Seminar Nr.4, Discrete Random Variables and Discrete Random Vectors

Theory Review

Bernoulli Distribution with parameter $p \in (0,1)$: $X \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

Binomial Distribution with parameters $n \in \mathbb{N}, p \in (0,1)$: $X \begin{pmatrix} k \\ C_n^k p^k q^{n-k} \end{pmatrix}_{k=\overline{0,n}}$

Hypergeometric Distribution with parameters $a, b, n \in \mathbb{N}, n \le a$: $X \begin{pmatrix} k \\ p_k \end{pmatrix}_{k=\overline{0,n}}$, where $p_k = \dfrac{C_a^k C_b^{n-k}}{C_{a+b}^n}$

Poisson Distribution with parameter $\lambda > 0$: $X \begin{pmatrix} k \\ p_k \end{pmatrix}_{k\in\mathbb{N}}$, where $p_k = \dfrac{\lambda^k}{k!} e^{-\lambda}$

Pascal Distribution with parameters $n \in \mathbb{N}, p \in (0,1)$: $X \begin{pmatrix} k \\ C_{n+k-1}^k p^n q^k \end{pmatrix}_{k\in\mathbb{N}}$

Geometric Distribution with parameter $p \in (0,1)$: $X \begin{pmatrix} k \\ pq^k \end{pmatrix}_{k\in\mathbb{N}}$

Discrete Uniform Distribution with parameter $m \in \mathbb{N}$: $X \begin{pmatrix} k \\ \dfrac{1}{m} \end{pmatrix}_{k=\overline{1,m}}$

Cumulative Distribution Function $F_X : \mathbb{R} \to \mathbb{R}$, $F_X(x) = P(X < x)$

$(X, Y) : S \to \mathbb{R}^2$ discrete random vector with pdf $p_{ij} = P(X = x_i, Y = y_j)$, $(i,j) \in I \times J$ and cdf $F = F_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = P(X < x, Y < y) = \sum\limits_{x_i<x} \sum\limits_{y_j<y} p_{ij}$, $\forall (x,y) \in \mathbb{R}^2$.

$p_i = P(X = x_i) = \sum\limits_{j\in J} p_{ij}$, $\forall i \in I$, $q_j = P(Y = y_j) = \sum\limits_{i\in I} p_{ij}$, $\forall j \in J$ (marginal densities)

Let $X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i\in I}$, $Y \begin{pmatrix} y_j \\ q_j \end{pmatrix}_{j\in J}$ be discrete random variables. Then

X and Y are **independent** $<=>$ $p_{ij} = P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j) = p_i q_j$.

$X + Y \begin{pmatrix} x_i + y_j \\ p_{ij} \end{pmatrix}_{(i,j)\in I\times J}$, $\alpha X \begin{pmatrix} \alpha x_i \\ p_i \end{pmatrix}_{i\in I}$, $XY \begin{pmatrix} x_i y_j \\ p_{ij} \end{pmatrix}_{(i,j)\in I\times J}$, $X/Y \begin{pmatrix} x_i/y_j \\ p_{ij} \end{pmatrix}_{(i,j)\in I\times J}$ $(y_j \neq 0)$

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1. A coin is flipped 3 times. Let X denote the number of heads that appear.
a) Find the probability distribution function (pdf) of X. What type of distribution does X have?
b) Find the cumulative distribution function (cdf) of X, F_X.
c) Find $P(X \le 2)$ and $P(X < 2)$.

2. It was found that the probability to log on to a computer from a remote terminal is 0.7. Let X denote the number of attempts that must be made to gain access to the computer:
a) Find the pdf of X. What type of distribution does X have?
b) Find the cdf of X, F_X.
c) Find the probability that at most 4 attempts must be made to gain access to the computer.
d) Find the probability that at least 3 attempts must be made to gain access to the computer.

3. A number is picked randomly out of 1, 2, 3, 4 and 5. Let X denote the number picked. Let Y be 1 if the number picked was 1, 2 if the number was prime and 3, otherwise.
a) Find the pdf of X, Y.
b) Find the pdf's of $X+Y$, XY.

4. Same problem with 2 numbers being picked randomly. Variable X refers to the 1^{st} number, variable Y to the 2^{nd}. Is there a difference in the answers, from the previous problem?

5. Eight letters are randomly distributed into 3 mailboxes. Let X be the number of letters in the 1^{st} mailbox. Find the pdf of X.

6. In an automobile factory two tasks are performed by robots: welding two joints and tightening three bolts. Let X denote the number of defective welds and Y the number of improperly tightened bolts produced per car. The joint pdf for (X, Y) is given in the following table:

$X\backslash Y$	0	1	2	3
0	.840	.030	.020	.010
1	.060	.010	.008	.002
2	.010	.005	.004	.001

Find
a) The marginal densities f_X, f_Y;
b) The probability that exactly two defective welds and one improperly tightened bolt will be produced by the robots (event A);
c) The probability that at least one defective weld and at least one improperly tightened bolt will be produced (event B);
d) The probability that at most one defective weld will be produced (event C).

7. Let $X_1, \ldots, X_n$ be independent and identically distributed, with a Bernoulli distribution with parameter p. Find the pdf of $Y = \sum_{i=1}^{n} X_i$. What type of distribution does Y have?

8. The independent variables X, Y have binomial distributions with parameters m, p and n, p, respectively. Find the pdf of $X+Y$.

Bonus Problems:

9. Let $X_1, \ldots, X_n$ be independent random variables having a discrete uniform distribution on the set $\{1, \ldots, m\}$. Find the pdf of $U = \max\{X_1, \ldots, X_n\}$ and $V = \min\{X_1, \ldots, X_n\}$.

10. A die is thrown 3 times. Let X_1, X_2 and X_3 be the number that shows on the 1^{st}, 2^{nd} and 3^{rd} throw, respectively.
a) Find $P(X_1 + X_2 = X_3)$ and $P(X_1 + X_2 + X_3 = 7)$.
b) Consider the equation $ax + by - c = 0$, where a, b, c are determined by throwing a die 3 times. Find the probability that the straight line defined by the equation passes through the point $(1, 1)$.