MA2501 - Assignment 2

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Problem 1

Part 1

Show that the function $f(x) = (x+1)(x-1)/3$ has a unique fixed point in the interval $[-1, 1]$. What can you say about the interval $[3, 4]$?

The definition of a fixed point is a point where $f(x) = x$ for a function f. This is equivalent with $f(x) - x = 0$. Observe that

$$\begin{aligned} f(x) - x &= \frac{(x+1)(x-1)}{3} - x \\ &= \frac{x^2-1}{3} - x \\ &= \frac{x^2-3x-1}{3} \end{aligned}$$

Let $g(x) = f(x) - x$. Note that

$$\begin{aligned} g(-1) &= \frac{1+3-1}{3} > 0 \\ g(1) &= \frac{1-3-1}{3} < 0 \end{aligned}$$

As f (and therefore also g) is continuous, $g(-1) > 0$ and $g(1) < 0$, we have from the Intermediate Value Theorem that g has at least one zero on the interval $[-1, 1]$. This means that $f(x) = x$ for at least one $x \in [-1, 1]$, meaning that f has a unique fixed point in the interval.

For the interval $[3, 4]$, observe that

$$\begin{aligned} g(3) &= \frac{9-9-1}{3} < 0 \\ g(4) &= \frac{16-12-1}{3} > 0 \end{aligned}$$

g is still continuous, meaning that the Intermediate Value Theorem gives us the same result here - g has at least one zero on the interval $[3, 4]$, meaning that f has a unique fixed point in the interval.

Part 2

Compute the spectral radius of the matrices

$$T_1 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -1 & 0 & -1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{pmatrix}$$

The spectral radius is defined as the largest absolute value of its eigenvalues. We therefore start by finding the eigenvalues of the matrices.

$$\begin{aligned} |T_1 - \lambda I| &= \begin{vmatrix} -\lambda & 1/2 & -1/2 \\ -1 & -\lambda & -1 \\ 1/2 & 1/2 & -\lambda \end{vmatrix} \\ &= -\lambda(\lambda^2 + 1/2) - 1/2(\lambda + 1/2) - 1/2(-1/2 + \lambda/2) \\ &= -\lambda^3 - \lambda/2 - \lambda/2 - 1/4 + 1/4 - \lambda/4 \\ &= -\lambda^3 - \frac{5}{4}\lambda \\ &= -\lambda(\lambda^2 + \frac{5}{4}) \end{aligned}$$

The roots of the characteristic polynomal, and thus the eigenvalues of T_1 are 0 and $\pm i\sqrt{\frac{5}{4}}$. The spectral radius of T_1 is given as

$$\begin{aligned} \rho(T_1) &= \max\left(|0|, \left|i\sqrt{\frac{5}{4}}\right|, \left|-i\sqrt{\frac{5}{4}}\right|\right) \\ &= \max\left(0, \sqrt{\frac{5}{4}}, \sqrt{\frac{5}{4}}\right) \\ &= \sqrt{\frac{5}{4}} \end{aligned}$$

We do the same for T_2:

$$\begin{aligned} |T_2 - \lambda I| &= \begin{vmatrix} -\lambda & 1/2 & -1/2 \\ 0 & -1/2 - \lambda & -1/2 \\ 0 & 0 & -1/2 - \lambda \end{vmatrix} \\ &= -\lambda(-1/2 - \lambda)(-1/2 - \lambda) - 0 + 0 \\ &= -\lambda(1/2 + \lambda)(1/2 + \lambda) \end{aligned}$$

The roots of the characteristic polynomal, and thus the eigenvalues of T_1 are 0 and $\pm\frac{1}{2}$. The spectral radius of T_2 is given as

$$\begin{aligned} \rho(T_2) &= \max\left(|0|, \left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|\right) \\ &= \max\left(0, \frac{1}{2}, \frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

Part 3

Show that for any matrix $A \in \mathbb{R}^{nxn}$

$$\|A\|_F := \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$$

defines a matrix norm (the so-called Frobenius norm.) Use the Cauchy-Schwarz inequality to show that for any matrix $A \in \mathbb{R}^{nxn}$ and any vector $x \in \mathbb{R}^n$

$$\|Ax\|_2 \leq \|A\|_F\|x\|_2.$$

We need to show four properties that must be satisfied for $\|A\|_F$ to be considered a matrix norm:

1. $\|A\|_F \geq 0$
2. $\|A\|_F = 0 \Leftrightarrow A = 0_{n,n}$
3. $\|\alpha A\|_F = |\alpha|\|A\|_F$ for scalar α
4. $\|A+B\|_F \leq \|A\|_F + \|B\|_F$ for other matrix B

1. $\|A\|_F \geq 0$

A squared scalar is always ≥ 0. This gives the following results:

$$\begin{aligned} |a_{ij}|^2 &\geq 0 \\ \Rightarrow \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right) &\geq 0 \\ \Rightarrow \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} &\geq 0 \\ \Rightarrow \|A\|_F &\geq 0 \end{aligned}$$

2. $\|A\|_F = 0 \Leftrightarrow A = 0_{n,n}$

We know from (1) that $|a_{ij}|^2 \geq 0$. This means that $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right) \geq |a_{ij}|^2$.

