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Bayesian Statistics and Data Analysis
Lecture 8b

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Thanks to Aki Vehtari, Aalto University

Section 1

Model assessment and selection

Predictive performance

- Modeling complex phenomena with models that are simplified
  All models are wrong... but some are useful.

Predictive performance

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Predictive performance

- Modeling complex phenomena with models that are simplified
  All models are wrong... but some are useful.
- True predictive performance is found out by using it to make predictions and comparing predictions to true observations
  - external validation
- Expected predictive performance
  - approximates the external validation

Goal of model evaluation

- Model choice is a (model-)decision-theoretic problem

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- Evaluate the utility of a model M for new unseen data $\tilde{y}$:

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where $\tilde{y}$ is an unseen observation generated from the true data generating process $p_{\text{text}}(\tilde{y})$, and y are observed data and $u(\tilde{y})$ is a utility function.

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- The expectation is with respect to p_{true} (f in BDA3)
- Choose the model function to maximize our utility

Model choice utility

- Application specific utility/cost functions are important
  - eg. money, life years, quality adjusted life years, etc.

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$$\log p_M(y^{\text{rep}}|y)$$

- We want the "best" model to explain the data

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Subsection 1

Measures of predictive accuracy

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(Bayesian) Points Prediction Accuracy

- Point residuals

$$e_i = y_i - E(\tilde{y}_i|y)\,,$$

where

$$E(\tilde{y}|y) = \int \tilde{y}_i p(\tilde{y}_i|y) d\tilde{y}\,,$$

i.e. the expected predicted value

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i.e. the expected predicted value

- Mean squared (prediction) error (MSE)

$$\text{MSE} = \frac{1}{n}\sum_{i}^{n}[y_i - E(\tilde{y}_i|y)]^2\,.$$

Probabilistic predictions

- The log score (a local and proper scoring rule)*

$$\log p(y|\theta)$$

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- The log predictive density (lpd)

$$\begin{aligned} \text{lpd} &= \log p(y|y) \\ &= \log \int p(y|\theta)p(\theta|y)d\theta \end{aligned}$$

Probabilistic predictions

- The lpd is usually approximated with the log point predictive density (lppd or just lpd)

$$\begin{aligned} \text{lppd} &= \sum_{i}^{n} \log\, p(y_i | y) \\ &\approx \log p(y|y) \end{aligned}$$

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Probabilistic predictions

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$$\text{lppd} = \sum_{i}^{n} \log\, p(y_i|y)$$
$$\approx \log p(y|y)$$

- Estimation using MCMC

$$\text{lppd} = \sum_{i}^{n} \log \left(\frac{1}{S} \sum_{s}^{S} p(y_i|\theta_s)) \right)$$

Subsection 2

Model selection

Model selection

- Evaluate how model M generalizes to unseen data $\tilde{y}$ (the *expected log predictive density*):

$$\mathrm{elpd}_M = \int \log p_M(\tilde{y}|y) p_{\mathrm{true}}(\tilde{y}) d\tilde{y}\,,$$

  where $\tilde{y}$ is an unseen observation generated from the true data generating process $p_{\mathrm{true}}(\tilde{y})$, and y are observed data.
- $\log p_M(\tilde{y}|y)$ is the log score (the utility of the model)

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- The expectation is with respect to p_{true}
- p_{true} is (almost always) unknown
- The utility function is the log scoring rule.

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Section 2

Cross-validation

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Leave-one-out cross-validation (LOO-CV)

- Evaluate how model M *generalizes* to unseen data (the *expected log predictive density*):

$$\mathrm{elpd}_M = \int \log p_M(\tilde{y}_i | y) p_{\mathrm{true}}(\tilde{y}_i) d\tilde{y}_i \,,$$

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where $\tilde{y}_i$ is an unseen observation generated from the true data generating process $p_{\text{true}}(\tilde{y}_i)$, and y are observed data.

- Can we approximate $p_{\text{true}}(\tilde{y}_i)$?

Leave-one-out cross-validation (LOO-CV)

- Approximate $p_{\text{true}}(\tilde{y}_i)$ with data y

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\text{elpd}_{\text{loo}} &= \sum_{i=1}^{n} \log p_M(y_i|\mathbf{y}_{-i}) \\
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- Similar to jack-knife resampling

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- Analogy: Monte Carlo approximation using our data
- Similar to jack-knife resampling
- The elpd, lpd and efficient number of parameters (p_{loo})

$$\text{elpd}_{\text{loo}} = \text{lpd} + p_{\text{loo}}$$

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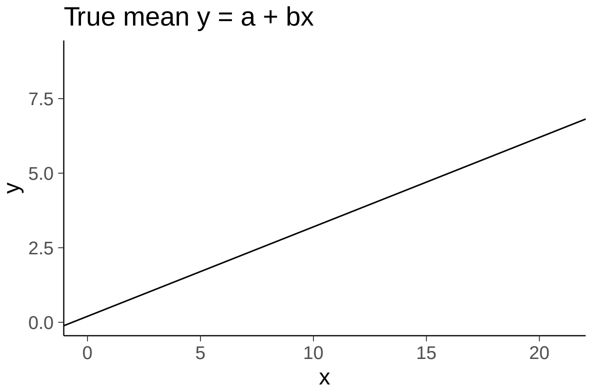
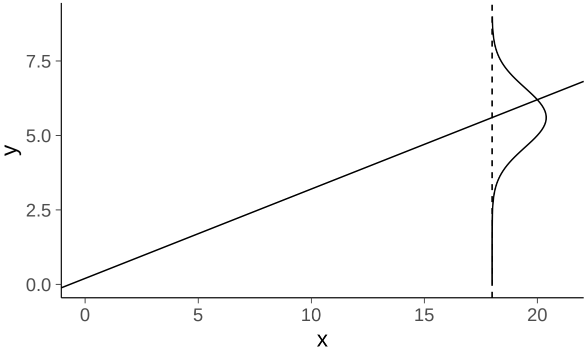


True mean and sigma



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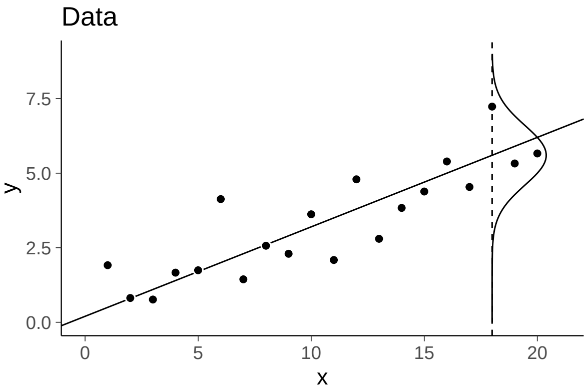

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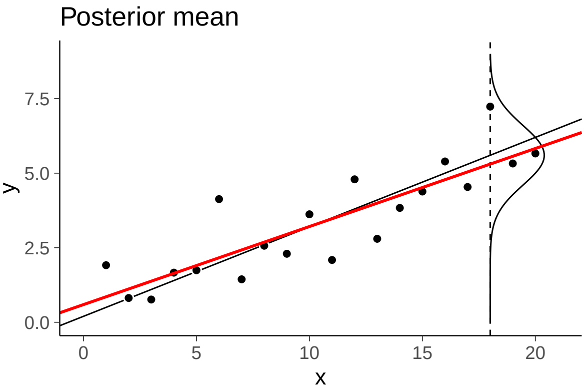

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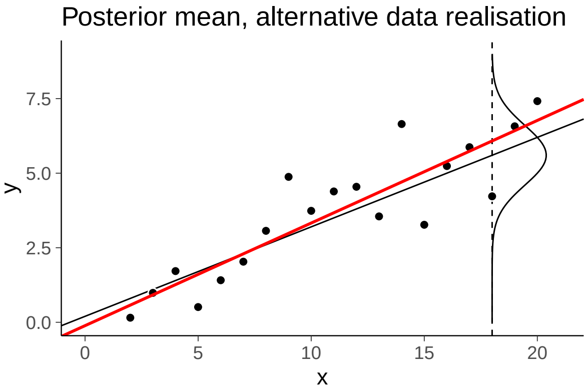

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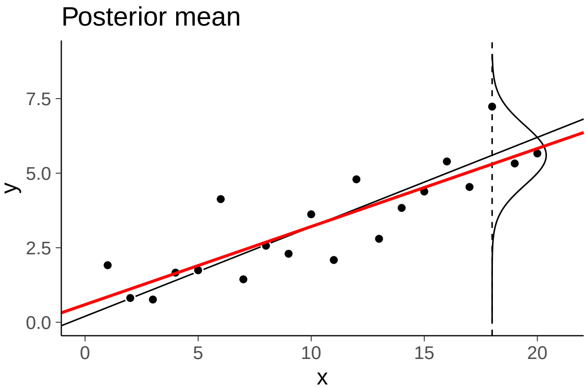

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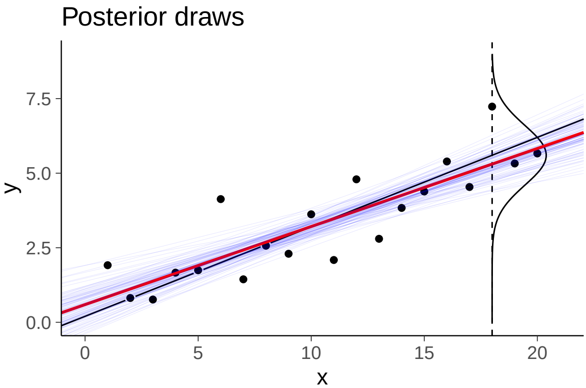
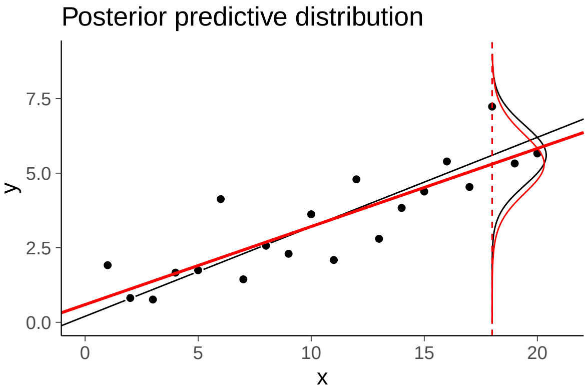
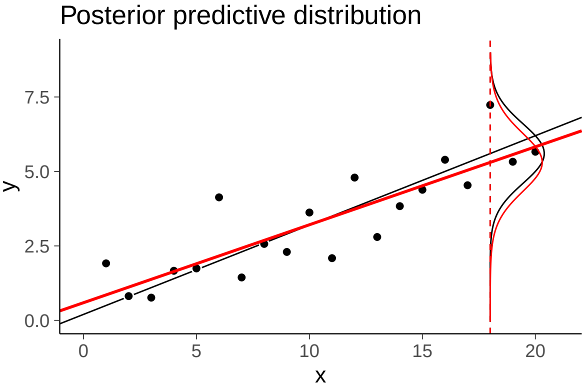
- Model assessment and selection
  - Measures of predictive accuracy
  - Model selection
- Cross-validation
  - When is LOO applicable
  - Fast LOO-CV
  - K-fold cross-validation
  - Comparison and selection
  - Additional reading
- Information criteria
- Model averaging
- Summary




