ØVING 7 side 1 Andreas B. Berg

1) $\vec{r}(x,y) = [x\ ,\ y\ ,\ x^2+y^2]$, $x, y \in \mathbb{R}$

$x = r\cos\Theta \quad y = r\sin\Theta \quad z = x^2+y^2 = r^2\cos^2 + r^2\sin^2 = r^2$

$\vec{r}(r,\Theta) = [r\cos\Theta\ ,\ r\sin\Theta\ ,\ r^2]$ $r>0,\ \Theta\in[0,2\pi)$

2) Param. for $x^2+y^2+z^2=16$ der $x\geq 0,\ y\geq 0,\ z\geq 0$

$\vec{r}(x,y) = [x\ ,\ y\ ,\ \sqrt{16-(x^2+y^2)}]$, $x\geq 0,\ y\geq 0,\ x^2+y^2\leq 16$

evt $r(\phi,\Theta) = [4\sin\phi\cos\Theta\ ,\ 4\sin\phi\sin\Theta\ ,\ 4\cos\phi]$

$\phi\in[0,\frac{\pi}{2}]\ ,\ \Theta\in[0,\frac{\pi}{2}]$

3) Param for $x^2+z^2=4$, mellom $y=0$ og $y=1$

$\vec{r}(\Theta,y) = [2\cos\Theta\ ,\ y\ ,\ 2\sin\Theta]$, $0<y<1,\ \Theta\in[0,2\pi)$

4) Param for $x^2+y^2+z^2=4$ over xy-planet & inni kjeglen $z^2=3(x^2+y^2) \Rightarrow z^2\leq 3(x^2+y^2)$

$\vec{r}(\phi,\Theta) = [2\sin\phi\cos\Theta\ ,\ 2\sin\phi\sin\Theta\ ,\ 2\cos\phi]$

$\phi\in[0,\frac{\pi}{6}] \qquad \Theta\in[0,2\pi)$

Forklaring ϕ: $z^2=3(x^2+y^2)$ gir

$x^2+y^2+3(x^2+y^2) = 4x^2+4y^2 = 4$

$\Rightarrow x^2+y^2 = 1$

$\Rightarrow 4\sin^2\phi\cos^2\Theta + 4\sin^2\phi\sin^2\Theta = 4\sin^2\phi = 1$

$\Rightarrow \sin^2\phi = \frac{1}{4} \Rightarrow \sin\phi = \frac{1}{2}$

$\Rightarrow \phi = \frac{1}{6}\pi,\ \frac{5}{6}\pi$

$\Rightarrow$ over xy-planet og $z^2\leq 3(x^2+y^2) \Rightarrow \phi\in[0,\frac{\pi}{6}]$

ØVING 7 side 2 Andreas B. Berg

5) Vis at $F(x,y) = (x^2+y+1\,,\ x-y-2)$ har omvendt funk. G def om $(1,-2)$ s.a. $G(1,-2) = (0,0)$. Finn G' i $(1,-2)$

Vis at F har omvendt funk. H om $(1,-2)$ s.a. $H(1,-2) = (-1,-1)$

Finn $H'(1,-2)$

Ser at F er injektiv på $x \in [0,\infty)$. Ser og at $F(0,0) = (1,-2)$. DEF 5.7.1 gir da at

$\exists F^{-1} = G$ s.a. $G(x) = y \Leftrightarrow F(y) = x$

Mao. $\exists G$ s.a. $G(1,-2) = (0,0)$

Finner Jakobimatr. til F: $DF(x,y) = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}$

Vet fra Teorem 5.7.2 at $DG(F(x)) = (DF(x))^{-1}$:

$G'(1,-2) = G'(F(0,0)) = (F'(0,0))^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$\left[\begin{matrix} 0 & 1 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \sim \left[\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}\right]$

$G'(1,-2) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Ser at F injektiv på $x \in (-\infty, 0]$. Ser at $F(-1,-1) = (1,-2)$, som gir at det finnes

$H = F^{-1}$ s.a. $H(1,-2) = (-1,-1)$ (def 5.7.1)

