Seminar Nr.1, Euler's Functions

<u>Euler's Gamma Function</u>

Definition: $\Gamma : (0, \infty) \to (0, \infty)$

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

Properties:

1. $\Gamma(1) = 1$;
2. $\Gamma(a+1) = a\Gamma(a)$, $\forall a > 0$;
3. $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$;
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\int_0^\infty e^{-\frac{t^2}{2}} dt = \int_{\mathbb{R}} e^{-t^2} dt = \sqrt{\pi}.$

<u>Euler's Beta Function</u>

Definition: $\beta : (0, \infty) \times (0, \infty) \to (0, \infty)$

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

Properties:

1. $\beta(a, 1) = \frac{1}{a}$, $\forall a > 0$;
2. $\beta(a, b) = \beta(b, a)$, $\forall a, b > 0$;
3. $\beta(a, b) = \frac{a-1}{b}\beta(a-1, b+1)$, $\forall a > 1, b > 0$;
4. $\beta(a, b) = \frac{b-1}{a+b-1}\beta(a, b-1) = \frac{a-1}{a+b-1}\beta(a-1, b)$, $\forall a > 1,\ b > 1$;
5. $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $\forall a > 0, b > 0.$