Øving 5 side 1 Andreas B. Berg

11.3

4) Vis at hvis $\{f_n\}$ konvergerer uniformt mot f, konvergerer den også punktvis.

$\{f_n\}$ konvergerer uniformt mot f

$$\Rightarrow \lim_{n\to\infty} d_A(f, f_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} f_n(x) - f(x) = 0 \qquad \forall x$$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) \Rightarrow \underline{\{f_n\} \text{ konv. punktvis mot } f.}$$

11) Vis at hvis $\{f_n\}$ og $\{g_n\}$ konv. uniformt mot f og g, så konv. $\{f_n + g_n\}$ unif. mot $f+g$

Vet: $\lim_{n\to\infty} d_A(f, f_n) = \lim_{n\to\infty} d_A(g_n, g) = 0$

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|\} = \lim_{n\to\infty} \sup\{|g_n(x) - g(x)|\} = 0$$

Ser: $\lim_{n\to\infty} \sup\{|f_n(x) + g_n(x) - (f(x) + g(x))|\}$

$$= \lim_{n\to\infty} \sup\{|\underbrace{f_n(x) - f(x)} + \underbrace{g_n(x) - g(x)}|\}$$

$$= \lim_{n\to\infty} \sup\{|\ 0 \ + \ 0\ |\} \ \underline{= 0}$$

<u>Så $\{f_n + g_n\}$ konv. uniformt mot $f+g$</u>

 Andreas B.Berg

12.1

1). Finn sum til geometrisk rekke:

a) $1 - \frac{1}{3} + \frac{1}{9} - \dots = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n \underset{|r|<1}{=} \frac{1}{1 + 1/3} = \frac{1}{4/3} = \underline{\underline{\frac{3}{4}}}$

c) $4 - \frac{2}{3} + \frac{1}{9} - \frac{1}{54} + \dots = \sum_{n=0}^{\infty} 4\left(\frac{-1}{6}\right)^n \underset{|\frac{-1}{6}|<1}{=} \frac{4}{1 + 1/6} = \frac{4}{7/6} = \underline{\underline{\frac{24}{7}}}$

3) Vis at rekken er konv. geom. rekke og finn summen

b) $\sum_{n=0}^{\infty} x^{2n}$, $|x|<1 \Rightarrow |x^2|<1$

$\sum_{n=0}^{\infty} (x^2)^n = \underline{\underline{\frac{1}{1-x^2}}}$

7) a) Bruk $T_n \ln(1-x)$ om $x=0$ til å vise $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$

$f(x) = \ln(1-x)$ $\qquad f(0) = \ln 1 = 0$

$f'(x) = \frac{-1}{1-x}$ $\qquad f'(0) = \frac{-1}{1} = -1$

$f''(x) = \frac{-1}{(1-x)^2}$ $\qquad f''(0) = \frac{-1}{1} = -1$

$f'''(x) = \frac{-2}{(1-x)^3}$ $\qquad f'''(0) = -2$

$f^{(4)}(x) = \frac{-6}{(1-x)^4}$ $\qquad f^{(4)}(0) = -6$

$T_n f(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \forall x \in (-1,1)$

Merk: $f^{(n)}(0) = (n-1)!$, så ledd n i $T_n f(x) = \frac{(n-1)!\, x^n}{n!} = \underline{\underline{\frac{x^n}{n}}}$

b) Vis at $\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \dots$

Se på $T_n f(\frac{1}{2}) = \ln(1 - \frac{1}{2}) = \ln(\frac{1}{2})$

$\ln\left(\frac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{(\frac{1}{2})^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n2^n}$

$\ln 1 - \ln 2 = -\ln 2 = -\sum_{n=1}^{\infty} \frac{1}{n2^n}$

$\Downarrow$

$\underline{\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}}$

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12.2

2) Bruk integraltesten til å vise $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ konv. hvis $p > 1$, div hvis $p \leq 1$:

Ser at rekken er positiv $\left(\frac{1}{n(\ln n)^p} > 0 \quad \forall\ n \geq 2\right)$, kontinuerlig og avtagende $\left(\frac{1}{(n+1)(\ln(n+1))^p} < \frac{1}{n(\ln n)^p} \quad \forall\ n \geq 2\right)$

Da konvergerer rekken hvis og bare hvis integralet konv.

$$\int_2^\infty \frac{1}{n(\ln n)^p}\,dn = \int_2^\infty \frac{1}{n}(\ln n)^{-p}\,dn \qquad u = \ln n \quad du = \frac{1}{n}\,dn$$

$$\int \frac{1}{n(\ln n)^p}\,dn = \int \frac{1}{u^p}\,du = \int u^{-p}\,dn$$

$p = 1$:

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln n| + C$$

$p \neq 1$:

$$\int u^{-p}\,du = \frac{u^{-p+1}}{-p+1} + C = \frac{(\ln n)^{-p+1}}{-p+1} + C$$

$p = 1$:

$$\int_2^\infty \frac{1}{n(\ln n)}\,dn = \Big[\ln|\ln n|\Big]_2^\infty = \ln|\ln \infty| - \ln|\ln 2| = \underline{\infty}$$

$p < 1$:

$$\int_2^\infty \frac{1}{n(\ln n)^p}\,dn = \left[\frac{(\ln n)^{1-p}}{-p+1}\right]_2^\infty \overset{1-p>0}{=\!=} \frac{(\ln \infty)^{1-p} - (\ln 2)^{1-p}}{-p+1} = \underline{\infty}$$

