ØVING 3 side 1 Andreas B. Berg

T 1.2

43) Kurve $\vec{r}(t) = (\cos t, \ t\sin t)$. Finn $\vec{v}(t)$, $v(t)$, $\vec{a}(t)$, $a(t)$

$$\underline{\vec{v}(t)} = \vec{r}\,'(t) = \underline{\underline{(-\sin t, \ \sin t + t\cos t)}}$$

$$\underline{v(t)} = |\vec{v}(t)| = \sqrt{\sin^2 t + (\sin t + t\cos t)^2}$$

$$= \underline{\underline{\sqrt{2\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t}}}$$

$$\underline{\vec{a}(t)} = \vec{v}\,'(t) = (-\cos t, \ \cos t + \cos t - t\sin t)$$

$$= \underline{\underline{(-\cos t, \ 2\cos t - t\sin t)}}$$

$$\underline{a(t)} = v'(t) = \frac{2\sin(2t) + \sin(2t) + 2t\cos(2t) - t^2\sin(2t) + 2t\cos^2 t}{2\sqrt{2\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t}}$$

$$= \underline{\underline{\frac{(3 - t^2)\sin(2t) + 2t(\cos(2t) + \cos^2 t)}{2\sqrt{2\sin^2 t + t\sin(2t) + t^2\cos^2 t}}}}$$

46) $\vec{r}(t) = (t, \ \ln(\cos t))$, $t \in [0, \frac{\pi}{4}]$

a) Finn $\vec{v}(t)$ og $v(t)$

$$\underline{\vec{v}(t)} = \vec{r}\,'(t) = \left(1, \ \frac{1}{\cos t}\cdot(-\sin t)\right) = \underline{\underline{(1, \ -\tan t)}}$$

$$v(t) = |\vec{v}(t)| = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = \underline{\underline{\sec t}}$$

b) Finn buelengden $= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1^2 + (-\tan t)^2}\, dt = \int_0^{\frac{\pi}{4}} \sec t \, dt = \int_0^{\frac{\pi}{4}} \frac{\cos t}{1 - \sin^2 t}\, dt$$

$u = \sin t \qquad du = \cos t \, dt$

$$\int \frac{1}{1 - u^2}\, du = \int \frac{1}{(u+1)(u-1)}\, du = \int \frac{A}{u+1} + \frac{B}{u-1}\, du$$

forts. ↓

 Andreas [illegible]

T 1.2

46) forts.)

$$A(1+u) + B(1-u) = 1$$

$$A + B = 1 \qquad A - B = 0$$
$$A = B = \tfrac{1}{2} \qquad A = B$$

$$\int \frac{1}{1-u^2}\,du = \int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du = \frac{1}{2}\left(\int \frac{1}{1-u}\,du + \int \frac{1}{1+u}\,du\right)$$

$$= \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) + C = \frac{\ln|1+u|}{2} - \frac{\ln|1-u|}{2} + C$$

$$\int_0^{\frac{\pi}{4}} \frac{\cos t}{1-\sin^2 t}\,dt = \left[\frac{\ln|1+\sin t|}{2} - \frac{\ln|1-\sin t|}{2}\right]_0^{\pi/4}$$

$$= \frac{\ln\left|\frac{2+\sqrt{2}}{2}\right|}{2} - \frac{\ln\left|\frac{2-\sqrt{2}}{2}\right|}{2} - \frac{\ln|1|}{2} - \frac{\ln|1|}{2}$$

$$= \frac{1}{2}\ln\frac{1+\frac{\sqrt{2}}{2}}{1-\sqrt{2}/2} = \frac{1}{2}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}} = \underline{\underline{\ln(\sqrt{2}+1)}}$$

11.1

1) Finn $T_4\, e^{x^2}$ i $x = 0$

$$f(x) = e^{x^2} \qquad f(0) = 1$$
$$f'(x) = 2x\,e^{x^2} \qquad f'(0) = 0$$
$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} \qquad f''(0) = 2$$
$$f'''(x) = 4x\,e^{x^2} + 8x\,e^{x^2} + 8x^2e^{x^2} \qquad f'''(0) = 0$$
$$f^{(4)}(x) = 4e^{x^2} + 8x^2e^{x^2} + 8e^{x^2} + 16x^2e^{x^2} + 16x\,e^{x^2} + 16x^2e^{x^2}$$
$$f^{(4)}(0) = 12$$

