CURS 9

20. IV. 2011

The tangent to a circle at a given point

$\mathcal{C}: x^2+y^2=R^2$, $M(x_0, y_0) \in \mathcal{C}$

Assume that the line $y = mx \pm R\sqrt{1+m^2}$ is a tangent line to $\mathcal{C}$

$y - y_0 = m(x - x_0)$

$M(x_0, y_0) \in$ tangent line $\Rightarrow y_0 = mx_0 \pm R\sqrt{1+m^2}$

$(y_0 - mx_0)^2 = (\pm R\sqrt{1+m^2})^2$

$(y_0 - mx_0)^2 = R^2(1+m^2)$

$y_0^2 - 2mx_0 + m^2x_0^2 = R^2(1+m^2)$

$y_0^2 - 2mx_0 + m^2x_0^2 = (x_0^2+y_0^2)(1+m^2)$

$\cancel{y_0^2} - 2mx_0 + \cancel{m^2x_0^2} = x_0^2 + \cancel{y_0^2} + \cancel{m^2x_0^2} + m^2y_0^2$

$\cancel{-x_0^2 - 2mx_0 =}$

$x_0^2 + 2mx_0y_0 + y_0^2 = 0$

$(x_0 + my_0)^2 = 0 \Leftrightarrow m = -\dfrac{x_0}{y_0}$

$y - y_0 = -\dfrac{x_0}{y_0}(x - x_0)$

$y_0 y - y_0^2 = -x_0 x + x_0^2$

$x_0 x + y_0 y = x_0^2 + y_0^2$

$\boxed{x_0 x + y_0 y = R^2}$

$\mathcal{C}: F(x,y)=0$, $M(x_0, y_0) \in \mathcal{C} \Leftrightarrow F(x_0, y_0) = 0$

The gradient $(\nabla F)(x_0, y_0)$ is a normal vector to the curve $\mathcal{C}$ at $M(x_0, y_0)$

$(\nabla F)(x_0, y_0) = (F_x(x_0, y_0), F_y(x_0, y_0)) =$

$= \left(\frac{\partial F}{\partial x}(x_0, y_0) \frac{\partial F}{\partial y}(x_0, y_0)\right)$

$\mathcal{C} = F^{-1}(0)$

$T_M(\mathcal{C}) : F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) = 0$

$T_M(\mathcal{C}) : \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0) = 0$

$\mathcal{C}: x^2 + y^2 = R^2$, $M(x_0, y_0) \in \mathcal{C} \Leftrightarrow x_0^2 + y_0^2 = R^2$

$\mathcal{C} = F^{-1}(0)$ where $F(x, y) = x^2 + y^2 - R^2$

$\frac{\partial F}{\partial x}(M) = 2x_0$; $\frac{\partial F}{\partial y}(M) = 2y_0$

