ØVING 4 side 1

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1) a) $f(x,y,z) = 3x^2 + 2y^2 + z$

$f'(x) = 6x \qquad f'(y) = 4y \qquad f'(z) = 1$

$\underline{\underline{f_{xx}(x,y,z) = 6}} \qquad \underline{\underline{f_{yy}(x,y,z) = 4}}$

<u>Alle andre annenordens partiell deriverte = 0</u>

b) $f(x,y) = \sin(x^2 - 3xy)$

$f'(x) = (2x - 3y)\cos(x^2 - 3xy)$

$f'(y) = -3x\cos(x^2 - 3xy)$

$\underline{\underline{f_{xx}(x,y)}} = \underline{\underline{2\cos(x^2 - 3xy) - (2x - 3y)^2\sin(x^2 - 3xy)}}$

$\underline{f_{xy}(x,y)} = -3\cos(x^2 - 3xy) + (2x - 3y)3x\sin(x^2 - 3xy)$

$= \underline{\underline{-3\cos(x^2 - 3xy) + (6x^2 - 9xy)\sin(x^2 - 3xy)}}$

$\underline{\underline{f_{yy}(x,y) = -9x^2\sin(x^2 - 3xy)}}$

$\underline{f_{yx}(x,y)} = -3\cos(x^2 - 3xy) + 3x(2x - 3y)\sin(x^2 - 3xy)$

$= \underline{\underline{-3\cos(x^2 - 3xy) + (6x^2 - 9xy)\sin(x^2 - 3xy)}}$

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2) Finnes $f: \mathbb{R}^2 \to \mathbb{R}$ med kont. andregrads part. der., hvis

$$f_x(x,y) = 2x - 5y \qquad f_y(x,y) = 4x + y$$

Har fra Schwarz' teorem at kont. andregradskurver $\Rightarrow$

$$f_{xy} = f_{yx} \quad \text{evt} \quad \frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$$

$$f_{xy}(x,y) = -5 \qquad f_{yx}(x,y) = 4$$

Ser at $f_{xy} \neq f_{yx}$, så det finnes ingen funksjon $f: \mathbb{R}^2 \to \mathbb{R}$ med kont. annengrads partiellderiverte og f_x, f_y som over.

3) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) Vis at $f(x,0) = 0$, $f(0,y) = 0$. Vis at $f_x(0,0) = 0$, $f_y(0,0) = 0$

$f(x,0)$: $\quad x = 0 \Rightarrow \underline{f(0,0) = 0}$

$x \neq 0$:

$$f(x,0) = \frac{x^3 \cdot 0 - x \cdot 0}{x^2 + 0} = \frac{0}{x^2} = \underline{0}$$

$f(0,y)$: $\quad y = 0 \Rightarrow \underline{f(0,0) = 0}$

$y \neq 0$:

$$f(0,y) = \frac{0 \cdot y - 0y^3}{0 + y^2} = \frac{0}{y^2} = \underline{0}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = \underline{0}$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = \underline{0}$$

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3) b) Vis at for $(x,y) \neq (0,0)$ er $f_x(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}$

$f_y(x,y) = -\frac{x(y^4 + 4x^2y^2 - x^4)}{(x^2+y^2)^2}$

$$f_x(x,y) = \frac{\partial}{\partial x}(x^3y - y^3x)(x^2+y^2)^{-1}$$

$$= (3x^2y - y^3)(x^2+y^2)^{-1} - 2x(x^3y - y^3x)(x^2+y^2)^{-2}$$

$$= \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - y^3x)}{(x^2+y^2)^2}$$

$$= \frac{3x^4y - x^2y^3 + 3x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} = \underline{\underline{\frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}}}$$

$$f_y(x,y) = \frac{\partial}{\partial y}(x^3y - y^3x)(x^2+y^2)^{-1}$$

$$= (x^3 - 3y^2x)(x^2+y^2)^{-1} - 2y(x^3y - y^3x)(x^2+y^2)^{-2}$$

$$= \frac{(x^3 - 3y^2x)(x^2+y^2) - 2y(x^3y - y^3x)}{(x^2+y^2)^2}$$

$$= \frac{x^5 + x^3y^2 - 3x^3y^2 - 2x^3y^2 + 2xy^4 - 3xy^4}{(x^2+y^2)^2}$$

$$= \frac{-xy^4 - 4x^3y^2 + x^5}{(x^2+y^2)^2} = \underline{\underline{\frac{-x(y^4 + 4x^2y^2 - x^4)}{(x^2+y^2)^2}}}$$

c) Vis at $f_{xy}(0,0) = -1$ med $f_{yx} = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h(-h^4) - 0}{h^4 \cdot h} = \lim_{h \to 0} \frac{-h^5}{h} = \underline{\underline{-1}}$$

Tilsvarende vis $f_{xy}(0,0) = 1$

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{-h(-h^5) - 0}{h \cdot h^4} = \frac{h^5}{h^5} = \underline{\underline{1}}$$

