Line Assignment

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***Problem Statement* - For a given Parallelogram ABCD, show that for any point 'P' inside the parallelogram,**

1. $\boldsymbol{Ar(APD) + Ar(PBC) = \frac{1}{2} Ar(ABCD)}$
2. $\boldsymbol{Ar(APD) + Ar(PBC) = Ar(APB) + Ar(PCD)}$

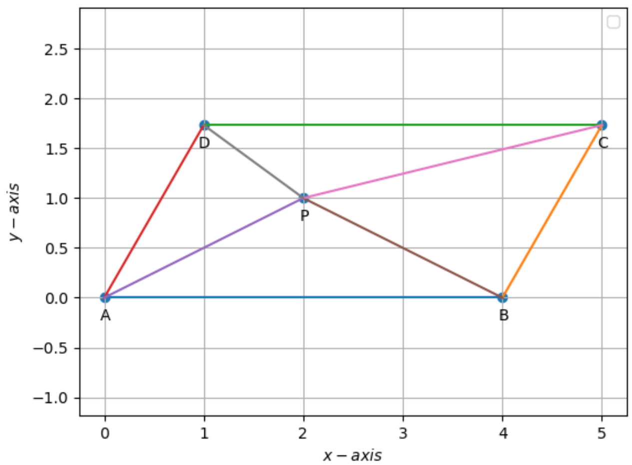

Figure 1: Parallelogram ABCD with interior point P

Solution

Part 1

WKT, area of a parallelogram with adjacent sides a & b is,

$$\text{Area of parallelogram} = \|\mathbf{a} \times \mathbf{b}\| \tag{1}$$

And, area of a triangle with adjacent sides p & q is,

$$\text{Area of triangle} = \frac{1}{2}\|\mathbf{p} \times \mathbf{q}\| \tag{2}$$

From Figure 1,
$(\mathbf{A}-\mathbf{D})$ & $(\mathbf{B}-\mathbf{C})$ are equal,

$$\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{B}-\mathbf{C}\| \tag{3}$$

Consider $\triangle$APD

$$\text{Area of } \triangle APD = \frac{1}{2}\|(\mathbf{A}-\mathbf{D}) \times (\mathbf{A}-\mathbf{P})\| \tag{4}$$

Consider $\triangle$PBC

$$\text{Area of } \triangle BPC = \frac{1}{2}\|(\mathbf{B}-\mathbf{C}) \times (\mathbf{P}-\mathbf{B})\| \tag{5}$$

On adding (4) & (5),

$$Ar(APD) + Ar(PBC) = \frac{1}{2}\|(\mathbf{A}-\mathbf{D}) \times (\mathbf{A}-\mathbf{P})\| + \frac{1}{2}\|(\mathbf{B}-\mathbf{C}) \times (\mathbf{P}-\mathbf{B})\| \tag{6}$$

From equation (3),

$$Ar(APD) + Ar(PBC) = \frac{1}{2}\|(\mathbf{A}-\mathbf{D}) \times (\mathbf{A}-\mathbf{P})\| + \frac{1}{2}\|(\mathbf{A}-\mathbf{D}) \times (\mathbf{P}-\mathbf{B})\| \tag{7}$$

$$\implies Ar(APD) + Ar(PBC) = \frac{1}{2}\|(\mathbf{A}-\mathbf{D}) \times [(\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B})]\| \tag{8}$$

Here, AP & PB are adjacent sides of $\triangle$ APB
From Triangle law of vector addition,
$(\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}) = (\mathbf{A}-\mathbf{B})$

$$\implies Ar(APD) + Ar(PBC) = \frac{1}{2}\|(\mathbf{A}-\mathbf{D}) \times (\mathbf{A}-\mathbf{B})\| \tag{9}$$

Since, $(\mathbf{A}-\mathbf{D})$ & $1(\mathbf{A}-\mathbf{B})$ are adjacent sides of paralleogram ABCD
With reference to (2),

$$Ar(ABCD) = \|(\mathbf{A}-\mathbf{D}) \times (\mathbf{A}-\mathbf{B})\| \tag{10}$$

From (9) & (10)

$$\therefore \quad Ar(APD) + Ar(PBC) = \frac{1}{2}Ar(ABCD) \tag{11}$$

Part 2

Similarly, we can prove taht,

$$Ar(APB) + Ar(PBD) = \frac{1}{2}Ar(ABCD) \tag{12}$$

On Comparing (11) and (12),

$$Ar(APD) + Ar(PBC) = Ar(APB) + Ar(PCD) \tag{13}$$

Hence Proved

Construction

A parallelogram ABCD is constructed unsing python,with the parameters that are mentioned in the table below.

Symbol	**Value**	**Description**
a	4	AB
b	2	AD
θ	$60°$	$\angle$A
$\mathbf{A}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Vertex A
$\mathbf{B}$	$\begin{pmatrix} a \\ 0 \end{pmatrix}$	Vertex B
$\mathbf{D}$	$b\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$	Vertex B
$\mathbf{C}$	$\mathbf{B}+\mathbf{D}$	Vertex C

Table 1: Parameter's Table

Proofs

Triangle law of vector addition
Consider a triangle APB with vertices,

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Vectors $(\mathbf{A}-\mathbf{P}),(\mathbf{P}-\mathbf{B})$ & $(\mathbf{A}-\mathbf{B})$ *are sides of* $\triangle APB$
Let's consider,

$$(\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad + \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{14}$$

$$\implies (\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}) = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad + \quad \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{15}$$

$$\implies (\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}) = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{16}$$

$$\implies (\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{17}$$

$$\therefore (\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}) = (\mathbf{A}-\mathbf{B}) \tag{18}$$

Thus, Triangle law of vector addition says taht sum of two adjacent side vectors of a triangle is equal to third side vector but in opposite direction.