Seminar Nr.6, Numerical Characteristics of Random Variables

Theory Review

Expectation:

- if $X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$ is discrete, then $E(X) = \sum_{i \in I} x_i p_i$.
- if X is continuous with pdf f, then $E(X) = \int_{\mathbb{R}} x f(x) dx$.

Variance: $V(X) = E\left((X - E(X))^2\right) = E\left(X^2\right) - (E(X))^2$.

Standard Deviation: $\sigma(X) = \sqrt{V(X)}$.

Moments:

- **moment of order k**: $\nu_k = E\left(X^k\right)$.
- **absolute moment of order k**: $\overline{\nu_k} = E\left(|X|^k\right)$.
- **central moment of order k**: $\mu_k = E\left((X - E(X))^k\right)$.

Properties:

1. $E(aX + b) = aE(X) + b$, $V(aX + b) = a^2 V(X)$
2. $E(X + Y) = E(X) + E(Y)$
3. if X and Y are independent, then $E(XY) = E(X)E(Y)$ and $V(X + Y) = V(X) + V(Y)$
4. if $h : \mathbb{R} \to \mathbb{R}$ is a measurable function, X a random variable;

- if X is discrete, then $E(h(X)) = \sum_{i \in I} h(x_i) p_i$
- if X is continuous, then $E(h(X)) = \int_{\mathbb{R}} h(x) f(x) dx$

Covariance: $\text{cov}(X, Y) = E((X - E(X))(Y - E(Y)))$

Correlation Coefficient: $\rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}}$

Properties:

1. $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$
2. $V\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum_{1 \le i < j \le n} a_i a_j \text{cov}(X_i, X_j)$
3. X, Y independent $=>$ $\text{cov}(X, Y) = \rho(X, Y) = 0$ (X and Y are *uncorrelated*)
4. $-1 \le \rho(X, Y) \le 1$; $\rho(X, Y) = \pm 1 \iff \exists\, a, b \in \mathbb{R},\ a \neq 0$ s.t. $Y = aX + b$

Let (X, Y) be a continuous random vector with pdf $f(x, y)$, let $h : \mathbb{R}^2 \to \mathbb{R}^2$ a measurable function, then

$$E(h(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) f(x, y) dx dy$$

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1. Find $E(X)$ and $V(X)$ for the following random variables:

a) $X \in B(n, p)$ (binomial);
b) $X \in G(p)$ (geometric);
c) $X \in \mathcal{P}(\lambda)$ (Poisson).

2. Find $E(X)$ and $V(X)$ for the random variables with the following pdf's:
a) $f_X(x) = \dfrac{1}{\pi\sqrt{a^2 - x^2}}$, $x \in (-a, a)$;

b) $f_X(x) = xe^{-x},\ x > 0$.

3. Find the k^{th} order central moments for a normally distributed random variable $X \in N(m, \sigma)$.

4. (Reduced Variables). Let X be a random variable with mean $E(X)$ and standard deviation $\sigma(X) = \sqrt{V(X)}$. Find the mean and variance of $Y = \dfrac{X - E(X)}{\sigma(X)}$.

5. The joint density function of the vector (X, Y) is $f(x, y) = k(x + y),\ (x, y) \in [0, 1] \times [0, 1]$. Find
a) the constant k;
b) the means and variances of X and Y;
c) the correlation coefficient $\rho(X, Y)$.

6. Let X be a discrete random variable with pdf $X \begin{pmatrix} -1 & 0 & 1 \\ \sin^2 a & \cos 2a & \sin^2 a \end{pmatrix}$, $a \in \left(0, \frac{\pi}{4}\right)$. For any $k \in \mathbb{N}^*$, find $\rho\left(X^{2k-1}, X^{2k}\right)$. (In particular, X and X^2 are uncorrelated, but *not* independent).

Bonus Problems

7. Let X and Y be independent random variables with a $N(0, 1)$ distribution. Find the expectation of the random variable

$$Z = e^{\frac{X^2+Y^2}{2}}(1 + X^2 + Y^2)^{-\frac{3}{2}}.$$

8. In an office n different letters are placed randomly into n envelopes with addresses. Let Z_n denote the random variable that shows the number of correct mailings. For each $k \in \{1, \ldots, n\}$, let X_k be the random variable defined by

$$X_k = \begin{cases} 1, & \text{if the } k\text{-th letter is placed correctly} \\ 0, & \text{otherwise .} \end{cases}$$

a) Find $E(X_k)$ and $V(X_k)$ for each $k \in \{1, \ldots, n\}$.
b) Find $E(Z_n)$ and $V(Z_n)$.
c) How many correct mailings are to be expected?