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12.4

3) For hvilke verdier av a konv. rekken?

a) $\sum_{n=1}^{\infty} \frac{a^n}{n} = \sum_{n=1}^{\infty} a_n$ Forholdstest:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{a^{n+1}\, n}{a^n(n+1)}\right| = \lim_{n\to\infty}\left|\frac{a\, n}{n+1}\right| = \lim_{n\to\infty}\left|\frac{a}{1+1/n}\right|$$

$$= \frac{|a|}{|1+0|} = |a|$$

Rekken konv. for $|a| < 1$

$a = 1$:

$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ divergere

$a = -1$:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Rightarrow$ konv.

$\Rightarrow \underline{\underline{\sum_{n=1}^{\infty} \frac{a^n}{n} \text{ konv. for } a \in [-1, 1)}}$

b) $\sum_{n=0}^{\infty} \frac{a^n}{n!} = \sum_{n=0}^{\infty} a_n$ Forholdstest:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{a^{n+1}\, n!}{(n+1)!\, a^n}\right| = \lim_{n\to\infty}\left|\frac{a}{n+1}\right| = 0$$

$\Rightarrow \underline{\underline{\sum_{n=0}^{\infty} \frac{a^n}{n!} \text{ konv. for alle } a \in \mathbb{R}}}$

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5) b) Anta $\sum_{n=1}^{\infty} a_n$ konv. Anta følgen $\{c_n\}$ begrenset.
Vil $\sum_{n=1}^{\infty} c_n a_n$ konvergere?

$\sum a_n$ konv. absolutt $\Rightarrow \sum c_n a_n$ konv. (fra a)

Anta $\sum a_n$ betinget konvergent:

- $\{c_n\}$ konvergerer $\Rightarrow \sum c_n a_n$ konvergerer
- $\{c_n\}$ konvergerer ikke $= \{c_n\}$ alternerende

$\Rightarrow$ Ikke gitt at $\sum c_n a_n$ konvergerer

12.5

1) For hvilke x konvergerer rekken?

b) $\sum_{n=0}^{\infty} (2x)^n$

$\sum_{n=0}^{\infty} a^n$ konv $\Leftrightarrow |a| < 1 \Rightarrow |2x| < 1 \Rightarrow -1 \leq 2x \leq 1$

$\Rightarrow \sum_{n=0}^{\infty} (2x)^n$ konv. på $x \in (-\frac{1}{2}, \frac{1}{2})$

c) $\sum_{n=0}^{\infty} (\ln x)^n$

$|\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e^{1}$

$\sum_{n=0}^{\infty} (\ln x)^n$ konv. på $x \in (e^{-1}, e^{1})$

e) $\sum_{n=1}^{\infty} \frac{2^n \sin^n x}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} (2\sin x)^n$ Forholdstest

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2\sin x)^{n+1} \cdot n^2}{(n+1)^2 (2\sin x)^n}\right| = \lim_{n\to\infty} \left|\frac{2\sin x \cdot n^2}{n^2 + 2n + 1}\right|$$

$$= \lim_{n\to\infty} \left|\frac{2\sin x}{1 + 2/n + 1/n^2}\right| = |2\sin x| < 1$$

$\Downarrow$

$$-1 < 2\sin x < 1 \Rightarrow -\frac{1}{2} < \sin x < 1$$

$$\pi n - \frac{\pi}{6} < x < \frac{\pi}{6} + \pi n$$

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12.5

1) e) konv for $-\frac{\pi}{6}+\pi n < x < \frac{\pi}{6}+\pi n$

$x = \pi/6 + \pi n$:

$$\sum_{n=1}^{\infty} \frac{2^n \sin x^n}{n^2} = \sum \frac{(2 \cdot 0{,}5)^n}{n^2} = \sum \frac{1}{n^2} \Rightarrow \text{konv.}$$

