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- Bayesian decision theory
- Examples
- Course Evaluation

Bayesian Statistics and Data Analysis
Lecture 9

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Thanks to Aki Vehtari, Aalto University

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Section 1

Bayesian decision theory

Bayesian decision theory

- Potential decisions (or actions) $d \in D$ (a)

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$$E[U(x)|d] = \int U(x)p(x|d)dx$$

- Optimal decision: d^*, which maximizes the expected utility

$$d^* = \arg\max_d E[U(x)|d]$$

Integrating inference and decisions

- To make an optimal decision we need $p(x|d)$

Integrating inference and decisions

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Integrating inference and decisions

- To make an optimal decision we need $p(x|d)$
- In many situations we can approximate $p(x|d) \approx p(x)$
- The benefit of Bayesian inference: We can use $p(x|d, y)$ i.e. integrating data in decision making

Challenges in decision making

- Actual utility functions are rarely linear

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- What is the cost of human life?

Challenges in decision making

- Actual utility functions are rarely linear
- What is the cost of human life?
- Multipel parties having different utilities

Section 2

Examples

Example of decision making: 2 choices

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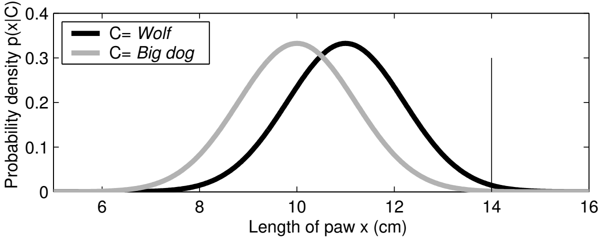
- Helen is going to pick mushrooms in a forest, while she notices a paw print which could made by a dog or a wolf

Example of decision making: 2 choices

- Helen is going to pick mushrooms in a forest, while she notices a paw print which could made by a dog or a wolf
- Helen measures that the length of the paw print is 14 cm and goes home to Google how big paws dogs and wolves have, and tries then to infer which animal has made the paw print



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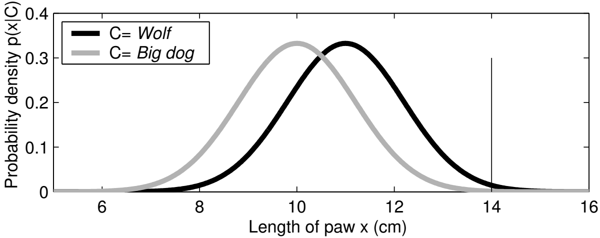

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- Likelihood of wolf is 0.92 (alternative being dog)

Example of decision making

- Helen assumes also that in her living area there are about one hundred times more free running dogs than wolves, that is *a priori* probability for wolf, before observation is 1%.

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- Likelihood and posterior

Animal	Likelihood	Posterior probability
Wolf	0.92	0.10
Dog	0.08	0.90

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Animal	Likelihood	Posterior probability
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Dog	0.08	0.90

- Posterior probability of wolf is 10%

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Decision d	Wolf	Dog
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Go to the forest	-1000	1

Utility matrix $U(x)$

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Utility matrix $U(x)$

Action d	Conditional utility $E[U(x)\|d]$
Stay home	0
Go to the forest	-100+0.9

Utilities for different actions

Example of decision making

- Maximum likelihood decision would be to assume that there is a wolf

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- Example illustrates that the uncertainties (probabilities) related to all consequences need to be carried on until final decision making

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Multi-stage decision making (Section 9.3)

- 95 year old has a tumor that is malignant with 90% probability

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  - expected lifetime is 34.8 months if no cancer

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  - Surgery: 0.35*0 + 0.65*(0.9*20.3 + 0.1*34.8 - 1) = 13.5mo

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  - No treatment: 0.9*5.6 + 0.1*34.8 = 8.5mo
- See the book for continuation of the example with additional test for cancer

Design of experiment

- Which experiment would give most additional information
  - decide values x_{n+1} for the next experiment
  - which values of x_{n+1} would reduce the posterior uncertainty most
- Example
  - Imagine that in bioassay the posterior uncertainty of LD50 is too large
  - which dose should be used in the next experiment to reduce the variance of LD50 as much as possible ?
    - this way less experiments need to be made (and less animals need to be killed)

Model selection as decision problem

- Evaluate how model M generalizes to unseen data $\tilde{y}$ (the *expected log predictive density*):

$$\text{elpd}_M = \int \log p_M(\tilde{y}|y) p_{\text{true}}(\tilde{y}) d\tilde{y} \,,$$

  where $\tilde{y}$ is an unseen observation generated from the true data generating process $p_{\text{true}}(\tilde{y})$, and y are observed data.
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- Formally, our decision is to choose the model $M^\star$.

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Section 3

Course Evaluation

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Course Evaluation

- What was good? What was fun?
- What can be improved? What was annoying?
- Did you get what you expected?
- How can I get you to speak more during class?