$T_M\,\mathcal{C}: \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0) = 0$

$T_M\,\mathcal{C}: 2x_0(x - x_0) + 2y_0(y - y_0) = 0$

$T_M\,\mathcal{C}: x_0 x - x_0^2 + y_0 y - y_0^2 = 0$

$T_M\,\mathcal{C}: x_0 x + y_0 y = x_0^2 + y_0^2$

$T_M\,\mathcal{C}: x_0 x + y_0 y = R^2$

Intersection of 2 circles

$\mathcal{C}_1 : x^2 + y^2 + 2a_1 x + 2b_1 y + c_1 = 0$

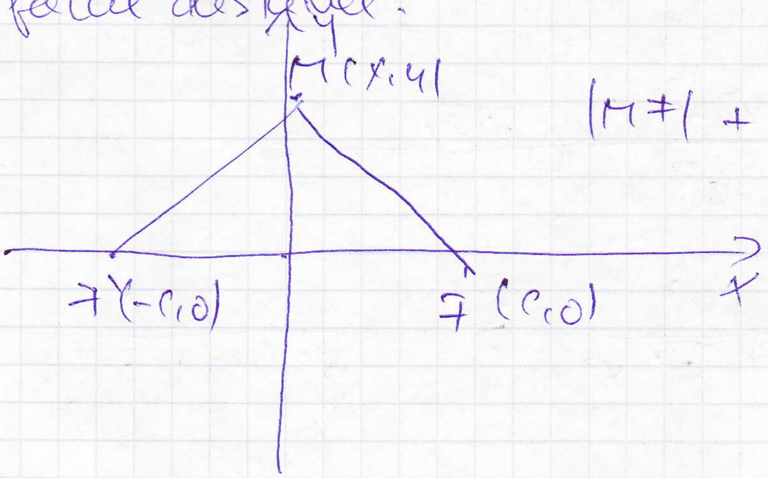
$\mathcal{C}_2 : x^2 + y^2 + 2a_2 x + 2b_2 y + c_2 = 0$

$$\mathcal{C}_1 \cap \mathcal{C}_2 : \begin{cases} x^2 + y^2 + 2a_1 x + 2b_1 y + c_1 = 0 \\ x^2 + y^2 + 2a_2 x + 2b_2 y + c_2 = 0 \end{cases}$$

$$\mathcal{C}_1 \cap \mathcal{C}_2 : \begin{cases} x^2 + y^2 + 2a_1 x + 2b_1 y + c_1 = 0 \\ 2(a_1 - a_2)x + 2(b_1 - b_2)y + c_1 - c_2 = 0 \end{cases}$$

The ellipse

<u>Def</u>: the ellipse is a closed curve in the plane defined as the locus of points in the plane whose distances to 2 fixed points have a constant sum. The 2 points are called the foci and the distance between them is called the focal distance.



$|MF| + |MF'| = 2a$ (const)

$a > c$

$$|MF| + |MF'| = 2a \Leftrightarrow$$

$$\Leftrightarrow \sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

$$\Leftrightarrow \sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2} \;|^2$$

$$\cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + \cancel{x^2} + 2xc + \cancel{c^2} + \cancel{y^2}$$

$$4a\sqrt{(x+c)^2+y^2} = 4a^2 + 4xc \;|:4$$

$$a\sqrt{(x+c)^2+y^2} = a^2 + xc \;|^2$$

$$a^2[(x+c)^2+y^2] = a^4 + 2a^2xc + x^2c^2$$

$$a^2x^2 + \cancel{2a^2xc} + a^2c^2 + a^2y^2 = a^4 + \cancel{2a^2xc} + x^2c^2$$

$$(a^2-c^2)x^2 + a^2y^2 = a^2(\overbrace{a^2-c^2}^{b^2})$$

$$b^2x^2 + a^2y^2 = a^2b^2 \;|: a^2b^2$$

$$\Rightarrow y^2 = \frac{1}{a^2}\left(a^2 s^2 - s^2 x^2\right)$$

$$y = \pm \frac{s}{a}\sqrt{a^2-x^2}$$

$$f(x) = \frac{s}{a}\sqrt{a^2-x^2}\,, \quad a^2-x^2 \geq 0 \iff |x| \leq a$$

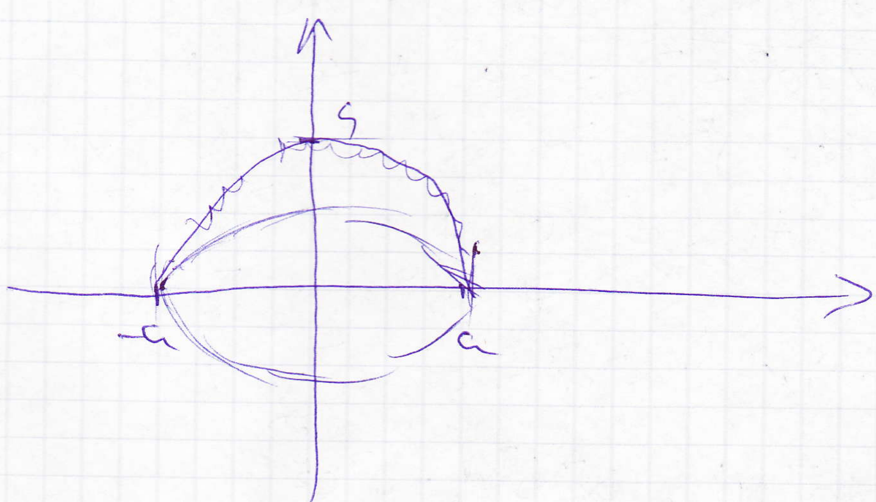
$$f'(x) = -\frac{s}{a}\,\frac{x}{\sqrt{a^2-x^2}}$$

