ØVING 3 side 1 Andreas B. Berg

1) Finnes det en kont. utvidelse av $f(x,y) = \frac{(x-y)^2}{x^2+y^2}$ (med $D_f = \mathbb{R}^2 \setminus 0$) til hele $\mathbb{R}^2$?

$$f(x,y) = \frac{x^2+y^2-2xy}{x^2+y^2} = 1 - \frac{2xy}{x^2+y^2}$$

Ser at $\lim_{(x,y)\to(0,0)} = 1$ i alle retninger:

$$\lim_{(x,0)\to(0,0)} 1 - \frac{0}{x^2} = 1 \qquad \lim_{(0,y)\to(0,0)} 1 - \frac{0}{y^2} = 1$$

Dermed har vi at

$$\underline{\underline{f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 1 & , (x,y) = (0,0) \end{cases} \text{ er kont. i } \mathbb{R}^2}}$$

2) La $f,g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ være funk. deriverbare i $a \in A$. Vis at

$$\nabla(fg)(a) = f(a)\nabla g(a) + g(a)\nabla f(a)$$

$$\nabla(fg)(a) = \left(\frac{\partial(fg)}{\partial x_1}(a), \ldots, \frac{\partial(fg)}{\partial x_n}(a)\right)$$

Ser på ledd i:

$$\frac{\partial(fg)}{\partial x_i}(a) = \frac{\partial(f(a)g(a))}{\partial x_i}$$

Vet at $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$\frac{\partial(fg)}{\partial x_i}(a) = \frac{\partial f(a)\cdot g(a) + f(a)\partial g(a)}{\partial x_i}$$

$$= \frac{\partial f(a)}{\partial x_i} g(a) + \frac{\partial g(a)}{\partial x_i} f(a) \quad \text{dermed:}$$

$$\nabla(fg)(a) = \left(\frac{\partial f(a)}{\partial x_1} g(a) + \frac{\partial g(a)}{\partial x_1} f(a), \ldots, \frac{\partial f(a)}{\partial x_n} g(a) + \frac{\partial g(a)}{\partial x_n} f(a)\right)$$

$$= \left(\frac{\partial f(a)}{\partial x_1}, \ldots, \frac{\partial f(a)}{\partial x_n}\right)\cdot g(a) + \left(\frac{\partial g(a)}{\partial x_1}, \ldots, \frac{\partial g(a)}{\partial x_n}\right) f(a)$$

$$\underline{\nabla(fg)(a) = g(a)\nabla f(a) + f(a)\nabla g(a)} \qquad \square$$

 Andreas B. Berg

3) Finn gradienten til

a) $f(x,y) = \sin(x^2+y^2)$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y),\ \frac{\partial f}{\partial y}(x,y)\right)$$

$$= \left(\frac{\partial}{\partial x}\sin(x^2+y^2),\ \frac{\partial}{\partial y}\sin(x^2+y^2)\right)$$

$$= \underline{\underline{\left(2x\cos(x^2+y^2),\ 2y\cos(x^2+y^2)\right)}}$$

b) $f(x,y,z) = z + x^2y + e^{y\cos(xz)}$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z),\ \frac{\partial f}{\partial y}(x,y,z),\ \frac{\partial f}{\partial z}(x,y,z)\right)$$

$$= \Big(2yx - yz\sin(xz)e^{y\cos(xz)},$$

$$x^2 + \cos(xz)e^{y\cos(xz)},$$

$$\underline{\underline{1 - yx\sin(xz)e^{y\cos(xz)}\Big)}}$$

4) Finn retningsderiverte $f'(a;r)$ til f i a, retning r:

a) $f(x,y,z) = x^2y + z^2$, $a = (1,0,1)$, $r = (1,1,-1)$

$$f'(a;r) = \lim_{h\to 0}\frac{f(a+hr) - f(a)}{h}$$

$$= \lim_{h\to 0}\frac{f(1+h,\ h,\ 1-h) - f(1,0,1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)^2h + (1-h)^2 - 1}{h}$$

