ØVING 8 side 1 — Andreas B. Berg

12.5

2) Bruk Weierstrass' M-test, vis konv. på mengden

a) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ på $\mathbb{R}$

$$|\sin(na)| \leq 1 \quad \forall n, a \Rightarrow \left|\frac{\sin(na)}{n^2}\right| \leq \frac{1}{n^2} \quad \forall a$$

Vet at $\sum M_n = \sum \frac{1}{n^2}$ konvergerer på $\mathbb{R}$

$\Rightarrow \underline{\underline{\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \text{ konv. uniformt og absolutt på } \mathbb{R}}}$

b) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^3}}$ på $[-1,1]$

$$\left|\frac{a^n}{\sqrt{n^3}}\right| = \frac{|a|^n}{|\sqrt{n^3}|} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} \quad \forall a \in [-1,1]$$

$\frac{1}{n^{3/2}}$ konvergerer, siden $\frac{3}{2} > 1$

$\Rightarrow \underline{\underline{\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^3}} \text{ konv. uniformt og absolutt på } [-1,1]}}$

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3) Vis at $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ kont på $[-1,1]$

Finner konvergensradien ved forholdstest:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}\cdot n^2}{(n+1)^2\cdot x^n}\right| = \lim_{n\to\infty}\left|\frac{x\cdot n^2}{n^2+2n+1}\right| = \lim_{n\to\infty}\left|\frac{x}{1+\frac{2}{n}+\frac{1}{n^2}}\right| = |x|$$

$\Rightarrow$ Konv. rad $= 1$

$x = 1$:

$f(1) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ konv.

$x = -1$:

$f(-1) = \sum_{-1}^{\infty} \frac{(-1)^n}{n^2}$ konv. absolutt $\Rightarrow$ konv.

$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ konv. på $x \in [-1,1]$

$\xrightarrow{\text{Lemma 12.6.8}}$ $\underline{\underline{f(x) \text{ kont på } [-1,1]}}$

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1) Finn taylorrekke i pkt. Finn konv. int I. Vis at rekken konv. mot funk. på I

d) $f(x) = \frac{1}{x}$ i pkt 1

$f'(x) = -1/x^2$ $\qquad f(1) = 1$

$f''(x) = 2/x^3$ $\qquad f'(1) = -1 \quad = -1!$

$f'''(x) = -6/x^4$ $\qquad f''(1) = 2 \quad = 2!$

$\qquad f'''(1) = -6 \quad = -3!$

$$T_n f(x) = 1 - (x-1) + \frac{2}{2}(x-1)^2 - \frac{3!}{3!}(x-1)^3 + \dots$$

$$= \underline{\underline{\sum_{n=0}^{\infty} (-1)^n (x-1)^n}}$$

Finner konv. rad med rottest:

$$\lim_{n\to\infty} \left| \sqrt[n]{(-1)^n (x-1)^n} \right| = \lim_{n\to\infty} \left| \sqrt[n]{(1-x)^n} \right| = \lim_{n\to\infty} |1-x| = |1-x|$$

$\Rightarrow$ konv. hvis $-1 < 1-x < 1 \Rightarrow -2 < -x < 0 \Rightarrow 2 > x > 0$

$x = 0$:

$$\sum_{n=0}^{\infty} (-1)^n (-1)^n = \sum_{n=0}^{\infty} 1 = 1 \Rightarrow \text{divergent}$$

$x = 2$:

$$\sum_{n=0}^{\infty} (-1)^n (1)^n = \sum_{n=0}^{\infty} (-1)^n \Rightarrow \text{divergent}$$

<u>Rekken har konv. int. $(0,2)$</u>

La $s(x)$ = summen av rekka på $(0,2) = \sum_{n=0}^{\infty} (1-x)^n$. La $y = (1-x)^n$

$$s(x) = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} = \frac{1}{1-(1-x)} = \frac{1}{x} = \underline{f(x)}$$

$$|y| < 1 \Rightarrow |1-x| < 1 \Rightarrow 0 < x < 2$$

<u>Dermed konvergerer $T_n f(x)$ mot $f(x)$ på $x \in (0,2)$</u>

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1) e) $f(x) = \ln(x+1)$; pkt 0

$$f'(x) = \frac{1}{x+1} \qquad f(0) = 0$$
$$f''(x) = -\frac{1}{(x+1)^2} \qquad f'(0) = 1$$
$$f'''(x) = \frac{2}{(x+1)^3} \qquad f''(0) = -1 = -1!$$
$$f^{(4)}(x) = \frac{-6}{(x+1)^4} \qquad f'''(0) = 2 = 2!$$
$$f^{(4)}(0) = -6 = -3!$$

$$T_n f(x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$
$$\underline{\underline{= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}}}$$

