E}: \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$$

(I) $1 - \dfrac{\lambda^2}{c^2} < 0 \Leftrightarrow |\lambda| > c \Rightarrow (z=\lambda) \cap \mathcal{E} = \emptyset$

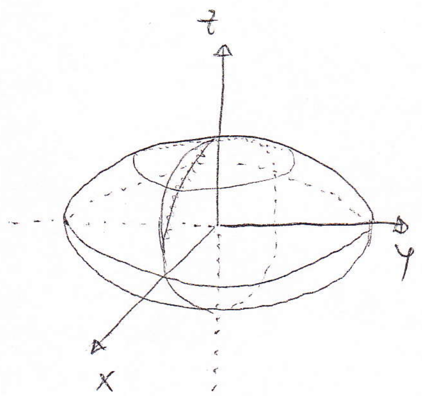
(II) $1 - \dfrac{\lambda^2}{c^2} = 0 \Leftrightarrow \lambda = \pm c \Rightarrow (z = \pm c) \cap \mathcal{E} = \{(0,0,\pm c)\}$

(III) $1 - \dfrac{\lambda^2}{c^2} > 0 \Leftrightarrow |\lambda| < c \Rightarrow (z=\lambda) \cap \mathcal{E} \begin{cases} \dfrac{x^2}{(\ldots)^2} + \dfrac{y^2}{\ldots} = 1 \end{cases}$

In particular $(xOy) \cap \mathcal{E} : \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ z = 0 \end{cases}$ ellipse

$(xOz) \cap \mathcal{E} : \begin{cases} \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1 \\ y = 0 \end{cases}$ ellipse

$(yOz) \cap \mathcal{E} : \begin{cases} \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \\ x = 0 \end{cases}$ ellipse



- <u>The hyperboloid of one sheet</u> is the quadric surface $\mathcal{H}_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

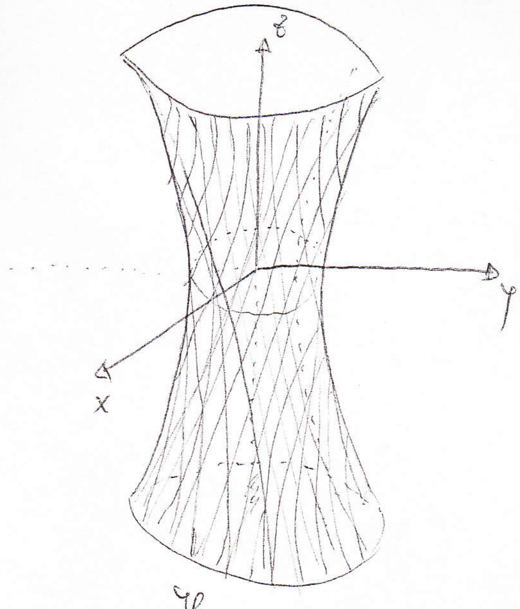
- The planes of coordinates are planes of symmetry of $\mathcal{H}_1$ ~~axes~~
- the axes — || — — || — axes — || — — || —

- $O(0,0,0)$ - center of symmetry

$(z = \lambda) \cap \mathcal{H}_1 \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \cancel{\dfrac{z^2}{c^2}} \dfrac{\lambda^2}{c^2} \\ z = \lambda \end{cases} \Leftrightarrow \begin{cases} \dfrac{x^2}{\left(a\sqrt{1+\frac{\lambda^2}{c^2}}\right)^2} + \dfrac{y^2}{\left(b\sqrt{1+\frac{\lambda^2}{c^2}}\right)^2} = 1 \\ z = \lambda \end{cases}$ ellipse

In particular $(xOy) \cap \mathcal{H}_1 : \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ z = 0 \end{cases}$ ellipse

$(xOz) \cap \mathcal{H}_1 : \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 \\ y = 0 \end{cases}$ - hyperbola



$\mathcal{H}_1$

$$\mathcal{H}_1: \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

$$\mathcal{H}_1: \left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right)$$

$$\Delta_\lambda : \begin{cases} \frac{x}{a} + \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right) \\ \lambda\left(\frac{x}{a} - \frac{z}{c}\right) = 1 + \frac{y}{b} \end{cases}$$

$$\Delta'_\mu : \begin{cases} \frac{x}{a} + \frac{z}{c} = \mu\left(1 + \frac{y}{b}\right) \\ \mu\left(\frac{x}{a} - \frac{z}{c}\right) = 1 - \frac{y}{b} \end{cases}$$

