 Andreas B. Berg

3.6

2) Finn $Var(Y)$ hvis $f_Y(y) = \begin{cases} 3/4, & 0 \le y \le 1 \\ 1/4, & 2 \le y \le 3 \\ 0, & \text{ellers} \end{cases}$

$$Var(Y) = E(Y^2) - E(Y)^2$$

$$E(Y^2) = \int_0^3 y^2 f_Y(y)\,dy = \int_0^1 \frac{3}{4} y^2\,dy + \int_2^3 \frac{1}{4} y^2\,dy$$

$$= \left[\frac{1}{4} y^3\right]_0^1 + \left[\frac{1}{12} y^3\right]_2^3 = \frac{1}{4} + \frac{27}{12} - \frac{8}{12} = \underline{\frac{11}{6}}$$

$$E(Y) = \int_0^3 y f_Y(y)\,dy = \int_0^1 \frac{3}{4} y\,dy + \int_2^3 \frac{1}{4} y\,dy$$

$$= \left[\frac{3}{8} y^2\right]_0^1 + \left[\frac{1}{8} y^2\right]_2^3 = \frac{3}{8} + \frac{9}{8} - \frac{4}{8} = \underline{1}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{11}{6} - 1 = \underline{\underline{\frac{5}{6}}}$$

6) $f_Y(y) = \frac{2y}{k^2}$, $0 \le y \le k$. Hvilke k gir $Var(Y) = 2$?

$$E(Y) = \int_0^k y \frac{2y}{k^2}\,dy = \int_0^k \frac{2y^2}{k^2}\,dy = \left[\frac{2}{3}\frac{y^3}{k^2}\right]_0^k = \frac{2}{3} k$$

$$E(Y^2) = \int_0^k y^2 \frac{2y}{k^2}\,dy = \int_0^k \frac{2y^3}{k^2}\,dy = \left[\frac{y^4}{2k^2}\right]_0^k = \frac{1}{2} k^2$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{1}{2}k^2 - \frac{4}{9}k = 2$$

$$9k^2 - 8k - 36 = 0$$

$$k = \frac{8 \pm \sqrt{64 - 4 \cdot 9 \cdot (-36)}}{18} = \frac{8 \pm 4\sqrt{85}}{18} \underline{\underline{\approx 2{,}49}}$$

7) $f_Y(y) = \begin{cases} \frac{1}{2}, & 2 \le y \le 3 \\ -y+1, & 0 \le y \le 1 \end{cases}$

$$E(Y) = \int_0^1 -y^2 + y\,dy + \int_2^3 \frac{1}{2} y\,dy = \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2\right]_0^1 + \left[\frac{1}{4}y^2\right]_2^3 = \frac{17}{12}$$

$$E(Y^2) = \int_0^1 -y^3 + y^2\,dy + \int_2^3 \frac{1}{2} y^2\,dy = \left[-\frac{1}{4}y^4 + \frac{1}{3}y^3\right]_0^1 + \left[\frac{1}{6}y^3\right]_2^3 = \frac{13}{4}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{13}{4} - \frac{289}{144} = \frac{179}{144}$$

$$\sigma = \sqrt{Var(Y)} = \underline{\underline{\frac{\sqrt{179}}{12}}} \approx \underline{1{,}115}$$

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3.6

16) $E(w)=\mu \quad Var(w)=\sigma^2$, vis at $E\left(\frac{w-\mu}{\sigma}\right)=0$ og $Var\left(\frac{w-\mu}{\sigma}\right)=1$

$$E\left(\frac{w-\mu}{\sigma}\right)=E\left(\frac{w}{\sigma}-\frac{\mu}{\sigma}\right)=E\left(\frac{w}{\sigma}\right)-E\left(\frac{\mu}{\sigma}\right)=E\left(\frac{1}{\sigma}w\right)-E\left(\frac{\mu}{\sigma}\right)$$

w stok var, μ, σ konst:

$$=\frac{1}{\sigma}E(w)-\frac{\mu}{\sigma}=\frac{\mu}{\sigma}-\frac{\mu}{\sigma}=\underline{0}$$

$$Var\left(\frac{w-\mu}{\sigma}\right)=E\left(\left(\frac{w-\mu}{\sigma}\right)^2\right)-E\left(\frac{w-\mu}{\sigma}\right)^2=E\left(\left(\frac{w-\mu}{\sigma}\right)^2\right)$$

$$=E\left(\frac{w^2-2w\mu+\mu^2}{\sigma^2}\right)=\frac{1}{\sigma^2}E(w^2)-\frac{2\mu}{\sigma^2}E(w)+\mu^2\cdot\frac{1}{\sigma^2}$$

$$=\frac{1}{\sigma^2}E(w^2)-\frac{2\mu^2+\mu^2}{\sigma^2}=\frac{1}{\sigma^2}E(w^2)-\frac{\mu^2}{\sigma^2}$$

$$E(w^2)=Var(w)+E(w)^2=\sigma^2+\mu^2$$

$$=\frac{\sigma^2+\mu^2-\mu^2}{\sigma^2}=\frac{\sigma^2}{\sigma^2}=\underline{1}$$

3.12

7) $X\sim poisson(\lambda) \iff p_X(k)=P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!} \quad k=0,1,\dots$

