MA2501 - Assignment 1

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Problem 1

Part 1

Let $x = 0.d_1...d_k... * 10^n$ in decimal representation (basis $b = 10$). Aiming at a k-digit floating point representation, we consider chopping instead of rounding, i.e. we keep the k first digits and throw away the rest.

$$fl(x) = 0.d_1...d_k d_{k+1}... * 10^n$$

Show that 10^{-k-1} is a bound for the relative error when using k-digit chopping.

Observe that relative error is given by

$$e_R = \frac{x - fl(x)}{x} \tag{1}$$

Observe that the numerator is given by

$$\begin{aligned} x - fl(x) &= 0.0...0d_{k+1}d_{k+2}... * 10^n \\ &= 0.d_{k+1}d_{k+1}... * 10^{n-k} \\ &< 1 * 10^{n-k} \end{aligned}$$

Assuming that $d_1 > 0$, the denominator is given by

$$\begin{aligned} x &\geq 0.d_1 * 10^n = d_1 * 10^{n-1} \\ &\geq 1 * 10^{n-1} \end{aligned}$$

Combining these, we get

$$\begin{aligned} e_R &= \frac{x - fl(x)}{x} \\ &< \frac{10^{n-k}}{10^{n-1}} = 10^{1-k} \end{aligned}$$

The relative error has an upper bound of 10^{1-k} when using k-digit chopping.

Part 2

Let s be a parameter. Show that the function $f(t) = t^3 + 2t + s$ crosses the t-axis exactly once for any value of s.

Observe that the derivate of f is $f'(t) = 3t^2 + 2$, and that $f'(t) > 0 \forall t$. f is therefore strictly monotone increasing, so can cross a horizontal line at most one time. This applies no matter the value of s.

Let $t_1 = -s$, $t_2 = s$ for $s > 0$. Note that f is continuous on the whole interval $[t_1, t_2]$. We then have

$$\begin{aligned} f(t_1) &= -s^3 - 2s + s = -s^3 - s < 0 \\ f(t_2) &= s^3 + 2s + s = s^3 + 3s > 0 \end{aligned}$$

The intermediate value theorem thus states that there must exist a number $u \in (t_1, t_2)$ such that $f(u) = 0$. This holds also for $s \leq 0$.

Because f is strictly monotone increasing, it can only cross the t-axis at most one time. Because there exists an u such that $f(u) = 0$, f must cross the t-axis at least one time. Combining these, we have proved that $f(s)$ crosses the t-axis exactly once for any value of s.

Part 3

Recall that Taylor's polynomial $p(t)$ is determined by requiring that the values of the polynomial and its first n derivates match those of a given function $f(t)$ at a single argument t_0, i.e. $p^{(i)}(t_0) = f^{(i)}(t_0)$ for $0 \leq i \leq n$. Find a formula for $R(t, t_0) = f(t) - p(t)$ in integral form. Assume that $f^{(n+1)}(t)$ is continuous between t and t_0.

By the Fundamental Theorem og Calculus, observe that

$$f(t) = f(t_0) + \int_{t_0}^{t} f'(x)dx$$

Choosing the following constants of integrations, we can integrate by parts:

$$\begin{aligned} u &= f' \\ du &= f''dx \\ v &= x - t \\ dv &= dx \end{aligned}$$

Then

$$\begin{aligned} f(t) &= f(t_0) + \int_{t_0}^{t} f'(x)dx \\ &= f(t_0) + f'(x)(x-t)|_{x=t_0}^{x=t} - \int_{t_0}^{t} f''(x)(x-t)dx \\ &= f(t_0) + f'(t_0)(t-t_0) + \int_{t_0}^{t} f''(x)(b-x)dx \end{aligned}$$

Repeating this integration with new constants

$$\begin{aligned} u &= f'' \\ du &= f'''dx \\ v &= \frac{-(t-x)^2}{2} \\ dv &= (t-x)dx \end{aligned}$$

Gives

$$
\begin{aligned}
f(t) &= f(t_0) + f'(t_0)(t-t_0) + \int_{t_0}^{t} f''(x)(b-x)dx \\
&= f(t_0) + f'(t_0)(t-t_0) - f''(x)\frac{(t-x)^2}{2}|_{x=t_0}^{x=t} + \int_{t_0}^{t} f'''(x)\frac{(t-x)^2}{2}dx \\
&= f(t_0) + f'(t_0)(t-t_0) + f''(t_0)\frac{(t-t_0)^2}{2} + \int_{t_0}^{t} f'''(x)\frac{(t-x)^2}{2}dx
\end{aligned}
$$

