MA2501 - Assignment 1

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Problem 1

Part 1

Let $x = 0.d_1...d_k... * 10^n$ in decimal representation (basis $b = 10$). Aiming at a k-digit floating point representation, we consider chopping instead of rounding, i.e. we keep the k first digits and throw away the rest.

$$fl(x) = 0.d_1...d_k d_{k+1}... * 10^n$$

Show that 10^{-k-1} is a bound for the relative error when using k-digit chopping.

Observe that relative error is given by

$$e_R = \frac{x - fl(x)}{x} \tag{1}$$

Observe that the numerator is given by

$$\begin{aligned} x - fl(x) &= 0.0...0d_{k+1}d_{k+2}... * 10^n \\ &= 0.d_{k+1}d_{k+1}... * 10^{n-k} \\ &< 1 * 10^{n-k} \end{aligned}$$

Assuming that $d_1 > 0$, the denominator is given by

$$\begin{aligned} x &\geq 0.d_1 * 10^n = d_1 * 10^{n-1} \\ &\geq 1 * 10^{n-1} \end{aligned}$$

Combining these, we get

$$\begin{aligned} e_R &= \frac{x - fl(x)}{x} \\ &< \frac{10^{n-k}}{10^{n-1}} = 10^{1-k} \end{aligned}$$

The relative error has an upper bound of 10^{1-k} when using k-digit chopping.

Part 2

Let s be a parameter. Show that the function $f(t) = t^3 + 2t + s$ crosses the t-axis exactly once for any value of s.

Observe that the derivate of f is $f'(t) = 3t^2 + 2$, and that $f'(t) > 0 \forall t$. f is therefore strictly monotone increasing, so can cross a horizontal line at most one time. This applies no matter the value of s.

Let $t_1 = -s$, $t_2 = s$ for $s > 0$. Note that f is continuous on the whole interval $[t_1, t_2]$. We then have

$$
\begin{aligned}
f(t_1) &= -s^3 - 2s + s = -s^3 - s < 0 \\
f(t_2) &= s^3 + 2s + s = s^3 + 3s > 0
\end{aligned}
$$

The intermediate value theorem thus states that there must exist a number $u \in (t_1, t_2)$ such that $f(u) = 0$. This holds also for $s \leq 0$.

Because f is strictly monotone increasing, it can only cross the t-axis at most one time. Because there exists an u such that $f(u) = 0$, f must cross the t-axis at least one time. Combining these, we have proved that $f(s)$ crosses the t-axis exactly once for any value of s.

Part 3

Recall that Taylor's polynomial $p(t)$ is determined by requiring that the values of the polynomial and its first n derivates match those of a given function $f(t)$ at a single argument t_0, i.e. $p^{(i)}(t_0) = f^{(i)}(t_0)$ for $0 \leq i \leq n$. Find a formula for $R(t, t_0) = f(t) - p(t)$ in integral form. Assume that $f^{(n+1)}(t)$ is continuous between t and t_0.

By the Fundamental Theorem og Calculus, observe that

$$
f(t) = f(t_0) + \int_{t_0}^{t} f'(x)dx
$$

Choosing the following constants of integrations, we can integrate by parts:

$$
\begin{aligned}
u &= f' \\
du &= f''dx \\
v &= x - t \\
dv &= dx
\end{aligned}
$$

Then

$$
\begin{aligned}
f(t) &= f(t_0) + \int_{t_0}^{t} f'(x)dx \\
&= f(t_0) + f'(x)(x-t)|_{x=t_0}^{x=t} - \int_{t_0}^{t} f''(x)(x-t)dx \\
&= f(t_0) + f'(t_0)(t-t_0) + \int_{t_0}^{t} f''(x)(b-x)dx
\end{aligned}
$$

Repeating this integration with new constants

$$
\begin{aligned}
u &= f'' \\
du &= f'''dx \\
v &= \frac{-(t-x)^2}{2} \\
dv &= (t-x)dx
\end{aligned}
$$

Gives

$$\begin{aligned}
f(t) &= f(t_0) + f'(t_0)(t-t_0) + \int_{t_0}^{t} f''(x)(b-x)dx \\
&= f(t_0) + f'(t_0)(t-t_0) - f''(x)\frac{(t-x)^2}{2}|_{x=t_0}^{x=t} + \int_{t_0}^{t} f'''(x)\frac{(t-x)^2}{2}dx \\
&= f(t_0) + f'(t_0)(t-t_0) + f''(t_0)\frac{(t-t_0)^2}{2} + \int_{t_0}^{t} f'''(x)\frac{(t-x)^2}{2}dx
\end{aligned}$$

Repeating this process n times gives

$$f(t) = f(t_0) + f'(t_0)(t-t_0) + f''(t_0)\frac{(t-t_0)^2}{2} + ... + f^{(n)}(t_0)\frac{(t-t_0)^n}{n!} + \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx$$

We have from the definition that $p^{(i)}(t_0) = f^{(i)}(t_0)$ for $0 \leq i \leq n$, thus we can rewrite this to be

$$\begin{aligned}
f(t) &= p(t_0) + p'(t_0)(t-t_0) + p''(t_0)\frac{(t-t_0)^2}{2} + ... + p^{(n)}(t_0)\frac{(t-t_0)^n}{n!} + \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx \\
&= p(t) + \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx
\end{aligned}$$

Rewriting this, we get

$$R(t,t_0) = f(t) - p(t) = \int_{t_0}^{t} f^{(n+1)}(x)\frac{(t-x)^n}{n!}dx \tag{2}$$

Part 4

Determine the Taylor polynomial $P_n(t)$ for $n = 2$ for the function $f(t) = e^t cos(t)$ around the point $t_0 = 0$. Find an upper bound for the remainder term for $t = 0.5$.