Finn mgf til poissonfordelt X:

$$M_X(t)=E(e^{tX})=\sum_{k=0}^{\infty}e^{tk}p_X(k)$$

$$=\sum_{k=0}^{\infty}e^{tk}\frac{e^{-\lambda}\lambda^k}{k!}=\sum_{k=0}^{\infty}\frac{e^{tk-\lambda}\lambda^k}{k!}=\sum_{k=0}^{\infty}\frac{(e^t\lambda)^k}{k!}e^{-\lambda}$$

$$=e^{-\lambda}\sum_{k=0}^{\infty}\frac{(e^t\lambda)^k}{k!}=e^{-\lambda}e^{e^t\lambda}=e^{-\lambda+e^t\lambda}\underline{\underline{=e^{\lambda(e^t-1)}}}$$

 Andreas B. Berg

3.1 2

9) Finn $E(Y^3)$ hvis $M_Y(t) = e^{\frac{1}{2}t^2}$

$$M_Y^{(3)}(t) = \frac{d^3}{(dt)^3} M_Y(t)$$

$$= \frac{d^2}{(dt)^2} e^{\frac{1}{2}t^2} \cdot t$$

$$= \frac{d}{dt} \quad e^{\frac{1}{2}t^2} + t^2 e^{\frac{1}{2}t^2}$$

$$= t e^{\frac{1}{2}t^2} + 2t e^{\frac{1}{2}t^2} + t^3 e^{\frac{1}{2}t^2}$$

$$= (t^3 + 3t) e^{\frac{1}{2}t^2}$$

$$E(Y^3) = M_Y^{(3)}(0) = (0 + 0)e^0 = \underline{\underline{0}}$$

4.2

17) X = feil per 30 sek

$X \sim$ poisson $(4{,}5)$

Finn sannsynlighet for fler enn 2 feil på 30 sek

$$P(X > 2) = 1 - (P(X=0) + P(X=1) + P(X=2))$$

$$= 1 - (p_X(0) + p_X(1) + p_X(2))$$

$$= 1 - \left(e^{-4.5} \left(\frac{4{,}5^0}{1} + \frac{4{,}5}{1} + \frac{4{,}5^2}{2}\right)\right)$$

$$= 1 - \left(e^{-4.5} \frac{125}{8}\right) \approx \underline{\underline{0{,}826}}$$

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4.2

27) X = dødsfall per dag $\sim$ Poisson$(0{,}1)$

Y = ventetid mellom dødsfall. $P(Y > y) = e^{-\lambda y}$

$$P(Y > 7) = e^{-0{,}1 \cdot 7} = \underline{e^{-0{,}7}} \approx \underline{\underline{0{,}50}}$$

Eks. K13

3)a) X kont, $f_X(x) = \begin{cases} e^{-(x-\theta)} & ; \ x \geq \theta > 0 \\ 0 & ; \ \text{ellers} \end{cases}$

Vis at $M_X(t) = \frac{e^{\theta t}}{1-t}, \ t < 1$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tk} f_X(k)\,dk$$

$$= \int_{\theta}^{\infty} e^{tk} e^{-k+\theta}\,dk$$

$$= \int_{\theta}^{\infty} e^{k(t-1)+\theta}\,dk$$

$$= \frac{1}{t-1}\left[e^{k(t-1)+\theta}\right]_{\theta}^{\infty}$$

$$= \frac{1}{t-1}\left(e^{\infty(t-1)+\theta} - e^{\theta(t-1)+\theta}\right)$$

$$= \frac{1}{t-1}\left(e^{\theta}\left(e^{(t-1)}\right)^{\infty} - e^{\theta t-\theta+\theta}\right)$$

$$t-1 < 0 \rightsquigarrow t < 1 \rightsquigarrow \left(e^{(t-1)}\right)^{\infty} = 0$$

$$= \frac{1}{t-1}\left(-e^{\theta t}\right) = \underline{\frac{e^{\theta t}}{1-t}, \ t < 1}$$

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{d}{dt}\left(e^{\theta t}\right)(1-t)^{-1} \Big|_{t=0}$$

$$= \frac{\theta e^{\theta t}}{1-t} + \frac{e^{\theta t}}{(1-t)^2} \Big|_{t=0}$$

$$= \frac{\theta}{1} + \frac{1}{1} = \underline{\underline{\theta + 1}}$$

 Andreas B. Berg

EKS K13

3) a) $E(X^2) = \frac{d}{dt}\left(\Theta e^{\Theta t}(1-t)^{-1} + e^{\Theta t}(1-t)^{-2}\right)\Big|_{t=0}$

$$= \frac{\Theta^2 e^{\Theta t}}{1-t} + \frac{\Theta e^{\Theta t}}{(1-t)^2} + \frac{\Theta e^{\Theta t}}{(1-t)^2} + \frac{2e^{\Theta t}}{(1-t)^3}\Big|_{t=0}$$

$$= \Theta^2 + \Theta + \Theta + 2 = \underline{\Theta^2 + 2\Theta + 2}$$

$$Var(X) = E(X^2) - E(X)^2 = \Theta^2 + 2\Theta + 2 - (\Theta + 1)^2$$

$$= \Theta^2 + 2\Theta + 2 - \Theta^2 - 2\Theta - 1 = \underline{\underline{1}}$$

EKS V16

2) a) Y = fisker på en time ~ Poisson (2)

X = fisk på t timer. $P(X=k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$

$P(X=4),\ t=2 \quad = \frac{(2\cdot 2)^4 e^{-4}}{4!} = \frac{4^3 e^{-4}}{3!} \approx \underline{\underline{0{,}195}}$

Siden $Y \sim$ poisson (2), forventes det å fås to (2) fisk per time. M.a.o. må det fiskes i <u>to timer</u> før man kan forvente å ha fått 4 fisk