Geometry
Anii I Informatică şi Ingineria Informaţiei

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Content

Generalities

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S = \{M(x, y, z) : F(x, y, z) = 0\},$$

where $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a real function and D is a domain, is called *surface* of implicit equation $F(x, y, z) = 0$ (the quadric surfaces, defined in the previous chapter for F a polynomial of degree two, are such of surfaces). On the other hand, the set

$$S_1 = \{M(x, y, z) : x = x(u, v), y = y(u, v), z = z(u, v)\},$$

where $x, y, z : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad (u, v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x, y, z) : F(x, y, z) = 0, G(x, y, z) = 0\},$$

where $F, G : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x, y, z) : x = x(t), y = y(t), z = z(t)\},$$

where $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ and I is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Consider a family of curves, depending on the parameter λ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x, y, z; \lambda) = 0 \\ F_2(x, y, z; \lambda) = 0 \end{cases} .$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter λ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.
Suppose now that the family of curves depends on two parameters λ, μ,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x, y, z; \lambda, \mu) = 0 \\ F_2(x, y, z; \lambda, \mu) = 0 \end{cases} ,$$

and that the parameters are related through $\varphi(\lambda, \mu) = 0$ (one can choose only the sub-family corresponding to such λ and μ). If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

Cylindrical Surfaces

Definition 1.1

The surface generated by a variable line (the *generatrix*), which remains parallel to a fixed line d and intersects a given curve $\mathcal{C}$, is called *cylindrical surface*. The curve $\mathcal{C}$ is called the *director curve* of the cylindrical surface.

Theorem 1.2

The cylindrical surface, with the generatrix parallel to the line d, where

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

and having the director curve $\mathcal{C}$, where

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

(suppose that d and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form $\varphi(\pi_1, \pi_2) = 0$.

Proof.

An arbitrary line, which is parallel to
$d : \begin{cases} \pi_1(x, y, z) = 0 \\ \pi_2(x, y, z) = 0 \end{cases}$, will be of equations

$$d_{\lambda,\mu} : \begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases} .$$

Of course, not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}$$

.

is compatible. By eliminating x, y and z between the four equations of the system, one obtains a *compatibility condition* for the parameters λ and μ, $\varphi(\lambda, \mu) = 0$. The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1, \pi_2) = 0$. $\square$

Remark: Any equation of the form $\varphi(\pi_1, \pi_2) = 0$, where π_1 and π_2 are linear function of x, y and z, represents a cylindrical surface, having the generatrices parallel to $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

Example: Let us find the equation of the cylindrical surface having the generatrices parallel to

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases} .$$

The equations of the generatrices d are

$$d_{\lambda,\mu} : \begin{cases} x + y = \lambda \\ z = \mu \end{cases} .$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \\ x + y = \lambda \\ z = \mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

$$\begin{cases} x = 1 \\ y = \lambda - 1 \\ z = \mu \end{cases}$$

and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda - 1)^2 + \mu - 1 = 0.$$

The equation of the surface is obtained by eliminating the parameters in

$$\begin{cases} x + y = \lambda \\ z = \mu \\ 2(\lambda - 1)^2 + \mu - 1 = 0 \end{cases}.$$

Then,

$$2(x + y - 1)^2 + x - 1 = 0.$$

Conical Surfaces

Definition 1.4

The surface generated by a variable line, which passes through a fixed point V and intersects a given curve $\mathcal{C}$, is called *conical surface*. The point V is called the *vertex* of the surface and the curve $\mathcal{C}$ *director curve.*

Theorem 1.5

The conical surface, of vertex $V(x_0, y_0, z_0)$ and director curve

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

(suppose that V and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0. \tag{1.1}$$

Proof.

The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu}: \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases}.$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters λ and μ, which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

□

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obtained by eliminating x, y and z in the the previous system of equations.
In these conditions, the surface is generated and its equation rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases}.$$

It follows that

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0..$$

□

Remark: If φ is a polinomial function, then the equation (1.1) can be written in the form $\phi(x - x_0, y - y_0, z - z_0) = 0$, where ϕ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$.
If φ is algebraic and V is the origin of the coordinate system, then the equation of the conical surface is $\phi(x, y, z) = 0$, with ϕ a homogeneous polynomial.
Conversely, an algebraic homogeneous equation in x, y and z represents a conical surface with the vertex at the origin.
Example: Let us determine the equation of the conical surface, having the vertex $V(1, 1, 1)$ and the director curve

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases} .$$

The family of lines passing through V has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases} .$$