Assume $\|A\|_F = 0$. Then

$$\begin{aligned} \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} &= 0 \\ \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 &= 0 \\ \Rightarrow |a_{ij}|^2 &\leq 0 \end{aligned}$$

As we know from (1) that $|a_{ij}|^2 \geq 0$. This means that

$$\begin{aligned} |a_{ij}|^2 &= 0 \\ \Rightarrow |a_{ij}| &= 0 \\ \Rightarrow A &= 0_{n,n} \end{aligned}$$

3. $\|\alpha A\|_F = |\alpha|\|A\|_F$ for scalar α

Observe that

$$
\begin{aligned}
\|\alpha A\|_F &= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha a_{ij}|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(|a_{ij}||\alpha|)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2|\alpha|^2\right)^{1/2} \\
&= \left(|\alpha|^2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} \\
&= |\alpha|\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} \\
&= |\alpha|\|A\|_F
\end{aligned}
$$

4. $\|A+B\|_F \leq \|A\|_F + \|B\|_F$ for other matrix B

Observe that

$$
\begin{aligned}
\|A+B\|_F &= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}+b_{ij}|^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + 2|a_{ij}||b_{ij}| + |b_{ij}|^2\right)^{1/2}
\end{aligned}
$$

Because $|a_{ij}| \geq 0$ and $|b_{ij}| \geq 0$, $2|a_{ij}||b_{ij}| \geq 0$, and we have

$$
\begin{aligned}
\|A+B\|_F &\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + 2|a_{ij}||b_{ij}| + |b_{ij}|^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + |b_{ij}|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} + \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\right)^{1/2} \\
&= \|A\|_F + \|B\|_F
\end{aligned}
$$

Use the Cauchy-Schwarz inequality to show that for any matrix $A \in \mathbb{R}^{nxn}$ and any vector $x \in \mathbb{R}^n$

$$\|Ax\|_2 \leq \|A\|_F \|x\|_2.$$

Note that we can write $\|Ax\|_2$ as

$$\|Ax\|_2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} h_j \right|^2$$

Using the Cauchy-Schwarz inequality, observe that

$$\begin{aligned}
\|Ax\|_2 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} h_j \right|^2 \\
&\leq \sum_{i=1}^{n} \left\{ \left(\sum_{j=1}^{n} |a_{ij}|^2 \right) \left(\sum_{j=1}^{n} |h_j|^2 \right) \right\} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \|x\|_2 \\
&= \|A\|_F \|x\|_2
\end{aligned}$$

Problem 2

Consider the system

$$
\begin{aligned}
10x_1 - x_2 + 2x_3 &= 6 \\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\
2x_1 - x_2 - 10x_3 - x_4 &= -11 \\
3x_2 - x_3 + x_4 &= 15
\end{aligned}
$$

Find its exact solution. Write down the Jacobi iterative method and generate the first 3 entries, $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ in the sequence of approximations $\{x^{(n)}\}_{n>0}$, $x^{(0)} = (0,0,0,0)^T$. Repeat the approximation using the Gauss-Seidel iterative method.

We start by finding the exact solution of the matrix form of the system:

$$
\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & -10 & -1 \\ 0 & 3 & -1 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}
$$

$$
\begin{bmatrix} -1 & 11 & -1 & 3 & 25 \\ 10 & -1 & 2 & 0 & 6 \\ 2 & -1 & -10 & -1 & -11 \\ 0 & 3 & -1 & 1 & 15 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & -11 & 1 & -3 & -25 \\ 0 & 109 & -8 & 30 & 256 \\ 0 & 21 & -12 & 5 & 39 \\ 0 & 3 & -1 & 1 & 15 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 & -8/3 & 2/3 & 30 \\ 0 & 1 & -1/3 & 1/3 & 5 \\ 0 & 0 & 85/3 & -19/3 & -289 \\ 0 & 0 & -5 & -2 & -66 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 26/15 & 326/5 \\ 0 & 1 & 0 & 7/15 & 47/5 \\ 0 & 0 & 1 & 2/5 & 66/5 \\ 0 & 0 & 0 & -53/3 & -663 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 8/53 \\ 0 & 1 & 0 & 0 & -430/53 \\ 0 & 0 & 1 & 0 & -96/53 \\ 0 & 0 & 0 & 1 & 1989/53 \end{bmatrix}
$$

Which gives the exact solution

$$
\begin{aligned}
x_1 &= 8/53 \\
x_2 &= -430/53 \\
x_3 &= -96/53 \\
x_4 &= 1989/53
\end{aligned}
$$

