ØVING 4 side 1 Andreas B. Berg

2.4

16) La B være $n \times n$-invertibel matrise. $\Phi :: M_{n\times n}(F) \to M_{n\times n}(F)$ s.a.:
$\Phi(A) = B^{-1}AB$. Vis at Φ er en isomorfi

Φ på: Vis $\forall\ C\ \exists\ A$ s.a. $\Phi(A) = C$

$$\Phi(A) = B^{-1}AB = C$$
$$A = BCB^{-1} \Rightarrow \Phi \text{ er på}$$

Φ en-til-en: Vis at hvis $\Phi(A_1) = \Phi(A_2) \Rightarrow A_1 = A_2$

$$\Phi(A_1) = B^{-1}A_1B = B^{-1}A_2B = \Phi(A_2)$$

multipliser B fra venstre, B^{-1} fra høyre:

$$\left.\begin{array}{l} BB^{-1}A_1BB^{-1} = A_1 \\ BB^{-1}A_2BB^{-1} = A_2 \end{array}\right\} A_1 = A_2 \Rightarrow \Phi \text{ er på}$$

Φ lineær:

$$\Phi(A+C) = B^{-1}(A+C)B = B^{-1}AB + B^{-1}CB = \Phi(A) + \Phi(C)$$
$$\Phi(rA) = B^{-1}rAB = rB^{-1}AB = r\Phi(A)$$

$\underline{\underline{\Phi \text{ lineær, en-til-en og på} \Rightarrow \Phi \text{ isomorfi}}}$

19) $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$, $T(M) = M^t$. $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$

a) Finn $[T]_\beta^\beta$.

$$T(E^{11}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad T(E^{12}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad T(E^{21}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad T(E^{22}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}}$$

2.4

1a) b) Sjekk at $L_{A\phi_\beta}(M) = \phi_\beta T(M)$, $A = [T]_\beta^\beta$, $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$L_{A\phi_\beta}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

$$\phi_\beta T(M) = \phi_\beta \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} = L_{A\phi_\beta}(M)$$

$$\underline{\underline{L_{A\phi_\beta}(M) = \phi_\beta T(M)}}$$

22) $c_0, c_1, \ldots, c_n \in F$. $T : P_n(F) \to F^{n+1}$, $T(f) = (f(c_0), f(c_1), \ldots, f(c_n))$

Vis at T er isomorf.

T lineær:

$$T(f+g) = ((f+g)(c_0), (f+g)(c_1), \ldots, (f+g)(c_n))$$
$$= (f(c_0)+g(c_0), f(c_1)+g(c_1), \ldots, f(c_n)+g(c_n))$$
$$= (f(c_0), f(c_1), \ldots, f(c_n)) + (g(c_0), g(c_1), \ldots, g(c_n))$$
$$= T(f) + T(g)$$

$$T(rf) = (rf(c_0), \ldots, rf(c_n)) = r(f(c_0), \ldots, f(c_n)) = rT(f)$$

T en-til-en:

$T(f) = 0 \Rightarrow f = 0$

Hvis $T(f) = 0$ må f ha $n+1$ nullpunkter.

f er polynom av grad n. Har tidligere vist at polynom av grad n maks kan ha n nullpunkter, ergo må $f = 0$

T på:

Siden $\dim(P_n(F)) = \dim(F^{n+1})$ og T er en-til-en, må T være på.

$\underline{\underline{T \text{ lineær, på og en-til-en} \Rightarrow T \text{ isomorf.}}}$

