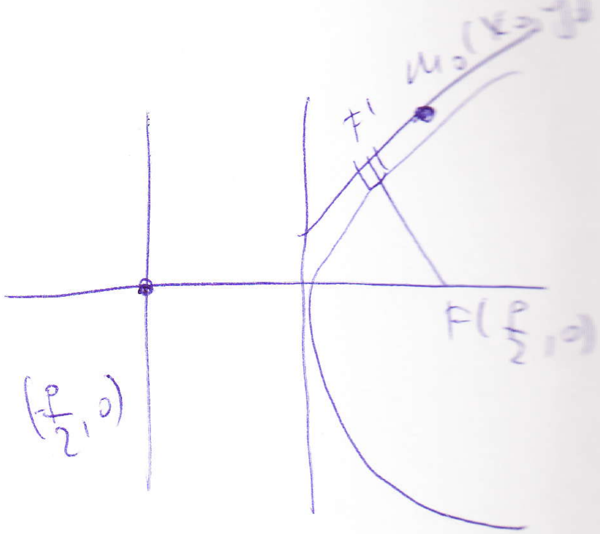
SEMINAR 10

Find the locus of the orthogonal projections of the focus of the parabola on the tangent lines to that parabola



$P: y^2 = 2px$

$F(\frac{p}{2}, 0)$

$M_0(x_0, y_0) \in P$

$T_{M_0}(P): y_0 y = p(x + x_0)$

$px - y_0 y + px_0 = 0$

$\vec{n}(p, -y_0)$

$$\frac{x - \frac{p}{2}}{p} = \frac{y - 0}{-y_0} \;;\; -y_0 x + y_0 \frac{p}{2} = p \cdot y$$

$\Rightarrow -2y_0 x + y_0 p - 2py = 0$

$$\begin{cases} px - y_0 y + px_0 = 0 \mid 2 \cdot y_0 \\ -2y_0 x - 2py + y_0 p = 0 \mid p \end{cases}$$

$$\begin{cases} 2y_0 px - 2y_0^2 y + 2py_0 x_0 = 0 \\ -2y_0 p\cancel{x} - 2p^2 y + y_0 p^2 = 0 \end{cases}$$

$-2y(y_0^2 + p^2) + y_0 p(2x_0 + p) = 0$

$\Rightarrow y = \frac{-y_0 p(2x_0 + p)}{\ldots} \Leftrightarrow y = y_0 p(p + 2x_0)$

$\frac{y_0}{2} = \frac{y_0 p(2x_0+p)}{2(y_0^2+p^2)}$

$\Rightarrow px - y_0\left(\frac{y_0 p(2x_0+p)}{2(y_0^2+p^2)}\right) + px_0 = 0$

$\Rightarrow x = \frac{y_0\left(\frac{y_0 p(2x_0+p)}{2(y_0^2+p^2)}\right) - px_0}{p}$

$x = \frac{\frac{y_0^2 \cancel{p}(2x_0+p)}{2(y_0^2+p^2)}}{\cancel{p}} - \frac{\cancel{p}x_0}{\cancel{p}}$

$= \frac{y_0(2x_0+p)}{2(y_0^2+p^2)} - x_0$

$x = \frac{2p^2x_0(2x_0+p)}{2p(2px_0+p^2)} - x_0$

$y = \frac{p y_0(2x_0+p)}{2(y_0^2+p^2)} = \frac{y_0}{2} \Rightarrow x = 0 \Rightarrow y = \frac{y_0}{2}$

2. Find the locus of the orthogonal projections of a focus of a hyperbola on its tangent lines

$\mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $F(\sqrt{a^2+b^2}, 0)$, $F'(-\sqrt{a^2+b^2}, 0)$

$M(x_0, y_0) \in \mathcal{H}$

$T_{M_0}(\mathcal{H}): \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$; $\bar{n} \,\|\, \left(\frac{x_0}{a^2}, -\frac{y_0}{b^2}\right)$

$$D: \frac{x-\sqrt{e^2+b^2}}{\frac{x_0}{a^2}} = \frac{-y}{\frac{y_0}{b^2}} \iff y \cdot \frac{x_0}{a^2} = (x-\sqrt{e^2+b^2})$$

$$-y x_0 b^2 = a^2 y_0 (x-\sqrt{a^2+b^2})$$

$$-y x_0 \cdot b^2 - a^2 y_0 x + a^2 y_0 \sqrt{a^2+b^2} = 0$$

$$\begin{cases} a^2 y_0 x + x_0 b^2 y - a^2 y_0 \sqrt{a^2+b^2} = 0 \\ b^2 x_0 x - a^2 y_0 y - a^2 b^2 = 0 \end{cases}$$

$$a^4 y_0^2 x + a^2 b y_0 x_0 y - a^4 y_0^2 \sqrt{a^2+b^2} = 0$$

$$b^2 x_0^2 x - a^2 b y_0 x_0 y - a^2 (y_0^2 \sqrt{a^2+b^2} + b^2 x_0) = 0$$

$$x = \frac{a^2(a^2 y_0^2 \sqrt{a^2+b^2} + b^4 x_0)}{(a^4 y_0^2 + b^4 x_0^2}$$

$$b^2 x_0 = \frac{a^2(a^2 y_0^2 \sqrt{a^2+b^2} + b^4 x_0)}{(a^4 y_0 + b^4 x_0^2) \cdot y_0} - a^2 b^2 = y$$

$$y = \frac{b^2 x_0 (a^2 y_0 \sqrt{a^2+b^2} + b^4 x_0)}{(a^4 y_0^2 + b^4 x_0^2) y_0} - \frac{b^2}{y_0}$$

$$y = \frac{b^2 (a^2 y_0^2 \sqrt{a^2+b^2} + b^4 x_0^2 - a^4 y_0^2 - b^4 x_0^2)}{y_0 (a^4 y_0^2 + b^4 x_0^2)}$$

$$= \frac{a^2 b^2 y_0^2 x_0 \sqrt{a^2+b^2} - a^2}{y_0 (a^4 y_0^2 + b^4 x_0^2)}$$

$$= \frac{a^2 b^2 (x_0 y_0 \sqrt{a^2+b^2} - a^2 y_0)}{a^2(\ldots}$$

$$\frac{x_0}{a^2}\sqrt{a^2+b^2}+\left(\frac{x_0^2}{a^2}\sqrt{a^2+b^2}+\left(\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}\right)t=1\right) \Leftrightarrow$$

$$t=\frac{1-\sqrt{a^2+b^2}\frac{x_0^2}{a^2}}{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^2}}$$

$$x=\sqrt{a^2+b^2}+\frac{x_0}{a^2}$$