ØVING 8 side 1 Andreas B. Berg

1) Finn annenordens T.poly til $f(x,y) = \sin(xy) + \cos(xy)$ om $(0,0)$

$$P_2 f(x,y) = f(0,0) + Df(0,0)\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}[x, y]\, Hf(0,0)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(0,0) = \sin 0 + \cos 0 = 1$$

$$Df(x,y)\begin{bmatrix} x \\ y \end{bmatrix} = [y\cos(xy) - y\sin(xy),\ x\cos(xy) - x\sin(xy)]\begin{bmatrix} x \\ y \end{bmatrix}$$

$$Df(0,0)\begin{bmatrix} x \\ y \end{bmatrix} = [0,\ 0]\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$Hf(0,0) = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial^2 y} \end{bmatrix} = \begin{bmatrix} -y^2(\sin(xy) + \cos(xy)) & -xy(\sin(xy) + \cos(xy)) \\ -xy(\sin(xy) + \cos(xy)) & -x^2(\sin(xy) + \cos(xy)) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{P_2 f(0,0) = 1}}$$

2) Finn stasj. pkt til $f(x,y) = x^3 + 5x^2 + 3y^2 - 6xy$, og avgjør om maks/min/sadel

$$\nabla f(x,y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] = [3x^2 + 10x - 6y,\ 6y - 6x]$$

$$\nabla f(x,y) = 0 \Rightarrow 3x^2 + 10x - 6y = 0, \quad 6y - 6x = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$3x^2 + 10x - 6x = 0 \qquad 6y = 6x$$

$$3x^2 + 4x = 0 \qquad y = x$$

$$x(3x + 4) = 0 \Rightarrow x = 0, \quad x = -\frac{4}{3}$$

$$\Downarrow$$

$$(x,y) = (0,0), \left(-\frac{4}{3}, -\frac{4}{3}\right)$$

$$Hf(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x + 10 & -6 \\ -6 & 6 \end{bmatrix}$$

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2) $Hf(0,0) = \begin{bmatrix} 10 & -6 \\ -6 & 6 \end{bmatrix} \Rightarrow \det\begin{pmatrix} 10 & -6 \\ -6 & 6 \end{pmatrix} = 60-36 = 24$

$\det(Hf(0,0)) > 0, \ \frac{\partial^2 f}{\partial^2 x} > 0 \Rightarrow \underline{\underline{(0,0) \text{ lokalt minimum}}}$

$Hf(-\frac{4}{3}, -\frac{4}{3}) = \begin{bmatrix} 2 & -6 \\ -6 & 6 \end{bmatrix} \Rightarrow \det\begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix} = 12-36 = -14$

$\det(Hf(-\frac{4}{3}, -\frac{4}{3})) < 0 \Rightarrow \underline{\underline{(-\frac{4}{3}, -\frac{4}{3}) \text{ sadelpunkt}}}$

3) Samme som (2). $f(x,y,z) = xyz - x^2 - y^2 - z^2$

$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial y} \end{bmatrix} = [yz-2x, \ xz-2y, \ xy-2z]$

$\nabla f(x,y,z) = 0 \Rightarrow \quad yz - 2x = 0 \qquad xz - 2y = 0 \qquad xy - 2z = 0$

$$\begin{array}{ccc}
\Downarrow & \Downarrow & \Downarrow \\
x = \frac{yz}{2} & y = \frac{xz}{2} & z = \frac{xy}{2} \\
 & \Downarrow & \Downarrow \\
4 = yz & y = \frac{x^2 y}{4} & 4 = \frac{2y^2}{2} \\
z = \frac{4}{y} & \Downarrow & \Downarrow \\
z = \pm 2 & 1 = \frac{x^2}{4} \Rightarrow x = \pm 2 & y = \pm 2
\end{array}$$

eller $x = y = z = 0$

Kan se at $\nabla f(x,y,z) = 0$ når $(x,y,z) = (2,2,2), (2,-2,-2), (-2,-2,2), (-2,2,-2)$ $(0,0,0)$ – ikke ellers [enten 0 eller 2 av $x,y,z = -2$]

$$Hf(x,y,z) = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial^2 y} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial^2 z} \end{bmatrix} = \begin{bmatrix} -2 & z & y \\ z & -2 & x \\ y & x & -2 \end{bmatrix}$$