Posterior predictive distribution
y
x
7.5
5.0
2.5
0.0
0
5
10
15
20



$$p(\tilde{y}|\tilde{x}=18,x,y)=\int p(\tilde{y}|\tilde{x}=18,\theta)p(\theta|x,y)d\theta$$

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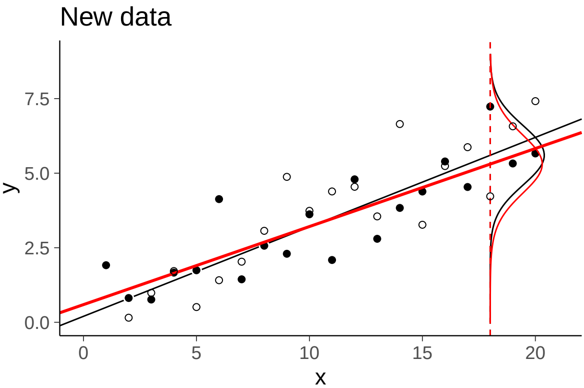

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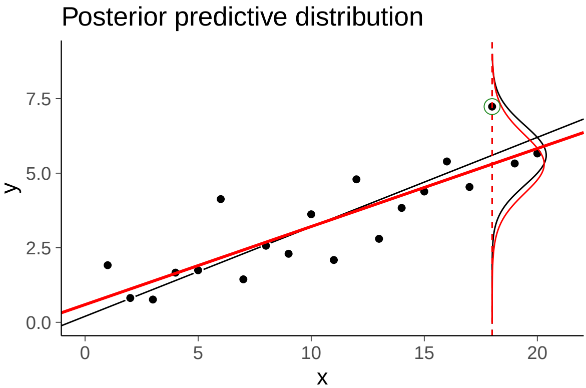

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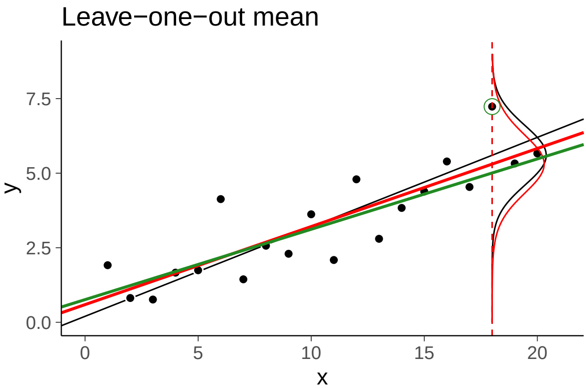

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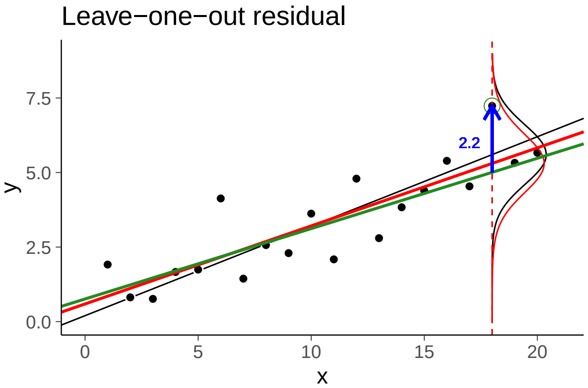
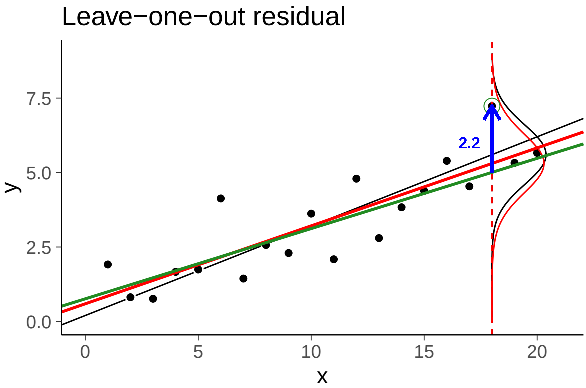
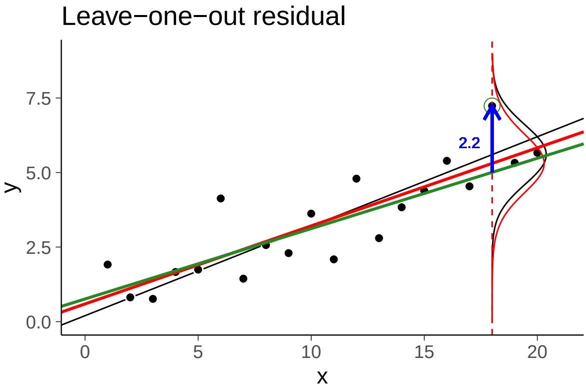




$$y_{18} - E[p(\tilde{y}|\tilde{x} = 18, x_{-18}, y_{-18})]$$



$y_{18} - E[p(\tilde{y}|\tilde{x} = 18, x_{-18}, y_{-18})]$

Can be use to compute, e.g., RMSE, R^2, 90% error

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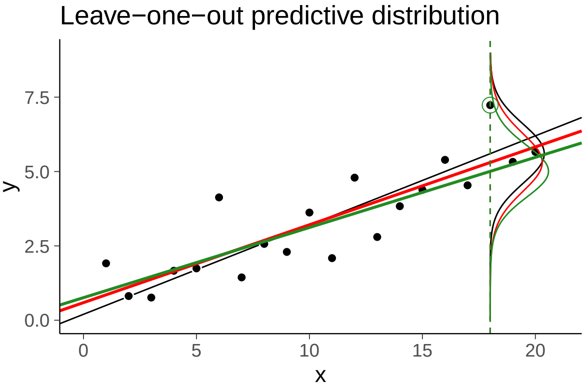
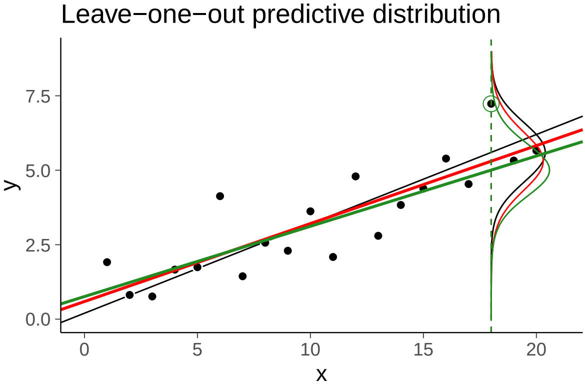
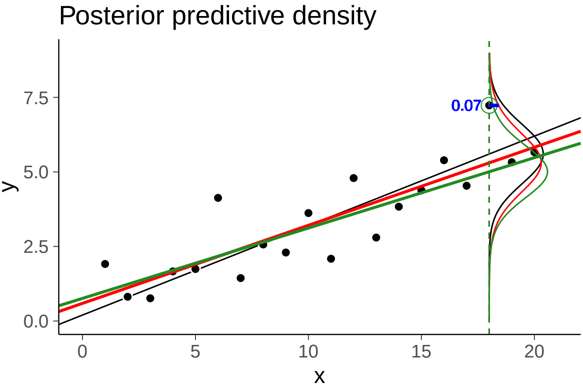




$$p(\tilde{y}|\tilde{x}=18, x_{-18}, y_{-18}) = \int p(\tilde{y}|\tilde{x}=18,\theta)p(\theta|x_{-18}, y_{-18})d\theta$$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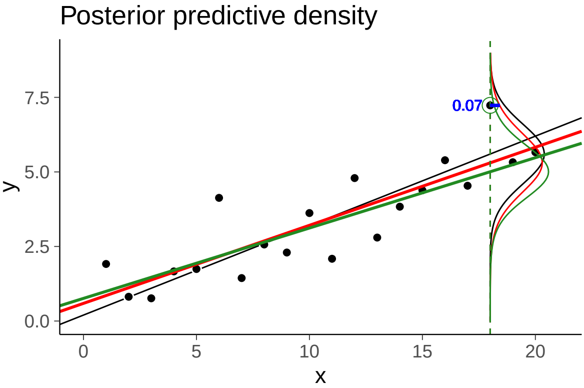
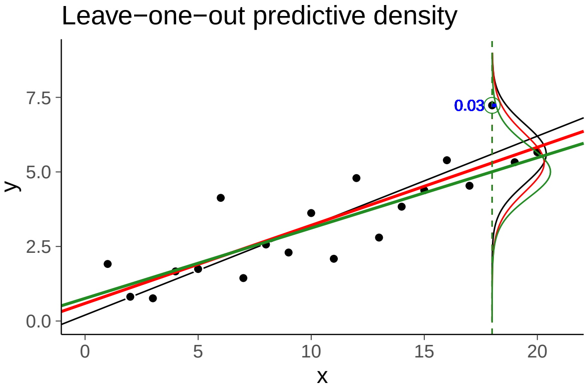


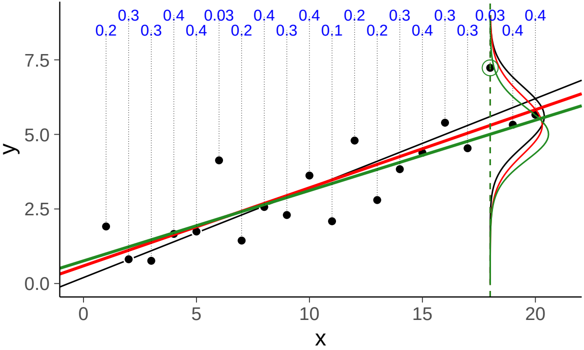
$$p(\tilde{y} = y_{18} | \tilde{x} = 18, x, y) \approx 0.07$$



