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1) Bruk Greens teorem, regn ut $\int_C F \cdot dr$ der C er pos. orient.

a) $F(x,y) = (x^2+y,\ x^2y)$ og C omkr. kv. hjørner $(0,0), (2,0), (2,2), (0,2)$
C stykkevis glatt, La R være kvadratet i C. På er

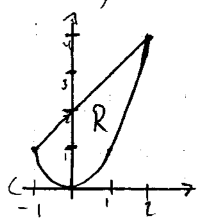
$$\int_C F\,dr = \iint_R \frac{\partial F_2}{dx} - \frac{\partial F_1}{\partial y}\, d(x,y)$$

$$= \int_0^2 \int_0^2 2xy - 1\, dx\,dy$$

$$= \int_0^2 \left[x^2y - x\right]_{x=0}^{2} dy = \int_0^2 4y-2\,dy$$

$$= \left[2y^2 - 2y\right]_0^2 = 8 - 4 = \underline{\underline{4}}$$

b) $F(x,y) = (x^2y + xe^x,\ xy^3 + e^{\sin(y)})$, C omkr. til omr.
avgr. av parabel $y = x^2$ og linjestykket med endep. $(-1,1), (2,4)$
C stykkevis glatt. La R være omr. avgr. av C



$$\frac{\partial F_2}{\partial x} = y^3 \qquad \frac{\partial F_1}{\partial y} = x^2$$

$$\int_C F\,dr = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dx\,dy = \int_{-1}^{2}\int_{x^2}^{x+2} y^3 - x^2\,dy\,dx$$

$$= \int_{-1}^{2} \left[\frac{1}{4}y^4 - x^2y\right]_{y=x^2}^{y=x+2} dx$$

$$= \int_{-1}^{2} \frac{1}{4}(x+2)^4 - x^2(x+2) - \frac{1}{4}x^8 + x^4\,dx$$

$$= \int_1^4 \frac{1}{4}u^4\,du - \int_{-1}^{2} x^3 + 2x^2\,dx - \int_{-1}^{2}\frac{1}{4}x^8\,dx + \int_{-1}^{2} x^4\,dx$$

$$= \left[\frac{1}{20}u^5\right]_1^4 - \left[\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{36}x^9 - \frac{1}{5}x^5\right]_{-1}^{2}$$

$$= \frac{1023}{20} - \left(4 + \frac{16}{3} + \frac{128}{9} - \frac{32}{5}\right) + \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{36} + \frac{1}{5}\right) = \underline{\underline{\frac{135}{4}}}$$

2) Regn ut arealet begrenset av $r(t) = (a\cos^3(t), b\sin^3(t))$ $t \in [0, 2\pi]$ der $a, b > 0$.

La R være området avgrenset av r. Vil finne

$\iint_R dxdy = \oint_C F dr$ der C er den lukkede kurven fra r og $F(x,y)$ s.a. $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$

La $F(x,y) = (0, x)$

$$\iint_R dxdy = \oint_C F dr = \int_0^{2\pi} F(r(t)) r'(t)\, dt$$

$$= \int_0^{2\pi} (0, a\cos^3(t))(-3a\cos^2(t)\sin(t), 3b\sin^2(t)\cos(t))dt$$

$$= \int_0^{2\pi} 3ab\sin^2(t)\cos^4(t)\, dt$$

$$= 3ab \int_0^{2\pi} \sin^2(t)\cos^4(t)\, dt$$

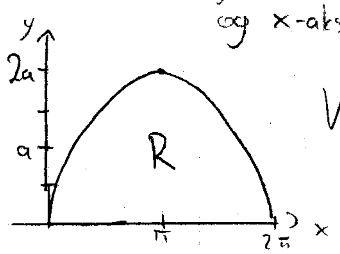
$$= 3ab\left[\frac{1}{6}\sin^3(t)\cos^3(t) + \frac{1}{16}\left(t - \frac{1}{4}\sin(4t)\right)\right]_0^{2\pi}$$

$$= 3ab\left(0 + \frac{1}{8}\pi\right) = \underline{\underline{\frac{3ab}{8}\pi}}$$

3) Finn arealet begrenset av x-aksen og sykloidebuen r gitt ved

$$x = a(\theta - \sin(\theta)),\ y = a(1 - \cos(\theta)),\ 0 \le \theta \le 2\pi,\ a > 0$$

La C være sykloidebuen, og la R være omr. avgr. av C og x-aksen



Vil finne $\iint_R dxdy = \oint_C F dr$ der

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

La $F(x,y) = (-y, 0)$

forts.