Jacobi iterative method

Let b be the target vector $(6, 25, -11, 15)$, D be the diagonal of the matrix form of the system and $L + U =$ matrix form $-D$. With the Jacobi iterative method, we have that

$$x^{(k)} = D^{-1}\left(b - (L+U)x^{(k-1)}\right)$$
$$x_i^{(k)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j\neq i} a_{ij}x_j^{(k-1)}\right)$$

Remember that $x^{(0)} = (0, 0, 0, 0)^T$.

$$x^{(1)} = \begin{bmatrix} \frac{1}{10}(6 - (-1*0 + 2*0 + 0*0)) \\ \frac{1}{11}(25 - (-1*0 - 1*0 + 3*0)) \\ \frac{1}{-10}(-11 - (2*0 - 1*0 - 1*0)) \\ 1(15 - (0*0 + 3*0 - 1*0)) \end{bmatrix} = \begin{bmatrix} 6/10 \\ 25/11 \\ 11/10 \\ 15 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} \frac{1}{10}(6 - (-1*25/11 + 2*11/10 + 0*15)) \\ \frac{1}{11}(25 - (-1*6/10 - 1*11/10 + 3*15)) \\ \frac{1}{-10}(-11 - (2*6/10 - 1*25/11 - 1*15)) \\ 1(15 - (0*6/10 + 3*25/11 - 1*11/10)) \end{bmatrix} = \begin{bmatrix} 334/550 \\ -183/110 \\ -279/550 \\ 1021/110 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} \frac{1}{10}(6 - (-1*-183/110 + 2*-279/550 + 0*1021/110)) \\ \frac{1}{11}(25 - (-1*334/550 - 1*-279/550 + 3*1021/110)) \\ \frac{1}{-10}(-11 - (2*334/550 - 1*-183/110 - 1*1021/110)) \\ 1(15 - (0*334/550 + 3*-183/110 - 1*1021/110)) \end{bmatrix} = \begin{bmatrix} 2943/5500 \\ -151/605 \\ 632/1375 \\ 322/11 \end{bmatrix}$$

Gauss-Seidel iterative method

With the Gauss-Seidel iterative method, we have that

$$x_i^{(k)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{k-1}\right)$$

Remember that $x^{(0)} = (0, 0, 0, 0)^T$. Note that the Gauss-Seidel method computes each element $x_i^{(k)}$ one at a time, using the previously calculated elements for later calculations with the same k

We can compute the general formula for $x^{(k)}$ ahead of time:

$$x^{(k)} = \begin{bmatrix} \frac{1}{10}\left(6 - (0) - (-1*x_2^{(k-1)} + 2*x_3^{(k-1)} + 0*x_4^{(k-1)}))\right) \\ \frac{1}{11}\left(25 - (-1*x_1^{(k)}) - (-1*x_3^{(k-1)} + 3*x_4^{(k-1)})\right) \\ \frac{1}{-10}\left(-11 - (2*x_1^{(k)} - 1*x_2^{(k)}) - (1*x_4^{(k-1)})\right) \\ 1\left(15 - (0*x_1^{(k)} + 3*x_2^{(k)} - 1*x_3^{(k)})\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{10}\left(6 + x_2^{(k-1)} - 2*x_3^{(k-1)}\right) \\ \frac{1}{11}\left(25 + x_1^{(k)} + x_3^{(k-1)} - 3*x_4^{(k-1)}\right) \\ \frac{1}{-10}\left(-11 - 2*x_1^{(k)} + x_2^{(k)} + x_4^{(k-1)}\right) \\ \left(15 - 3*x_2^{(k)} + x_3^{(k)}\right) \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} \frac{1}{10}(6 + 0 - 2*0) \\ \frac{1}{11}(25 + 6/10 + 0 - 3*0) \\ \frac{1}{-10}(-11 - 2*6/10 + 128/55 + 0) \\ (15 - 3*128/55 + 543/550) \end{bmatrix} = \begin{bmatrix} 6/10 \\ 128/55 \\ 543/550 \\ 4953/550 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} \frac{1}{10}\left(6 + 128/55 - 2 * 543/550\right) \\ \frac{1}{11}\left(25 + 1747/2750 + 543/550 - 3 * 4953/550\right) \\ \frac{1}{-10}\left(-11 - 2 * 1747/2750 + 1181/6050 + 4953/550\right) \\ \left(15 - 3 * 1181/6050 + 0.306988\right) \end{bmatrix} = \begin{bmatrix} 1747/2750 \\ 1181/6050 \\ 0.306988 \\ 14.721368 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} \frac{1}{10}\left(6 + 1181/6050 - 2 * 0.306988\right) \\ \frac{1}{11}\left(25 + 0.558123 + 0.306988 - 3 * 14.721368\right) \\ \frac{1}{-10}\left(-11 - 2 * 0.558123 - 1.663545 + 14.721368\right) \\ \left(15 - 3 * -1.663545 - 0.094158\right) \end{bmatrix} = \begin{bmatrix} 0.558123 \\ -1.663545 \\ -0.094158 \\ 19.896477 \end{bmatrix}$$