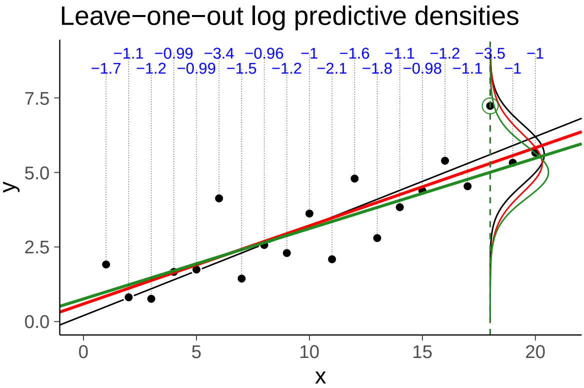
$$p(\tilde{y} = y_{18} | \tilde{x} = 18, x, y) \approx 0.07$$

$$p(\tilde{y} = y_{18} | \tilde{x} = 18, x_{-18}, y_{-18}) \approx 0.03$$

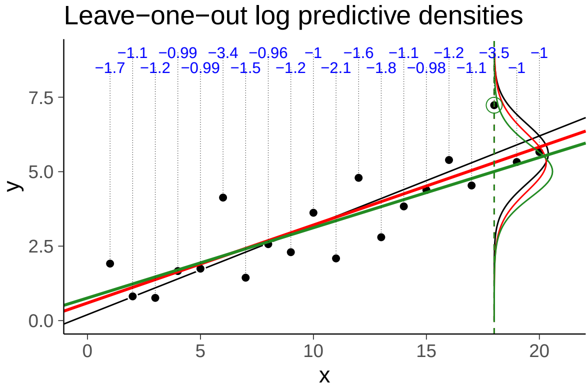
Leave−one−out predictive densities



$p(y_i|x_i, x_{-i}, y_{-i}), \quad i = 1, \ldots, 20$



$$\log p(y_i | x_i, x_{-i}, y_{-i}), \quad i = 1, \ldots, 20$$



$\sum_{i=1}^{20} \log p(y_i|x_i, x_{-i}, y_{-i}) \approx -29.5$

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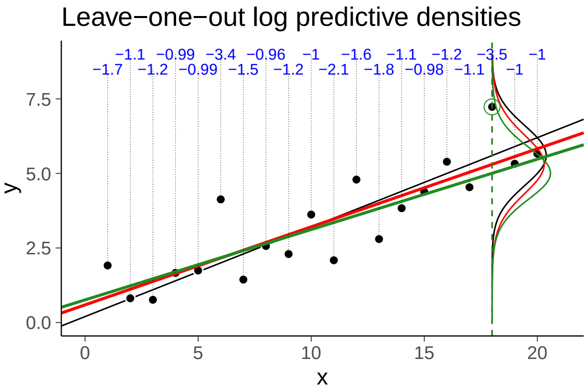
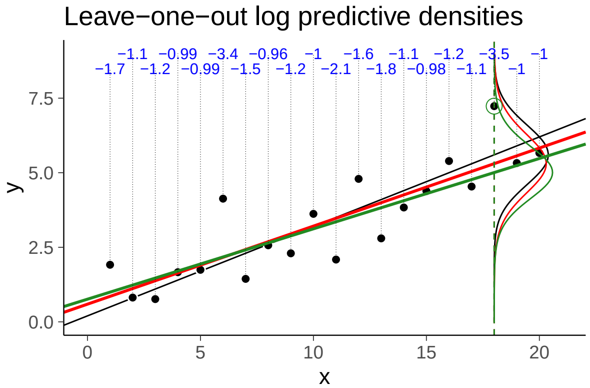
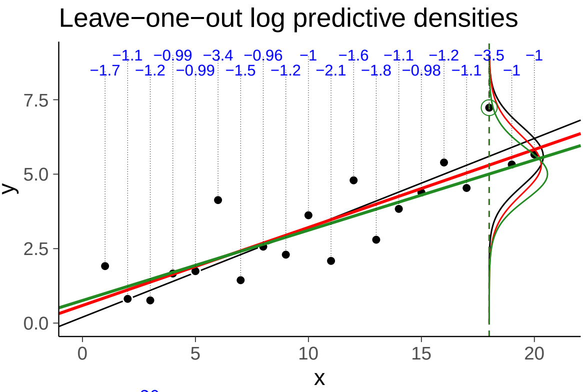


$$\text{elpd\_loo} = \sum_{i=1}^{20} \log p(y_i|x_i, x_{-i}, y_{-i}) \approx -29.5$$



$\text{elpd\_loo} = \sum_{i=1}^{20} \log p(y_i|x_i, x_{-i}, y_{-i}) \approx -29.5$

unbiased estimate of log posterior pred. density for new data



$\text{elpd\_loo} = \sum_{i=1}^{20} \log p(y_i|x_i, x_{-i}, y_{-i}) \approx -29.5$

$\text{lpd} = \sum_{i=1}^{20} \log p(y_i|x_i, x, y) \approx -26.8$

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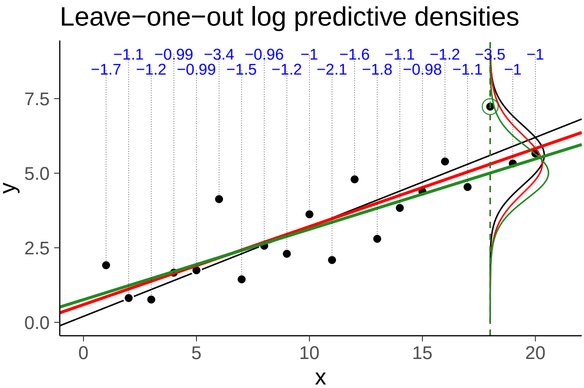
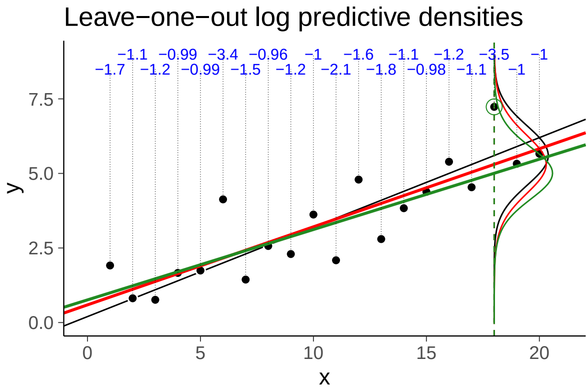


$$\text{elpd\_loo} = \sum_{i=1}^{20} \log p(y_i|x_i, x_{-i}, y_{-i}) \approx -29.5$$

$$\text{lpd} = \sum_{i=1}^{20} \log p(y_i|x_i, x, y) \approx -26.8$$

$$\text{p\_loo} = \text{lpd} - \text{elpd\_loo} \approx 2.7$$



$$\text{elpd\_loo} = \sum_{i=1}^{20} \log p(y_i|x_i, x_{-i}, y_{-i}) \approx -29.5$$

$$\text{SE} = \text{sd}(\log p(y_i|x_i, x_{-i}, y_{-i})) \cdot \sqrt{20} \approx 3.3$$

Stan and loo package

```
 Computed from 4000 by 20 log-likelihood matrix

          Estimate  SE
elpd_loo    -29.5 3.3
p_loo         2.7 1.0
------
Monte Carlo SE of elpd_loo is 0.1.

Pareto k diagnostic values:
                         Count Pct.    Min. n_eff
(-Inf, 0.5]   (good)     18    90.0%   899
 (0.5, 0.7]   (ok)        2    10.0%   459
   (0.7, 1]   (bad)       0     0.0%   <NA>
   (1, Inf)   (very bad)  0     0.0%   <NA>

All Pareto k estimates are ok (k < 0.7).
See help('pareto-k-diagnostic') for details.
```

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Subsection 1

When is LOO applicable

Pro and cons with LOO-CV

+ Intuitive
+ Robust
+ Good theoretical properties

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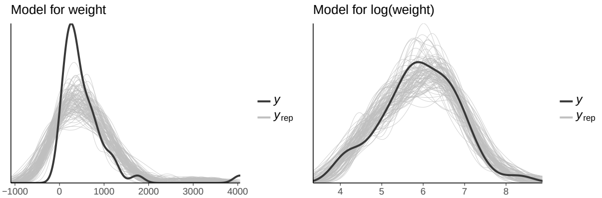
Pro and cons with LOO-CV

+ Intuitive
+ Robust
+ Good theoretical properties
- Can be costly (naive LOO-CV mean n posterior computations)

Sometimes cross-validation is not needed

- Posterior predictive checking can be sufficient



Predicting the yields of mesquite bushes.

Gelman, Hill & Vehtari (2020, Ch. 11)

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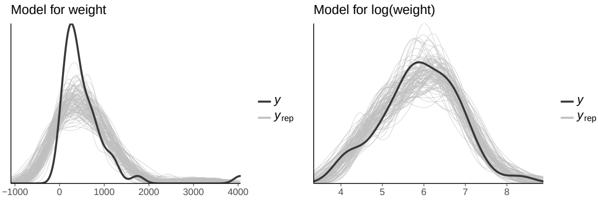

Predicting the yields of mesquite bushes.
Gelman, Hill & Vehtari (2020, Ch. 11)

- In nested case, often easier and more accurate to analyse posterior distribution of more complex model directly

Data generating mechanisms and prediction tasks

1. You have to make some assumptions on data generating mechanism p_{true}

$$\text{elpd}_M = \int \log p_M(\tilde{y}|y) p_{\text{true}}(\tilde{y}) d\tilde{y} \,,$$

2. Use the knowledge of the prediction task if available
3. Cross-validation can be used to analyse different parts, even if there is no clear prediction task

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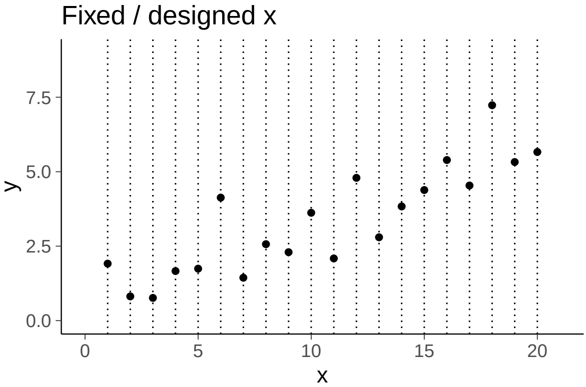
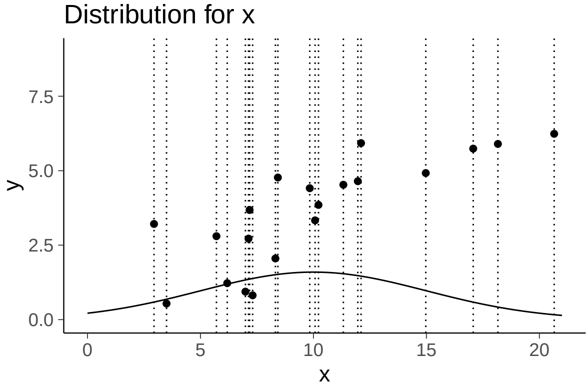
x in p_{true}