The Taylor polynomial for $f(t)$ is given by

$$\begin{aligned}
P_2(t) &= f(t_0) + f'(t_0)(t-t_0) + f''(t_0)\frac{(t-t_0)^2}{2} \\
&= f(0) + f'(0)(t) + f''(0)\frac{t^2}{2}
\end{aligned}$$

Observe the following

$$\begin{aligned}
&f(0) = e^0 cos(0) = 1 \\
&f'(t) = e^t cos(t) - e^t sin(t) \\
&f'(0) = e^0(cos(0) - sin(0)) = 1 \\
&f''(t) = e^t(cos(t) - sin(t)) - e^t(sin(t) + cos(t)) = -2e^t sin(t) \\
&f''(0) = 1 - 1 = 0
\end{aligned}$$

We thus have the Taylor polynomial

$$\begin{aligned}
P_2(t) &= f(0) + f'(0)(t) + f''(0)\frac{t^2}{2} \\
&= 1 + t
\end{aligned}$$

We have from the Remainder Estimation Theorem that if there is a positive constant M such that $|f'''(t)| \leq M$ for all $t \in [0, 0.5]$, then the remainder term can be written

$$|R_n(t)| \leq M\frac{|t - t_0|^{n+1}}{(n+1)!}$$
$$|R_2(0.5)| \leq M\frac{|0.5|^3}{3!} = \frac{M}{48}$$

Observe the following

$$f'''(t) = -2e^t sin(t) - 2e^t cos(t) = -2e^t(sin(t) + cos(t))$$
$$f^{(4)}(t) = -2e^t(sin(t) + cos(t)) - 2e^t(cos(t) - sin(t)) = -4e^t cos(t)$$

Note that $e^t > 0\,\forall t$ and $cos(t) > 0\,\forall t \in [0, 0.5]$. Then $f^{(4)}(t) < 0\,\forall t \in [0, 0.5]$, and $f'''(t)$ is strictly monotone decreasing in the same interval. This means that $f(t)on[0, 0.5]$ has two extremas - at $t = 0$ or $t = 0.5$:

$$|f'''(0)| = \left|-2e^0(sin(0) + cos(0))\right| = 2$$
$$|f'''(0.5)| = \left|-2e^{0.5}(sin(0.5) + cos(0.5))\right| \approx 3.3261$$

We therefore let $M = |f'''(0.5)| \approx 3.3261$, and get

$$|R_2(0.5)| \leq \frac{M}{48} \approx \frac{3.3261}{48} \approx 0.069$$

The upper bound for the remainder term for $t = 0.5$ is ≈ 0.069.

Problem 2

Consider the equation $t^2 = a$ written in fixed point form $t = F(t)$. It turns out that several $F(t)$ are possible:

$$F_1(t) = 0.5(t + at^{-1})$$
$$F_2(t) = at^{-1}$$
$$F_3(t) = 2t - at^{-1}.$$

Verify that this is true and discuss the (non-)convergence behavior for the corresponding iteration $t_{n+1} = F(t_n), n \geq 0$, for each of the three cases. If possible, determine the order of convergence.

Observe that

$$\begin{aligned} F_1(t) &= 0.5(t + at^{-1}) \\ &= 0.5(t + t^2t^{-1}) \\ &= 0.5(t + t) = t \end{aligned}$$

$$\begin{aligned} F_2(t) &= at^{-1} \\ &= t^2t^{-1} = t \end{aligned}$$

$$\begin{aligned} F_3(t) &= 2t - at^{-1} \\ &= 2t - t^2t^{-1} \\ &= 2t - t = t \end{aligned}$$

Thus, $F_1(t) = t$, $F_2(t) = t$ and $F_3(t) = t$.

As all functions $F_n(t)$ are defined and continuous on $\mathbb{R} \setminus 0$, we have from the Contraction Mapping Theorem that they will converge if they are *contractions*, i.e. if there exists a constant L such that $0 < L < 1$ and $|F_n(t_1) - F_n(t_0)| \leq L|t_1 - t_0| \, \forall t_1, t_0 \in \mathbb{R} \setminus 0$.

Note that all functions are continuous and differentiable on $\mathbb{R} \setminus 0$. We therefore have from the Mean Value Theorem that for any $x, y \in \mathbb{R} \setminus 0$

$$|F_n(t_1) - F_n(t_0)| = |F_n'(\beta)(t_1 - t_0)| = |F_n'(\beta)||t_1 - t_0|$$

for some β between t_1 and t_0.

Problem 3

Süli-Mayers: Ex. 1.8, 2.8, 4.8