Repeating this process n times gives

$$
f(t) = f(t_0) + f'(t_0)(t-t_0) + f''(t_0)\frac{(t-t_0)^2}{2} + ... + f^{(n)}(t_0)\frac{(t-t_0)^n}{n!} + \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx
$$

We have from the definition that $p^{(i)}(t_0) = f^{(i)}(t_0)$ for $0 \leq i \leq n$, thus we can rewrite this to be

$$
\begin{aligned}
f(t) &= p(t_0) + p'(t_0)(t-t_0) + p''(t_0)\frac{(t-t_0)^2}{2} + ... + p^{(n)}(t_0)\frac{(t-t_0)^n}{n!} + \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx \\
&= p(t) + \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx
\end{aligned}
$$

Rewriting this, we get

$$
R(t,t_0) = f(t) - p(t) = \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx \tag{2}
$$

Part 4

Determine the Taylor polynomial $P_n(t)$ for $n = 2$ for the function $f(t) = e^t cos(t)$ around the point $t_0 = 0$. Find an upper bound for the remainder term for $t = 0.5$.

The Taylor polynomial for $f(t)$ is given by

$$
\begin{aligned}
P_2(t) &= f(t_0) + f'(t_0)(t-t_0) + f''(t_0)\frac{(t-t_0)^2}{2} \\
&= f(0) + f'(0)(t) + f''(0)\frac{t^2}{2}
\end{aligned}
$$

Observe the following

$$
\begin{aligned}
&f(0) = e^0 cos(0) = 1 \\
&f'(t) = e^t cos(t) - e^t sin(t) \\
&f'(0) = e^0(cos(0) - sin(0)) = 1 \\
&f''(t) = e^t(cos(t) - sin(t)) - e^t(sin(t) + cos(t)) = -2e^t sin(t) \\
&f''(0) = 1 - 1 = 0
\end{aligned}
$$

We thus have the Taylor polynomial

$$
\begin{aligned}
P_2(t) &= f(0) + f'(0)(t) + f''(0)\frac{t^2}{2} \\
&= 1 + t
\end{aligned}
$$

We have from the Remainder Estimation Theorem that if there is a positive constant M such that $|f'''(t)| \leq M$ for all $t \in [0, 0.5]$, then the remainder term can be written

$$|R_n(t)| \leq M\frac{|t-t_0|^{n+1}}{(n+1)!}$$
$$|R_2(0.5)| \leq M\frac{|0.5|^3}{3!} = \frac{M}{48}$$

Observe the following

$$f'''(t) = -2e^t sin(t) - 2e^t cos(t) = -2e^t(sin(t) + cos(t))$$
$$f^{(4)}(t) = -2e^t(sin(t) + cos(t)) - 2e^t(cos(t) - sin(t)) = -4e^t cos(t)$$

Note that $e^t > 0 \, \forall t$ and $cos(t) > 0 \, \forall t \in [0, 0.5]$. Then $f^{(4)}(t) < 0 \, \forall t \in [0, 0.5]$, and $f'''(t)$ is strictly monotone decreasing in the same interval. This means that $f(t) on [0, 0.5]$ has two extremas - at $t = 0$ or $t = 0.5$:

$$|f'''(0)| = \left|-2e^0(sin(0) + cos(0))\right| = 2$$
$$|f'''(0.5)| = \left|-2e^{0.5}(sin(0.5) + cos(0.5))\right| \approx 3.3261$$

We therefore let $M = |f'''(0.5)| \approx 3.3261$, and get

$$|R_2(0.5)| \leq \frac{M}{48} \approx \frac{3.3261}{48} \approx 0.069$$

The upper bound for the remainder term for $t = 0.5$ is ≈ 0.069.

Problem 2

Consider the equation $t^2 = a$ written in fixed point form $t = F(t)$. It turns out that several $F(t)$ are possible:

$$F_1(t) = 0.5(t + at^{-1})$$
$$F_2(t) = at^{-1}$$
$$F_3(t) = 2t - at^{-1}.$$

Verify that this is true and discuss the (non-)convergence behavior for the corresponding iteration $t_{n+1} = F(t_n), n \geq 0$, for each of the three cases. If possible, determine the order of convergence.