LOO is ok for fixed / designed x: $p_{\text{true}}(y|x)$

x in $p_{\rm true}$



LOO is ok for random x. $p_{\rm true}(y, x)$

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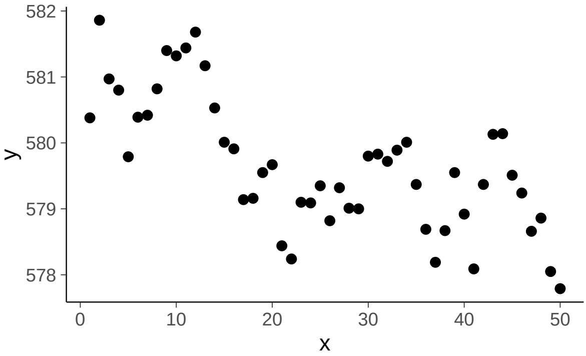
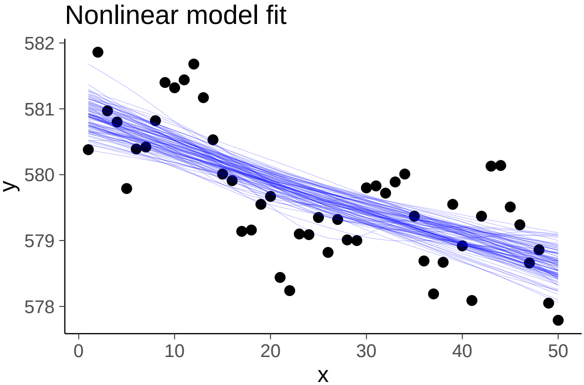
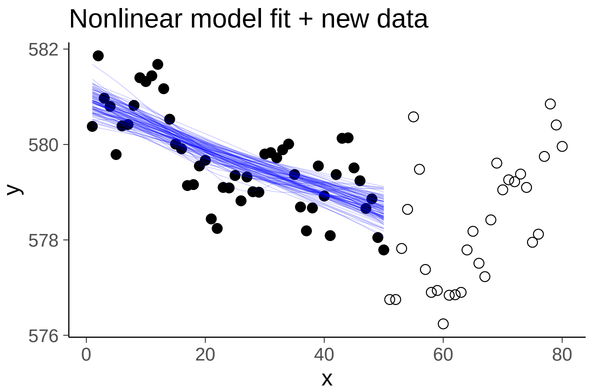
p_{true} extrapolation



p_{true} extrapolation

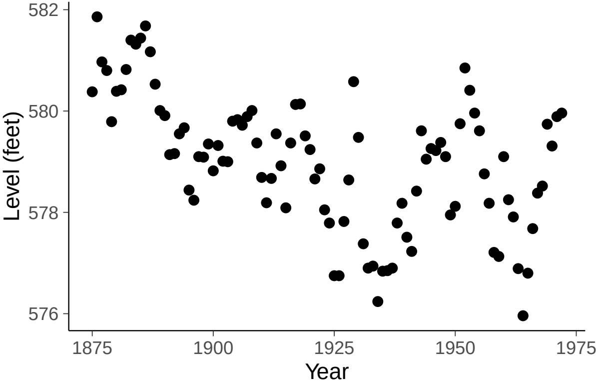


p_{true} extrapolation



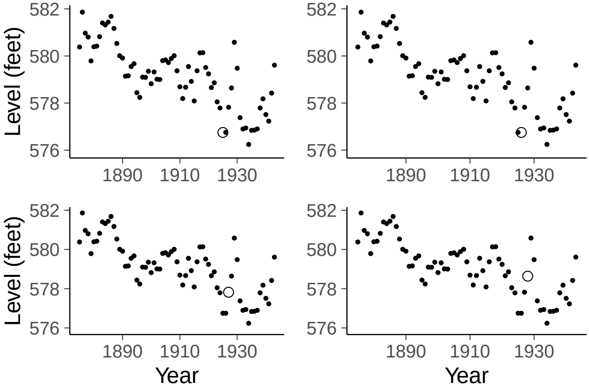
Extrapolation is more difficult

LOO for time series data



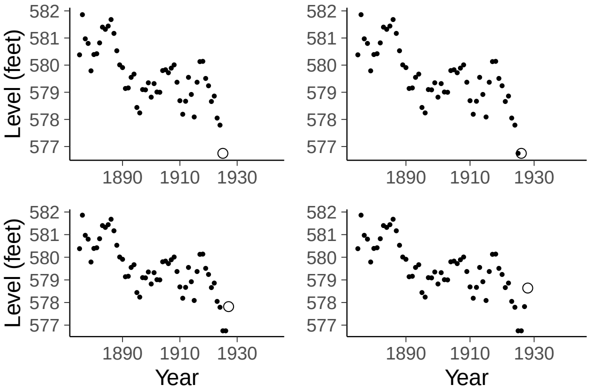
p_{true} for time series?

LOO for time series



Leave-one-out cross-validation is ok for assessing conditional model

LOO for time series



Leave-future-out cross-validation is better for predicting future

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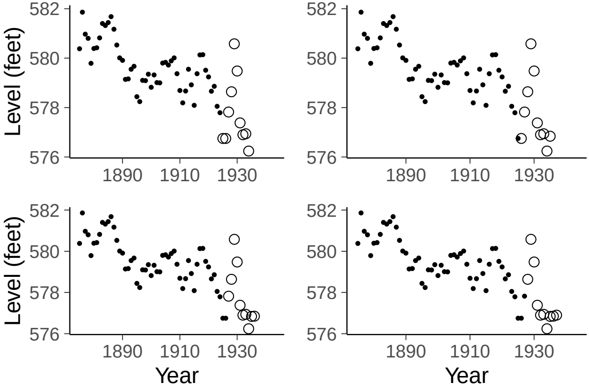
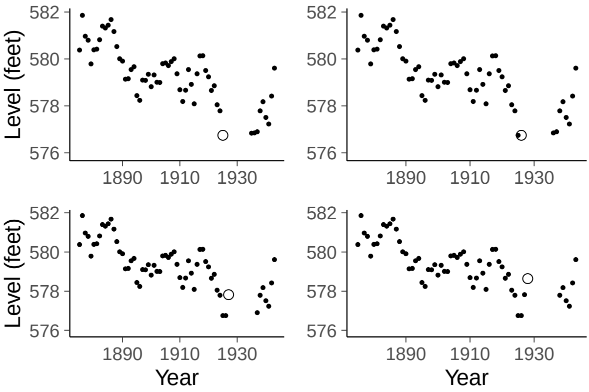
LOO for time series



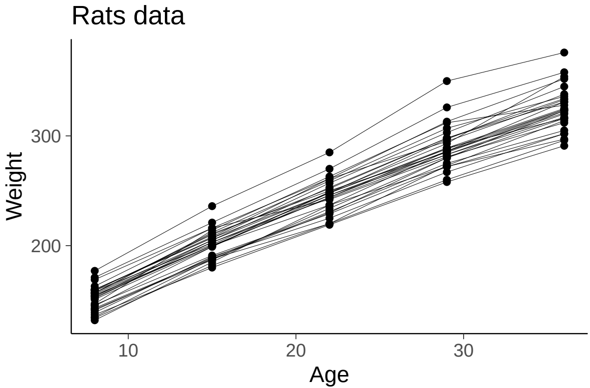
m-step-ahead cross-validation is better for predicting further future

LOO for time series



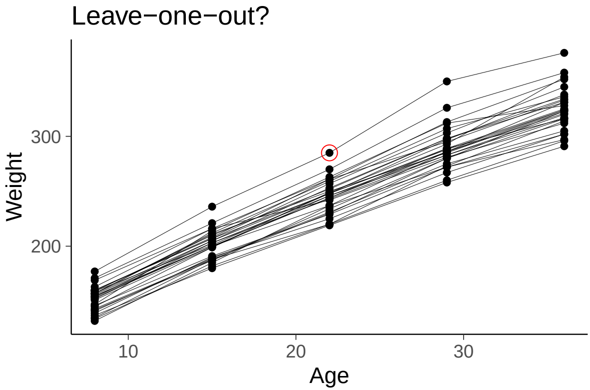
m-step-ahead leave-a-block-out cross-validation

LOO for hiearchical data



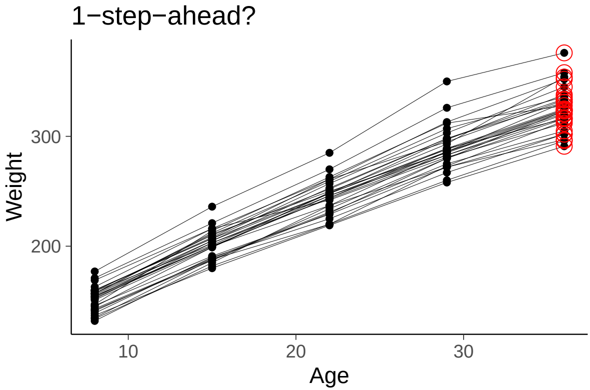
Can LOO or other cross-validation be used with hierarchical data?

LOO for hiearchical data



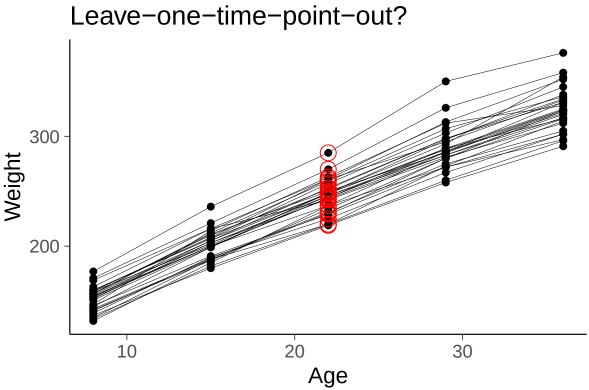
Yes!

LOO for hiearchical data



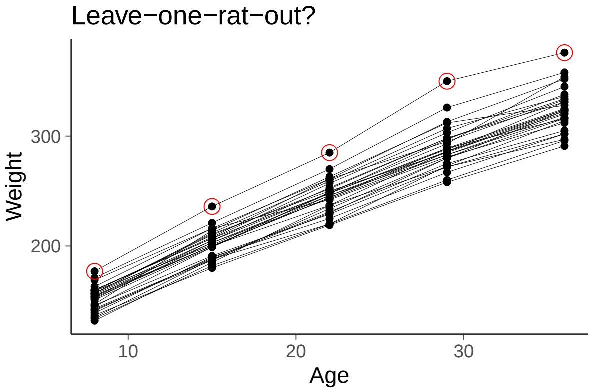
Yes!