$x = -\pi/6 + \pi n$:

$$\sum_{n=1}^{\infty} \frac{2^n \sin x^n}{n^2} = \sum \frac{(2 \cdot -0{,}5)^n}{n^2} = \sum \frac{(-1)^n}{n^2} \Rightarrow \text{konv. abs.} \Rightarrow \text{konv.}$$

$$\underline{\underline{\sum_{n=1}^{\infty} \frac{2^n \sin^n x}{n^2} \text{ konv. på } x \in \left[-\frac{\pi}{6}+\pi n, \frac{\pi}{6}+\pi n\right] \text{ for heltall } n}}$$

3) a) Vis at $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ konv. unif. mot en f på hele $\mathbb{R}$. Forholdstest:

Sjekker abs. konvergens:

$$\sum_{n=1}^{\infty} \frac{|\cos(nx)|}{|n^2|} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ konv.} \qquad \forall x$$

$$\underline{\underline{\text{Da må } \sum \frac{\cos(nx)}{n^2} \text{ konv. (absolutt) mot en f } \forall x.}}$$

b) $\frac{\cos(nx)}{n^2}$ er kont. $\forall n$. Da må

$$f = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \underline{\underline{\text{sum av kont. funk.}}} \text{ være kontinuerlig.}$$

c) Vis at $\int_0^x f(t)\,dt = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \quad \forall x$

$$\frac{d}{dx}\int_0^x f(t)\,dt \overset{\Updownarrow}{=} \frac{d}{dx}\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \sum_{n=1}^{\infty} \frac{d}{dx}\frac{\sin(nx)}{n^3}$$

$$\underline{f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}}$$

<u>Som vi jo viste i oppg. 1</u>

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12.6

1): Finn konvergensintervallet:

a) $\sum_{n=0}^{\infty}(x-2)^n$ konv $\Leftrightarrow |x-2|<1 \Rightarrow -1<x-2<1$

Konv. intervall $x \in (1,3)$

b) $\sum_{n=0}^{\infty}\frac{x^n}{3^n} = \sum\left(\frac{x}{3}\right)^n$ konv $\Leftrightarrow \left|\frac{x}{3}\right|<1 \Rightarrow -1<\frac{x}{3}<1$

Konv. int. $(-3,3)$

d) $\sum_{n=1}^{\infty}\frac{(x+1)^n}{\sqrt{n}}$ Rottest:

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(x+1)^n}{\sqrt{n}}\right|} = \lim_{n\to\infty}\frac{|x+1|}{\left|\sqrt[n]{n^{1/2}}\right|} = |x+1|$$

$|x+1|<1 \Rightarrow -1<x+1<1 \Rightarrow \underline{-2<x<0}$

$x=-2$:

$\sum_{n=1}^{\infty}\frac{(-2+1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ $\quad \lim_{n\to\infty}\frac{1}{\sqrt{n}}=0 \Rightarrow$ konv.

$x=0$:

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \Rightarrow$ divergent $\left[\sum\frac{1}{n^p}\text{ konv} \Leftrightarrow p>1\right]$

$\sum_{n=1}^{\infty}\frac{(x+1)^n}{\sqrt{n}}$ konv. int $[-2,0)$

g) $\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}x^n$ $\quad x=0 \Rightarrow$ konv.

$x\neq 0$: Forholdstest:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)!)^2x^{n+1}(2n)!}{(2n+2)!(n!)^2x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x}{(2n+1)(2n+2)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{2(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{nx+x}{4n+2}\right| = \lim_{n\to\infty}\left|\frac{x+x/n}{4+2/n}\right| = \frac{|x|}{4}$$

$\frac{|x|}{4}<1 \Rightarrow -1<\frac{x}{4}<1 \Rightarrow \underline{-4<x<4}$

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12.6

1)g) $x = 4$:

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} 4^n \Rightarrow \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} 4^n = \infty \Rightarrow \text{divergerer}$$

$$\left[\begin{aligned} 4^n (n!)^2 &= 4^n (1\cdot 2\cdot 3\cdots n)(1\cdot 2\cdot 3\cdots n) = (2\cdot 4\cdot 6\cdots 2n)^2 \\ &= 2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots 2n\cdot 2n > 1\cdot 2\cdot 3\cdot 4\cdots 2n = (2n)! \end{aligned}\right]$$

$x = -4$:

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-4)^n \Rightarrow \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} (-4)^n = \infty \Rightarrow \text{divergerer}$$

$$\underline{\underline{\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n \text{ konv. int } (-4, 4)}}$$

2) Finn konv. int.

f) $\sum_{n=0}^{\infty} \left(\frac{nx}{1+2n}\right)^n$ konv Rottest:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{nx}{1+2n}\right)^n} = \lim_{n\to\infty} \left|\frac{nx}{1+2n}\right| = \lim_{n\to\infty} \left|\frac{x}{2+1/n}\right| = \frac{|x|}{2}$$

$$\frac{|x|}{2} < 1 \Rightarrow -1 < \frac{|x|}{2} < 1 \Rightarrow \underline{-2 < x < 2}$$

$\underline{x = 2}$:

$$\sum_{n=0}^{\infty} \left(\frac{2n}{2n+1}\right)^n \sim \lim_{n\to\infty} \left(\frac{2n}{2n+1}\right)^n = \frac{1}{\sqrt{e}} \neq 0 \Rightarrow \text{divergerer}$$

$x = -2$

$$\sum_{n=0}^{\infty} \left(\frac{-2n}{2n+1}\right)^n = \sum (-1)^n \left(\frac{2n}{2n+1}\right)^n \sim \lim_{n\to\infty} \left(\frac{2n}{2n+1}\right)^n \neq 0 \Rightarrow \text{divergerer}$$

$$\underline{\underline{\sum_{n=0}^{\infty} \left(\frac{nx}{2n+1}\right)^n \text{ konv. int } (-2, 2)}}$$

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12.6

7) La $\{a_n\}_{n=0}^{\infty}$ pos. følge

a) Vis at $\sum a_n$ konv $\Leftrightarrow \sum \ln(1+a_n)$ konv

$\Rightarrow$: Anta $\sum a_n$ konv. Grensesammenlikning!

$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} \quad \text{Vet } \lim_{n\to\infty} a_n = 0: \quad x = a_n$$

$$= \lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{\text{L'Hôp}}{=} \lim_{x\to 0} \frac{1}{1+x} = 1 < \infty$$

$\sum a_n$ konv $\Rightarrow \sum \ln(1+a_n)$ konv.

$\Leftarrow$: Anta $\sum \ln(1+a_n)$ konv:

Vet at $\lim_{n\to\infty} \ln(1+a_n) = 0 = \ln(1)$

$\Rightarrow \lim_{n\to\infty} a_n = 0$ Grensesammenlikn.:

$$\lim_{n\to\infty} \frac{a_n}{\ln(1+a_n)} = \lim_{x\to 0} \frac{x}{\ln(1+x)} \overset{\text{L'Hôp}}{=} \lim_{x\to 0} \frac{1+x}{1} = 1 < \infty$$

$\sum \ln(1+a_n)$ konv $\Rightarrow \sum a_n$ konv

$\Rightarrow \underline{\underline{\sum a_n \text{ konv.} \Leftrightarrow \sum \ln(1+a_n) \text{ konv.}}}$

b) For hvilke p konv. $\sum_{n=1}^{\infty} \ln\left[1 + \ln\left(1+\frac{1}{n^p}\right)\right]$?

$\sum \ln\left[1+\ln\left(1+\frac{1}{n^p}\right)\right]$ konv $\Leftrightarrow \sum \ln\left(1+\frac{1}{n^p}\right)$ konv

$\Leftrightarrow \sum \frac{1}{n^p}$ konv, $\Leftrightarrow \underline{\underline{p > 1}}$

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12.6

7) c) Hva er konv. int til $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)x^n$. Forholdstest:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\ln\left(1+\frac{1}{n+1}\right)x^{n+1}}{\ln\left(1+1/n\right)x^n}\right| = \lim_{n\to\infty}\left|\frac{\ln\left(1+1/{n+1}\right)}{\ln\left(1+1/n\right)}x\right| = |x|$$

$$|x|<1 \Rightarrow -1<|x|<1$$

$x=1$:

$$\sum \ln\left(1+\frac{1}{n}\right) \text{ konv} \Leftrightarrow \sum\frac{1}{n} \text{ konv. IKKE} \Rightarrow \text{div.}$$

