TMA 4120

Øving 1, side 1

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6.1 4) $\mathcal{L}(\cos^2 \omega t) = \mathcal{L}\left(\frac{1}{2} + \frac{\cos(2\omega t)}{2}\right) \stackrel{\text{linear}}{=} \mathcal{L}\left(\frac{1}{2}\right) + \mathcal{L}\left(\frac{\cos(2\omega t)}{2}\right)$

$= \frac{1}{2}s + \frac{1}{2}\mathcal{L}(\cos(2\omega t)) \qquad \left[\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}\right]$

$= \underline{\underline{\frac{1}{2}\left(s + \frac{s}{s^2+4\omega^2}\right)}}$

13) $f(t) = \begin{cases} 1 & : 0 < t < 1 \\ -1 & : 1 < t < 2 \\ 0 & : \text{ellers} \end{cases}$

$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt + \int_1^2 -e^{-st}\,dt + \int_2^\infty 0\,dt$

$= \left[-\frac{1}{s}e^{-st}\right]_0^1 + \left[\frac{1}{s}e^{-st}\right]_1^2$

$= -\frac{1}{s}e^{-s} + \frac{1}{s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-s} = \underline{\underline{\frac{1}{s}\left(e^{-2s} - 2e^{-s} + 1\right)}}$

26) $F(s) = \mathcal{L}(f) = \frac{5s+1}{s^2-25} = \frac{5s}{s^2-25} + \frac{1}{s^2-25} = 5\frac{s}{s^2-25} + \frac{1}{5}\frac{5}{s^2-25}$

Vet at $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$ og $\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$, så ser at

$f(t) = \mathcal{L}^{-1}\left(\frac{5s+1}{s^2-25}\right) = \mathcal{L}^{-1}\left(5\frac{s}{s^2-25}\right) + \mathcal{L}^{-1}\left(\frac{1}{5}\frac{5}{s^2-25}\right)$

$= 5\mathcal{L}^{-1}\left(\frac{s}{s^2-25}\right) + \frac{1}{5}\mathcal{L}^{-1}\left(\frac{5}{s^2-25}\right)$

$= \underline{\underline{5\cosh(5t) + \frac{1}{5}\sinh(5t)}}$

32) $F(s) = \frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$

$1 = A(s+b) + B(s+a)$

$s = -b \Rightarrow 1 = B(a-b) \Rightarrow B = \frac{1}{a-b}$

$s = -a \Rightarrow 1 = A(b-a) \Rightarrow A = \frac{1}{b-a} = -\frac{1}{a-b}$

$F(s) = \frac{1}{a-b}\frac{1}{s+b} - \frac{1}{a-b}\frac{1}{s+a}$

$f(t) = \mathcal{L}^{-1}\left(\frac{1}{a-b}\left(\frac{1}{s+b} - \frac{1}{s+a}\right)\right) \stackrel{\text{linear}}{=} \frac{1}{a-b}\left(\mathcal{L}^{-1}\left(\frac{1}{s+b}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+a}\right)\right)$

$= \frac{1}{a-b}\left(e^{-bt} - e^{-at}\right)$

$= \underline{\underline{\frac{e^{-bt} - e^{-at}}{a-b}}} \stackrel{\text{ev.}}{=} \underline{\underline{\frac{e^{-at} - e^{-bt}}{b-a}}}$

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6.1. 40) $F(s) = \mathcal{L}(f) = \frac{4}{s^2-2s-3} = \frac{4}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$

$4 = A(s+1) + B(s-3)$

$s=-1 \Rightarrow 4 = B(-4) \Rightarrow B=-1$

$s=3 \Rightarrow 4 = A(4) \Rightarrow A=1$

$F(s) = \frac{1}{s-3} - \frac{1}{s+1}$

$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{1}{s+1}\right) \overset{\text{linear}}{=} \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

Vet at $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, ergo får vi

$\underline{\underline{f(t) = e^{3t} - e^{-t}}}$

6.2. 5) $y'' - \frac{1}{4}y = 0 \qquad y(0) = 12 \qquad y'(0) = 0$

$[s^2 Y - sy(0) - y'(0)] - \frac{1}{4}Y = \mathcal{L}(0) = 0$

$(s^2 - \frac{1}{4})Y = 12s$

$Y(s) = \frac{12s}{s^2 - 1/4} = 12\frac{s}{s^2-1/4}$

Vet at $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$. Har dermed

$y(t) = \mathcal{L}^{-1}(Y(s)) = \underline{\underline{12\cosh\left(\frac{1}{2}t\right)}}$