LOO for hiearchical data



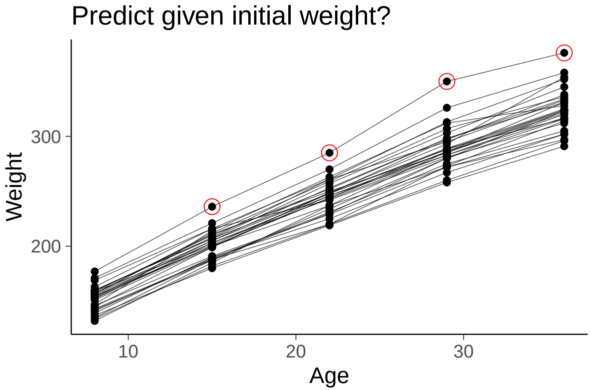
Yes!

LOO for hiearchical data



Yes!

LOO for hiearchical data



Yes!

Subsection 2

Fast LOO-CV

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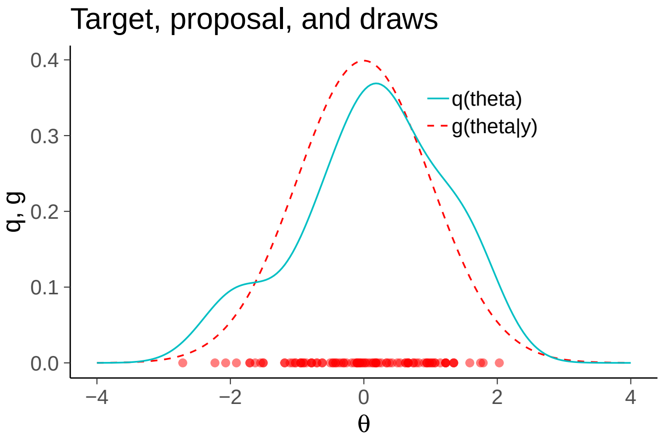
Fast cross-validation

1. Pareto smoothed importance sampling LOO (PSIS-LOO)

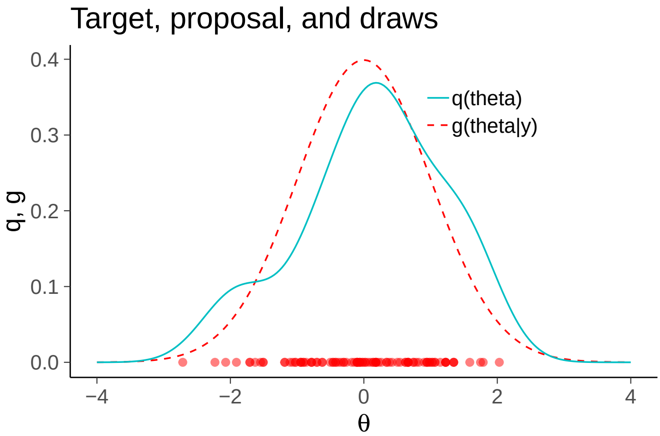
Fast cross-validation

1. Pareto smoothed importance sampling LOO (PSIS-LOO)
2. K-fold cross-validation

Recap: Importance sampling

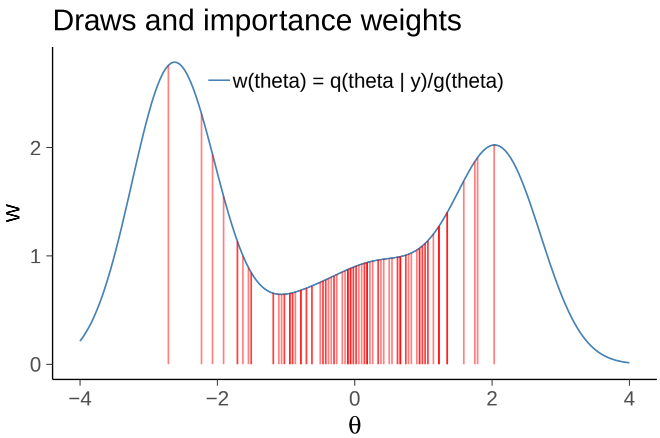


Recap: Importance sampling



$$\mathrm{E}[f(\theta)] \approx \frac{\sum_s w_s f(\theta^{(s)})}{\sum_s w_s}, \qquad \text{where} \quad w_s = \frac{q(\theta^{(s)})}{g(\theta^{(s)})}$$

Recap: Importance sampling



$$\mathrm{E}[f(\theta)] \approx \frac{\sum_s w_s f(\theta^{(s)})}{\sum_s w_s}, \qquad \text{where} \quad w_s = \frac{q(\theta^{(s)})}{g(\theta^{(s)})}$$

Importance sampling for LOO-CV

- Having samples θ^s from $p(\theta^s|D)$

$$p(\tilde{y}_i|x_i, D_{-i}) \approx \frac{\sum_{s=1}^{S} p(\tilde{y}_i|\theta^s) w_i^s}{\sum_{s=1}^{S} w_i^s},$$

where w_i^s are importance weights and

$$w_i^s = \frac{p(\theta^s|x_i, D_{-i})}{p(\theta^s|D)} \propto \frac{1}{p(y_i|\theta^s)}.$$

Importance sampling for LOO-CV

- Having samples θ^s from $p(\theta^s|D)$

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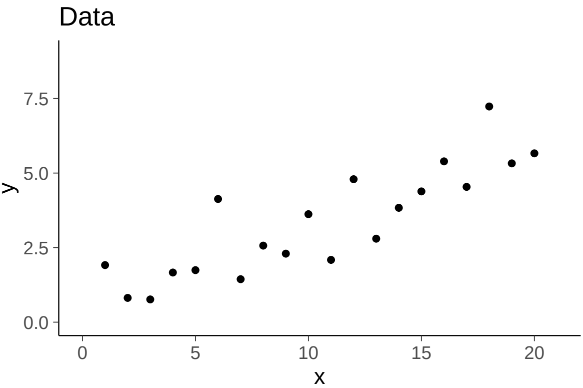
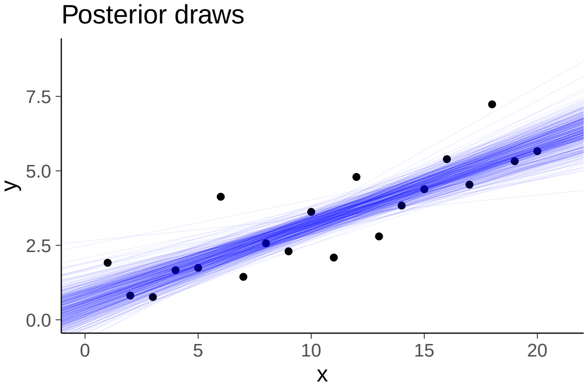
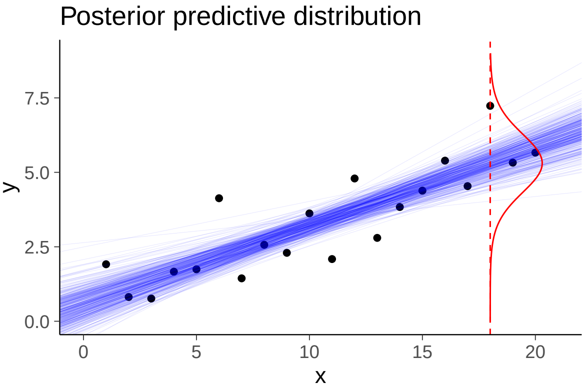
- If evaluated with $\tilde{y}_i = y_i$

$$p(y_i|x_i, D_{-i}) \approx \frac{1}{\sum_{s=1}^{S} \frac{1}{p(y_i|\theta^s)}},$$


Data
7.5
5.0
2.5
0.0
y
0
5
10
15
20
x



$$\theta^{(s)} \sim p(\theta|x, y)$$



$$\theta^{(s)} \sim p(\theta|x, y), \quad p(\tilde{y}|\tilde{x}, x, y) \approx \frac{1}{S}\sum_{s=1}^{S} p(\tilde{y}|\tilde{x}, \theta^{(s)})$$

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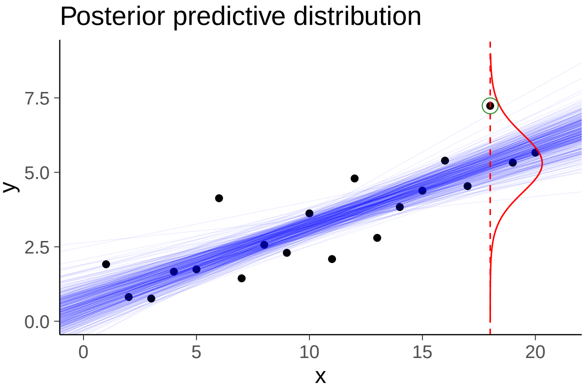

$$\theta^{(s)} \sim p(\theta|x, y), \quad p(\tilde{y}|\tilde{x}, x, y) \approx \frac{1}{S}\sum_{s=1}^{S} p(\tilde{y}|\tilde{x}, \theta^{(s)})$$

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- Model assessment and selection
  - Measures of predictive accuracy
  - Model selection
- Cross-validation
  - When is LOO applicable
  - Fast LOO-CV
  - K-fold cross-validation
  - Comparison and selection
  - Additional reading
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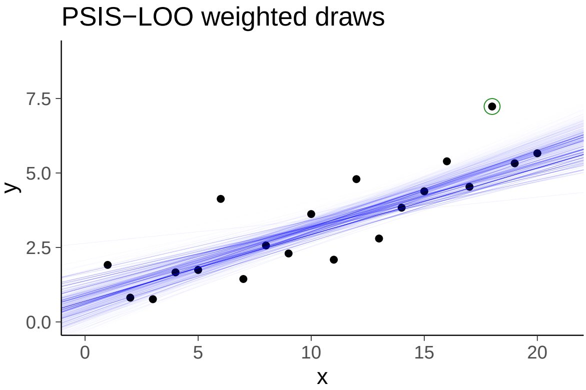
- Summary



$$\theta^{(s)} \sim p(\theta|x, y)$$

$$r_i^{(s)} = p(\theta^{(s)}|x_{-i}, y_{-i})/p(\theta^{(s)}|x, y)$$

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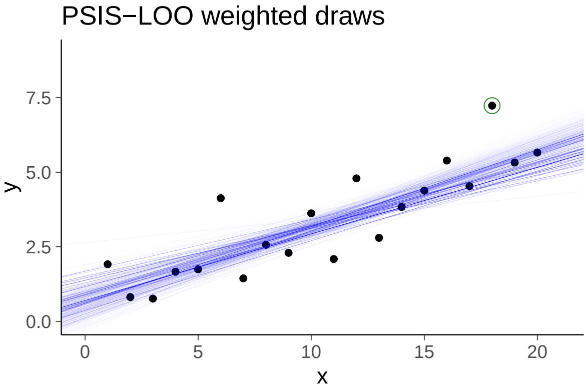
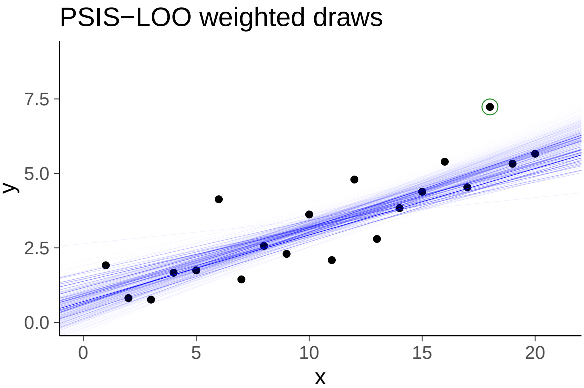