$p > 1$:

$$\int_2^\infty \frac{1}{n(\ln n)^p}\,dn = \frac{1}{1-p}\left[\frac{1}{(\ln n)^{p-1}}\right]_2^\infty \overset{p-1>0}{=\!=} \frac{1}{1-p}\left(\frac{1}{\infty} - \frac{1}{\ln 2}\right)$$

$$= \underline{\frac{1}{1-p} - \frac{1}{\ln 2} < \infty}$$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ konv. for $p > 1$, div. for $p \leq 1$

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12.2

3) Avgjør om rekken konv. eller div. med sammenlikn/grensesammenlikn:

a) $\sum_{n=1}^{\infty} \frac{7n^2+3}{4n^3-2} > \sum_{n=1}^{\infty} \frac{7n^2}{4n^3} = \sum_{n=1}^{\infty} \frac{7}{4} \cdot \frac{1}{n}$

$7n^2+3 > 7n^2 \qquad 4n^3-2 < 4n^3$

Vet at $\sum_{n=1}^{\infty} \frac{1}{n}$ divergerer, så da divergerer $\sum_{n=1}^{\infty} \frac{7n^2+3}{4n^3-2}$ også

f) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ positiv rekke ($\sin\frac{1}{n} > 0 \ \forall \ n \geq 1$)

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{1/n} \underset{=}{\overset{r=\frac{1}{n}}{=}} \lim_{r\to 0} \frac{\sin r}{r} \underset{=}{\overset{\text{L'Hôpital}}{=}} \lim_{r\to 0} \frac{\cos r}{0} = 1$$

Vet at $\sum_{n=1}^{\infty} \frac{1}{n}$ divergere, så når $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{1/n} > 0$ må $\sum_{n=1} \sin\frac{1}{n}$ divergere

i) $\sum_{n=1}^{\infty} \left(\sqrt{n^3+1} - n^{3/2}\right) = b_n$

$$\lim_{n\to\infty} \frac{\sqrt{n^3+1} - \sqrt{n^3}}{1/n^{3/2}} = \lim_{n\to\infty} n^{\frac{3}{2}}\left(\sqrt{n^3+1} - \sqrt{n^3}\right)$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\left(\sqrt{n^3+1} - \sqrt{n^3}\right)\left(\sqrt{n^3+1} + \sqrt{n^3}\right)}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \lim_{n\to\infty} \frac{n^{\frac{3}{2}}\left(n^3+1-n^3\right)}{\sqrt{n^3+1} - \sqrt{n^3}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}\left(\sqrt{1+1/n^3} + 1\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^3}+1} = \frac{1}{2}$$

Siden $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ konv. og $\lim_{n\to\infty} \frac{b_n}{1/n^{3/2}} < \infty$ må $\sum_{n=1}^{\infty} b_n$ konvergere.

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5) Avgjør om rekken konv. eller div. med forhold/rottetesten

d) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ Forholdstesten:

$$\lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \lim_{n\to\infty} \frac{e^{n+1} \cdot n!}{e^n \cdot (n+1)!} = \lim_{n\to\infty} \frac{e^1}{n+1} = 0 < 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{e^n}{n!}$ konvergerer

e) $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ Rottetesten:

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2^n}}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n^n}$ konvergerer

g) $\sum_{n=1}^{\infty} \frac{n! \cdot 4^n}{n^n}$ Forholdstesten:

$$\lim_{n\to\infty} \frac{\frac{(n+1)! \cdot 4^{n+1}}{(n+1)^{n+1}}}{\frac{n! \cdot 4^n}{n^n}} = \lim_{n\to\infty} \frac{(n+1)! \cdot 4^{n+1} \cdot n^n}{(n+1)^{n+1} \cdot n! \cdot 4^n} = \lim_{n\to\infty} \frac{4n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{4n^n}{n^n(1+1/n)^n} = \lim_{n\to\infty} \frac{4}{(1+1/n)^n} = \frac{4}{1} = 4 > 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n! \cdot 4^n}{n^n}$ divergerer

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12.2

9) Avgjør om $\sum_{n=1}^{\infty} \frac{1+2+\ldots+n}{2^n}$ konv. eller div.

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} + 2 \cdot \frac{1}{2^n} + 3\frac{1}{2^n} + \ldots + n\frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} 2\frac{1}{2^n} + \ldots + \sum_{n=1}^{\infty} n\frac{1}{2^n}$$

Vet fra 12.1.7 at hvis Σa_n er konvergent er $\Sigma c a_n$ konv., og summen av to konvergente rekker er konvergente.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{2}{2^n} + \ldots$ er konvegent. Hvis $\sum_{n=1}^{\infty} \frac{n}{2^n}$ er konv. vil dermed hele rekken være en sum av konvergente rekker, og dermed konvergent.

$\sum_{n=1}^{\infty} \frac{n}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n}$. Vet at $\sum_{n=1}^{\infty} \frac{1}{2^n}$ konvergere. Dermed konv. alle $\sum_{n=1}^{\infty} \frac{x}{2^n}$ for $x \in \{1, 2, \ldots, n\}$, så

$$\underline{\underline{\sum_{n=1}^{\infty} \frac{1+2+\ldots+n}{2^n} \text{ konvergerer}}}$$