10) $y'' + 0{,}04y = 0{,}02t^2 \qquad y(0) = -25 \qquad y'(0) = 0$

$s^2Y - sy(0) - y'(0) + 0{,}04Y = \mathcal{L}(0{,}02t^2) = 0{,}04/s^3$

$(s^2 + 0{,}04)Y = \frac{0{,}04}{s^3} - 25s$

$Y = \frac{1}{s^2+0{,}04}\left(\frac{0{,}04}{s^3} - 25s\right) = \frac{0{,}2^2}{s^3(s^2+0{,}2^2)} - \frac{25s}{s^2+0{,}2^2}$

Ser ved delbrøksoppspalting at

$Y = \frac{1}{s^3} - \frac{5}{s}\left(\frac{0{,}2}{s^2+0{,}2^2}\right) - 25\left(\frac{s}{s^2+0{,}2^2}\right)$

Vet at $\mathcal{L}\left(\frac{1}{2}t^2\right) = \frac{1}{2}\cdot\frac{2}{s^3} = \frac{1}{s^3} \qquad \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$

$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \qquad \mathcal{L}^{-1}\frac{1}{s}F(s) = \int_0^t f(\tau)\,d\tau$

$\Rightarrow y(t) \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) - 25\mathcal{L}^{-1}\left(\frac{s}{s^2+0{,}2^2}\right) - 5\mathcal{L}^{-1}\left(\frac{1}{s}\left(\frac{0{,}2}{s^2+0{,}2^2}\right)\right)$

$= \frac{1}{2}t^2 - 25\cos(0{,}2t) - 5\int_0^t \sin(0{,}2\tau)\,d\tau$

$= \frac{1}{2}t^2 - 25\cos(0{,}2t) + 5\cos(0{,}2t) - 5$

$\Rightarrow \underline{\underline{y(t) = \frac{1}{2}t^2 - 20\cos(0{,}2t) - 5}}$

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6.2 26) $\mathcal{L}(f) = \frac{1}{s^4 - s^2} = \frac{1}{s^2}\left(\frac{1}{s^2-1}\right)$

Vet at $\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$, ergo er $\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = \sinh(t)$

Har fra teorem 3 at:

$$\mathcal{L}^{-1}\left(\frac{1}{s}\left(\frac{1}{s^2-1}\right)\right) = \int_0^t \sinh(\tau)\,d\tau = \frac{1}{2}\int_0^t e^{\tau} - e^{-\tau}\,d\tau$$
$$= \frac{1}{2}\left(\int_0^t e^{\tau}\,d\tau - \int_0^t e^{-\tau}\,d\tau\right)$$
$$= \frac{1}{2}\left(\left[e^{\tau}\right]_0^t + \left[e^{-\tau}\right]_0^t\right)$$
$$= \frac{1}{2}\left(e^t - 1 + e^{-t} - 1\right) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1$$

Bruker teorem 3 igjen, og ser at

$$\mathcal{L}^{-1}\left(\frac{1}{s}\left[\frac{1}{s}\left(\frac{1}{s^2-1}\right)\right]\right) = \int_0^t \frac{1}{2}e^{\tau} + \frac{1}{2}e^{-\tau} - 1\,d\tau$$
$$= \frac{1}{2}\int_0^t e^{\tau}d\tau + \frac{1}{2}\int_0^t e^{-\tau}d\tau - \int_0^t 1\,d\tau$$
$$= \frac{1}{2}\left[e^{\tau}\right]_0^t + \frac{1}{2}\left[e^{-\tau}\right]_0^t - \left[\tau\right]_0^t$$
$$= \frac{1}{2}\left(e^t - 1 - e^{-t} + 1\right) - t$$
$$= \underline{\underline{\frac{1}{2}e^t - \frac{1}{2}e^{-t} - t}}$$

6.3. 21) $y'' + 9y = \begin{cases} 8\sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } t > \pi \end{cases}$ $\quad y(0) = 8/3, \quad y'(0) = 1$

$$= 8\sin(t)\left[1 - u(t-\pi)\right]$$

$$s^2Y - s\,y(0) - y'(0) + 9Y = \mathcal{L}\left(8\sin(t) - 8\sin(t)u(t-\pi)\right)$$
$$(s^2+9)Y - \frac{8}{3}s - 1 = \frac{8}{s^2+1} - \mathcal{L}\left(8\sin(t)u(t-\pi)\right)$$