$$= \lim_{h\to 0}\frac{h^3 + 2h^2 + h + h^2 + 2h + 1 - 1}{h}$$

$$= \lim_{h\to 0} h^2 + 2h + 1 + h + 2 = \underline{\underline{3}}$$

 Andreas B. Berg

4) b) $f(x,y,z) = z\sin(xy)$, $a = (\frac{\pi}{2}, 1, 0)$, $r = (2, 0, -1)$

$$f'(a;r) = \lim_{h\to 0} \frac{f(a+hr) - f(a)}{h} =$$

$$= \lim_{h\to 0} \frac{f(\frac{\pi}{2}+2h, 1, -h) - f(\frac{\pi}{2}, 1, 0)}{h}$$

$$= \lim_{h\to 0} \frac{-h\sin(\frac{\pi}{2}+2h) - 0\sin(\frac{\pi}{2})}{h}$$

$$= \lim_{h\to 0} -\sin(\frac{\pi}{2}+2h) = -\sin(\frac{\pi}{2}) = \underline{\underline{-1}}$$

5) La $f(x,y) = 100 - 2x^2 - 3y^2$

a) I hvilken retning vokser f raskest i $(2, 3)$?

$$\nabla f(x,y) = (-4x, -6y)$$
$$\nabla f(2,3) = (-8, -18)$$

I punktet $(2,3)$ vokser f raskest i y-retning.

b) I hvilket punkt (x,y) har f størst verdi?

Ser fra ∇f at f synker i alle retninger fra et toppunkt.

f har dermed størst verdi når $\nabla f(x,y) = (0,0)$

$$-4x = 0 \Rightarrow x = 0$$
$$-6y = 0 \Rightarrow y = 0$$

f har størst verdi i punktet $(0,0)$

c) Vis at gradienten til f er null i punktet i b).

Allerede vist:

$$\nabla f(0,0) = (-4\cdot 0, -6\cdot 0) = \underline{(0,0)}$$

6) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$

a) Vis at $\frac{\partial f}{\partial x}(0,0) = 0$ og $\frac{\partial f}{\partial y}(0,0) = 0$. Hva er $\nabla f(0,0)$?

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial}{\partial x} 0 = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \frac{\partial}{\partial y} 0 = 0$$

$$\nabla f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = \underline{\underline{(0,0)}}$$

b) Vis at selv om retningsderiverte til f eksisterer i (0,0), er f verken kont. eller deriverbar i (0,0)

f kont. $\Leftrightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

Se på $\lim_{(x,x^2)\to(0,0)} f(x,x^2)$:

$$\lim_{(x,x^2)\to(0,0)} f(x,x^2) = \lim_{x\to 0} \frac{x^2 x^2}{x^4+x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

<u>Dette gir at $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$, så f ikke kont. i (0,0).</u>

<u>f ikke kont. i (0,0) $\Rightarrow$ f ikke deriverbar i (0,0)</u>

c) Vis at $f'(0;r) = \frac{r_1^2}{r_2}$, $r = (r_1, r_2)$, $r_2 \neq 0$

$$f'((0,0);r) = \lim_{h\to 0} \frac{f((0,0)+h(r_1,r_2)) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3 r_1^2 r_2}{h^4 r_1^4 + h^2 r_2^2} \Big/ h$$

$$= \lim_{h\to 0} \frac{r_1^2 r_2}{h^2 r_1^4 + r_2^2} = \frac{r_1^2 r_2}{r_2^2} = \underline{\underline{\frac{r_1^2}{r_2}}}$$

