Circle Assignment

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***Problem Statement* - Draw a circle of radius 3 cm. Take two points P and Q on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points P and Q.**

1. Radius of circle OA = OB = 3cm
2. Distance from center to points OP = OQ = 7cm

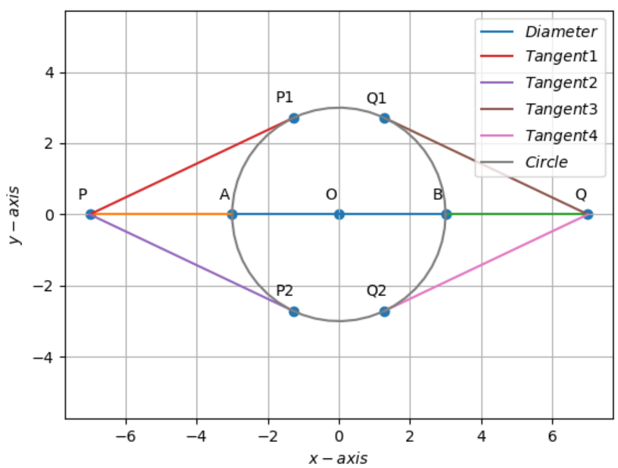

Figure 1: Circle with center O and points P & Q

Solution

Consider the circle of radius 3cm whose center is at origin and points P & Q each at a distance of 7cm from the center.
Two tangents can be drawn from point P on to the circle and let the point of contacts be P1 and P2, and other two tangents can be drawn from point Q on to the circle and let the point of contacts be Q1 and Q2

The point of intersection of line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{1}$$

with the conic section

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2}$$

is given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{3}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) \pm\sqrt{[\mathbf{m}^\top(\mathbf{V}\mathbf{q}\mathbf{u})]^2 - (\mathbf{q}^\top\mathbf{V}\mathbf{q}+2\mathbf{u}^\top\mathbf{q}+f)(\mathbf{m}^\top\mathbf{V}\mathbf{m})}\Big) \tag{4}$$

If the line L touches the conic at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{5}$$

In this case, the conic intercept has exactly one root.Hence,

$$\left[\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{q}^\top\mathbf{V}\mathbf{q}+2\mathbf{u}^\top\mathbf{q}+f\right) = 0 \tag{6}$$

So, the equation of conic $x^2 + y^2 = 9$ can be

written in the form of eq (2) as,

$$\mathbf{x}^\top\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}0 & 0\end{pmatrix}\mathbf{x} + 9 = 0 \tag{7}$$

Let us conisder the direction vector of **L** as m,

$$\mathbf{m} = \begin{pmatrix}1\\ \lambda\end{pmatrix} \tag{8}$$

and $\mathbf{q}$ be the point P,

$$\mathbf{q} = \begin{pmatrix}-7\\ 0\end{pmatrix} \tag{9}$$

Substituting (7),(8) and (9) in eq (6), we get

$$\left[\mathbf{m}^\top(\mathbf{I}\mathbf{q})\right]^2 - \left(\mathbf{m}^\top\mathbf{I}\mathbf{m}\right)\left(\mathbf{q}^\top\mathbf{I}\mathbf{q}+(-9)\right) = 0$$

$$\left[\begin{pmatrix}1 & \lambda\end{pmatrix}\begin{pmatrix}-7\\ 0\end{pmatrix}\right]^2 - \left(\begin{pmatrix}1 & \lambda\end{pmatrix}\begin{pmatrix}1\\ \lambda\end{pmatrix}\right)\left(\begin{pmatrix}-7 & 0\end{pmatrix}\begin{pmatrix}-7\\ 0\end{pmatrix} - 9\right) = 0 \tag{10}$$

$$(-7)^2 - \left(1+\lambda^2\right)(49-9) = 0$$
$$49 - \left(1+\lambda^2\right)(40) = 0$$
$$\left(1+\lambda^2\right)(40) = 49$$
$$1+\lambda^2 = \frac{49}{40}$$
$$\lambda^2 = \frac{9}{40}$$
$$\lambda = \pm\frac{3}{\sqrt{40}}$$

$$\lambda = \pm 0.4743 \tag{11}$$

That is,

$$\mathbf{m} = \begin{pmatrix} 1 \\ \pm 0.4743 \end{pmatrix} \tag{12}$$

From (4) and (6)

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right) \tag{13}$$

$$\mu_i = \frac{1}{\begin{pmatrix}1 & \frac{3}{\sqrt{40}}\end{pmatrix}\mathbf{I}\begin{pmatrix}1 \\ \frac{3}{\sqrt{40}}\end{pmatrix}}\left(-\begin{pmatrix}1 & \sqrt{40}\end{pmatrix}\left(\mathbf{I}\begin{pmatrix}-7 \\ 0\end{pmatrix}\right)\right)$$
$$\mu_i = \frac{1}{1+\frac{9}{40}}(-7+0)$$
$$\mu_i = \frac{-7}{\left(\frac{49}{40}\right)}$$
$$\mu_i = \frac{40}{7}$$

$$\mu_i = 5.714 \tag{14}$$

Now (3) becomes,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} + 5.714 * \begin{pmatrix} 1 \\ \pm 0.4743 \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} + \begin{pmatrix} 5.714 \\ \pm 2.710 \end{pmatrix} \tag{16}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1.285 \\ 2.710 \end{pmatrix} \quad or \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1.285 \\ -2.710 \end{pmatrix} \tag{17}$$

Therefore,

$$\mathbf{P_1} = \begin{pmatrix} -1.285 \\ 2.710 \end{pmatrix}$$
$$\mathbf{P_2} = \begin{pmatrix} -1.285 \\ -2.710 \end{pmatrix}$$

In the similar way, points of contact from the other point Q that are Q_1 & Q_2 can be

$$\mathbf{Q_1} = \begin{pmatrix} 1.285 \\ 2.710 \end{pmatrix}$$
$$\mathbf{Q_2} = \begin{pmatrix} 1.285 \\ -2.710 \end{pmatrix}$$

Construction

A Circle with center O and radius 3cm is constructed unsing python,with the parameters that are mentioned in the table below.

Symbol	**Value**	**Description**
r	3	Radius
d	7	distance from O to P & Q
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center
$\mathbf{e_1}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Unit Vector
$\mathbf{P}$	-d∗**e1**	Point P
$\mathbf{Q}$	d∗**e1**	Point Q
θ	$\angle P_1OP$	Angle P_1PO
$\mathbf{P_1}$	$r\begin{pmatrix} cos(\pi-\theta) \\ sin(\pi-\theta) \end{pmatrix}$	Point of contact P1
$\mathbf{P_2}$	$r\begin{pmatrix} cos(\pi-\theta) \\ -sin(\pi-\theta) \end{pmatrix}$	Point of contact P2
$\mathbf{Q_1}$	$r\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$	Point of contact Q1
$\mathbf{Q_2}$	$r\begin{pmatrix} cos\theta \\ -sin\theta \end{pmatrix}$	Point of contact Q2

Table 1: Parameter's Table