The system of equations

$$\begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \\ x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}$$

must be compatible. A solution is

$$\begin{cases} x = 1 - \lambda \\ y = 1 - \mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1 - \lambda)^2 + (1 - \mu)^2]^2 - (1 - \lambda)(1 - \mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters λ and μ in

$$\begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \\ ((1 - \lambda)^2 + (1 - \mu)^2)^2 - (1 - \lambda)(1 - \mu) = 0 \end{cases}.$$

Expressing $\lambda = \dfrac{x-1}{z-1}$ and $\mu = \dfrac{y-1}{z-1}$ and replacing in the compatibility condition, one obtains

$$\left[\left(\frac{z-x}{z-1}\right)^2 + \left(\frac{z-y}{z-1}\right)^2\right]^2 - \left(\frac{z-x}{z-1}\right)\left(\frac{z-y}{z-1}\right) = 0,$$

or

$$[(z-x)^2 + (z-y)^2]^2 - (z-x)(z-y)(z-1)^2 = 0.$$

Conoidal Surfaces.

Definition 1.6

The surface generated by a variable line, which intersects a given line d and a given curve $\mathcal{C}$, and remains parallel to a given plane π, is called *conoidal surface*. The curve $\mathcal{C}$ is the *director curve* and the plane π is the *director plane* of the conoidal surface.

Theorem 1.7

The conoidal surface whose generatrix intersects the line

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$

and the curve

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

and has the director plane $\pi = 0$*, (suppose that* π *is not parallel to* d *and that* $\mathcal{C}$ *is not contained into* π*), is characterized by an equation of the form*

$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{1.2}$$

Proof.

An arbitrary generatrix of the conoidal surface is contained into a plane parallel to π and, on the other hand, comes from the bundle of planes containing d. Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases} .$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

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and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu \pi_2 \\ \varphi(\lambda, \mu) = 0 \end{cases}.$$

By expressing λ and μ, one obtains (1.2). □

Example: Let us find the equation of the conoidal surface, whose generatrices are parallel to *xOy* and intersect *Oz* and the curve

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}.$$

The equations of *xOy* and *Oz* are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases},$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases} .$$

From the compatibility of the system of equations

$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases} ,$$

one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

and, replacing $\lambda = \dfrac{y}{x}$ and $\mu = z$, the equation of the conoidal surface is

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0.$$

Revolution Surfaces

Definition 1.8

The surface generated after the rotation of a given curve $\mathcal{C}$ around a given line d is said to be a *revolution surface.*

Theorem 1.9

The equation of the surface generated by the curve

$$\mathcal{C}: \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

in its rotation around the line

$$d: \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r},$$

is of the form

$$\varphi((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2, px+qy+rz) = 0. \quad (1.3)$$

Proof.

An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around d, a circle situated into a plane orthogonal on d and having the center on the line d. This circle can be seen as the intersection between a sphere, having the center on d and of variable radius, and a plane, orthogonal on d, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases} .$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

must be compatible. One obtains the compatibility

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$$\varphi(\lambda, \mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (1.3). □

Example: Let us determine the equation of the *torus* (the surface generated by a circle $\mathcal{C}$, which turns around an exterior line, lying in the plane of the circle). Choose the system of coordinates such that *Oz* is the line *d* and *Ox* is the orthogonal line on *d*, passing through the center of $\mathcal{C}$. Let *r* be the radius of the circle and $(0, a, 0)$ the coordinates of its center. Since the line is exterior to the circle, then $a > r > 0$. In this system of coordinates, the equations of the circle and of the line are, respectively,

$$\mathcal{C} : \begin{cases} (y-a)^2 + z^2 = r^2 \\ x = 0 \end{cases} \quad \text{and} \quad d : \begin{cases} x = 0 \\ z = 0 \end{cases} .$$

The equations of the family of circles generating the surface are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases} .$$

The system of equations

$$\begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \\ x = 0 \\ (y-a)^2 + z^2 = r^2 \end{cases}$$

must be compatible. Choose the first three equations in order to obtain a solution of the system

$$\begin{cases} x = 0 \\ y = \pm\sqrt{\lambda - \mu^2} \\ z = \mu \end{cases} .$$

Replacing in the remained equation, one obtains the compatibility condition

$$(\pm\sqrt{\lambda-\mu^2}-a)^2+\mu^2=r^2.$$

The equation of the torus is

$$(\pm\sqrt{x^2+y^2}-a)^2+z^2=r^2,$$

or

$$(x^2+y^2+z^2+a^2-r^2)^2=4a^2(x^2+y^2).$$

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