 Andreas B. Berg

2.5

3) For β og β' basis for $P_2(\mathbb{R})$, finn $[T]_{\beta'}^{\beta}$

b) $\beta = \{1, x, x^2\}$ $\quad \beta' = \{\overset{v_1}{a_2x^2 + a_1x + a_0}, \overset{v_2}{b_2x^2 + b_1x + b_0}, \overset{v_3}{c_2x^2 + c_2x + c_0}\}$

$$T(v_1) = (a_0, a_1, a_2)$$
$$T(v_2) = (b_0, b_1, b_2)$$
$$T(v_3) = (c_0, c_1, c_2)$$

$$\underline{\underline{[T]_{\beta'}^{\beta} = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}}}$$

c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ $\quad \beta' = \{\overset{v_1}{1}, \overset{v_2}{x}, \overset{v_3}{x^2}\}$

$$T(v_1) = (0, 1, -3)$$
$$T(v_2) = (-1, 0, 2)$$
$$T(v_3) = (0, 0, 1)$$

$$\underline{\underline{[T]_{\beta'}^{\beta} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}}}$$

d) $\beta = \{\overset{w_1}{x^2 - x + 1}, \overset{w_2}{x + 1}, \overset{w_3}{x^2 + 1}\}$ $\quad \gamma = \{1, x, x^2\}$

$\beta' = \{\underset{v_1}{x^2 + x + 4}, \underset{v_2}{4x^2 - 3x + 2}, \underset{v_3}{2x^2 + 3}\}$

$$[T]_{\beta'}^{\beta} = [A]_{\gamma}^{\beta} [B]_{\beta'}^{\gamma} \qquad [A]_{\gamma}^{\beta} = \left([C]_{\beta}^{\gamma}\right)^{-1}$$

$$B(v_1) = (4, 1, 1) \qquad C(w_1) = (1, -1, 1)$$
$$B(v_2) = (2, -3, 4) \qquad C(w_2) = (1, 1, 0)$$
$$B(v_3) = (3, 0, 2) \qquad C(w_3) = (1, 0, 1)$$

$$B = \begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

 Andreas B. Berg

2.5

3) d) forts. Finner $A = C^{-1}$:

$$[C\ I_3] = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{pmatrix} = [I_3\ A]$$

$$[T]_{\beta'}^{\beta} = [A]_{\gamma}^{\beta}[B]_{\beta'}^{\gamma} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$$

$$\underline{\underline{[T]_{\beta'}^{\beta} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}}}$$

4) La T lin. op. $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix}$

La $\beta = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$, $\beta' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

Finn $[T]_{\beta'}^{\beta'}$

$$[T]_{\beta}^{\beta} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

La $A = [\cdot]_{\beta'}^{\beta}$

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1, 1)$ $\quad A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1, 2)$

$$A = [\cdot]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[T]_{\beta'}^{\beta'} = [\cdot]_{\beta}^{\beta'}[T]_{\beta}^{\beta}[\cdot]_{\beta'}^{\beta} = \left([\cdot]_{\beta'}^{\beta}\right)^{-1}[T]_{\beta}^{\beta}[\cdot]_{\beta'}^{\beta}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix}$$

$$\underline{\underline{= \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}}}$$

2.5

8) La $T: V \to W$, V, W endelig-dim, β og β' ordn. base V, γ og γ' ord. base W.

Vis at da er $[T]_{\beta'}^{\gamma'} = P^{-1} [T]_\beta^\gamma Q$, der $[Q]_{\beta'}^{\beta}$ og $[P]_{\gamma'}^{\gamma}$.

$$[T]_{\beta'}^{\gamma'} = \left([P]_{\gamma'}^{\gamma}\right)^{-1} [T]_\beta^\gamma [Q]_{\beta'}^{\beta}$$

$$= [P^{-1}]_\gamma^{\gamma'} [T]_\beta^\gamma [Q]_{\beta'}^{\beta}$$

Se på $[T]_{\beta'}^{\gamma'}(x_{\beta'})$ $\quad x_{\beta'} \in V$

Først gjør Q om basen: $Q(x_{\beta'}) = x_\beta$

Så gjør T om vektorrommet: $T(x_\beta) = y_\gamma$, $y \in W$

Så gjør P^{-1} om basen: $P^{-1}(y_\gamma) = y_{\gamma'}$

Altså er $[T]_{\beta'}^{\gamma'} = P^{-1} [T]_\beta^\gamma Q$ den lineære transformasjonen T på en vektor med base β' som gir ut en vektor i W med base γ'