Øving 5 side 1 Andreas B. Berg

1) $\vec{r}(t) = (t^3, t^2)$

a) $\underline{\vec{v}(t)} = \vec{r}'(t) = \underline{\underline{(3t^2, 2t)}}$

$\underline{v(t)} = |v(t)| = \sqrt{9t^4 + 4t^2} = \underline{\underline{t\sqrt{9t^2+4}}}$

$\underline{\vec{a}(t)} = \vec{v}'(t) = \underline{\underline{(6t, 2)}}$

$\underline{a(t)} = v'(t) = \underline{\underline{\frac{1}{2t\sqrt{9t^2+4}}}}$

b) $L_{0-10} = \int_0^{10} v(t)\,dt = \int_0^{10} t\sqrt{9t^2+4}\,dt \quad (*)$

$u = 9t^2 + 4 \qquad du = 18t\,dt \qquad t\,dt = \frac{1}{18}du$

$\int t\sqrt{9t^2+4}\,dt = \int \frac{1}{18}\sqrt{u}\,du = \frac{1}{27}u^{\frac{3}{2}} + C$

$(*) = \frac{1}{27}\left[(9t^2+4)^{\frac{3}{2}}\right]_0^{10} = \underline{\underline{\frac{1}{27}\left(904^{\frac{3}{2}} + 8\right)}} \approx 1007$

2)a) $\vec{r}(t) = (a\cos t, b\sin t) \qquad t \in [0, 2\pi]$

$\frac{x(t)^2}{a^2} + \frac{y(t)^2}{b^2} = \frac{a^2\cos(t)^2}{a^2} + \frac{b^2\sin(t)^2}{b^2}$

$= \cos^2 t + \sin^2 t = 1$

$\Rightarrow \underline{\vec{r}(t) \text{ er ellipsen av } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

b) Hastighet: $\underline{r'(t)} = \underline{\underline{(a\sin t, -b\cos t)}}$

Fart: $|r'(t)| = \underline{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$

$= \underline{\sqrt{a^2 + (b^2 - a^2)\cos^2 t}} = \underline{\sqrt{(a^2-b^2)\sin^2 t + b^2}}$

Akselerasjon: $a(t) = \vec{v}'(t) = r''(t) = \underline{\underline{(-a\cos t, -b\sin t)}}$

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2) c) Omkrets $= L_{0\to 2\pi} = \int_0^{2\pi} |r'(t)|\,dt$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

3) $\vec{r}(t) = (2\cos t,\ \sqrt{2}\sin t,\ \sqrt{2}\sin t)$

a) Hastighet: $v(t) = r'(t) = (-2\sin t,\ \sqrt{2}\cos t,\ \sqrt{2}\cos t)$

Fart: $|v(t)| = \sqrt{4\sin^2 t + 2\cos^2 t + 2\cos^2 t} = \sqrt{4} = 2$

Akselerasjon: $a(t) = v'(t) = (-2\cos t,\ -\sqrt{2}\sin t,\ -\sqrt{2}\sin t)$

b) Buelengde: $L_{0\to 2\pi} = \int_0^{2\pi} |v(t)|\,dt = \int_0^{2\pi} 2\,dt = 4\pi$

c) Ser at $x^2 + y^2 + z^2 = 4\cos^2 t + 4\sin^2 t = 4$, så kurven ligger i kuleflate om origo med $r = 2$, også $x^2 + y^2 + z^2 = 4 \quad \forall t$ ....