$$\theta^{(s)} \sim p(\theta|x,y)$$

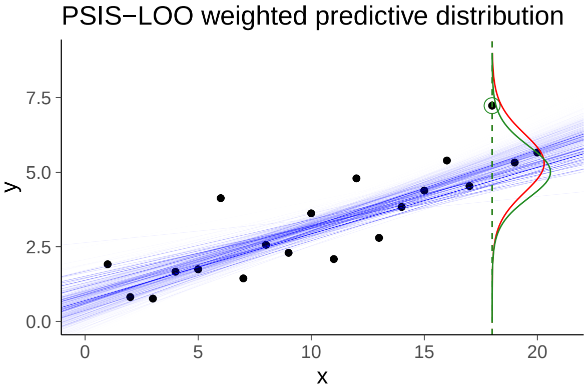
$$r_i^{(s)} = p(\theta^{(s)}|x_{-i},y_{-i})/p(\theta^{(s)}|x,y) \propto 1/p(y_i|x_i,\theta^{(s)})$$



$$\theta^{(s)} \sim p(\theta|x, y)$$

$$r_i^{(s)} = p(\theta^{(s)}|x_{-i}, y_{-i})/p(\theta^{(s)}|x, y) \propto 1/p(y_i|x_i, \theta^{(s)})$$

$$\log(1/p(y_i|x_i, \theta^{(s)})) = -\mathrm{log\_lik}[i]$$



$$\theta^{(s)} \sim p(\theta|x,y)$$
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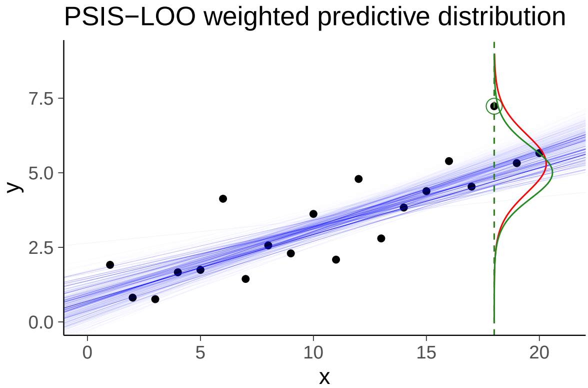

$$\theta^{(s)} \sim p(\theta|x, y)$$
$$r_i^{(s)} = p(\theta^{(s)}|x_{-i}, y_{-i})/p(\theta^{(s)}|x, y) \propto 1/p(y_i|x_i, \theta^{(s)})$$
$$p(y_i|x_i, x_{-i}, y_{-i}) \approx \sum_{s=1}^{S}[w_i^{(s)} p(y_i|x_i, \theta^{(s)})]$$

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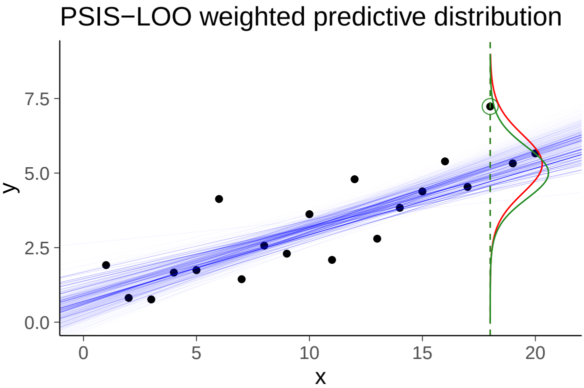
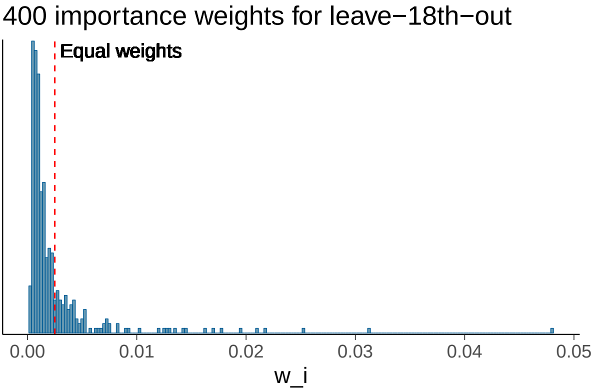
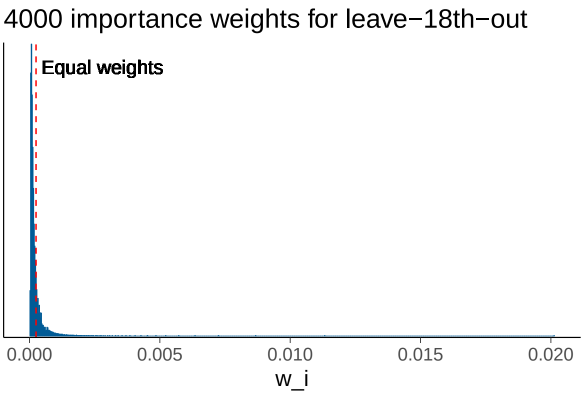


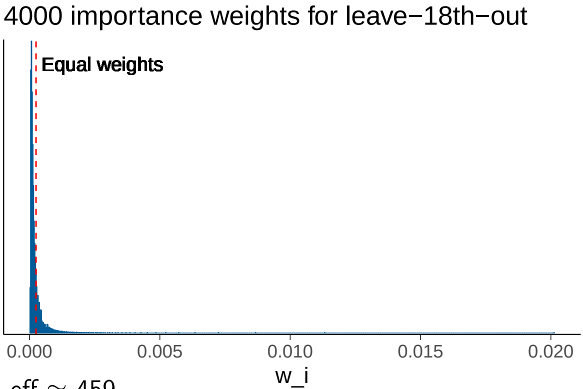
$\theta^{(s)} \sim p(\theta|x, y)$

$r_i^{(s)} = p(\theta^{(s)}|x_{-i}, y_{-i})/p(\theta^{(s)}|x, y) \propto 1/p(y_i|x_i, \theta^{(s)})$

$p(y_i|x_i, x_{-i}, y_{-i}) \approx \sum_{s=1}^{S}[w_i^{(s)} p(y_i|x_i, \theta^{(s)})]$, where $w \leftarrow \mathrm{PSIS}(r)$


400 importance weights for leave−18th−out
Equal weights
0.00
0.01
0.02
0.03
0.04
0.05
w_i


4000 importance weights for leave−18th−out
Equal weights
0.000
0.005
0.010
0.015
0.020
w_i



n_eff ≈ 459

see Vehtari, Gelman & Gabry (2017b)

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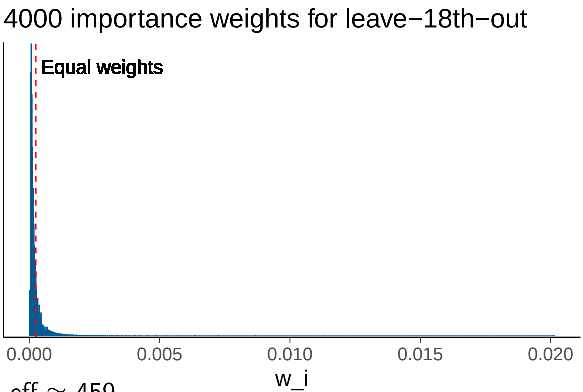

$n\_eff \approx 459$

Pareto $\hat{k} \approx 0.52$

1. Pareto $\hat{k}$ estimates the tail shape which determines the convergence rate of PSIS. Less than 0.7 is ok.

see Vehtari, Gelman & Gabry (2017b)

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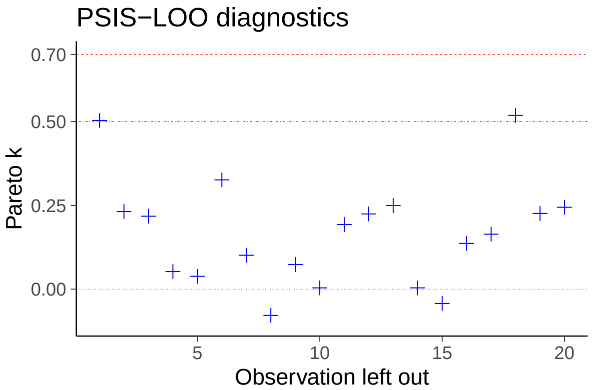
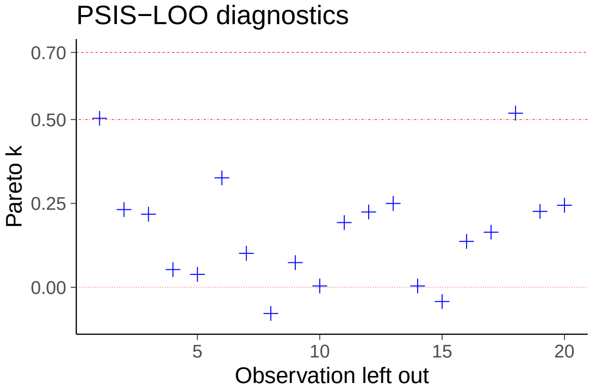
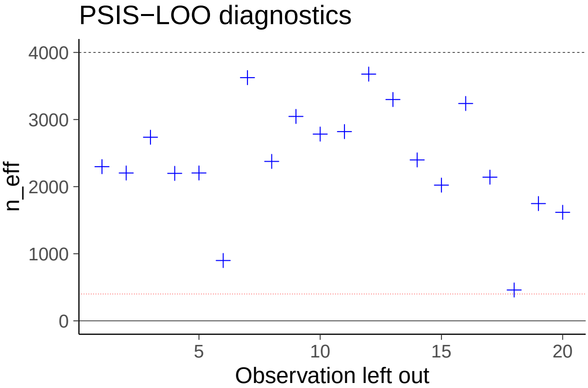





```
Pareto k diagnostic values:
                         Count  Pct.    Min. n_eff
(-Inf, 0.5]   (good)     18     90.0%   899
 (0.5, 0.7]   (ok)        2     10.0%   459
   (0.7, 1]   (bad)       0      0.0%   <NA>
   (1, Inf)   (very bad)  0      0.0%   <NA>
```




```
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```

loo package

```
 Computed from 4000 by 20 log-likelihood matrix

         Estimate  SE
elpd_loo    -29.5 3.3
p_loo         2.7 1.0

------
Monte Carlo SE of elpd_loo is 0.1.

