MA2501 - Assignment 2

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Problem 1

Part 1

Show that the function $f(x) = (x+1)(x-1)/3$ has a unique fixed point in the interval $[-1, 1]$. What can you say about the interval $[3, 4]$?

The definition of a fixed point is a point where $f(x) = x$ for a function f. This is equivalent with $f(x) - x = 0$. Observe that

$$\begin{aligned} f(x) - x &= \frac{(x+1)(x-1)}{3} - x \\ &= \frac{x^2 - 1}{3} - x \\ &= \frac{x^2 - 3x - 1}{3} \end{aligned}$$

Let $g(x) = f(x) - x$. Note that

$$\begin{aligned} g(-1) &= \frac{1+3-1}{3} > 0 \\ g(1) &= \frac{1-3-1}{3} < 0 \end{aligned}$$

As f (and therefore also g) is continuous, $g(-1) > 0$ and $g(1) < 0$, we have from the Intermediate Value Theorem that g has at least one zero on the interval $[-1, 1]$. This means that $f(x) = x$ for at least one $x \in [-1, 1]$, meaning that f has a unique fixed point in the interval.

For the interval $[3, 4]$, observe that

$$\begin{aligned} g(3) &= \frac{9-9-1}{3} < 0 \\ g(4) &= \frac{16-12-1}{3} > 0 \end{aligned}$$

g is still continuous, meaning that the Intermediate Value Theorem gives us the same result here - g has at least one zero on the interval $[3, 4]$, meaning that f has a unique fixed point in the interval.

Part 2

Compute the spectral radius of the matrices

$$T_1 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -1 & 0 & -1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{pmatrix}$$

The spectral radius is defined as the largest absolute value of its eigenvalues. We therefore start by finding the eigenvalues of the matrices.

$$\begin{aligned}
|T_1 - \lambda I| &= \begin{vmatrix} -\lambda & 1/2 & -1/2 \\ -1 & -\lambda & -1 \\ 1/2 & 1/2 & -\lambda \end{vmatrix} \\
&= -\lambda(\lambda^2 + 1/2) - 1/2(\lambda + 1/2) - 1/2(-1/2 + \lambda/2) \\
&= -\lambda^3 - \lambda/2 - \lambda/2 - 1/4 + 1/4 - \lambda/4 \\
&= -\lambda^3 - \frac{5}{4}\lambda \\
&= -\lambda(\lambda^2 + \frac{5}{4})
\end{aligned}$$

The roots of the characteristic polynomal, and thus the eigenvalues of T_1 are 0 and $\pm i\sqrt{\frac{5}{4}}$. The spectral radius of T_1 is given as

$$\begin{aligned}
\rho(T_1) &= \max\left(|0|, \left|i\sqrt{\frac{5}{4}}\right|, \left|-i\sqrt{\frac{5}{4}}\right|\right) \\
&= \max\left(0, \sqrt{\frac{5}{4}}, \sqrt{\frac{5}{4}}\right) \\
&= \sqrt{\frac{5}{4}}
\end{aligned}$$

We do the same for T_2:

$$\begin{aligned}
|T_2 - \lambda I| &= \begin{vmatrix} -\lambda & 1/2 & -1/2 \\ 0 & -1/2 - \lambda & -1/2 \\ 0 & 0 & -1/2 - \lambda \end{vmatrix} \\
&= -\lambda(-1/2 - \lambda)(-1/2 - \lambda) - 0 + 0 \\
&= -\lambda(1/2 + \lambda)(1/2 + \lambda)
\end{aligned}$$

The roots of the characteristic polynomal, and thus the eigenvalues of T_1 are 0 and $\pm\frac{1}{2}$. The spectral radius of T_2 is given as

$$\begin{aligned}
\rho(T_2) &= \max\left(|0|, \left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|\right) \\
&= \max\left(0, \frac{1}{2}, \frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Part 3