Observe that

$$\begin{aligned} F_1(t) &= 0.5(t + at^{-1}) \\ &= 0.5(t + t^2 t^{-1}) \\ &= 0.5(t + t) = t \end{aligned}$$

$$\begin{aligned} F_2(t) &= at^{-1} \\ &= t^2 t^{-1} = t \end{aligned}$$

$$\begin{aligned} F_3(t) &= 2t - at^{-1} \\ &= 2t - t^2 t^{-1} \\ &= 2t - t = t \end{aligned}$$

Thus, $F_1(t) = t$, $F_2(t) = t$ and $F_3(t) = t$.

As all functions $F_n(t)$ are defined and continuous on $\mathbb{R} \setminus 0$, we have from the Contraction Mapping Theorem that they will converge if they are *contractions*, i.e. if there exists a constant L such that $0 < L < 1$ and $|F_n(t_1) - F_n(t_0)| \leq L|t_1 - t_0| \, \forall t_1, t_0 \in \mathbb{R} \setminus 0$.

Note that all functions are continuous and differentiable on $\mathbb{R} \setminus 0$. We therefore have from the Mean Value Theorem that for any $x, y \in \mathbb{R} \setminus 0$

$$|F_n(t_1) - F_n(t_0)| = |F_n'(\beta)(t_1 - t_0)| = |F_n'(\beta)||t_1 - t_0|$$

for some β between t_1 and t_0.

Problem 3

Süli-Mayers: Ex. 1.8, 2.8, 4.8

Exercise 1.8

Suppose that the function f has a continuous second derivative, that $f(\xi) = 0$, and that in the interval $[X, \xi]$, with $X < \xi$, $f'(x) > 0$ and $f''(x) < 0$. Show that the Newton iteration, starting from any x_0 in $[X, \xi]$, converges to ξ.

As $f'(x) > 0$ for all $x \in [X, \xi]$, f must be absolutely monotonically increasing over the interval. As $f(\xi) = 0$, this means that $f(x_0) < 0$ for all $x_0 \in [X, \xi)$.

The Newton iteration states that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. For $x_0 = \xi$, observe that $f(\xi) = 0$, thus $f(x_0) = 0$, thus

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n$$
$$\lim_{n \to \infty} x_n = x_0 = \xi$$

As $f(x_n) < 0$ and $f'(x_n) > 0$ for all $x_n \in [X, \xi]$, we have that $\frac{f(x_n)}{f'(x_n)} < 0$ for all $x_n \in [X, \xi)$. This means that, for $x_n \in [X, \xi)$:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n + \left| \frac{f(x_n)}{f'(x_n)} \right| > x_n \end{aligned}$$

This means that for $x_0 \in [X, \xi)$:

$$\lim_{n \to \infty} x_n = \xi$$

As $\lim_{n \to \infty} x_n = \xi$ for both $x_0 = \xi$ and $x_0 \in [X, \xi)$, it follows that

$$\lim_{n \to \infty} x_n = \xi$$

for all $x_0 \in [X, \xi]$, and that the Newton iteration converges to ξ for any $x_0 \in [X, \xi]$.

Exercise 2.8

(i) Show that, for any vector $\mathbf{v} = (v_1, ..., v_n)^T \in \mathbb{R}^n$, $\|\mathbf{v}\|_\infty \leq \|\mathbf{v}\|_2$ *and* $\|\mathbf{v}\|_2^2 \leq \|\mathbf{v}\|_1 \|\mathbf{v}\|_\infty$.

In each case give an example of a nonzero vector $\mathbf{v}$ *for which equality is attained. Deduce that* $\|\mathbf{v}\|_\infty \leq \|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_1$. *Show also that* $\|\mathbf{v}\|_2 \leq \sqrt{n}\|\mathbf{v}\|_\infty$.

Observe that

$$\|v\|_\infty = \max_i |v_i| \geq v_i \ \forall i,$$

thus

$$\|v\|_2 = \sqrt{\sum_{i=1}^n |v_i|^2} \leq \sqrt{\sum_{i=1}^n \|v\|_\infty^2} = \sqrt{n\|v\|_\infty^2} = \sqrt{n}\|v\|_\infty$$

and

$$\|v\|_\infty = \sqrt{(\max_i |v_i|)^2} \leq \sqrt{\sum_{i=1}^n |v_i|^2} = \|v\|_2$$

If $n = 1$, then $v_i = \|v\|_\infty \ \forall i$, giving $\|v\|_2 = \sqrt{n}\|v\|_\infty$. It follows that $\|v\|_2 = \sqrt{1}\|v\|_\infty = \|v\|_\infty$, thus any vector with length 1 (such as $v = (\pi)$) is an example where equality is attained. This is proven as $\|(\pi)\|_\infty = max([\pi]) = \pi$, and $\|(\pi)\|_2 = \sqrt{\pi^2} = \pi$.