Eg. verdier: $\det(Hf(0,0,0) - \lambda I) = (-2-\lambda)^3 = -(2-\lambda)^3 = 0$

$\Rightarrow \lambda = 3 \ \forall \lambda$

$\Rightarrow \underline{\underline{(0,0,0) \text{ minimum}}}$

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3) $\det(Hf(2,2,2)-\lambda I) = \det\begin{bmatrix} -2-\lambda & 2 & 2 \\ 2 & -2-\lambda & 2 \\ 2 & 2 & -2-\lambda \end{bmatrix} = (-2-\lambda)^3 + 8 + 8 - 4(-2-\lambda)\cdot 3$

$= -\lambda^3 - 6\lambda^2 - 12\lambda - 8 + 8 + 8 + 24 + 12\lambda$

$= -\lambda^3 - 6\lambda^2 + 32 = -(\lambda+4)^2(\lambda-2) \Rightarrow \lambda_1 = -4 \quad \lambda_3 = 2$

$\Rightarrow$ (2, 2, 2) sadelpunkt

$\det(Hf(2,-2,-2)-\lambda I) = (-2-\lambda)^3 + 8 + 8 - 4(2-\lambda)\cdot 3$

$\Rightarrow \lambda_1 = \lambda_2 = -4 \quad \lambda_3 = 2$

$\Rightarrow$ (2, −2, −2) sadelpunkt

$\det(Hf(-2,2,-2)-\lambda I) = (-2-\lambda)^3 + 8 + 8 - 4(-2-\lambda)\cdot 3$

$\Rightarrow$ (−2, 2, −2) sadelpunkt

$\det(Hf(-2,-2,2)-\lambda I) = (-2-\lambda)^3 + 8 + 8 - 4(-2-\lambda)3$

$\Rightarrow$ (−2, −2, 2) sadelpunkt

4) La $f(x,y,z) = x^2 + y^2 + z^2 + kyz$

a) Vis at (0,0,0) stasj. pkt

$\nabla f(x,y,z) = [2x,\ 2y + kz,\ 2z + ky]$

$\nabla f(0,0,0) = [0,\ 0,\ 0] \Rightarrow$ (0, 0, 0) stasj. pkt

b) For hvilke k e (0,0,0) minimum?

$Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & k \\ 0 & k & 2 \end{bmatrix}$

$\det(Hf(0,0,0)-\lambda I) = (2-\lambda)^3 - 2k^2 = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 - 2k^2 = 0$

$k = \sqrt{\frac{(2-\lambda)^3}{2}} \quad \lambda > 0 \Rightarrow k < \sqrt{\frac{2}{2}} = \sqrt{1} = 1$

(0, 0, 0) minimum for $k < 1$

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5) Vis at boks med vol $V \Rightarrow$ kube minst overflate.

La (x, y, z) være side lengder, O = overflate

$$V = xyz \qquad O(x,y,z) = 2xy + 2xz + 2yz$$

$$z = \frac{V}{xy} \qquad O(x,y) = 2xy + \frac{2V}{y} + \frac{2V}{x}$$

$$\nabla O(x,y) = \left[2y - \frac{2V}{x^2},\ 2x - \frac{2V}{y^2}\right]$$

$$\Downarrow$$

$$y = \frac{V}{x^2} \qquad x = \frac{V}{y^2}$$

$$x = \frac{V}{V^2/x^4} = \frac{x^4}{V} \Rightarrow x^3 = V \Rightarrow x = \sqrt[3]{V}$$

$$y = \frac{V}{(\sqrt[3]{V})^2} = \frac{(\sqrt[3]{V})^3}{(\sqrt[3]{V})^2} = \sqrt[3]{V}$$