d) Ser at $y = z \ \forall t$, $\Rightarrow$ $y - z = 0 \ \forall t$

$\Rightarrow$ kurven ligger i planet $y - z = 0$

4) $c(t) = (\cos^2 t,\ 3t - t^2,\ t)$. Finn param. tangent i $t=0$

$v(t) = c'(t) = (-\sin(2t),\ 3 - 2t,\ 1)$

$|v(t)| = \sqrt{\sin^2(2t) + 9 - 12t + 4t^2 + 1}$

$|v(0)| = \sqrt{\sin^2(0) + 9 + 1} = \sqrt{10}$

$$T_0 = \left(\frac{-\sin(0)}{\sqrt{10}},\ \frac{3}{\sqrt{10}},\ \frac{1}{\sqrt{10}}\right) = \left(0,\ \frac{3}{\sqrt{10}},\ \frac{1}{\sqrt{10}}\right)$$

evt $= \left(0,\ \frac{3\sqrt{10}}{10},\ \frac{\sqrt{10}}{10}\right)$

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5) Anta $f: \mathbb{R}^2 \to \mathbb{R}$ har kont. annenordens partiellderiverte, og at $r(t) = x(t)i + y(t)j$ der x og y er to ganger deriverbare. La $g(t) = f(r(t))$. Vis at:

$$g''(t) = \frac{\partial^2 f}{\partial x^2}(r(t))x'(t)^2 + 2\frac{\partial^2 f}{\partial x \partial y} r(t)x'(t)y'(t) + \frac{\partial^2 f}{\partial y^2} r(t)y'(t)^2$$
$$+\frac{\partial f}{\partial x} r(t)x''(t) + \frac{\partial f}{\partial y} r(t)y''(t)$$

$$g'(t) = f'(r(t)) \cdot r'(t)$$
$$= \left(\frac{\partial f}{\partial x}(r(t)), \frac{\partial f}{\partial y}(r(t))\right) \cdot (x'(t), y'(t))$$
$$= \frac{\partial f}{\partial x}(r(t))x'(t) + \frac{\partial f}{\partial y}(r(t))y'(t)$$

La $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$ Merk: $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$

$$g''(t) = f_x'(r(t)) \cdot x'(t) \cdot r'(t) + f_x(r(t)) \cdot x''(t)$$
$$+ f_y'(r(t)) \cdot y'(t) \cdot r'(t) + f_y(r(t)) \cdot y''(t)$$
$$= \left(\frac{\partial^2 f}{\partial x^2}(r(t))x'(t), \frac{\partial^2 f}{\partial x \partial y}(r(t))x'(t)\right) \cdot (x'(t), y'(t))$$
$$+\frac{\partial f}{\partial x}(r(t)) \cdot x''(t) + \frac{\partial f}{\partial y}(r(t)) \cdot y''(t)$$
$$+\left(\frac{\partial^2 f}{\partial x \partial y}(r(t))y'(t), \frac{\partial^2 f}{\partial y^2}(r(t))y'(t)\right) \cdot (x'(t), y'(t))$$
$$= \frac{\partial^2 f}{\partial x^2}(r(t))x'(t)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(r(t))x'(t)y'(t)$$
$$+\frac{\partial^2 f}{\partial y^2}(r(t))y'(t)^2 + \frac{\partial f}{\partial x}(r(t))x''(t) + \frac{\partial f}{\partial y}(r(t))y''(t) \quad \blacksquare$$

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6) a) $f(x,y) = xy \qquad r(t) = (e^t, \cos t)$

$$g(t) = f(r(t)) = e^t \cos t$$

$$g'(t) = \underline{\underline{e^t \cos t - e^t \sin t}} = e^t(\cos t - \sin t)$$

$$\nabla f(r(t)) \cdot r'(t) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (x'(t), y'(t))$$

$$= (\cos t, e^t) \cdot (e^t, -\sin t)$$

$$= \underline{\underline{e^t \cos t - e^t \sin t}} = \underline{g'(t)}$$

b) $f(x,y) = (x^2 + y^2) \log\left(\sqrt{x^2+y^2}\right) \qquad r(t) = (e^t, e^{-t})$

$$g(t) = f(r(t)) = (e^{2t} + e^{-2t}) \log\left(\sqrt{e^{2t}+e^{-2t}}\right)$$

$$g'(t) = (2e^{2t} - 2e^{-2t}) \log\left(\sqrt{e^{2t}+e^{-2t}}\right)$$

$$+ (e^{2t} + e^{-2t}) \frac{e^{2t} - e^{-2t}}{\ln 10 \cdot (e^{2t} + e^{-2t})}$$