If $n > 1$, let j be the index there $v_j = \max_i |v_i|$. It then follows that $(\max_i |v_i|)^2 = |v_j|^2 < \sum_{i=1}^n |v_i|^2$ (as $\sum_{i=1}^n |v_i|^2 = |v_j|^2 + \sum_{i \neq j} |v_i|^2$), thus inequality is attained and $\|v\|_\infty \leq \|v\|_2$ for all vectors $v \in \mathbb{R}^n$.

Observe that

$$\begin{aligned}\|v\|_2^2 = \sqrt{\sum_{i=1}^n |v_i|^2}^2 = \sum_{i=1}^n |v_i|^2 \leq \sum_{i=1}^n |v_i| \max_i |v_i| \\ = \sum_{i=1}^n |v_i| \|v\|_\infty = n\|v\|_\infty \sum_{i=1}^n |v_i| = n\|v\|_\infty \|v\|_1\end{aligned}$$

If $\|v\|_\infty = v_i \ \forall i$, then it follows that $\|v\|_2^2 = \|v\|_1 \|v\|_\infty$, thus any vector where this is true (such as any vector with length 1, such as $v = (\pi)$) is an example where equality is attained. This is proven as $\|(\pi)\|_2^2 = \sqrt{\pi^2}^2 = \pi^2$, and $\|(\pi)\|_1 * \|(\pi)\|_\infty = \pi * \pi = \pi^2$.

If $\|v\|_\infty > v_i$ for any i, then it follows that $\|v\|_2^2 < \|v\|_1 \|v\|_\infty$. This can be proven by the vector $v = (1, 2)$, where $\|v\|_2^2 = \sqrt{1+4}^2 = 5$, and $\|v\|_1 * \|v\|_\infty = (1+2) * 2 = 6$.

It follows from this that $\|v\|_2^2 \leq \|v\|_1 \|v\|_\infty$ for all vectors $v \in \mathbb{R}^n$.

As $\|v\|_\infty > 0$ and $\|v\|_2 \geq \|v\|_\infty$, we have that

$$\|v\|_2 = \frac{\|v\|_2^2}{\|v\|_2} \leq \frac{\|v\|_1 \|v\|_\infty}{\|v\|_2} \leq \frac{\|v\|_1 \|v\|_\infty}{\|v\|_\infty} = \|v\|_1$$

Combining this with the earlier answer, it follows that

$$\|v\|_\infty \leq \|v\|_2 \leq \|v\|_1$$

(ii) Show that, for any matrix $A \in \mathbb{R}^{m*n}$, $\|A\|_\infty \leq \sqrt{n}\|A\|_2$ and $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$.

In each case give an example of a matrix A for which equality is attained.

Observe that

$$\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|$$

i.e. $\|A\|_\infty$ is the maximum row-sum in the matrix. Let A_k be the row which satisfies

$$\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}| = \|A_k\|_1$$

i.e. the row with the maximum row-sum in A. Then we have

$$\begin{aligned} \|A\|_\infty = \|A_k\|_1 &\leq \sqrt{n}\|A_k\|_2 \\ &\leq \sqrt{n}\sum_{i=1}^{n}\|A_i\|_2 = \sqrt{n}\|A\|_2 \end{aligned}$$

where the first inequality was proven in *(i)*, and the second follows from the fact that $\sum_{i=1}^{n}\|A_i\|_2 = \|A_k\|_2 + \sum_{i\neq k}\|A_i\|_2$.

Note that equality is attained when A is a square matrix with $n = 1$, such as $A = \begin{bmatrix} 2 \end{bmatrix}$. In this case, $\|A\|_\infty = \max(2) = 2$, and $\sqrt{n}\|A\|_2 = 1 * 2 = 2$.

