SEMINAR 11

9.V.2011

① a) Det the coord of the ellipse:

$\mathcal{E}: 9x^2+25y^2-225=0$

b) Sketch the graph of $y=-\frac{3}{4}\sqrt{16-x^2}$

$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

$c^2=a^2-b^2$

$F(c,0), F(-c,0)$

$\frac{x^2}{25}+\frac{y^2}{9}-1=0$

$a=5$
$b=3$ $\Rightarrow c=4$

$F(4,0)$, $F'(-4,0)$

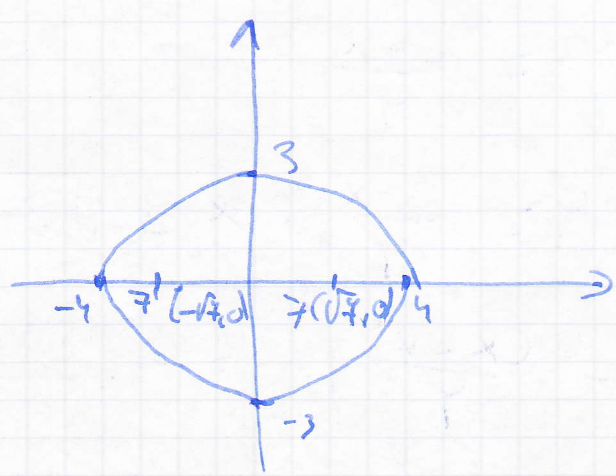
b) $y^2=\frac{9}{16}(16-x^2)$

$y^2=9-\frac{9x^2}{16} \Rightarrow$

$\frac{9x^2}{16}+y^2-9=0$

$\frac{x^2}{16}+\frac{y^2}{9}-1=0$

$a=4,\ b=3,\ c=\sqrt{7}$



② Find the pos of $d: 2x+y-10=0$ relative to the ellipse $\mathcal{E}: \frac{x^2}{9}+\frac{y^2}{4}-1=0$

③ Find the geometric locus of the orthogonal projection of a focus of an ellipse on the tangent lines of the ellipse

$$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

let slope of tg = m

$$tg: \; y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$m_{FF_1} = -\frac{1}{m}$$

$$FF_1: \; y - y_F = m_{FF_1}(x - x_F)$$

$$FF_1: \; y = -\frac{1}{m}(x - c)$$

$$\{F_1\} = FF_1 \cap tg$$

$\Rightarrow$ eliminate m ~~fr~~ between eq of the tg and FF_1

$$m = -\frac{1}{y}(x - c)$$

$$m = \frac{c - x}{y}$$

$$y = \frac{c - x}{y} \cdot x \pm \sqrt{a^2 \cdot \frac{c^2 - 2cx + x^2}{y^2} + b^2}$$

$$\left(y - \frac{cx - x^2}{y}\right)^2 = a^2 \cdot \frac{c^2 - 2cx + x^2}{y^2} \, b^2$$

$$y^2 - 2y \cdot \frac{cx - x^2}{y} + \frac{c^2x^2 - 2cx^3 + x^4}{y^2} =$$

$$= a^2(c - x)^2 + (a^2 - c^2)y^2$$

$$y^4 - 2cxy^2 + x^2y^2 + c^2x^2 - 2cx^3 + x^4 =$$
$$= a^2c^2 - 2a^2cx + a^2x^2 + a^2y^2 - c^2y^2$$

$$y^2(y^2 + x^2 - a^2) - 2cx(x^2 + y^2 - a^2) +$$
$$+ x^2(y^2 + x^2 - a^2) + c^2(-a^2 + x^2 + y^2) = 0$$
$$(x^2 + y^2 - a^2)(y^2 - 2cx + x^2 + c^2) = 0$$

i $x^2 + y^2 - a^2 = 0$ - eq. of a circle $(0,0)$, $R = 0$

ii $x^2 + y^2 + c^2 - 2cx = 0$

$(x-c)^2 + y^2 = 0$

$x = c$

$y = 0$ $\Rightarrow F(c, 0)$ not possible.

④ Let d_1, d_2 be 2 var. orthogonal lines passing through $A(a,0)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and P_1, P_2 the intersection points of the d_1, d_2 with $\mathcal{E}$. S.t. $P_1 P_2$ passes through a fixed point.