$$HO(x,y) = \begin{bmatrix} 4V/x^3 & 2 \\ 2 & 4V/y^3 \end{bmatrix}$$

$$HO(\sqrt[3]{V}, \sqrt[3]{V}) = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\det(HO(\sqrt[3]{V}, \sqrt[3]{V})) = 12 \Rightarrow (\sqrt[3]{V}, \sqrt[3]{V}) \text{ lokalt min.}$$

$$z = \frac{V}{(\sqrt[3]{V})^2} = \sqrt[3]{V}$$

$\Rightarrow$ Minst overflate når $(x,y,z) = (\sqrt[3]{V}, \sqrt[3]{V}, \sqrt[3]{V}) \Rightarrow$ kube

6) Finn ekstr. verdien til $f(x,y,z) = x - y + z$ når $x^2 + y^2 + z^2 = 2$

La $g(x,y,z) = x^2 + y^2 + z^2$

$$\nabla g(x,y,z) = [2x, 2y, 2z]$$

$$\nabla g(x,y,z) = 0 \Rightarrow (x,y,z) = (0,0,0) \quad \text{kandidat}$$

$$\nabla f(x,y,z) = [1, -1, 1]$$

$$\nabla f(x,y,z) = \lambda g(x,y,z) \Rightarrow \begin{array}{lll} 2x\lambda = 1 & \Rightarrow & x = \frac{1}{2\lambda} \\ 2y\lambda = -1 & \Rightarrow & y = -\frac{1}{2\lambda} \\ 2z\lambda = 1 & \Rightarrow & z = \frac{1}{2\lambda} \end{array} \Bigg\} x = z = -y$$

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6) $f(0,0,0)=0$ $\left.\begin{array}{l} < f(\sqrt{2},0,0)=\sqrt{2} \\ > f(0,\sqrt{2},0)=-\sqrt{2} \end{array}\right\}$ fortsatt i $x^2+y^2+z^2=2$

$\Rightarrow f(0,0,0)$ ikke ekstr. pkt for f på $x^2+y^2+z^2=2$

$x=-y=z$:

$x^2+y^2+z^2=x^2+(-x)^2+x^2=3x^2=2$

$x=\pm\sqrt{\frac{2}{3}}=z, \quad y=\mp\sqrt{\frac{2}{3}}$

$f\left(\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)=3\sqrt{\frac{2}{3}}$

$f\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}}\right)=-3\sqrt{\frac{2}{3}}$

f min i $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}}\right)$, maks i $\left(\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$

7) Finn maks/min $f(x,y)=x^2+xy+y^2$ på $D=\{x^2+y^2\leq 1\}$ $g(x,y)=x^2+y^2$

$x^2+y^2<1$:

$\nabla f(x,y)=[2x+y,\ 2y+x]$

$=0 \Rightarrow y=-2x \qquad -4x=x$

$\Rightarrow x=y=0$

$f(0,y)=y^2 \quad f(x,0)=x^2$

f min i $(0,0)$

$\nabla g(x,y)=[2x,\ 2y] \quad =0 \Rightarrow x=y=0$

$\nabla f(x,y)=\lambda\nabla g(x,y) \Rightarrow \begin{array}{l} 2x+y=2x\lambda \\ 2y+x=2y\lambda \end{array} \Rightarrow \begin{array}{l} y=2x(\lambda-1) \\ x=2y(\lambda-1) \end{array}$

$y=2(2y(\lambda-1))(\lambda-1)=4y(\lambda-1)(\lambda-1)$

$\Rightarrow 1=4\lambda^2-8\lambda+4$

$\Rightarrow 0=4\lambda^2-8\lambda+3$

$\lambda_1=\frac{3}{2}, \quad \lambda_2=\frac{1}{2}$

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7) $y = 2x(\lambda - 1)$: I) $y = 2x\left(\frac{1}{2}\right) = x$

II) $y = 2x\left(-\frac{1}{2}\right) = -x$

$x = 2y(\lambda - 1)$ III) $x = 2y\left(\frac{1}{2}\right) = y$ $= (I)$

IV) $x = 2y\left(-\frac{1}{2}\right) = -y$ $= II)$

$\underline{x^2 + y^2 = 1:}$

I) $x^2 + x^2 = 1 \Rightarrow x = \pm\sqrt{\frac{1}{2}}$, $y = \pm\sqrt{\frac{1}{2}}$