$$f''(x) = -\frac{s}{a}\,\frac{\sqrt{a^2-x^2} - x\,\frac{-x}{\sqrt{a^2-x^2}}}{a^2-x^2} =$$

$$= -\frac{s}{a}\,\frac{a^2-x^2+x^2}{(a^2-x^2)\sqrt{a^2-x^2}} = -\frac{as}{(a^2-x^2)\sqrt{a^2-x}}$$

Intersection of 2 circles

x	$-a$			0			a
$f'(x)$	$\|+\infty$	$+$	$+$	0	$-$	$-$	$-\infty\|$
$f(x)$	0	$\nearrow$		s		$\searrow$	0
$f''(x)$	$\|$	$-$	$-$			$-$	$\|$



$b^2 = a^2 - c^2$

$c^2 = a^2 - b^2$

$c = \sqrt{a^2 - b^2} < a$

The intersection of an ellipse with a line

$$\begin{cases} \mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ d: y = mx + n \end{cases}$$

$\Leftrightarrow b^2x^2 + a^2(mx+n)^2 = a^2b^2$

$\Leftrightarrow b^2x^2 + a^2m^2x^2 + 2a^2mnx + n^2a^2 = a^2b^2$

$\Leftrightarrow (a^2m^2 + b^2)x^2 + 2a^2mnx + a^2(n^2 - b^2) = 0$

$\Delta = 4a^4m^2n^2 - 4a^2(n^2 - b^2)(a^2m^2 + b^2)$

- If $\Delta < 0$, then $\mathcal{E} \cap d = \emptyset \Rightarrow d$ is exterior to $\mathcal{E}$
- If $\Delta = 0$, then d is tangent to $\mathcal{E}$ $\Rightarrow \mathcal{E} \cap d$ is a singleton
- If $\Delta > 0$, then $\mathcal{E} \cap d$ is formed by 2 points and d is secant to $\mathcal{E}$

$\Delta = 0 \Leftrightarrow a^4m^2n^2 - a^2(n^2 - b^2)(a^2m^2 + b^2) = 0$

$\Leftrightarrow a^2m^2n^2 - a^2m^2n^2 + n^2b^2 - b^2a^2m^2 - b^4 = 0$

$\Rightarrow n^2 = a^2m^2 + b^2$

$\Leftrightarrow n = \pm\sqrt{a^2m^2 + b^2}$

Remarks:

Consider the ellipse $\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

- If $a = b$ then $\mathcal{E}: x^2 + y^2 = a^2$ - circle $(F = F' = O(0,0))$

- $e = \frac{c}{a}$ - eccentricity of $\mathcal{E}$ since

$c < a \Rightarrow e < 1$

$0 < e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2 < 1$

The tangent to the ellipse $\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $M(x_0, y_0) \in \mathcal{E}$

$\mathcal{E} = F^{-1}(0)$, $F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

$T_M(\mathcal{E}) \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0) = 0$

$T_M(\mathcal{E}): \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) = 0$

$T_M(\mathcal{E}): \frac{x_0 x}{a^2} - \frac{x_0^2}{a^2} + \frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} = 0$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0}{b^2}$$

$T_M(\mathcal{E}): \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$