SEMINAR 9

(1) Find the intersection points between the line (d) $2x+y-10=0$ and the ellipse $\mathcal{E}: x^2+3y^2-25=0$

$$\begin{cases} 2x+y=10 \\ x^2+3y^2=25 \end{cases} \Rightarrow y=10-2x$$

$$x^2+3(10-2x)^2=25$$

$$\Rightarrow 13x^2-120x+245=0$$

$$\Delta = 14400-14300=100 \Rightarrow \sqrt{\Delta}=10$$

$$x_1=\frac{-120+10}{2}=-55$$

$$x_2=\frac{-130}{2}=-65 \Rightarrow y_1=120;\ y_2=140; \Rightarrow A_1(-55,120)$$

$$A_2(-65,140)$$

(2) Find the equation of the tangent to the ellipse $\mathcal{E}: x^2+4y^2-20=0$ orthogonal to the line (d) $2x-2y-13=0$

$$y=-\frac{2x}{2}+\frac{13}{-2} \Rightarrow m_d=1 \Rightarrow m_h=1$$

$$h: \begin{cases} y=-x+n \\ x^2+4y^2-20=0 \end{cases}$$

$$x^2+4(n-x)^2-20=0$$

$$x^2+4(n^2-2nx+x^2)-20=0$$

$$5x^2+4n^2-8nx-20=0$$

$$\Rightarrow \Delta = 64-20(4n^2-20)=0$$

$$\Rightarrow -80n^2+64n^2+400=0 \Rightarrow -16n^2=-400 \Rightarrow$$

$$\Rightarrow n=\sqrt{\frac{100}{16}}=\pm\sqrt{5}$$

(4) Let $C_\lambda: x^2+y^2+\lambda x+(2\lambda+3)y=0, \lambda\in\mathbb{R}$ be a family of circles. Prove that the circles from the family have two fixed points.

$$\begin{cases} x^2+y^2+\lambda(x+2y)+5y=0 \\ x^2+y^2+3y+\lambda(x+2y)=0 \end{cases} \qquad \begin{cases} x^2+y^2+3y=0 \\ x+2y=0 \Rightarrow x=-2y \end{cases}$$

$$x^2+3y=0 \Leftrightarrow 4y^2+y^2+3y=0 \Rightarrow 5y^2+3y=0 \Rightarrow y(5y+3)=0$$

$$\Rightarrow y_1=0 \quad x_1=0 \quad \Rightarrow O(0,0)$$

$$y_2=-\frac{3}{5} \quad x_2=\frac{6}{5} \quad A\left(\frac{6}{5},-\frac{3}{5}\right)$$

$$\sqrt{k^2-\frac{k^2}{4}}=\sqrt{\frac{3}{2}}k=\frac{\sqrt{3}}{2}k;\ BC=Ox;\ y=0$$

$$AB=\frac{x}{-k}=\frac{y-\frac{\sqrt{3}}{2}k}{-\frac{\sqrt{3}}{2}k} \Leftrightarrow AB: -k\left(y-\frac{\sqrt{3}}{2}k\right)=$$

$$=\frac{\sqrt{3}}{2}kx$$

$$\frac{\sqrt{3}}{2}kx-ky+\frac{\sqrt{3}}{2}k^2=0;\quad AC: \frac{x}{k}=\frac{y-\frac{\sqrt{3}}{2}k}{-\frac{\sqrt{3}}{2}k} \Leftrightarrow$$

$$AB: \sqrt{3}kx-2ky+\sqrt{3}k^2=0$$

$$\frac{\sqrt{3}}{2}kx=ky-\frac{\sqrt{3}}{2}k^2 \Leftrightarrow -\frac{\sqrt{3}}{2}kx-ky+\frac{\sqrt{3}}{2}k^2=0$$

$$AB: \sqrt{3}kx-2ky+\sqrt{3}k^2=0$$

$$AC: -\sqrt{3}\cdot kx-2ky+\sqrt{3}k^2=0$$

$$BC: y=0$$

$$\delta(M,AB)^2+\delta(M,AC)^2+\delta(M,BC)^2=\rho^2$$

$$\frac{(\sqrt{3}kx-2ky+\sqrt{3}k^2)^2}{3k^2+4k^2}=\frac{(-\sqrt{3}kx-2ky+\sqrt{3}k^2)^2}{3k^2+4k^2}=\rho^2$$

$$\boxed{(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\alpha\gamma+2\beta\gamma}$$

$$3k^2x^2+4k^2y^2+3k^4-4\sqrt{3}k^2+6k^2x-4\sqrt{3}k^2y+3k^2x^2+4k^2y^2+3k^2+$$

$$4\sqrt{3}k^2xy-6k^3x-4\sqrt{3}k^2y+3k^2y^2+4k^2y^2=\rho^2(7k^2)$$

$$\Rightarrow 6k^2x^2+8k^2y^2-8\sqrt{3}k^3y+6k^4=7k^2\rho^2.$$