Conic Assignment

Bole Manideep

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Problem Statement **-Find the radius of the circle passing through the foci of the elipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having its centre at (0,3)**

Solution

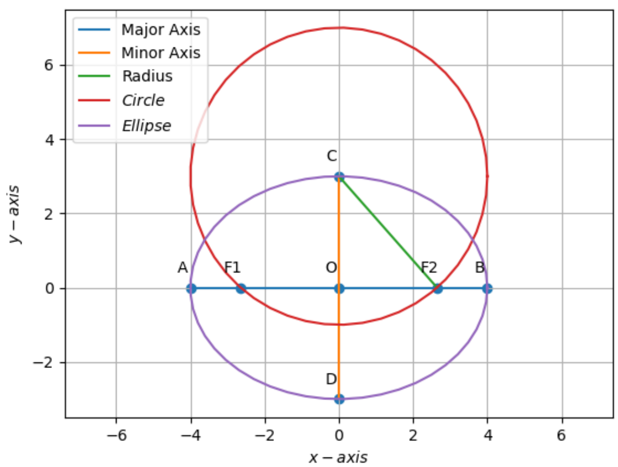

Figure 1: Ellpise with center O along with Circle C

Given an ellipse with center **O** and semi major axis lenth a = 4cm and semi minor axis length b = 3cm.

Let $\mathbf{F_1}$ & $\mathbf{F_2}$ be the Focii of the ellipse, where $\mathbf{A}, \mathbf{B}$ & $\mathbf{C}, \mathbf{D}$ be the extreme points on major & minor axis respectively.

The equation of a conic with diretrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and Focus **F** is given by,

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

For the given equation of ellipse,

$$\mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \& \quad f = -144 \tag{2}$$

The eigenvalue decomposition of a symmetric matrix **V** is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D} \qquad \mathbf{P} = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} \end{pmatrix} \tag{3}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{4}$$

On solving (3) with $\mathbf{P_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\mathbf{P_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get

$$\mathbf{D} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \tag{5}$$

where,

$$\lambda_1 = 9 \;\; and \;\; \lambda_2 = 16 \tag{6}$$

We have,

$$eccentricity, \quad e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{7}$$

from (6),

$$e = 0.6614 \tag{8}$$

Normal vector of diretrix **n** is given by

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{P_1} \tag{9}$$

This gives,

$$\mathbf{n} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{10}$$

For e $\neq$ 1, we have

$$c = \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 \left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2 \left(e^2 - 1\right) \left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e \left(e^2 - 1\right)} \tag{11}$$

On solving we get,

$$c = \pm 24.1911 \tag{12}$$

Focii of a conic is given by the equation,

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \tag{13}$$

Yelding,

$$\mathbf{F} = \pm 2.6456 \tag{14}$$

Therefore, focii of the ellipse are,

$$\mathbf{F_1} = \begin{pmatrix} -2.6456 \\ 0 \end{pmatrix} \quad \& \quad \mathbf{F_2} = \begin{pmatrix} 2.6456 \\ 0 \end{pmatrix} \tag{15}$$

Given $\mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is the center of the circle and is passing through focii of the ellipse.
Let the circle be,

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{16}$$

where,

$$\mathbf{V} = \mathbf{I} \quad \& \quad \mathbf{u_1} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \tag{17}$$

Since circle is passing through $\mathbf{F_1}$

$$\mathbf{F_1}^\top \mathbf{V} \mathbf{F_1} + 2\mathbf{u_1}^\top \mathbf{F_1} + f_1 = 0 \tag{18}$$

$$\begin{pmatrix}-2.6456 & 0\end{pmatrix}\begin{pmatrix}-2.6456\\0\end{pmatrix} + 2\begin{pmatrix}0 & -3\end{pmatrix}\begin{pmatrix}-2.6456\\0\end{pmatrix} + f_1 = 0 \tag{19}$$

$$6.99 + 0 + f_1 = 0$$
$$\implies f_1 = -6.99$$

Hence, Equation of the circle is given as,

$$\mathbf{x}^\top \mathbf{I}\mathbf{x} + 2\begin{pmatrix}0\\-3\end{pmatrix}\mathbf{x} - 6.99 = 0 \tag{20}$$

We have, radius of the circle,

$$r = \sqrt{\|\mathbf{u}\|^2 - f_1} \tag{21}$$

$$r = \sqrt{\left(\sqrt{0^2 + (-3)^2}\right)^2 - (-6.99)}$$
$$r = \sqrt{9 + 6.99}$$
$$r = \sqrt{15.99}$$
$$\therefore \quad Radius, r = 3.99cm \tag{22}$$

Construction

An ellipse with center O and major,minor axis a,b respectively along with circle with center C is constructed unsing python,with the parameters that are mentioned in the table below.

Symbol	**Value**	**Description**
a	4	Semi Major Axis
b	3	Semi Minor Axis
O	$\begin{pmatrix}0\\0\end{pmatrix}$	Center of Ellipse
e1	$\begin{pmatrix}1\\0\end{pmatrix}$	Unit Vector along x-axis
e2	$\begin{pmatrix}1\\0\end{pmatrix}$	Unit Vector along y-axis
A	-a∗**e1**	Vertex A
B	a∗**e1**	Vertex B
C	b∗**e2**	Center of circle (C)
D	-b∗**e2**	Vertex D
d	$\sqrt{a^2 - b^2}$	distance between Center (O) & Focii
$\mathbf{F_1}$	-d∗e1	Focus 1 of Ellipse
$\mathbf{F_2}$	d∗e1	Focus 2 of Ellipse

Table 1: Parameter's Table