Pareto k diagnostic values:
                         Count Pct.    Min. n_eff
(-Inf, 0.5]   (good)     18    90.0%   899
 (0.5, 0.7]   (ok)        2    10.0%   459
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   (1, Inf)   (very bad)  0     0.0%   <NA>

All Pareto k estimates are ok (k < 0.7).
See help('pareto-k-diagnostic') for details.
```

see more in Vehtari, Gelman & Gabry (2017b)

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Stan and loo package

```
 Computed from 4000 by 20 log-likelihood matrix

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All Pareto k estimates are ok (k < 0.7).
See help('pareto-k-diagnostic') for details.

Model comparison:
(negative 'elpd_diff' favors 1st model, positive favors 2nd)

elpd_diff        se
     -0.2       0.1
```

loo package

```
Computed from 4000 by 20 log-likelihood matrix

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p_loo         2.7 1.0

------
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All Pareto k estimates are ok (k < 0.7).
See help('pareto-k-diagnostic') for details.
```

Stan code

$$\log(r_i^{(s)}) = \log(1/p(y_i|x_i, \theta^{(s)})) = -\text{log\_lik}[i]$$

Stan code

$$\log(r_i^{(s)}) = \log(1/p(y_i|x_i, \theta^{(s)})) = -\text{log\_lik}[i]$$

```
...
model {
  alpha ~ normal(pmualpha, psalpha);
  beta ~ normal(pmubeta, psbeta);
  y ~ normal(mu, sigma);
}
generated quantities {
  vector[N] log_lik;
  for (i in 1:N)
    log_lik[i] = normal_lpdf(y[i] | mu[i], sigma);
}
```

Pareto smoothed importance sampling LOO

1. PSIS-LOO for hierarchical models
   1.1 leave-one-group out is challenging for PSIS-LOO

   see Merkel, Furr and Rabe-Hesketh (2018) for an approach using quadrature integration

Pareto smoothed importance sampling LOO

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2. PSIS-LOO for non-factorizable models
   2.1 mc-stan.org/loo/articles/loo2-non-factorizable.html

Pareto smoothed importance sampling LOO

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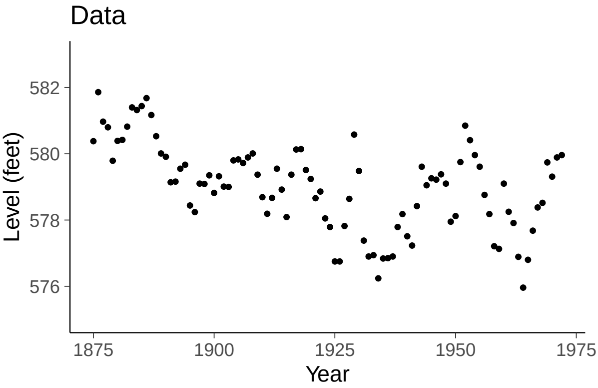
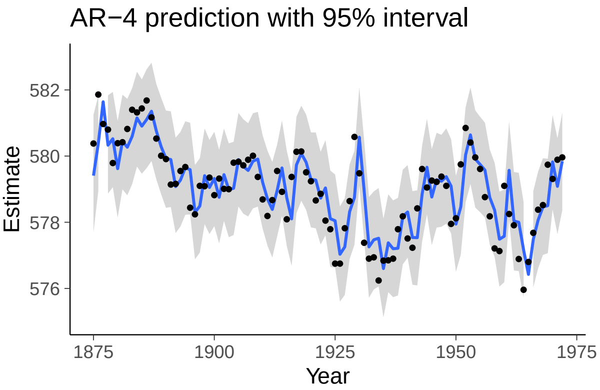
2. PSIS-LOO for non-factorizable models
   2.1 mc-stan.org/loo/articles/loo2-non-factorizable.html
3. PSIS-LOO for time series
   3.1 Approximate leave-future-out cross-validation
   mc-stan.org/loo/articles/loo2-lfo.html

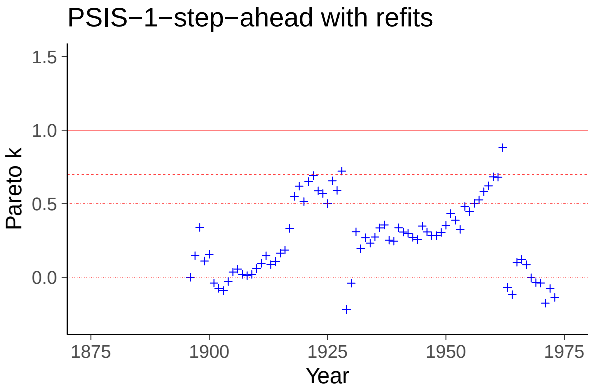
PSIS-LOO for time series



PSIS-LOO for time series



PSIS-LOO for time series



mc-stan.org/loo/articles/loo2-lfo.html

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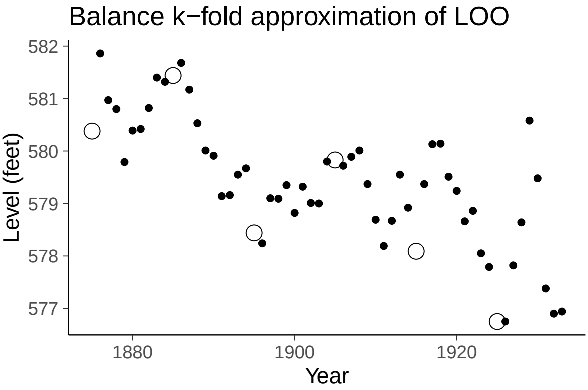
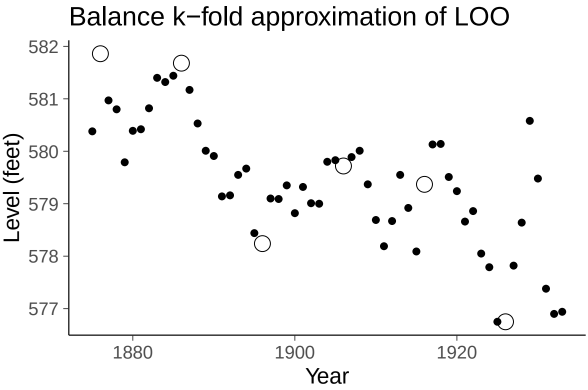
K-fold cross-validation

1. K-fold cross-validation can approximate LOO
   1.1 all uses for LOO
2. K-fold cross-validation can be used for hierarchical models
   2.1 good for leave-one-group-out
3. K-fold cross-validation can be used for time series
   3.1 with leave-block-out


Balance k−fold approximation of LOO
Level (feet)
582
581
580
579
578
577
1880
1900
1920
Year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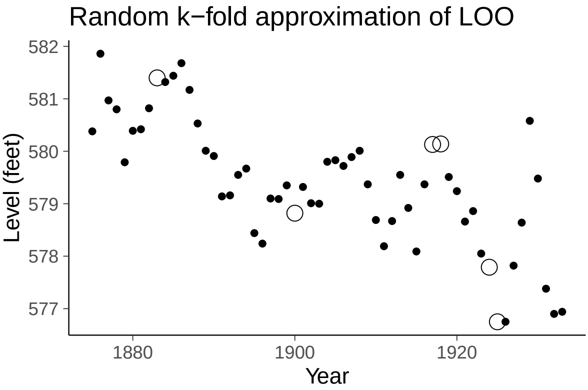
Random k−fold approximation of LOO
582
581
580
579
578
577
Level (feet)
1880
1900
1920
Year

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- Model assessment and selection
  - Measures of predictive accuracy
  - Model selection
- Cross-validation
  - When is LOO applicable
  - Fast LOO-CV
  - K-fold cross-validation
  - Comparison and selection
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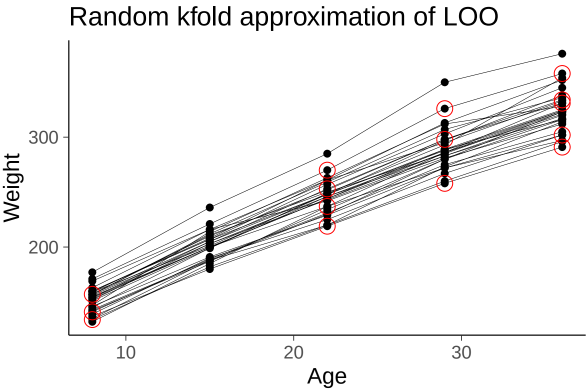

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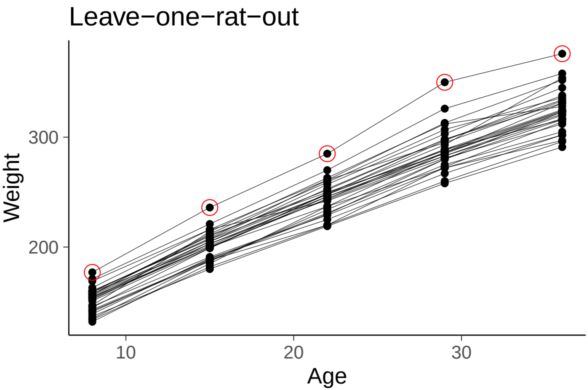

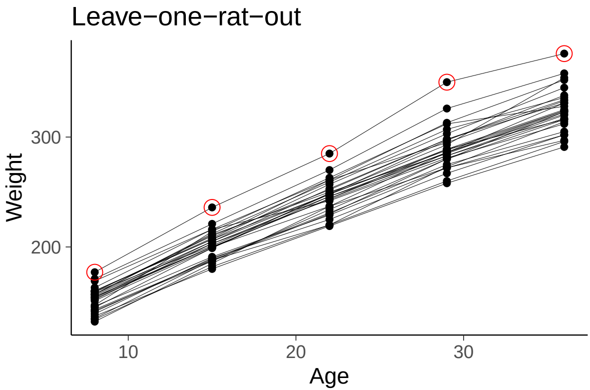

kfold_split_random()

kfold_split_balanced()

kfold_split_stratified()

Cross-validation for model assessment

1. CV is good for model assessment when application specific utility/cost functions are used
   1.1 e.g. 90% absolute error

Cross-validation for model assessment

1. CV is good for model assessment when application specific utility/cost functions are used
   1.1 e.g. 90% absolute error
2. Also useful in model checking in similar way as posterior predictive checking (PPC)
   2.1 model misspecification diagnostics (e.g. Pareto-k and p_loo)
   2.2 checking calibration of leave-one-out predictive posteriors (ppc_loo_pit in bayesplot)

see demos avehtari.github.io/modelselection/

Model comparison

1. "A popular hypothesis has it that primates with larger brains produce more energetic milk, so that brains can grow quickly" (from Statistical Rethinking)
   1.1 Model 1: formula $=$ kcal.per.g $\sim$ neocortex
   1.2 Model 2: formula $=$ kcal.per.g $\sim$ neocortex $+$ log(mass)

mc-stan.org/loo/articles/loo2-example.html

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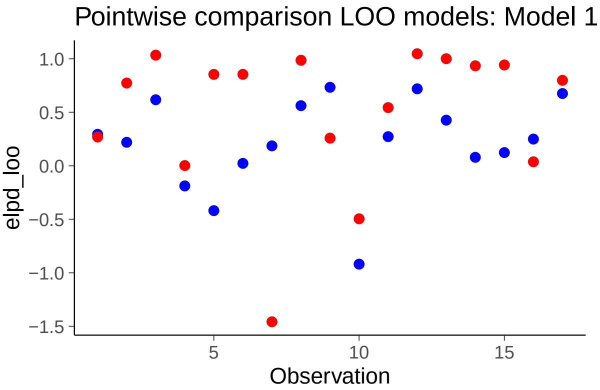

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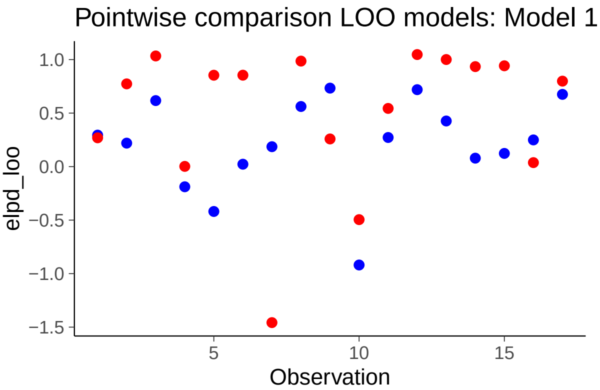
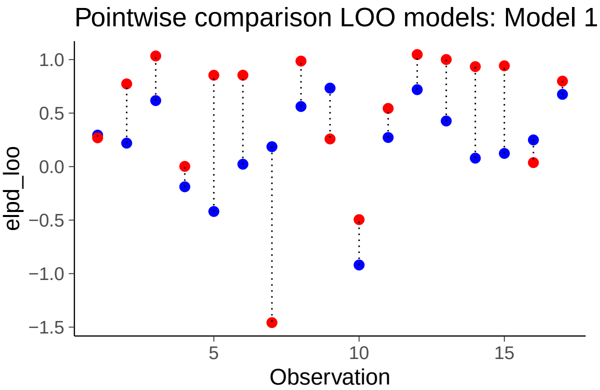


Model 1 elpd_loo ≈ 3.7, SE$=1.8$
Model 2 elpd_loo ≈ 8.4, SE$=2.8$



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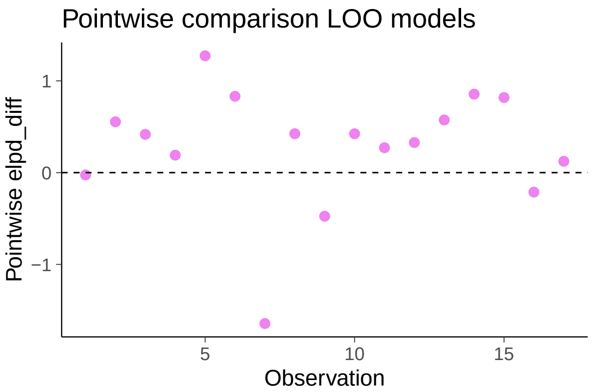
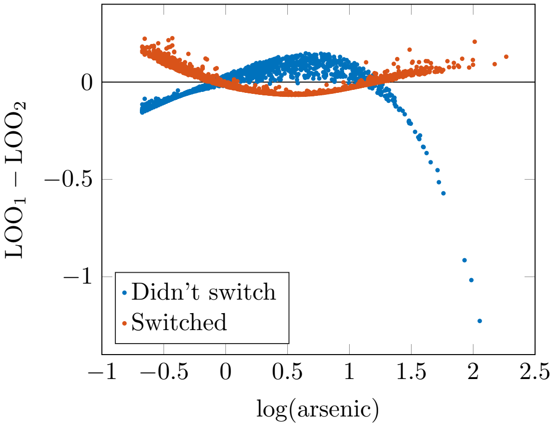