$x=-1$:

$$\sum \ln\left(1+\frac{1}{n}\right)(-1)^n \text{ konv} \Leftrightarrow \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)=0 \Rightarrow \text{konv.}$$

$$\underline{\underline{\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)x^n \text{ konv. int. } [-1,1)}}$$

12.7

1) Finn $f'(x)$ og $F(x)=\int_a^x f(t)\,dt$ som rekker:

a) $f(x)=\sum_{n=0}^{\infty} n^2x^n$, $a=0$

$$f'(x)=\sum_{n=1}^{\infty} n\cdot n^2x^{n-1} = \underline{\underline{\sum_{n=1}^{\infty} n^3x^{n-1}}} = \underline{\underline{\sum_{n=0}^{\infty}(n+1)^3x^n}}$$

$$F(x)=\int_0^x\sum_{n=0}^{\infty}n^2t^n\,dt=\sum_{n=0}^{\infty}\int_0^x n^2t^n\,dt = \underline{\underline{\sum_{n=0}^{\infty}\frac{n^2}{n+1}t^{n+1}}} = \underline{\underline{\sum_{n=1}^{\infty}\frac{(n-1)^2}{n}t^n}}$$

d) $f(x)=\sum_{n=0}^{\infty}\frac{3^n(x-4)^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n}{n!}(x-4)^n$ $a=4$

$$f'(x)=\sum_{n=0}^{\infty}\frac{d}{dx}\frac{3^n}{n!}(x-4)^n = \underline{\underline{\sum_{n=1}^{\infty}\frac{3^n}{(n-1)!}(x-4)^{n-1}}} = \underline{\underline{\sum_{n=0}^{\infty}\frac{3^{n+1}}{n!}(x-4)^n}}$$

$$F(x)=\sum_{n=0}^{\infty}\int_4^x\frac{3^n}{n!}(t-4)^n\,dt = \underline{\underline{\sum_{n=0}^{\infty}\frac{3^n}{(n+1)!}(x-4)^{n+1}}} = \underline{\underline{\sum_{n=1}^{\infty}\frac{3^{n-1}}{n!}(x-4)^n}}$$

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12.7

3) a) Forklar hvorfor $\frac{x^2}{1-x^3} = \sum_{n=0}^{\infty} x^{3n+2}$ når $|x| < 1$

Vet: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1$

$$\Rightarrow \frac{x^2}{1-x^3} = x^2 \frac{1}{1-x^3} = x^2 \sum_{n=0}^{\infty} (x^3)^n = \underline{\sum_{n=0}^{\infty} x^{3n+2}}$$

$$|x^3| < 1 \Rightarrow \underline{|x| < 1}$$

b) Vis at $\ln(1-x^3) = -\sum_{n=1}^{\infty} \frac{x^{3n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} (x^3)^n$ når $|x| < 1$

Deriver begge sider:

$$V.S. = \frac{-3x^2}{1-x^3} = -3x^2 \sum_{n=1}^{\infty} x^{3n} = -3\sum_{n=1}^{\infty} x^{3n+2}$$

$$H.S. = -\sum_{n=2}^{\infty} (x^3)^{n-1} \cdot 3x^2 = -3\sum_{n=1}^{\infty} x^{3n+2} = V.S.$$

Integrer for å få tilbake originalllikningen. $\Rightarrow \underline{\ln(1-x^3) = -\sum_{n=1}^{\infty} \frac{x^{3n}}{n}}$

c) Vis at $\sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{n} = \ln 2$, og sammenlign med eks. 12.7.4

Vet at $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $x \in (-1,1)$

Integrerer: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n$

$x = 1$:

$\sum \frac{(-1)^{n-1}}{n}$ konv. $\Leftrightarrow \lim_{n\to\infty} \left|\frac{(-1)^{n-1}}{n}\right| = 0 \rightsquigarrow \lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow$ konv for $x = 1$

$$(-1)^{n-1} = (-1)^{n+1} = (-1)^{3n+1}$$

La $x = 1$. Da er

$$\underline{\underline{\ln(1+1) = \ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{n}}}$$

Dette er det samme som blir bekreftet i eks. 12.7.4