ØVING 2, side 1 — Andreas B. Berg

7.3.2) Find the moment-generating function for a chi square random variable and use it to show that $E(X_n^2) = n$, $Var(X_n^2) = 2n$

Finner i formelheftet at $M_X(t) = \left(\frac{1}{1-2t}\right)^{n/2}$ for $X = \chi_n^2$

$$E(X) = M^{(1)}(t)\big|_{t=0} \qquad \left[\text{kjerneregel: } \frac{1}{1-2t} = u\right]$$

$$= \frac{d}{du} u^{\frac{n}{2}} \frac{d}{dt}\left(\frac{1}{1-2t}\right) = \frac{d}{du} u^{\frac{n}{2}} \frac{d}{dt}\left((1-2t)^{-1}\right)$$

$$= \frac{n u^{\frac{n-2}{2}}}{2} (-1)(1-2t)^{-2}(-2)$$

$$= n \left(\frac{1}{1-2t}\right)^{\frac{n-2}{2}} \left(\frac{1}{1-2t}\right)^2 = \frac{n}{(1-2t)^{\frac{n+2}{2}}}$$

$$[t=0]: \quad = \frac{n}{(1-0)^{(n+2)/2}} = \underline{\underline{n}}$$

$$E(X^2) = M^{(2)}(t)\big|_{t=0}$$

$$M^{(2)}(t) = \frac{d}{dt} \frac{n}{(1-2t)^{\frac{n+2}{2}}} = n \frac{d}{dt} (1-2t)^{\frac{-n-2}{2}}$$

$$= n\left(\frac{-n-2}{2}\right)(1-2t)^{\frac{-n-4}{2}}(-2)$$

$$= n(n+2)(1-2t)^{\frac{-n-4}{2}}$$

$$[t=0]: \quad = n(n+2)(1-0)^{\frac{-n-4}{2}} = n(n+2)$$

$$Var(X) = E(X^2) - E(X)^2 = n^2 + 2n - n^2 = \underline{\underline{2n}}$$

7.3.4) Use the fact that $(n-1)S^2/\sigma^2$ is a chi square rand. var. with $n-1$ df to prove that

$$Var(S^2) = \frac{2\sigma^4}{n-1}$$

$$\frac{(n-1)S^2}{\sigma^2} = \chi^2_{n-1} \Rightarrow S^2 = \frac{\chi^2_{n-1}\cdot\sigma^2}{n-1}$$

$$Var(S^2) = Var\left(\frac{\chi^2_{n-1}\cdot\sigma^2}{n-1}\right) = Var\left((n-1)^{-1}\cdot\sigma^2\cdot\chi^2_{n-1}\right)$$

$[$Vet: $Var(aX) = a^2 Var(X)$, $Var(\chi^2_a) = 2a]$

$$= (n-1)^{-2}\sigma^4 Var(\chi^2_{n-1}) = \frac{\sigma^4}{(n-1)^2}2(n-1) = \underline{\underline{\frac{2\sigma^4}{(n-1)}}}$$

7.3.8) V, U indep. chi square random var. with 7 and 9 df, respectively. Is it more likely that $\frac{V/7}{U/9}$ will be between (1) $(2.51, 3.29)$ or (2) $(3.29, 4.20)$

Def 7.3.3 gir at $F_{m,n} = \frac{V/m}{U/n}$ har F-distribusjon med m og n df. $F_{7,9} = \frac{V/7}{U/9}$

~~La $F_{7,9} = \frac{V/7}{U/9}$, F-distribuert. T 7.3.3 gir pdf~~

$$\cancel{f_{F_{7,9}}(w) = \frac{\Gamma(\frac{7+9}{2})7^{7/2}9^{9/2}w^{(7/2)-1}}{\Gamma(\frac{7}{2})\Gamma(\frac{9}{2})(9+7w)^{(7+9)/2}} = \frac{\Gamma(8)7^{7/2}9^{9/2}w^{9/2}}{\Gamma(7/2)\Gamma(9/2)(9+7w)^8}}$$

$$\cancel{= \frac{\Gamma(8)7^{7/2}9^{9/2}}{\Gamma(7/2)\Gamma(9/2)}\cdot\frac{w^{9/2}}{(9+7w)^8} \overset{def}{=} a\cdot\frac{w^{9/2}}{(9+7w)^8}}$$

$$\cancel{P(2.51 < F < 3.29) = \int_{2.51}^{3.29} a\frac{w^{9/2}}{(9+7w)^8}dw = a\int_{2.51}^{3.29}\frac{w^{9/2}}{(9+7w)^8}dw}$$