Videre: $H'(1,-2) = H'(F(-1,-1)) = (F'(-1,-1))^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$\left[\begin{matrix} -2 & 1 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \sim \left[\begin{matrix} -1 & 0 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} -1 & -1 \\ 0 & 1 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} -1 & -1 \\ -1 & -2 \end{matrix}\right]$

$H'(1,-2) = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$

ØVING 7 side 3 Andreas B. Berg

6) Vis at gjennom hver (x_0, y_0) i $x^3 + y^3 + y = 1$ går en $y = f(x)$ som tilfredsstiller likningen. Finn $f'(x_0)$

La $F(x,y) = x^3 + y^3 + y - 1$. Teorem 5.7.8:

$F: \mathbb{R}^2 \to \mathbb{R}$. La $(x_0, y_0) \in \mathbb{R}^2$ s.a. $F(x_0, y_0) = 0$

(Merk: Da oppfyller (x_0, y_0) likn. i oppgaven).

$F_y(x_0, y_0) = 3y^2 + 1 \neq 0 \quad \forall y \Rightarrow$ inverterbar

Da sier teorem 5.7.8 at $\exists$ omegn U kring x_0 slik at $\forall x \in U$ finnes $f(x)$ s.a. $F(x, f(x)) = 0$

M.a.o. finnes $y = f(x)$ s.a. $(x_0, f(x_0))$ oppfyller likningen $\forall (x_0, y_0)$

$$f'(x_0) = -\left(\frac{\partial F}{\partial y}(x_0, y_0)\right)^{-1}\left(\frac{\partial F}{\partial x}(x_0, y_0)\right)$$

$$= -(3y_0^2 + 1)^{-1}(3x_0^2) = -\frac{3x^2}{3y^2 + 1}$$

7) La $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xy^2e^z + z$. Vis at $\exists\, g(x,y)$ om $(-1,2)$ s.a. $g(-1,2) = 0$ og $f(x, y, g(x,y)) = -4$. Finn $\frac{\partial g}{\partial x}(-1,2)$ og $\frac{\partial g}{\partial y}(-1,2)$. La $h(x,y,z) = f(x,y,z) + 4$

$h: \mathbb{R}^{2+1} \to \mathbb{R}^1$ med kont. part. deriverte. La $r = (x,y)$. Anta $(r_0, z_0) \in \mathbb{R}^3$ s.a. $h(r_0, z_0) = h(x_0, y_0, z_0) = 0$

$\frac{\partial h}{\partial y}(x_0, y_0, z_0) = 2xe^zy \neq 0$ for $x, y \neq 0 \Rightarrow$ inverterbar.

Da (teorem 5.7.8) $\exists$ omegn U kring r_0 s.a. for hver $r \in U$ $\exists\, g(r)$ s.a. $h(r, g(r)) = h(x, y, g(x,y)) = 0$

Ser at $h(-1, 2, 0) = f(-1, 2, 0) + 4 = -4 + 4 = 0$.

$\Rightarrow \exists\, g(x,y)$ om $(-1,2)$ s.a. $g(x,y) = 0$. Da er

$$f(x, y, g(x,y)) = h(x, y, g(x,y)) - 4 = -4$$

forts.

7) $\frac{\partial g}{\partial x}(-1,2) = -\left(\frac{\partial h}{\partial z}(-1,2,g(-1,2))\right)^{-1}\left(\frac{\partial h}{\partial x}(-1,2,g(-1,2)\right)$

$\left[\frac{\partial h}{\partial z} = xy^2e^z + 1 \quad \frac{\partial h}{\partial x} = y^2e^z \quad g(-1,2)=0\right]$

$= -\frac{2^2 e^0}{(-1)2^2e^0 + 1} = \frac{-4}{-3} = \underline{\underline{\frac{4}{3}}}$

$\frac{\partial g}{\partial y}(-1,2) = -\left(\frac{\partial h}{\partial z}(-1,2,g(-1,2))\right)^{-1}\left(\frac{\partial h}{\partial y}(-1,2,g(-1,2)\right)$