[Vet at $\sin(t-\pi) = -\sin(t)$]

$$= \frac{8}{s^2+1} + \mathcal{L}\left(8\sin(t-\pi)u(t-\pi)\right)$$

[Teorem 1:] $$= \frac{8}{s^2+1} + e^{-\pi s}\frac{8}{s^2+1} = \left(1 + e^{-\pi s}\right)\frac{8}{s^2+1}$$

$$(s^2+9)Y = \left(1+e^{-\pi s}\right)\frac{8}{s^2+1} + \frac{8}{3}s + 1$$
$$Y = \frac{1+e^{-\pi s}}{s^2+9}\cdot\frac{8}{s^2+1} + \frac{8}{3s^2+27}s + \frac{1}{s^2+9}$$

$$y = 8\mathcal{L}^{-1}\left(\frac{1+e^{-\pi s}}{(s^2+9)(s^2+1)}\right) + \frac{8}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)$$
$$y(t) = 8\mathcal{L}^{-1}\left(\frac{1}{(s^2+9)(s^2+1)} + \frac{e^{-\pi s}}{(s^2+9)(s^2+1)}\right) + \frac{8}{3}\cos(3t) + \frac{1}{3}\sin(3t)$$

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Forts. 6.3.21)

$$\frac{1}{(s^2+9)(s^2+1)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+1}$$

$$1 = (As+B)(s^2+1) + (Cs+D)(s^2+9)$$

$$1 = (A+C)s^3 + (B+D)s^2 + (A+9C)s + (B+9D)$$

$$\Rightarrow A = C = 0$$

$$B = -D$$

$$B+9D = 9D-D = 8D = 1 \Rightarrow D = \frac{1}{8} \Rightarrow B = -\frac{1}{8}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+9)(s^2+1)}\right) = \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin(3t)\right)$$

Teorem 1 gir:

$$\mathcal{L}^{-1}\left(e^{-\pi s}\left[\frac{1}{(s^2+9)(s^2+1)}\right]\right) = \frac{1}{8}\left(\sin(t-\pi) - \frac{1}{3}\sin(3(t-\pi))\right)u(t-\pi)$$

$$\Rightarrow y(t) = \sin t - \frac{1}{3}\sin(3t) + \left[\sin(t-\pi) - \frac{1}{3}\sin(3t-3\pi)\right]u(t-\pi)$$
$$+ \frac{8}{3}\cos(3t) + \frac{1}{3}\sin(3t)$$

$$\underline{\underline{y(t) = \sin(t) + \frac{8}{3}\cos(3t) + \sin(t-\pi)u(t-\pi) - \frac{1}{3}\sin(3(t-\pi))u(t-\pi)}}$$

38) $R = 4\,\Omega$, $L = 1\,H$, $C = 0.05\,F$, $v = \begin{cases} 34e^{-t}\,V, & 0<t<4 \\ 0, & t>4 \end{cases}$

Har $Li' + Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t)$

$$\Rightarrow i' + 4i + 20\int_0^t i(\tau)\,d\tau = 34e^{-t}, \quad 0<t<4$$

$$i(0) = 0, \quad i'(0) = 0$$

$$\mathcal{L}(i') + 4\mathcal{L}(i) + 20\mathcal{L}\left(\int_0^t i(\tau)\,d\tau\right) = \mathcal{L}(34e^{-t})$$

$$sI - i(0) + 4I + 20\frac{I}{s} = \frac{34}{s+1}$$

$$\left(s+4+\frac{20}{s}\right)I = \frac{34}{s+1}$$

$$I = \frac{34s}{s+1}\cdot\frac{1}{s^2+4s+20}$$

$$I = \frac{34s}{(s+1)(s^2+4s+20)}$$

$$= \frac{2s+40}{s^2+4s+20} - \frac{2}{1+s}$$

$$= \frac{2s+40}{(s+2)^2+16} - \frac{2}{s+1}$$

$$= \frac{2s+4}{(s+2)^2+16} + \frac{36}{(s+2)^2+16} - \frac{2}{s+1}$$

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Øving 1, side 5

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forts. 6.3.38) $i(s) = \mathcal{L}^{-1}\left(\frac{2s+4}{(s+2)^2+16} + \frac{36}{(s+2)^2+16} - \frac{2}{s+1}\right)$

$= 2\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+16}\right) + 36\,\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+16}\right) - 2\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

$= 2e^{-2t}\cos(4t) + 9e^{-2t}\sin(4t) - 2e^{-t}$

$$\underline{\underline{= e^{-2t}\left[2\cos(4t) + 9\sin(4t)\right] - 2e^{-t}, \quad 0 < t < 4}}$$