Slope $d_1 = m \Rightarrow$ slope $d_2 = -\frac{1}{m}$

d_1: $(y - y_A) = m(x - x_A)$

d_1: $y = m(x-a)$

d_2: $y = -\frac{1}{m}(x-a)$

$\{P_1\} = \mathcal{E} \cap d_1$

$$\Rightarrow \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = m(x-a) \end{cases}$$

$$\frac{x^2}{a^2} + \frac{m^2(x-a)^2}{b^2} = 1 \quad \Big| \cdot a^2b^2$$

$$b^2x^2 + a^2m^2(x^2 - 2xa + a^2) = a^2b^2$$

$$\cancel{b^2x^2 + a^2m^2x^2 - a^2m^2}$$

$$x^2(b^2 + a^2m^2) - 2a^3m^2x + a^4m^2 - a^2b^2 = 0$$

$$\Delta = 4a^6m^4 - 4(a^4m^2 - a^2b^2)(b^2 + a^2m^2)$$

$$= \cancel{4a^6m^4} - \cancel{4a^4m^2b^2} + 4a^2b^4 - \cancel{4a^6m^4} + \cancel{4a^4b^2m^2} = 4a^2b^4$$

$$x_{1,2} = \frac{2a^3m^2 \pm 2ab^2}{2(b^2 + a^2m^2)} = \frac{a^3m^2 \pm ab^2}{b^2 + a^2m^2} = \frac{a(a^2m^2 \pm b^2)}{b^2 + a^2m^2}$$

$\cancel{x = a}$ $\quad x_1 = a \Rightarrow A$

$\cancel{y_{1,2} =}$

$$x_2 = \frac{a(a^2m^2 - b^2)}{a^2m^2 + b^2}$$

$$y_2 = \frac{-2mab}{a^2m^2 + b^2}$$

$\{P_2\} = d_2 \cap \mathcal{E}$

$$P_2\left(\frac{a(a^2 - m^2b^2)}{a^2 + m^2b^2} ; \frac{2mab}{a^2 + m^2b^2}\right.$$

$$\frac{x - \frac{a(a^2m^2 - s^2)}{a^2m^2 + s^2}}{\frac{a(a^2m^2 - s^2)}{a^2 + m^2s^2} - \frac{a(a^2m^2 - s^2)}{a^2m^2 + s^2}} = \frac{y + \frac{2mas}{a^2m^2 + s^2}}{\frac{2mas}{a^2m^2 + s^2} + \frac{2mas}{a^2m^2 + s^2}}$$

$$N_1 = \frac{a^2(1-m^2)(1+m^2)}{(a^2 + m^2s^2)(a^2m^2 + s^2)}$$

$$N_2 = \frac{m(a^2+s^2)(1+m^2)}{(a^2 + m^2s^2)(a^2m^2 + s^2)}$$

$$P_1P_2: \quad \frac{x - \frac{a(a^2m^2 - s^2)}{a^2m^2 + s^2}}{a^2(1-m^2)} = \frac{y + \frac{2mas}{a^2m^2 + s^2}}{m(a^2+s^2)}$$

$m_1, m_2 \Rightarrow$ intersection point

$$\begin{cases} m_1(a^2+s^2)x - m_1(a^2+s^2)\frac{a(a^2m_1^2 - s^2)}{a^2m_1^2 + s^2} = y\,a^2(1-m_1^2) + \\ \qquad + a^2(1-m_1^2)\frac{2mas}{a^2m_1^2 + s^2} \\ \ldots m_2 \end{cases}$$

...

$$y = 0$$

$$x = \frac{a(a^2 - s^2)}{(a^2 + s^2)}$$

$$\Rightarrow P_1P_2 \cap P_1'P_2' = \{M\}$$

$$M\left(\frac{a(a^2 - s^2)}{a^2 + s^2} ; 0\right)$$

(5) Find eq of the tangent lines to $H: \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$ which are $\perp$ to $d: 4x + 3y - 7 = 0$

$$tg: y = m_{tg}\, x \pm \sqrt{a^2 m_{tg}^2 - b^2}$$

$$m_d = -\frac{4}{3} \Rightarrow m_{tg} = \frac{3}{4}$$

$$y = \frac{3}{4}x \pm \sqrt{\cancel{20}^{5} \cdot \frac{9}{\cancel{16}_{4}} - 5}$$

$$tg_{1,2}: y = \frac{3}{4}x \pm \frac{5}{2}$$

(6) Find eq. of tg lines to $H\ \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$ passing through $P(1, -5)$

$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

$P \in tg$

$$-5 = m \pm \sqrt{3m^2 - 5}$$

$$-5 - m = \pm\sqrt{3m^2 - 5} \quad |^2$$

$$m^2 + 10m + 25 = \pm(3m^2 - 5)$$

$$m^2 + 10m + 25 = 3m^2 - 5$$

$$-2m^2 + 10m + 30 = 0 \quad | : (-2)$$

$$m^2 - 5m - 15 = 0$$

$$\Delta = 25 + 60 = 85$$

$$m_{1,2} = \frac{5 \pm \sqrt{85}}{2}$$

$tg_{1,2}$:

$$y = \frac{5 + \sqrt{85}}{2}x \pm \sqrt{3\left(\frac{5 \pm \sqrt{85}}{2}\right)^2 - 5}$$