$\Delta_\lambda \subseteq \mathcal{H}_1, \forall \lambda \in \mathbb{R}$ and $\Delta'_\mu \subseteq \mathcal{H}_1, \forall \mu \in \mathbb{R}$

- The hyperboloid of two sheets is the quadric surface $\mathcal{H}_2: \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

$a, b, c > 0$

- the planes of coordinates are planes of symmetry of $\mathcal{H}_2$
- axes —"— axes —"—
- $O(0,0,0)$ - center of symmetry for $\mathcal{H}_2$

$$(z = \lambda) \cap \mathcal{H}_2 \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1 \\ z = \lambda \end{cases}$$

(I) $\frac{\lambda^2}{c^2} - 1 < 0 \iff |\lambda| < c \Rightarrow (z = \lambda) \cap \mathcal{H}_2 = \emptyset$

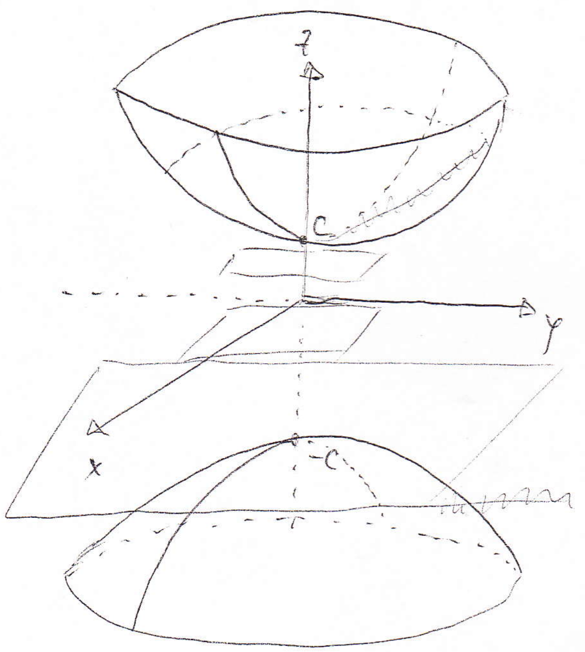
(II) $\frac{\lambda^2}{c^2} - 1 = 0 \iff \lambda = \pm c \Rightarrow (z = \pm c) \cap \mathcal{H}_2 = \{(0,0,\pm c)\}$

(III) $\frac{\lambda^2}{c^2} - 1 > 0 \iff |\lambda| > c \Rightarrow (z = \lambda) \cap \mathcal{H}_2 : \begin{cases} \frac{x^2}{\left(a\sqrt{\frac{\lambda^2}{c^2} - 1}\right)^2} + \frac{y^2}{\left(b\sqrt{\frac{\lambda^2}{c^2} - 1}\right)^2} = 1 \\ z = \lambda \end{cases}$

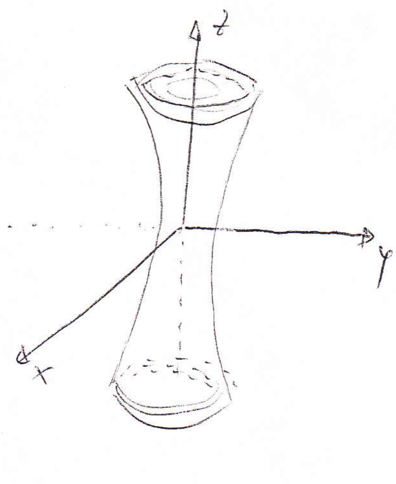
ellipse

$$(xOz) \cap \mathcal{H}_2 : \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -1 \\ y = 0 \end{cases}$$ – hyperbola

$$(yOz) \cap \mathcal{H}_2 : \begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1 \\ x = 0 \end{cases}$$ – hyperbola



$\mathcal{H}_2$



The hyperboloids $\mathcal{H}_1$, $\mathcal{H}_2$ and their common asymptotic cone