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4) Finn Jakobi-matrisen til F:

a) $F(x,y) = \begin{pmatrix} x^2 y \\ x + y^2 \end{pmatrix}$

$$J = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)$$

$$F_x(x,y) = \begin{pmatrix} 2xy \\ 1 \end{pmatrix} \qquad F_y(x,y) = \begin{pmatrix} x^2 \\ 2y \end{pmatrix}$$

$$\underline{\underline{J = \begin{pmatrix} 2xy & x^2 \\ 1 & 2y \end{pmatrix}}}$$

b) $F(x,y,z) = \begin{pmatrix} e^{x^2y+z} \\ xyz^2 \end{pmatrix}$

$$F_x(x,y,z) = \begin{pmatrix} 2xy\,e^{x^2y+z} \\ yz^2 \end{pmatrix}$$

$$F_y(x,y,z) = \begin{pmatrix} x^2 e^{x^2y+z} \\ xz^2 \end{pmatrix}$$

$$F_z(x,y,z) = \begin{pmatrix} e^{x^2y+z} \\ 2xyz \end{pmatrix}$$

$$\underline{\underline{J = \begin{pmatrix} 2xe^{x^2y+z} & x^2 e^{x^2y+z} & e^{x^2y+z} \\ yz^2 & xz^2 & 2xyz \end{pmatrix}}}$$

5) La $f: A \subset \mathbb{R}^n \to \mathbb{R}$ være deriverbar i $a \in A$. Er $g(x) = f^2(x) + 2f(x)$ deriverbar i $a \in A$? Bestem $\nabla g(a)$

$$\frac{\partial}{\partial x_i} g(a) = 2f(a) f_{x_i}(a) + 2 f_{x_i}(a) \quad \text{[Kjerneregel]}$$

Dette gjelder for alle $i \in \{1, 2, .., n\}$, så g er deriverbar i $a \in A$.

$$\nabla g(a) = \left(\frac{\partial}{\partial x_1} g(a), \dots, \frac{\partial}{\partial x_n} g(a) \right)$$

$$\underline{\underline{= \left(2f(a)f_{x_1}(a) + 2f_{x_1}(a), \dots, 2f(a)f_{x_n}(a) + 2f_{x_n}(a) \right)}}$$

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6) La $f(u,v) = ue^{-v}$, $g(x,y,z) = 2xy + z$ $h(x,y,z) = 2y(z+x)$

Bruk kjerneregel, finn part. deriv. av $f(g(x,y,z), h(x,y,z))$

La $k(x,y,z) = f(g(x,y,z), h(x,y,z))$

$$k_x(x,y,z) = f_u(g,h)\,g_x + f_v(g,h)\,h_x$$
$$= e^{-h(x,y,z)}\,2y - g(x,y,z)e^{-h(x,y,z)}\,2y$$
$$\underline{\underline{= 2ye^{-2y(z+x)}(1 - 2xy - z)}}$$

$$k_y(x,y,z) = f_u(g,h)\,g_y + f_v(g,h)\,h_y$$
$$= e^{-2y(z+x)}\,2x - (2xy+z)e^{-2y(z+x)}\,2(z+x)$$
$$= e^{-2y(z+x)}\left(2x - (2xy+z)(2z+2x)\right)$$
$$\underline{\underline{= 2e^{-2y(z+x)}\left(2x^2y + 2xyz + xz + z^2 - x\right)}}$$

$$k_z(x,y,z) = f_u(g,h)\,g_z + f_v(g,h)\,h_z$$
$$= e^{-2y(z+x)} - (2xy+z)e^{-2y(z+x)}\,2y$$
$$\underline{\underline{= 2e^{-2y(z+x)}\left(\tfrac{1}{2} - 2xy^2 - yz\right)}}$$

7) $G: \mathbb{R}^2 \to \mathbb{R}^3$, $F: \mathbb{R}^3 \to \mathbb{R}^2$, $G(1,-2) = (1,2,3)$

$$G'(1,-2) = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}, \quad F'(1,2,3) = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & 2 \end{pmatrix}$$

Finn Jacobi-matrisen til $H(x) = F(G(x))$ i $(1,-2)$

$$H'(x) = F'(G(x))G'(x)$$

$$J(1,-2) = H'(1,-2) = F'(G(1,-2))G'(1,-2)$$

$$= F'(1,2,3)G'(1,-2)$$

$$= \begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 13 & -7 \\ 10 & 0 \end{pmatrix}}}$$

8) Finn lineariseringen til:

a) $f(x,y) = e^{x+y}$; $(x,y) = (0,0)$

$$T_{\vec{x}_0} f(\vec{x}) = f(\vec{x}_0) + f'(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

$$T_{(0,0)} f(x,y) = f(0,0) + f'(0,0)\begin{pmatrix} x-0 \\ y-0 \end{pmatrix}$$

$$f(0,0) = e^0 = 1$$

$$f'(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(e^{x+y}, e^{x+y}\right)$$

$$T_{(0,0)} f(x) = 1 + \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \underline{\underline{1 + x + y}}$$

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8) b) $F(x,y) = (x^2y,\ xy + x)$; $(x,y) = (-2,1)$

$$T_{(-2,1)}F(x,y) = F(-2,1) + \nabla F(-2,1)\begin{pmatrix} x+2 \\ y-1 \end{pmatrix}$$

$$F(-2,1) = (4,\ -4)$$

$$\nabla F(x,y) = \left(\frac{\partial F}{\partial x},\ \frac{\partial F}{\partial y}\right)$$

$$= \begin{pmatrix} 2xy & x^2 \\ y+1 & x \end{pmatrix}$$

$$\nabla F(-2,1) = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$$

$$\underline{\underline{T_{(-2,1)}F(x,y) = (4,\ -4) + \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x+2 \\ y-1 \end{pmatrix}}}$$

$$= (4,\ -4) + \left(-4(x+2)+4(y-1),\ 2(x+2)-2(y-1)\right)$$

$$= (4,-4) + (-4x-8+4y-4,\ 2x+4-2y+2)$$

$$\underline{\underline{= (-4x+4y-8,\ 2x-2y+2)}}$$