$$T_4 f(x) = 1 + 0x + \frac{2x^2}{2} + \frac{0x^3}{6} + \frac{12x^4}{24}$$

$$\underline{\underline{T_4\, e^{x^2} = 1 + x^2 + \frac{1}{2}x^4}}$$

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11.1

2) Finn $T_3 \sqrt{x}$ i $x = 1$

$$f(x) = \sqrt{x} \qquad f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}} \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \qquad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} \qquad f'''(1) = \frac{3}{8}$$

$$T_3 f(x) = 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8} + \frac{3(x-1)^3}{8 \cdot 6}$$

$$\underline{\underline{T_3 f(x) = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16}}}$$

5) Finn $T_5 \sinh x$ i $x = 0$

$$f(x) = \sinh x \qquad f(0) = 0$$

$$f'(x) = \cosh x \qquad f'(0) = 1$$

$$f''(x) = \sinh x \qquad f''(0) = 0$$

$$f'''(x) = \cosh x \qquad f'''(0) = 1$$

$$f^{(4)}(x) = \sinh x \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cosh x \qquad f^{(5)}(0) = 1$$

$$T_5 f(x) = 0 + x + \frac{0x^2}{2!} + \frac{x^3}{3!} + \frac{0x^4}{4!} + \frac{x^5}{5!}$$

$$\underline{\underline{T_5 f(x) = x + \frac{x^3}{6} + \frac{x^5}{120}}}$$

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1.1.1

10) Finn $T_3(x^4 - 3x^2 + 2x - 7)$ i $x = 1$

$$f(1) = 1 - 3 + 2 - 7 = -7$$

$$f'(x) = 4x^3 - 6x + 2 \qquad f'(1) = 4 - 6 + 2 = 0$$

$$f''(x) = 12x^2 - 6 \qquad f''(1) = 12 - 6 = 6$$

$$f'''(x) = 24x \qquad f'''(1) = 24$$

$$T_3 f(x) = -7 + 0x + \frac{6(x-1)^2}{2} + \frac{24(x-1)^3}{6}$$

$$\underline{\underline{T_3 f(x) = -7 + 3(x-1)^2 + 4(x-1)^3}}$$

11.2

2) Finn $T_4 \sin x$ i $x = 0$. Vis at $R_4 \sin b \leq \frac{|b|^5}{120}$ for alle b

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f'''(0) = 0$$

$$f^{(4)}(x) = -\cos x \qquad f^{(4)}(0) = -1$$

$$T_4 \sin x = 0 + x + 0x^2 - \frac{x^3}{3!}$$

$$\underline{\underline{T_4 \sin x = x - \frac{x^3}{6}}}$$

$$|R_4 f(b)| = \left|\frac{f^{(5)}(c)}{5!}(x-0)^5\right| = \frac{|x^5 \sin c|}{120} = \frac{|x^5|\,|\sin c|}{120}$$

$|\sin c| \leq 1$ for alle c, så

$$\underline{\underline{|R_4 \sin b| \leq \frac{|x|^5}{120}}}$$

11.2

6) Finn $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ ved passende Taylor-polynom

Bruker $T_2 f(x)$ i $x=0$, $f(x) = e^x - 1 - x$ $\quad f(0) = 0$

$f'(x) = e^x - 1 \qquad f'(0) = 0$

$f''(x) = e^x \qquad f''(0) = 1$

$$\left[R_2 f(x) = \frac{f'''(c)}{6}(x^3) = \frac{e^c}{6}x^3\right]$$

$$T_2 f(x) = 0 + 0x + \frac{x^2}{2} = \frac{1}{2}x^2$$

$$f(x) = T_2 f(x) + R_2 f(x) = \frac{1}{2}x^2 + \frac{e^c}{6}x^3 = x^2\left(\frac{1}{2} + \frac{e^c}{6}x\right)$$

$$\Rightarrow \lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{x^2\left(\frac{1}{2} + \frac{e^c}{6}x\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{1}{2} + \frac{e^c}{6}x = \underline{\underline{\frac{1}{2}}}$$