$$= \underline{\underline{(2e^{2t} - 2e^{-2t}) \log\left(\sqrt{e^{2t}+e^{-2t}}\right) + \frac{e^{2t} - e^{-2t}}{\ln 10}}}$$

$$\nabla f(r(t)) \cdot r'(t) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (x'(t); y'(t))$$

$$= \Bigg(2e^t \log\left(\sqrt{e^{2t}+e^{-2t}}\right) + (\cancel{e^{2t} + e^{-2t}}) \frac{e^t}{(\ln 10)(\cancel{e^{2t}+e^{-2t}})},$$

$$2e^{-t} \log\left(\sqrt{e^{2t}+e^{-2t}}\right) + \frac{e^{-t}}{(\ln 10)}\Bigg) \cdot (e^t, -e^{-t})$$

$$= 2e^{2t} \log\left(\sqrt{e^{2t}+e^{-2t}}\right) + \frac{e^{2t}}{\ln 10} - 2e^{-2t} \log\left(\sqrt{e^{2t}+e^{-2t}}\right)$$

$$- \frac{e^{-2t}}{\ln 10}$$

$$= \underline{\underline{(2e^{2t} - 2e^{-2t}) \log\left(\sqrt{e^{2t}+e^{-2t}}\right) + \frac{e^{2t} - e^{-2t}}{\ln 10}}} = \underline{g'(t)}$$

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7) Regn ut $\int_C f\,ds$ når $f(x,y,z) = z\cos(xy)$ og

C param. ved $r(t) = 3t\,i + 4t\,j + 5t\,k$, $t \in [0, \sqrt{\pi}]$

$\forall$ r er glatt i intervallet

$$\int_C f\,ds = \int_{s_0}^{s_1} f(r(s))\,ds = \int_0^{\sqrt{\pi}} f(r(t))\,|r'(t)|\,dt$$

$$= \int_0^{\sqrt{\pi}} 5t\cos(12t^2)\cdot\sqrt{50}\;dt$$

$$= \int_0^{\pi} 25\sqrt{2}\;t\;\cos(12t^2)\,dt \quad (\star)$$

$u = 12t^2$, $du = 24t\,dt$, $t\,dt = \frac{1}{24}du$

$$\int 25\sqrt{2}\,t\cos(12t^2)\,dt = \frac{25\sqrt{2}}{24}\int \cos(u)\,du$$

$$= \frac{25\sqrt{2}}{24}\sin(u) + C = \frac{25\sqrt{2}}{24}\sin(12t^2) + C$$

$$(\star) = \frac{25\sqrt{2}}{24}\left[\sin(12t^2)\right]_0^{\sqrt{\pi}} = \frac{25\sqrt{2}}{24}(0-0) = \underline{\underline{0}}$$

8) Regn ut $\int_C f\,ds$ når $f(x,y,z) = xyz$, C param. ved

$r(t) = (e^t, -e^{-t}, \sqrt{2}\,t)$ $t \in [0,1]$

r glatt i intervallet

$$\int_C f\,ds = \int_0^1 f(r(t))\,|r'(t)|\,dt$$

$$= \int_0^1 \frac{e^t}{-e^t}\sqrt{2}\,t \cdot \sqrt{e^{2t} + e^{-2t} + 2}\;dt$$

$$= \int_0^1 -\sqrt{2}\,t\cdot\sqrt{e^{2t}+e^{-2t}+2}\;dt \qquad \left[\begin{aligned} &e^{2t} + 2 + e^{-2t} \\ &= (e^t)^2 + 2(e^t)(e^{-t}) + (e^{-t})^2 \end{aligned}\right.$$

$$= \int_0^1 -\sqrt{2}\,t\sqrt{(e^t + e^{-t})^2}\;dt$$

$$= \int_0^1 -\sqrt{2}\,t\,(e^t + e^{-t})\,dt \qquad \left[\begin{aligned} &u = t & u' = 1 \\ &v' = (e^t + e^{-t}) & v = e^t - e^{-t} \end{aligned}\right.$$

$$= -\sqrt{2}\left(\left[t(e^t - e^{-t})\right]_0^1 - \int_0^1 e^t - e^{-t}\,dt\right)$$

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forts.