```
Model comparison:
(negative 'elpd_diff' favors 1st model, positive favors 2nd)

elpd_diff        se
      4.7       2.7
```

Arsenic well example – Model comparison



An estimated difference in $\mathrm{elpd}_{\mathrm{loo}}$ of 16.4 with SE of 4.4.

see Vehtari, Gelman & Gabry (2017a)

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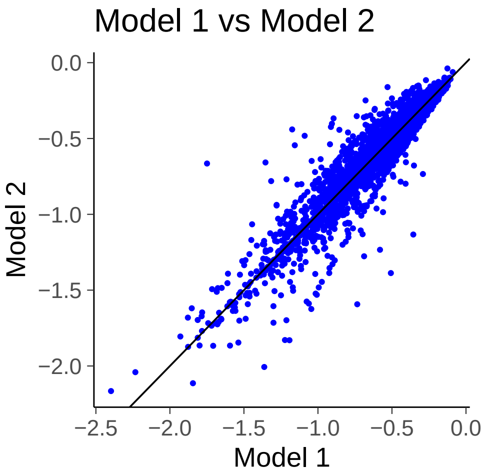
Arsenic well example – Model comparison

1. Probability of switching well with high arsenic level in rural Bangladesh
   1.1 Model 1 covariates: log(arsenic) and distance
   1.2 Model 2 covariates: log(arsenic), distance and education level

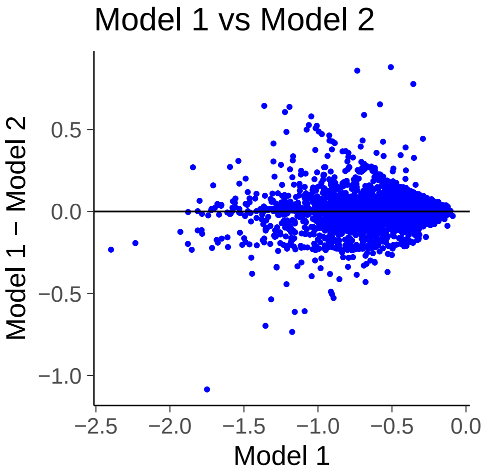
Gelman, Hill & Vehtari (2020): Regression and Other Stories, Chapter 13.

Arsenic well example – Model comparison



Model 1 elpd_loo ≈ -1952, SE=16
Model 2 elpd_loo ≈ -1938, SE=17

Arsenic well example – Model comparison




```
> loo_compare(model1, model2)
       elpd_diff  se_diff
model2    0.0       0.0
model1  -14.4       6.1
```

see Vehtari, Gelman & Gabry (2017a)

Arsenic well example – Model comparison

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`se_diff` and normal approximation for the uncertainty in the difference is good only if models are well specified and the number of observations is relatively big (more details in a forthcoming article).

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What if one is not clearly better than others?

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1. Continuous expansion including all models?
   1.1 and then analyse the posterior distribution directly
   avehtari.github.io/modelselection/betablockers.html
   1.2 sparse priors like regularized horseshoe prior instead of variable selection
   video, refs and demos at
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4. In a nested case choose more complex if you want to take into account all the uncertainties.
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Cross-validation and model selection

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   2.1 selection process leads to overfitting
3. Overfitting in selection process is not unique for cross-validation

Selection induced bias and overfitting

- Selection induced bias in cross-validation
  - same data is used to assess the performance and make the selection
  - the selected model fits more to the data
  - the CV estimate for the selected model is biased
  - recognized already, e.g., by Stone (1974)

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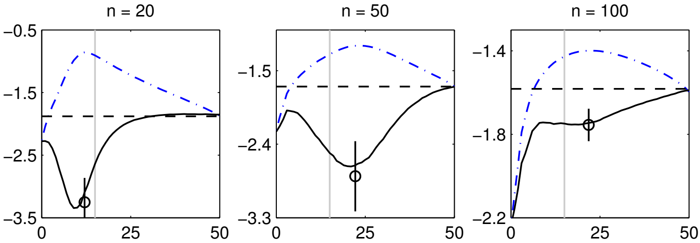
- Performance of the selection process itself can be assessed using two level cross-validation, but it does not help choosing better models
- Bigger problem if there is a large number of models as in covariate selection

Selection induced bias in variable selection



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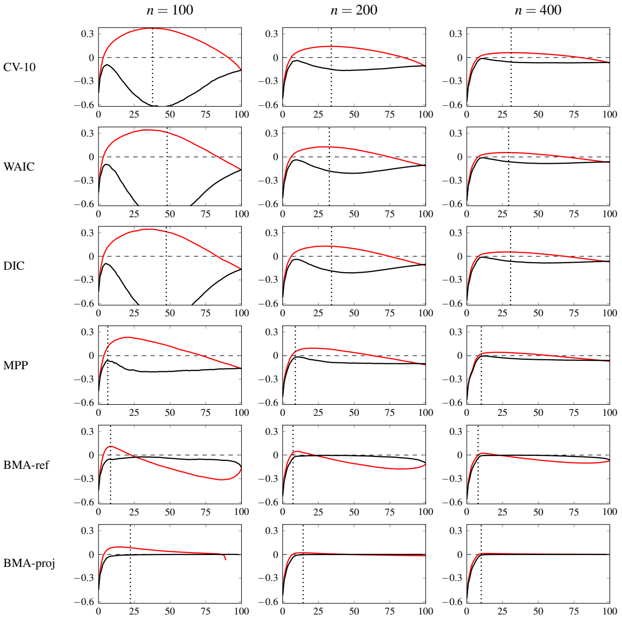

Section 3

Information criteria

WAIC vs PSIS-LOO

see Vehtari, Gelman & Gabry (2017a)

WAIC vs PSIS-LOO

1. WAIC has same assumptions as LOO

see Vehtari, Gelman & Gabry (2017a)

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5. Multiplying by -2 doesn't give any benefit (Watanabe didn't multiply by -2)

see Vehtari, Gelman & Gabry (2017a)

*IC

1. AIC uses maximum likelihood estimate for prediction
2. DIC uses posterior mean for prediction
3. BIC is an approximation for marginal likelihood
4. TIC, NIC, RIC, PIC, BPIC, QIC, AICc, ...

Section 4

Model averaging

Section 5

Summary

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Take-home messages

1. It's good to think predictions of observables, because observables are the only ones we can observe
2. Cross-validation can simulate predicting and observing new data
3. Cross-validation is good if you don't trust your model
4. Different variants of cross-validation are useful in different scenarios
5. Cross-validation has high variance, and **if** you trust your model you can beat cross-validation in accuracy

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