Konv. rad med forholdstest:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n x^{n+1} \cdot n}{(-1)^{n-1} x^n (n+1)}\right| = \lim_{n\to\infty} \left|\frac{x \cdot n}{n - n}\right| = \lim_{n\to\infty} \left|\frac{x}{\frac{1}{n} - 1}\right|$$

$$= |x| \quad : \quad |x| < 1 \Rightarrow \text{Konv. rad} = 1$$

$x = 1$:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \sum \frac{(-1)^{n-1}}{n}$$

$$\lim_{n\to\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0 \quad \text{og} \quad \frac{1}{n} > 0 \Rightarrow \text{konv.}$$

$x = -1$:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum \frac{(-1)^{2n-1}}{n} = \sum \frac{-1}{n} = -\sum \frac{1}{n} \Rightarrow \text{div}$$

<u>Konv. omr. $I = (-1, 1]$</u>

La $s(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

$$s'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}$$

$$s(x) = \int_0^x \frac{1}{1+t}\,dt = \ln(1+x) + \ln(1+0) = \ln(x+1) = f(x)$$

<u>Dermed konv. $T_n f(x)$ mot $f(x)$ på $I = (-1, 1]$</u>

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3) Bruk $T_n g(x)$ om 0 til å finne $T_n f(x)$:

b) $g(x) = e^x \qquad f(x) = e^{-x^3} \qquad g(0) = e^0 = 1$

$g'(x) = g''(x) = \dots = e^x \qquad g'(0) = g''(0) = \dots = 1$

$$T_n g(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$T_n f(x) = T_n g(-x^3) = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \underline{\underline{\sum_{n=0}^{\infty} \frac{(-x)^{3n}}{n!}}}$$

d) $g(x) = \ln(1+x) \qquad f(x) = \ln(1-x^3) \qquad$ Vet fra 12.8.1 e:

$$T_n g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$T_n f(x) = T_n g(-x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-x^3)^n}{n} = \sum_{n=1}^{\infty} - \frac{x^{3n}}{n} = \underline{\underline{-\sum_{n=1}^{\infty} \frac{x^{3n}}{n}}}$$

f) $g(x) = e^x \qquad f(x) = x^2 e^x$

$$T_n g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$T_n f(x) = x^2 T_n g(x) = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underline{\underline{\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}}} \overset{evt}{=} \underline{\underline{\sum_{n=2}^{\infty} \frac{x^n}{(n-1)!}}}$$

14) a) Finn konv. omr. til $\sum_{n=0}^{\infty} \frac{3^n x^n}{n+1}$. Forholdstest:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1} (n+1)}{(n+2) 3^n x^n} \right| = \lim_{n\to\infty} \left| \frac{3x(n+1)}{n+2} \right| = \lim_{n\to\infty} \left| \frac{3x + \frac{3x}{n}}{1 + \frac{2}{n}} \right|$$

$$= |3x| < 1 \quad \Rightarrow \quad -1 < 3x < 1 \quad \Rightarrow \quad -\frac{1}{3} < x < 1$$

$x = -\frac{1}{3}$:

$$\sum_{n=0}^{\infty} \frac{3^n (-\frac{1}{3})^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Rightarrow \text{konv.}$$

$x = \frac{1}{3}$:

$$\sum_{n=0}^{\infty} \frac{3^n (\frac{1}{3})^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \Rightarrow \text{div.}$$

$\underline{\underline{\sum_{n=0}^{\infty} \frac{3^n x^n}{n+1} \text{ konv. på } [-\frac{1}{3}, \frac{1}{3})}}$

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14) b) Finn summen $s(x)$ til rekken

$$x\,s(x) = x\sum_{n=0}^{\infty} \frac{3^n x^n}{n+1} = \sum_{n=0}^{\infty} 3^n \frac{x^{n+1}}{n+1}$$

$$\left(x\,s(x)\right)' = \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$$

$$x\,s(x) = \int_0^x \frac{1}{1-3t}\,dt = \ln(1-3x)$$

$$\underline{\underline{s(x) = \frac{\ln(1-3x)}{x} \quad \text{på } \left[-\tfrac{1}{3}, \tfrac{1}{3}\right)}}$$

15) a) Finn konv. omr. til $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$. Forholdstest:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2n+2}(2n+1)}{(2n+3)(-1)^n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{-x^2(2n+1)}{2n+3}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-x^2 + \frac{1}{2n}}{1 + 3/2n}\right| = |-x^2|$$

$$|-x^2| < 1 \Rightarrow |x^2| < 1 \Rightarrow -1 < x^2 < 1 \Rightarrow -1 < x < 1$$