8) Har: $= -\sqrt{2}\left(\left[t(e^t - e^{-t})\right]_0^1 - \int_0^1 e^t - e^{-t}\,dt\right)$

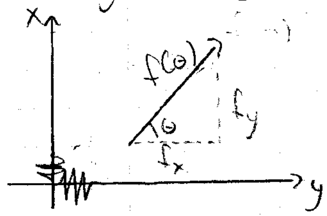
$= -\sqrt{2}\left(\left[t(e^t - e^{-t})\right]_0^1 - \left[e^t + e^{-t}\right]_0^1\right)$

$= -\sqrt{2}\left(e^1 - e^{-1} - e^1 - e^{-1} + e^0 + e^{-0}\right)$

$= -\sqrt{2}\left(2 - 2e^{-1}\right) = \underline{\underline{2\sqrt{2}\left(e^{-1} - 1\right)}}$

9) En kurve $r = f(\theta)$, $\theta \in [a, b]$

a) Kurven har parametrisering $r(\theta) = (f_x, f_y)$, der f_x er kurven i x-retning og f_y i y-retning. Se på en øyeblikksskisse av $f(\theta)$:



Ser at $f_x = f(\theta)\cos\theta$ og $f_y = f(\theta)\sin(\theta)$

$\Rightarrow \underline{\underline{r(\theta) = \left(f(\theta)\cos\theta,\ f(\theta)\sin\theta\right)}}$

b) $|v(\theta)| = |r'(\theta)| =$

$= \left|\left(f'(\theta)\cos\theta - f(\theta)\sin(\theta),\ f'(\theta)\sin\theta + f(\theta)\cos(\theta)\right)\right|$

$= \sqrt{f'(\theta)^2\cos^2\theta - 2f'(\theta)\cos\theta f(\theta)\sin\theta + f(\theta)^2\sin^2(\theta)}$

forts: $\sqrt{+ f'(\theta)^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\cos^2\theta}$

$\cos^2 x + \sin^2 x = 1$: $\underline{\underline{= \sqrt{f(\theta)^2 + f'(\theta)^2}}}$

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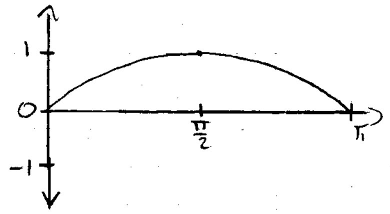
9) c) Anta $f(\theta) = \sin\theta$, $\theta \in [0, \pi]$

Skisse :



$$L = \int_0^\pi |r'(\theta)|\, d\theta = \int_0^\pi |v(\theta)|\, d\theta$$

$$= \int_0^\pi \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^\pi \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0^\pi 1\, d\theta = \underline{\underline{\pi}}$$

d) Regn ut $\int_C g\, ds$ der C er r fra (c) og $g(x,y) = xy$

r glatt i intervallet, $r(\theta) = (\sin\theta\cos\theta,\ \sin^2\theta)$

$$\int_C g\, ds = \int_0^\pi g(r(\theta)) \cdot |r'(\theta)|\, d\theta$$

$$= \int_0^\pi \sin^3\theta \cos\theta \cdot 1\, d\theta$$

$$= \int_0^\pi \sin^3\theta\cos\theta\, d\theta = (*) \qquad u = \sin\theta \qquad du = \cos\theta\, d\theta$$

$$\int \sin^3\theta\cos\theta\, d\theta = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{\sin^4\theta}{4} + C$$

$$(*) = \left[\frac{\sin^4\theta}{4}\right]_0^\pi = \frac{0}{4} - \frac{0}{4} = \underline{\underline{0}}$$