—

Definition 2.10 gives that

$$\|A\| = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$$

Combining this with the proofs found in *(i)*, observe that

$$\begin{aligned} \|A\|_2 = \max_x \frac{\|Ax\|_2}{\|x\|_2} &\leq \max_x \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_2} \\ &\leq \max_x \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{m}\|A\|_\infty \end{aligned}$$

where the first inequality follows from $\|v\|_2 \leq \sqrt{n}\|v\|_\infty$, as found in *(i)*, and the second inequality follows from $\|v\|_2 \geq \|v\|_\infty$, also found in *(i)*.

Equality is attained, also here, when A is a square matrix with $n = 1$, such as $A = \begin{bmatrix} 2 \end{bmatrix}$, where $\|A\|_2 = 2$ and $\sqrt{m}\|A\|_\infty = 2$.

Exercise 4.8

Suppose that $\xi = \lim_{k\to\infty} \mathbf{x}^{(k)}$ in $\mathbb{R}$. Following Definition 1.4, explain what is meant by saying that "the sequence $\mathbf{x}^{(k)}$ converges to ξ linearly, with asymptotic rate $-\log_{10}\mu$", where $0 < \mu < 1$.

When a sequence $\mathbf{x}$ converges linearly to a constant C with an asymptotic rate of γ, this means that the sequence satisfies the equation

$$\lim_{k\to\infty} \frac{\left|\mathbf{x}^{(k+1)} - C\right|}{\left|\mathbf{x}^{(k)} - C\right|} = \gamma \tag{3}$$

In this case, the sequence $\mathbf{x}$ satisfies the following equation:

$$\lim_{k\to\infty} \frac{\left|\mathbf{x}^{(k+1)} - \xi\right|}{\left|\mathbf{x}^{(k)} - \xi\right|} = \log_{10}\mu$$

Given the vector function $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ of two real variables x_1 and x_2 defined by $f_1(x_1, x_2) = x_1^2 + x_2^2 - 2$ and $f_2(x_1, x_2) = x_1 + x_2 - 2$, show that $\mathbf{f}(\xi) = \mathbf{0}$ when $\xi = (1, 1)^T$.

$$\begin{aligned}
\mathbf{f}(\xi) &= \mathbf{f}(1,1)^T \\
&= (f_1(1,1), f_2(1,1)) \\
&= (1+1-2, 1+1-2) = (0,0) = \mathbf{0}
\end{aligned}$$

Suppose that $x_1^{(0)} \neq x_2^{(0)}$; show that one iteration of Newton's method for the solution $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ with starting value $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)})^T$ then gives $\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)})^T$ such that $x_1^{(1)} + x_2^{(1)} = 2$.

Newton's method states that $x^{(n+1)} = x^{(n)} - J_f(\mathbf{x}^{(n)})^{-1} f(\mathbf{x}^{(n)})$, where $J_f(\mathbf{x}^{(n)})^{-1}$ is the inverse of the Jacobian matrix of f. We compute this as

$$\begin{aligned}
J_f(\mathbf{x}^{(n)}) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 & 2x_2 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2x_1 - 2x_2} & \frac{-x_2}{x_1 - x_2} \\ \frac{-1}{2x_1 - 2x_2} & \frac{x_1}{x_1 - x_2} \end{bmatrix}
\end{aligned}$$

It follows that

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \mathbf{x}^{(0)} - J_f(\mathbf{x}^{(0)})^{-1} f(\mathbf{x}^{(0)}) \\
&= (x_1^{(0)}, x_2^{(0)}) - \begin{bmatrix} \frac{1}{2x_1-2x_2} & \frac{-x_2}{x_1-x_2} \\ \frac{-1}{2x_1-2x_2} & \frac{x_1}{x_1-x_2} \end{bmatrix} * (x_1^2 + x_2^2 - 2, x_1 + x_2 - 2)^T \\
&= (x_1^{(0)}, x_2^{(0)}) - \begin{bmatrix} \frac{x_1^2+x_2^2-2}{2x_1-2x_2} + \frac{-(x_1x_2+x_2^2-2x_1)}{x_1-x_2} \\ \frac{-(x_1^2+x_2^2-2)}{2x_1-2x_2} + \frac{x_1^2+x_1x_2-2x_1}{x_1-x_2} \end{bmatrix} \\
&= (x_1^{(0)}, x_2^{(0)}) - \begin{bmatrix} \frac{x_1^2+x_2^2-2-2x_1x_2-2x_2^2+4x_1}{2x_1-2x_2} \\ \frac{-x_1^2-x_2^2+2+2x_1^2+2x_1x_2-4x_1}{2x_1-2x_2} \end{bmatrix} \\
&= (x_1^{(0)}, x_2^{(0)}) - \begin{bmatrix} \frac{x_1^2-x_2^2-2x_1x_2+4x_1-2}{2x_1-2x_2} \\ \frac{x_1^2-x_2^2+2x_1x_2-4x_1+2}{2x_1-2x_2} \end{bmatrix} \\
&= \begin{bmatrix} x_1 - \frac{x_1^2-x_2^2-2x_1x_2+4x_1-2}{2x_1-2x_2} \\ x_2 - \frac{x_1^2-x_2^2+2x_1x_2-4x_1+2}{2x_1-2x_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2x_1^2-2x_1x_2-(x_1^2-x_2^2-2x_1x_2+4x_1-2)}{2x_1-2x_2} \\ \frac{2x_1x_2-2x_2^2-(x_1^2-x_2^2+2x_1x_2-4x_1+2)}{2x_1-2x_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{x_1^2+x_2^2-4x_1+2}{2x_1-2x_2} \\ \frac{-x_1^2-x_2^2+4x_1-2}{2x_1-2x_2} \end{bmatrix}
\end{aligned}
$$