$x = -1$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{3n}}{2n+1} = \sum \frac{(-1)^n}{2n+1} \;:\; \lim_{n\to\infty} \frac{1}{2n+1} = 0 \Rightarrow \text{konv.}$$

$x = 1$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \Rightarrow \text{konv.}$$

$$\underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1} \quad \text{konv. på } [-1, 1]}}$$

b) Finn summen $s(x)$ av rekken

$$x\,s(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\left(x\,s(x)\right)' = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2}$$

$$s(x) = \frac{1}{x}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{x}\cdot \arctan x = \frac{\arctan x}{x}, \quad x \neq 0$$

$$x = 0:\; s(x) = \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n}}{2n+1} = 1 + \sum_{n=1}^{\infty} (-1)^n 0 = 1$$

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15) b) $s(x) = \begin{cases} \dfrac{\arctan x}{x}, & x \in [-1,0) \cup (0,1] \\ 1, & x = 0 \end{cases}$

c) Bruk (b) til å regne $\arctan(\frac{1}{2})$ med nøyaktighet > 0.01

$$\arctan\left(\tfrac{1}{2}\right) = \tfrac{1}{2} s\left(\tfrac{1}{2}\right) = \tfrac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$$

Nøyaktighet bedre enn 0,01:

$0{,}01 > |S_{n+1}(\frac{1}{2})|$, der S_{n+1} er første ledd vi ikke tar med.

$$S_n\left(\tfrac{1}{2}\right) = (-1)^n \frac{(\frac{1}{2})^{2n}}{2n+1} = \frac{(-\frac{1}{4})^n}{2n+1}$$

$n = 1: |S_1(\frac{1}{2})| = 0{,}08.. > 0{,}01$

$n = 2: |S_2(\frac{1}{2})| = 0{,}012 > 0{,}01$

$n = 3: |S_3(\frac{1}{2})| = 0{,}002 < 0{,}01$

$$\arctan\left(\tfrac{1}{2}\right) = \tfrac{1}{2} \sum_{n=0}^{2} (-1)^n \frac{(\frac{1}{4})^n}{2n+1} = \tfrac{1}{2} \sum_{n=0}^{2} \frac{(-\frac{1}{4})^n}{2n+1}$$

$$= \tfrac{1}{2}\left(1 - \tfrac{1}{12} + \tfrac{1}{80}\right) = \tfrac{1}{2}\left(\tfrac{223}{240}\right) = \underline{\underline{\tfrac{223}{480}}}$$

Kommentar: fasiten har sagt $\arctan(\frac{1}{2}) = \frac{1}{2}\sum_{n=0}^{1} a_n$. Siden $a_2 = 0{,}0125 > 0{,}01$ mener jeg dette er feil, siden vi ikke med sikkerhet kan si at nøyaktigheten er bedre enn 0.01. Derfor bruker jeg $\arctan(\frac{1}{2}) = \frac{1}{2}\sum_{n=0}^{2} a_n$

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18) a) Avgjør for hvilke x $\sum_{n=1}^{\infty} \frac{n}{n+1} x^{n+1}$ konv. Forholdstest:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+2}(n+1)}{(n+2)\,x^{n+1}\cdot n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x}{(n^2+2n)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2x+2nx+x}{n^2+2n}\right| = \lim_{n\to\infty}\left|\frac{x+\frac{2x}{n}+\frac{x}{n^2}}{1+2/n}\right| = |x| \Rightarrow \text{konv.rad} = 1$$

$x = -1$:

$$\sum_{n=1}^{\infty} \frac{n}{n+1}(-1)^{n+1} \Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1} n}{n+1} \neq 0 \Rightarrow \text{div.}$$

$x = 1$:

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \Rightarrow \lim_{n\to\infty} \frac{n}{n+1} \neq 0 \Rightarrow \text{div}$$

$$\underline{\underline{\sum_{n=1}^{\infty} \frac{n}{n+1} x^{n+1} \text{ konv. for } x \in (-1,1)}}$$

b) Finn sum $s(x)$ av rekka

$$S'(x) = \sum_{n=1}^{\infty} n x^n$$

$$\frac{S'(x)}{x} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\int_0^x \frac{s'(t)}{t}\,dt = \sum_{n=1}^{\infty} \int_0^x n t^{n-1}\,dt = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