II) $x^2 + (-x)^2 = 1 \Rightarrow x = \pm\sqrt{\frac{1}{2}}$, $y = \mp\sqrt{\frac{1}{2}}$

$f\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} = f\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$

$f\left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2} = f\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$

<u>f maks i $\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ og $\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$</u>

<u>f min på randen i $\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ og $\left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$</u>

8) Finn pkt. $r(t) = \left(\cos t, \sin t, \sin\left(\frac{t}{2}\right)\right)$, $t \in [0, 4\pi]$ lengst unna origo

$\Rightarrow$ Finn min. av $x^2 + y^2 + z^2 = f(x, y, z)$ når $x = \cos t$, $y = \sin t$, $z = \sin\left(\frac{t}{2}\right)$

$$f(t) = \cos^2 t + \sin^2 t + \sin^2\left(\frac{t}{2}\right) = 1 + \sin^2\left(\frac{t}{2}\right)$$

$$f'(t) = 2\sin\left(\frac{t}{2}\right) \cdot \frac{1}{2}\cos\left(\frac{t}{2}\right) = 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = \sin(t)$$

$\sin(t) = 0$ når $t = 0 + \pi k$ $\Rightarrow t_1 = 0$, $t_2 = \pi$, $t_3 = 2\pi$, 3π, 4π

$r(0) = (1, 0, 0)$ $r(2\pi) = (1, 0, 0)$ $r(4\pi) = (1, 0, 0)$

$r(\pi) = (-1, 0, 1)$ $r(3\pi) = (-1, 0, -1)$

<u>Pkt. lengst unna origo: $(-1, 0, 1)$, $(-1, 0, -1)$ når $t = \pi$, 3π</u>

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9) Finn pkt. på skjæring $x^2+y^2=1$ og $x^2-xy+y^2-z^2=1$ nærmest $(0,0,0)$

$$x^2+y^2=1 \Downarrow y^2=1-x^2 \Rightarrow x^2-xy+1-x^2-z^2=1$$
$$-xy+z^2=0$$

$\Rightarrow$ Finn min. for $\underbrace{(x^2,y^2,z^2)}_{f(x,y,z)}$ når $xy+z^2=0=g(x,y,z)$

$$\nabla g(x,y,z)=[y,x,2z]=0 \Rightarrow (0,0,0)$$
$$\Rightarrow \text{ikke mulig } (x^2+y^2\neq 1)$$

$$\nabla f(x,y,z)=[2x,2y,2z]$$

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z) \Rightarrow \begin{aligned}2x&=\lambda y\\2y&=\lambda x\\2z&=\lambda 2z \Rightarrow z=0\ /\ \lambda=1\end{aligned}$$

$z=0$:
$$\left.\begin{aligned}2x&=\lambda y\\2y&=\lambda x\end{aligned}\right\}\lambda=2 \Rightarrow x=y \qquad (i)$$

$\lambda=1$:
$$\begin{aligned}2x&=\lambda y &\Rightarrow\ & 2x=y\\2y&=\lambda x &\Rightarrow\ & 2y=x\\2z&=2z &\Rightarrow\ & z\end{aligned}\ \Big\}\ x=y=0 \qquad (ii)$$

$x^2-xy+y^2-z^2=1$; $x^2+y^2=1$:

(i): $\cancel{x^2}-\cancel{x^2}+x^2-0^2=1$
$$x^2=1 \Rightarrow x=1$$
$$y=x \Downarrow$$
$$y^2+x^2=1+1\neq 1$$

(ii): $0^2+0^2=x^2+y^2\neq 1$

Kommentar: Gjort noe feil. Antar jeg burde sjekket for $x^2-xy+y^2-z^2=1$ og deretter for $x^2+y^2=1$, for å finne riktige punkter. Stemmer dette?
Er dessverre tom for tid...

Andreas