It is clear to see that

$$
\begin{aligned}
x_1^{(1)} + x_2^{(1)} &= \frac{x_1^2 + x_2^2 - 4x_1 + 2}{2x_1 - 2x_2} + \frac{-x_1^2 - x_2^2 + 4x_1 - 2}{2x_1 - 2x_2} \\
&= \frac{x_1^2 + x_2^2 - 4x_1 + 2 - x_1^2 - x_2^2 + 4x_1 - 2}{2x_1 - 2x_2} \\
&= \frac{0}{2x_1 - 2x_2} = 0
\end{aligned}
$$

This is a different result than what I was supposed to show, which is interesting. I have not succeeded in pinpointing exactly where I was wrong...

Determine $\mathbf{x}^{(1)}$ *when* $x_1^{(0)} = 1 + \alpha$, $x_2^{(0)} = 1 - \alpha$, *where* $\alpha \neq 0$. *Assuming that* $x_1^{(0)} \neq x_2^{(0)}$, *deduce that Newton's method converges linearly to* $(1, 1)^T$, *with asymptotic rate of convergence* $\log_{10} 2$. *Why is the convergence rate not quadratic?*

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \mathbf{x}^{(0)} - J_f(\mathbf{x}^{(0)})^{-1} f(\mathbf{x}^{(0)}) \\
&= (1+\alpha, 1-\alpha) - \begin{bmatrix} \frac{1}{2+2\alpha-2+2\alpha} & \frac{-1+\alpha}{1+\alpha-1+\alpha} \\ \frac{-1}{2+2\alpha-2+2\alpha} & \frac{1+\alpha}{1+\alpha-1+\alpha} \end{bmatrix} * (1 + 2\alpha + \alpha^2 + 1 - 2\alpha + \alpha^2 - 2, 1 + \alpha + 1 - \alpha - 2)^T \\
&= (1+\alpha, 1-\alpha) - \begin{bmatrix} \frac{1}{4\alpha} & \frac{-1+\alpha}{2\alpha} \\ \frac{-1}{4\alpha} & \frac{1+\alpha}{2\alpha} \end{bmatrix} * (2\alpha^2, 2\alpha)^T \\
&= (1+\alpha, 1-\alpha) - \begin{bmatrix} \frac{\alpha}{2} - 1 + \alpha \\ \frac{-\alpha}{2} + 1 + \alpha \end{bmatrix} \\
&= (1 + \alpha - \frac{\alpha}{2} + 1 - \alpha, 1 - \alpha + \frac{\alpha}{2} - 1 - \alpha) \\
&= (2 - \frac{\alpha}{2}, -2\alpha + \frac{\alpha}{2})
\end{aligned}
$$

For the convergence rate to be quadratic, it would have to satisfy the following inequation:

$$\lim_{k\to\infty} \frac{\left|x^{(k+1)} - (1,1)^T\right|}{\left|x^{(k)} - (1,1)^T\right|^2} < C \tag{4}$$

It does not, however, but it is easy to see that

$$\lim_{k\to\infty} \frac{\left|x^{(k+1)} - (1,1)^T\right|}{\left|x^{(k)} - (1,1)^T\right|} < \log_{10} 2, \tag{5}$$

which gives a linear convergence to $(1,1)^T$ with asymptotic rate of convergence $\log_{10} 2$.