3) $\iint_R dx\,dy = \oint_C F\,dr = \int_0^{2\pi} F(r(\theta))\,r'(\theta)\,d\theta$

$$= \int_0^{2\pi} (a\cos\theta - a,\ 0)(a - a\cos\theta,\ a\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} a^2\cos\theta - a^2 - a^2\cos^2\theta + a^2\cos\theta\ d\theta$$

$$= \int_0^{2\pi} 2a^2\cos\theta - a^2 - a^2\cos^2\theta\ d\theta$$

$$= a^2\left[2\sin\theta - \theta - \tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta)\right]_0^{2\pi}$$

$$= a^2\left(\frac{2\pi}{2}\right) = \underline{\underline{\pi a^2}}$$

4) En ellipse har ligningen $9x^2 + 4y^2 - 18x + 16y = 11$

a) Finn sentrum, halvakser til ellipsen, lag skisse

Vil ha på form $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

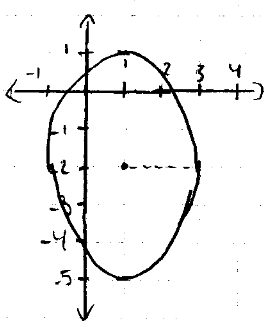
$$9x^2 - 18x + 4y^2 + 16y = 11$$

$$9(x^2 - 2x + 1) - 9 + 4(y^2 + 4y + 4) - 16 = 11$$

$$9(x-1)^2 + 4(y+2)^2 = 36 \quad \Big|\cdot\tfrac{1}{36}$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

<u>Sentrum: $(1, -2)$. Halvakser: $a = 2$, $b = 3$</u>



forts.

4) b) Vis at $r(t) = (1 + 2\cos(t), -2 + 3\sin(t))$, $t \in [0, 2\pi)$
param. av ellipsen.

Ser at x pendler mellom -1 og 3 $(-2 \le 2\cos(t) \le 2)$, så sentrum har $x = 1$ og halvakse $a = 2$.
Tilsvarende går y mellom -5 og 1, så sentrum har $y = -2$ og halvakse $b = 3$.

Mao. hvis r er ellipse, så stemmer det.
Når $t = 0$ er $(x, y) = (3, -2)$. Når t går mot $\frac{\pi}{2}$, går (x, y) mot $(1, 1)$.

$t \to \pi$, $(x, y) \to (-1, -2)$

$t \to \frac{3\pi}{2}$, $(x, y) \to (1, -5)$

$t \to 2\pi$, $(x, y) \to (3, -2)$, så r danner en ellipse $\Rightarrow$ $\underline{\underline{r \text{ er param. av ellipsen.}}}$

Regn ut $\int_C F\,dr$ der C er ellipsen med pos. orientering og $F(x, y) = (y^2, x)$.

$$\int_C F\,dr \overset{\text{ellipse}}{=} \int_0^{2\pi} F(r(t)) \cdot r'(t)\,dt$$

$$= \int_0^{2\pi} ((-2+3\sin t)^2, 1+2\cos t) \cdot (-2\sin t, 3\cos t)\,dt$$

$$= \int_0^{2\pi} -8\sin t + 24\sin^2 t - 18\sin^3 t + 3\cos t + 6\cos^2 t\,dt$$

$$\left.\begin{matrix} \int_0^{2\pi} \cos t = 0 \\ \int_0^{2\pi} \sin t = 0 \end{matrix}\right\} = \int_0^{2\pi} 24\sin^2 t + 6\cos^2 t\,dt$$

$$= \int_0^{2\pi} 6 + 18\sin^2 t\,dt$$

$$= \left[6t + 9t - \frac{9}{2}\sin(2t)\right]_0^{2\pi}$$

$$= 12\pi + 18\pi - 0 = \underline{\underline{30\pi}}$$

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4)c) Regn ut $\iint_R (1-2y)\,dx\,dy$ der R avgr. av ellipsen

Ser at $\iint_R (1-2y)\,dx\,dy = \underbrace{\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dx\,dy = \oint_C F\,dr}_{\text{Greens}} = \underline{\underline{30\pi}}$

5) Bruk Greens, vis at hvis F kons. felt, så er $\oint_C F\,dr = 0$ $\forall$ enkle, lukkede, stykkevis glatte c. La c være som beskrevet, i $\mathbb{R}^2$

$F = (F_1, F_2)$ kons. i $\mathbb{R}^2 \Rightarrow (F_1, F_2) = \nabla\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$

La R = omr. avgr. av C. Greens gir da:

$$\oint_C F\,dr = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = \iint_R \left(\frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y}\right)dx\,dy \underline{= 0}$$

6) Anta C er enkel, lukket kurve som oppfyller betingelsene i Greens. La $D \subset \mathbb{R}^2$ være omr. avgr. av C. Vis at areal $D = \oint_C (0,x)\,dr = -\oint_C (y,0)\,dr$

Areal $(D) = \iint_D 1\,dx\,dy \overset{\text{Greens}}{=} \oint_C F\,dr$ for en F s.a.