$$\frac{S'(x)}{x} = \frac{d}{dx}\frac{1}{1-x} = \ln(1-x)$$

$$S'(x) = x\ln(1-x)$$

$$S(x) = \int_0^x t\ln(1-t)\,dt$$

$$\int x\ln(1-x)\,dx \qquad u = 1-x \quad du = -dx \quad x\,dx = -x\,du = (u-1)\,du$$

$$= \int (u-1)\ln u\,du = \int \overset{v'}{u}\,\overset{w}{\ln u}\,du - \int \overset{z'}{1}\,\overset{y}{\ln u}\,du$$

$$= \frac{1}{2}u^2\ln u - \int \frac{u}{2}\,du - u\ln u + \int 1\,du$$

$$= \frac{2u^2\ln u - u^2}{4} - u\ln u + u \quad = \frac{2(1-x)^2\ln(1-x)}{4} - \frac{(1-x)^2}{4} - (1-x)\ln(1-x) + (1-x)$$

$$\underline{\underline{S(x) = \frac{(1-x)^2(2\ln x - 1)}{4} + (1-x)(1-\ln(1-x))}}$$

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1) Vis at likhetene holder for $|x| < 1$:

a) $$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

$$(1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n = \sum_{n=0}^{\infty} \frac{-2\cdot(-3)\cdot\ldots\cdot(-1-n)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{(n+1)!}{n!}\right) x^n = \underline{\sum_{n=0}^{\infty} (-1)^n (n+1) x^n}$$

b) $$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot2n} x^n$$

$$(1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n = \sum_{n=0}^{\infty} \frac{(-1/2)(-3/2)\cdot\ldots\cdot(1/2-n)\cdot2^n}{n!\cdot2^n} x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot3\cdot\ldots\cdot(2n-1)}{2^n(n!)} x^n = \underline{\sum_{n=0}^{\infty} (-1)^n \frac{1\cdot3\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot\ldots\cdot2n} x^n}$$

c) $$(1+x)^{1/3} = 1 + \sum_{n=1}^{\infty} \frac{(-1)\cdot2\cdot5\cdot\ldots\cdot(3n-4)}{3^n\, n!} x^n$$

$$\sum_{n=0}^{\infty} \binom{1/3}{n}(-x)^n = \sum_{n=1}^{\infty} \frac{(1/3)(-2/3)(-5/3)\cdot\ldots\cdot(4/3-n)\cdot3^n}{n!\cdot3^n} (-x)^n + 1$$

$$= \sum_{n=1}^{\infty} \frac{(1)(-2)(-5)\cdot\ldots\cdot(4-3n)}{3^n(n!)} (-x)^n + 1$$

$$= \underline{\sum_{n=1}^{\infty} \frac{(-1)(2)(5)\cdot\ldots\cdot(3n-4)}{3^n(n!)} x^n + 1}$$

d) $$\sqrt{1-x^2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)\cdot1\cdot3\cdot\ldots\cdot(2n-3)}{2\cdot4\cdot6\cdot\ldots\cdot2n} x^{2n}$$

$$(1+(-x^2))^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n}(-x^2)^n = 1 + \sum_{n=1}^{\infty} \frac{(1/2)(-1/2)(-3/2)\cdot\ldots\cdot(3/2-n)}{n!}(-x^2)^n$$

$$= \underline{1 + \sum_{n=1}^{\infty} \frac{(-1)\cdot1\cdot3\cdot\ldots\cdot(2n-3)}{2^n\, n!} x^{2n}}$$

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2) Finn $T_n f(x)$, $f(x) = \ln(x + \sqrt{1+x^2})$ om pkt. 0

$$f'(x) = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}}$$

$$= 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (x^2)^n = 1 + \sum_{n=1}^{\infty} \frac{(1/2)(-1/2)(-3/2)\cdot\ldots\cdot(3/2-n)}{n!} (x^2)^n$$

$$\Rightarrow T_n f(x) = \int_0^x \frac{1}{\sqrt{1+t^2}}\, dt = \int_0^x 1 + \sum_{n=1}^{\infty} \frac{(1)(-1)(-3)\cdot\ldots\cdot(3-2n)}{2^n\, n!}\, t^{2n}$$

$$\underline{\underline{T_n f(x) = x + \sum_{n=1}^{\infty} \frac{(-1)(1)(3)\cdot\ldots\cdot(2n-3)}{(-2)^n\, n!\,(2n+1)}\, x^{2n+1}}}$$

evt $$\underline{\underline{= x + \sum_{n=1}^{\infty} \frac{(-1)(1)(3)\cdot\ldots\cdot(2n-3)}{2\cdot 4\cdot\ldots\cdot 2n\cdot(2n+1)}\, (-1)^n x^{2n+1}}}$$