Ser fra tabell at $F_{0.05,7,9} = 3{,}29$. Med andre ord er

$P(F_{7,9} > 3{,}29) = 0{,}05$, altså er $P(F_{7,9} < 3{,}29) = 0{,}95$

Ser også at $P(F_{7,9} > 4{,}20) = 0{,}025$.

Ser at $P(3{,}29 < F_{7,9} < 4{,}20) = 0{,}05 - 0{,}025 = \underline{0{,}025}$

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cont

7.3.8) Ser fra tabell at $P(F_{7,9} < 2{,}51) = 0{,}90$

$$\Rightarrow P(2{,}51 < F_{7,9} < 3{,}29) = P(F_{7,9} < 3{,}29) - P(F_{7,9} < 2{,}51) = \underline{0{,}05}$$

<u>Lett å se at</u> $\underline{P(2{,}51 < F_{7,9} < 3{,}29) > P(3{,}29 < F_{7,9} < 4{,}20)}$

$\Rightarrow \frac{U/7}{V/9}$ er mest sannsynlig i (1)

7.3.9) a) $P(0{,}109 < F_{4,6} < x) = 0{,}95$

$$= P(0{,}109 < F_{4,6}) - P(F_{4,6} > x)$$

$$= 0{,}975 - P(F_{4,6} > x) = 0{,}95$$

$$P(F_{4,6} > x) = 0{,}025 \Rightarrow F(F_{4,6} < x) = 0{,}975$$

tabell $\Rightarrow$ $\underline{\underline{x = 6{,}23}}$

b) $P(0{,}427 < F_{11,7} < 1{,}69 = x$

$$= P(F_{11,7} < 1{,}69) - P(F_{11,7} < 0{,}427)$$

$$= 0{,}75 - 0{,}10 = \underline{\underline{0{,}65}} = x$$

c) $P(F_{x,x} > 5{,}35) = 0{,}01$

$$\Rightarrow P(F_{x,x} < 5{,}35) = 0{,}99$$

tabell $\Rightarrow$ $\underline{\underline{x = 9}}$

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7.3.11) If a random variable F has an F distribution with m and n df, show that $\frac{1}{F}$ has an F distribution with n and m df

Vet fra def. 7.3.3 at hvis F er F-fordelt, kan vi skrive $F = \frac{U/m}{V/n}$, der $U \sim \chi^2_m$ og $V \sim \chi^2_n$ chi square-fordelte med m og n df, respektivt.

Ser lett at $\frac{1}{F} = \frac{V/n}{U/m}$ med samme U, V. Da sier def 7.3.3 at $\frac{1}{F} \sim F_{n,m}$, altså at $\frac{1}{F}$ er F-distribuert med n og m df.

7.3.14) Evaluate the integral $\int_0^\infty \frac{1}{1+x^2}\,dx$ using student t distribution

Ser på pdf til student t-fordeling:

$$f_{T_n}(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)\left(1+t^2/n\right)^{(n+1)/2}}, \quad -\infty < t < \infty$$

La $n = 1$, og vi får

$$f_{T_1}(t) = \frac{\Gamma(1)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)(1+t^2)} = \frac{1}{\sqrt{\pi}\sqrt{\pi}\,(1+t^2)} = \frac{1}{\pi}\cdot\frac{1}{1+t^2}$$

Ser at $\frac{1}{1+x^2} = \pi f_{T_1}(x)$. Vet at $f_{T_n}(x)$ er symmetrisk, og at $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \;\forall X$

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2}\,dx = \pi\int_0^\infty f_{T_1}(x)\,dx = \pi\,\frac{1}{2}\int_{-\infty}^{\infty} f_{T_1}(x)\,dx = \underline{\underline{\frac{\pi}{2}}}$$