$\left[\frac{\partial h}{\partial y} = 2xye^z\right]$

$= -\frac{2(-1)\cdot 2e^0}{(-1)(2^2)e^0 + 1} = -\frac{-4}{-3} = \underline{\underline{-\frac{4}{3}}}$

8) Vis at $y'(x) = \frac{\frac{\partial \phi}{\partial x}(x,y(x))}{\frac{\partial \phi}{\partial y}(x,y(x))}$ gitt $\phi(x,y(x)) = C$, forutsatt de part. deriv. eksisterer og $\frac{\partial \phi}{\partial y}(x,y(x)) \neq 0$

La $\Theta(x,y) = \phi(x,y) - C$. Da vil $\Theta(x,y(x)) = 0$. De partiellderiv. til Θ = de part. deriv. til ϕ (C forsvinner fra derivasjon), så disse er kont. Da gir teorem 5.7.8 at $y(x)$ finnes, og at

$y'(x) = -\left(\frac{\partial \Theta}{\partial y}(x,y(x))\right)^{-1}\left(\frac{\partial \Theta}{\partial x}(x,y(x)\right)$

$= \frac{\frac{\partial \phi}{\partial x}(x,y(x))}{\frac{\partial \phi}{\partial y}(x,y(x))}$

9) Vis at $\exists\ u(x,y),\ v(x,y)$ om $(2,-1)$ s.a.

$$x^2+y^2-u^2+v^2=0$$

$$2xy+y^2-2u^2+3v^2+8=0$$

$$\Rightarrow x^2+y^2-u^2+v^2 = 2xy+y^2-2u^2+3v^2+8$$

$$\Rightarrow -2y^2+x^2-2xy+u^2-2v^2-8 = f(x,y,u,v)=0$$

$$f(2,-1,u(x,y),v(x,y)) = -2+4+4+u^2-2v^2-8 = -2+u^2-2v^2=0$$

$$\Rightarrow u^2 = 2v^2+2$$

$$x^2-y^2-u^2+v^2 = x^2-y^2-v^2-2 = g(x,y,v)=0$$

$g: \mathbb{R}^3 \to \mathbb{R}$, kont part. der. La $r=(x,y)$. Anta $(r_0, v_0)\in\mathbb{R}^2$

s.a. $g(r_0,v_0)=0$ $\frac{\partial g}{\partial v}(r_0,v_0) = -2v \neq 0$ for $v\neq 0$

Da $\exists\ U$ kring r_0 s.a. $\forall\ x\in U\ \exists\ v(r)$ s.a.

$$g(r,v(r)) = g(x,y,v(x,y))=0$$

Ser at $g(2,-1,v(2,-1)) = 4-1-v^2-2 = 1-v^2=0$

$$\Rightarrow \underline{\underline{v(2,-1)=1}}$$

$$\frac{\partial v}{\partial x}(2,-1) = \frac{\frac{\partial g}{\partial x}(2,-1,v(2,-1))}{\frac{\partial g}{\partial y}(2,-1,v(2,-1))} = \frac{4}{2} = 2 \quad ?$$

$$\left[\frac{\partial g}{\partial x}=2x \quad \frac{\partial g}{\partial y} = -2y\right]$$

$$u^2=2v^2+2 \Rightarrow \underline{\underline{u(2,-1)}} = \sqrt{2v^2(2,-1)+2} = \sqrt{4} = \underline{\underline{2}}$$

$$\frac{d}{dx}\left(u(2,-1)^2\right) = \frac{d}{dx}\left(2v(2,-1)^2+2\right) = \frac{d}{dx}\left(2v(2,-1)^2\right)$$

$$\Downarrow = 4v(2,-1)(v'(2,-1)) = 4\cdot 1\cdot \frac{5}{4} = 5$$

$$2u(2,-1)u'(2,-1) = 4u'(2,-1) \Rightarrow u'(2,-1) = 5/4$$

FÅR RIKTIG VERDI, MEN IKKE DERIVERT. HVORFOR? !!