 Andreas B. Berg

6) d) Vis at $f'((0,0);r) \neq \nabla f(0,0) \cdot r$

Har vist $f'((0,0);r) = \frac{r_1^2}{r_2}$

$\nabla f(0,0) \cdot r = (0,0) \cdot (r_1, r_2) = 0$

Såfremt $r_1 \neq 0$ vil $f'(\vec{0};r) \neq \nabla f(\vec{0}) \cdot r$
Setning 2.4.8 motsies ikke, fordi den antar at f er deriverbar i punktet a [her: $a = (0,0)$].
Siden f ikke er deriverbar i $(0,0)$, gjelder heller ikke setning 2.4.8 for f i $(0,0)$

7) Vis at $f(x,y) = \begin{cases} (x+y)^3 \sin((x+y)^{-1}) & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

er deriverbar i $(0,0)$
Fra def: f deriverbar i $(0,0)$ hvis $\nabla f(0,0)$ eksistere

og $\lim_{r \to \vec{0}} \frac{f(a+r) - f(a) - \nabla f(a) \cdot r}{\| r \|} = 0$, $a = (0,0)$

Finner $\nabla f(0,0)$:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$$

$$= \left(3(x+y)^2 \sin\left(\frac{1}{x+y}\right) + (x+y)^3 \cos\left(\frac{1}{x+y}\right)\frac{-1}{(x+y)^2},\right.$$
$$\left. 3(x+y)^2 \sin\left(\frac{1}{x+y}\right) + (x+y)^3 \cos\left(\frac{1}{x+y}\right)\frac{-1}{(x+y)^2}\right)$$

$$= \left(3(x+y)^2 \sin\left(\frac{1}{x+y}\right) - (x+y)\cos\left(\frac{1}{x+y}\right),\right.$$
$$\left. 3(x+y)^2 \sin\left(\frac{1}{x+y}\right) - (x+y)\cos\left(\frac{1}{x+y}\right)\right)$$

[Dette gjelder <u>ikke</u> i $(0,0)$! Se neste side...]

7) $\nabla f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = \left(\frac{\partial}{\partial x} 0, \frac{\partial}{\partial y} 0\right)$

$\underline{= (0, 0)}$ eksisterer

Regner ut $\lim\limits_{r \to (0,0)} \frac{f(a+r) - f(a) - \nabla f(a) \cdot r}{\| r \|}$ $\quad a = (0,0)$, $r = (r_1, r_2)$

$$= \lim_{r \to \vec{0}} \frac{f(r_1, r_2) - f(0,0) - \nabla f(0,0) \cdot r}{\| r \|}$$

$$= \lim_{r \to \vec{0}} \frac{(r_1 + r_2)^3 \sin\left(\frac{1}{r_1 + r_2}\right) - 0 - 0}{\sqrt{r_1^2 + r_2^2}} \rightsquigarrow \text{L'Hôpitals}, \frac{d}{d(r_1+r_2)}$$

$$= \lim_{r \to \vec{0}} 2\sqrt{r_1^2 + r_2^2} \left(3(r_1+r_2)^2 \sin\left(\frac{1}{r_1+r_2}\right) - (r_1+r_2)\cos\left(\frac{1}{r_1+r_2}\right)\right)$$

$\left|\sin\left(\frac{1}{r_1+r_2}\right)\right| \le 1$, $\left|\cos\left(\frac{1}{r_1+r_2}\right)\right| \le 1$, så

$(r_1 + r_2) \cdot 2\sqrt{r_1^2 + r_2^2}$ går fortere mot 0 :

$$= \lim_{r \to \vec{0}} 2\sqrt{r_1^2 + r_2^2}\,(r_1 + r_2)\left(3(r_1 + r_2)\sin\left(\frac{1}{r_1+r_2}\right) - \cos\left(\frac{1}{r_1+r_2}\right)\right)$$

$$\underline{\underline{= 0}}$$

Gradienten til f i $(0,0) = \nabla f(0,0)$ eksisterer og

$\lim\limits_{r \to \vec{r}} \frac{f((0,0) + r) - f(0,0) - \nabla f(0,0) \cdot r}{\| r \|} = 0$, så

$\underline{\underline{f \text{ er derivesbar i punktet } (0,0)}}$