Show that for any matrix $A \in \mathbb{R}^{nxn}$

$$\|A\|_F := \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$$

defines a matrix norm (the so-called Frobenius norm.) Use the Cauchy-Schwarz inequality to show that for any matrix $A \in \mathbb{R}^{nxn}$ and any vector $x \in \mathbb{R}^n$

$$\|Ax\|_2 \leq \|A\|_F \|x\|_2.$$

We need to show four properties that must be satisfied for $\|A\|_F$ to be considered a matrix norm:

1. $\|A\|_F \geq 0$
2. $\|A\|_F = 0 \Leftrightarrow A = 0_{n,n}$
3. $\|\alpha A\|_F = |\alpha| \|A\|_F$ for scalar α
4. $\|A + B\|_F \leq \|A\|_F + \|B\|_F$ for other matrix B

1. $\|A\|_F \geq 0$

A squared scalar is always ≥ 0. This gives the following results:

$$\begin{aligned} |a_{ij}|^2 &\geq 0 \\ \Rightarrow \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right) &\geq 0 \\ \Rightarrow \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} &\geq 0 \\ \Rightarrow \|A\|_F &\geq 0 \end{aligned}$$

2. $\|A\|_F = 0 \Leftrightarrow A = 0_{n,n}$

We know from (1) that $|a_{ij}|^2 \geq 0$. This means that $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right) \geq |a_{ij}|^2$.

Assume $\|A\|_F = 0$. Then

$$\begin{aligned} \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} &= 0 \\ \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 &= 0 \\ \Rightarrow |a_{ij}|^2 &\leq 0 \end{aligned}$$

As we know from (1) that $|a_{ij}|^2 \geq 0$. This means that

$$\begin{aligned} |a_{ij}|^2 &= 0 \\ \Rightarrow |a_{ij}| &= 0 \\ \Rightarrow A &= 0_{n,n} \end{aligned}$$

3. $\|\alpha A\|_F = |\alpha|\|A\|_F$ for scalar α

Observe that

$$
\begin{aligned}
\|\alpha A\|_F &= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha a_{ij}|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(|a_{ij}||\alpha|)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2|\alpha|^2\right)^{1/2} \\
&= \left(|\alpha|^2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} \\
&= |\alpha|\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} \\
&= |\alpha|\|A\|_F
\end{aligned}
$$

4. $\|A+B\|_F \leq \|A\|_F + \|B\|_F$ for other matrix B

Observe that

$$
\begin{aligned}
\|A+B\|_F &= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}+b_{ij}|^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + 2|a_{ij}||b_{ij}| + |b_{ij}|^2\right)^{1/2}
\end{aligned}
$$

Because $|a_{ij}| \geq 0$ and $|b_{ij}| \geq 0$, $2|a_{ij}||b_{ij}| \geq 0$, and we have

$$
\begin{aligned}
\|A+B\|_F &\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + 2|a_{ij}||b_{ij}| + |b_{ij}|^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + |b_{ij}|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} + \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\right)^{1/2} \\
&= \|A\|_F + \|B\|_F
\end{aligned}
$$

Use the Cauchy-Schwarz inequality to show that for any matrix $A \in \mathbb{R}^{nxn}$ and any vector $x \in \mathbb{R}^n$

$$\|Ax\|_2 \leq \|A\|_F \|x\|_2.$$

Note that we can write $\|Ax\|_2$ as

$$\|Ax\|_2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} h_j \right|^2$$

Using the Cauchy-Schwarz inequality, observe that

$$\begin{aligned}
\|Ax\|_2 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} h_j \right|^2 \\
&\leq \sum_{i=1}^{n} \left\{ \left(\sum_{j=1}^{n} |a_{ij}|^2 \right) \left(\sum_{j=1}^{n} |h_j|^2 \right) \right\} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \|x\|_2 \\
&= \|A\|_F \|x\|_2
\end{aligned}$$