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

La $G = (0,x)$, $H = (-y, 0)$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = 1 - 0 \underline{= 1} \qquad \frac{\partial H_2}{\partial x} - \frac{\partial H_2}{\partial y} = -(-1) \underline{= 1}$$

<u>Dermed ser vi at areal $(D) = \oint_C (0,x)\,dr = -\oint_C (y,0)\,dr$</u>

7) La C = enhetssirkel, pos. retn. F vektorfelt: $F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

La $C_\varepsilon := \{(x,y) \mid x^2 + y^2 \le \varepsilon\}$. Finn $\oint_C F\,dr$ på denne måten

Ser at $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0$, så

$$\oint_C F\,dr - \oint_{C_\varepsilon} F\,dr = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \Rightarrow \oint_C F\,dr = \oint_{C_\varepsilon} F\,dr$$

Param. C_ε ved $r(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi)$

$$F(r(t)) = \left(\frac{-r\sin t}{r^2\sin^2 t + r^2\cos^2 t}, \frac{r\cos t}{r^2\sin^2 t + r^2\cos^2 t}\right) = \left(\frac{-\sin t}{r}, \frac{\cos t}{r}\right)$$

forts.

7) $\int_{C_\varepsilon} F\,dr = \int_{C_\varepsilon} F(s(t))s'(t)\,dt = \int_0^{2\pi} \left(\frac{-\sin t}{r}, \frac{\cos t}{r}\right)(-r\sin t, r\cos t)\,dt$

$= \int_0^{2\pi} \frac{r\sin^2 t}{r} + \frac{r\cos^2 t}{r}\,dt = \int_0^{2\pi} 1\,dt = 2\pi$

$\int_C F\,dr = \int_{C_\varepsilon} F\,dr \Rightarrow \int_C F\,dr = \underline{\underline{2\pi}}$

Verifiser direkte linjeint.:

Enhetssirkel: $r(t) = (\cos t, \sin t),\ t \in [0, 2\pi)$

$\int_C F\cdot dr = \int_C F(r(t))r'(t)\,dt = \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right)(-\sin t, \cos t)\,dt$

$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} 1\,dt = \underline{\underline{2\pi}}$

8) Beregn $\iiint_A (xy + z)\,dx\,dy\,dz$ der $A = \{(x,y,z) \in \mathbb{R}^3 : x \in [0,1], y \in [0,2], z \in [0, x^2 y]\}$

$= \int_0^1 \int_0^2 \int_0^{x^2 y} xy + z\,dz\,dy\,dx = \int_0^1 \int_0^2 \left[xyz + \frac{1}{2}z^2\right]_0^{x^2 y} dy\,dx$

$= \int_0^1 \int_0^2 x^3 y^2 + \frac{1}{2}x^4 y^2\,dy\,dx = \int_0^1 \left[\frac{1}{3}x^3 y^3 + \frac{1}{6}x^4 y^3\right]_0^2 dx$

$= \int_0^1 \frac{8}{3}x^3 + \frac{4}{3}x^4\,dx = \left[\frac{2}{3}x^4 + \frac{4}{15}x^5\right]_0^1 = \frac{10}{15} + \frac{4}{15} = \underline{\underline{\frac{14}{15}}}$

9) Regn ut $\iiint_R \sqrt{x^2+y^2}\, d(x,y,z)$ der $R = \{(x,y,z) \in \mathbb{R}^3 :$
$x^2+y^2+z^2 \leq 4\}$

Ser at
$$R: \begin{cases} 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \rho^2 \leq 4 \quad \Rightarrow \quad 0 \leq \rho \leq 2 \end{cases}$$

$$\iiint_R \sqrt{x^2+y^2}\, dx\,dy\,dz$$

$$= \int_0^2 \int_0^{2\pi} \int_0^{\pi} \sqrt{\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi}\; \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \int_0^{\pi} \sqrt{\rho^2 \sin^2\phi}\; \rho^2 \sin\phi \; d\phi\, d\theta\, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \int_0^{\pi} \rho^3 \sin^2\phi \; d\phi\, d\theta\, d\rho$$

$$= \int_0^2 \rho^3 \int_0^{2\pi} \int_0^{\pi} \sin^2\phi \, d\phi\, d\theta\, d\rho$$

$$= \frac{1}{2}\int_0^2 \rho^3 \int_0^{2\pi} \left[\phi - \frac{1}{2}\sin(2\phi)\right]_{\phi=0}^{\pi} d\theta\, d\rho$$

$$= \frac{1}{2}\int_0^2 \rho^3 \int_0^{2\pi} \pi \; d\theta\, d\rho$$

$$= \frac{1}{2}\int_0^2 \rho^3 \left[\pi\,\theta\right]_{\theta=0}^{2\pi} d\rho = \frac{1}{2}\int_0^4 \rho^3\, 2\pi^2\, d\rho$$

$$= \frac{\pi^2}{4}\left[\rho^4\right]_0^2 = \underline{\underline{4\pi^2}}$$