10) a) Skriv opp $T_6 \sin(x)$ i $x = 0$

$$\underline{T_6 \sin x} = \underline{\underline{x - \frac{x^3}{3!} + \frac{x^5}{5!}}} = \underline{\underline{x - \frac{x^3}{6} + \frac{x^5}{120}}}$$

b) Bruk a) til å regne $\int_0^1 \sin(x^2)dx$ med feil < 0.00002

$$T_6 \sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}$$

$$\int_0^1 \sin(x^2)dx = \int_0^1 T_n \sin(x^2)\,dx + \int_0^1 R_n \sin(x^2)dx$$

Feil $< 0{,}00002 \Rightarrow \int_0^1 R_n \sin(x^2)\,dx < 0.00002$

$$\left|R_n \sin(x^2)\right| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \left|\frac{1}{(n+1)!}x^{n+1}\right| \qquad c \in (0,x)$$

siden abs. verdi til alle deriverte av $\sin(x^2) \le 1$

$$\int_0^1 R_n \sin(x^2)dx \le \frac{1}{(n+1)!}\int_0^1 x^{n+1}\,dx = \frac{1}{(n+1)!}\left[\frac{1}{n+2}x^{n+2}\right]_0^1$$

$$= \frac{1}{(n+2)!}$$

forts. ↓

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10) (forts.) Har at

$$\int_0^1 R_n f(x)\,dx \leq \frac{1}{(n+2)!} < 0.00002 = \frac{1}{50000}$$

Da må $(n+2)! > 50000 \Rightarrow n+2 = 9 \Rightarrow n = 7$

$$T_7 \sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{7!}$$

$$\int_0^1 \sin(x^2)\,dx \simeq \int_0^1 T_7 \sin(x^2)\,dx = \int_0^1 x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{7!}\,dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{6\cdot 7}x^7 + \frac{1}{11\cdot 120}x^{11} - \frac{1}{7!\cdot 15}x^{15}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} \approx \underline{\underline{0{,}310}}$$

15) $g(x) = \sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$

a) Finn $T_2 g(x)$ i $x = 0$

$g(0) = 1$

$g'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$ $\quad g'(0) = \frac{1}{3}$

$g''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}}$ $\quad g''(0) = -\frac{2}{9}$

$$T_2 g(x) = 1 + \frac{1}{3}\cdot x - \frac{2}{9}\cdot\frac{x^2}{2}$$

$$\underline{\underline{T_2 g(x) = 1 + \frac{x}{3} - \frac{x^2}{9}}}$$

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15) b) Vis at for $x \geq 0$ er $|R_2 g(x)| \leq \frac{5}{81} x^3$

$$|R_2 g(x)| = \frac{g'''(c)}{3!} x^3$$

$$g'''(x) = \frac{10}{27}(1+x)^{-\frac{8}{3}}$$

$$|R_2 g(x)| = \frac{10}{27 \cdot 6}(1+c)^{-\frac{8}{3}} x^3 = \frac{5}{81}(1+c)^{-\frac{8}{3}} x^3 \quad c \in (0,x)$$

$$x \geq 0 \Rightarrow c \geq 0 \Rightarrow (1+c) \geq 1 \Rightarrow (1+c)^{-\frac{8}{3}} \leq 1 \quad \text{dvs}$$

$$\underline{|R_2 g(x)|} = \frac{5}{81}(1+c)^{-\frac{8}{3}} x^3 \leq \underline{\frac{5}{81} x^3}$$

$$\Rightarrow \underline{\underline{|R_2 g(x)| \leq \frac{5}{81} x^3}}$$

c) Finn $\sqrt[3]{1003}$ med 7 gjeldende desimaler. $g(x) = \sqrt[3]{3+x}$

$$\Rightarrow \text{Feil} < 0.00000001 \Rightarrow |R_n g(x)| < 10^{-8}$$

$$R_n g(x) = \frac{g^{(n+1)}(c)}{(n+1)!} x^{n+1} \leq \frac{x^{n+1}}{(n+1)!}$$

$$g^{(n+1)}(c) < 1 \quad \forall \quad n \geq 0, \ c \geq -2$$

Vet at $x = 1000$:

$$|R_n g(1000)| \leq \frac{1000^{n+1}}{(n+1)!} < 10^{-8}$$

$$\frac{10^{3n+3} \cdot 10^8}{(n+1)!} < 1$$

$$10^{3n+11} < (n+1)!$$

$$\Rightarrow \text{python} \Rightarrow \underline{n = 2731}$$

Kommentar: Noe er galt. Veldig galt!
Problemet er at jeg ikke vet hva, og det begynner å bli sent på fredag, så hodet funker ikke. Ser fram til LF, for hvordan løser man 15 c) ??? -Andreas