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3.8

2) $f_X(x) = xe^{-x}$ $\quad f_Y(y) = e^{-y}$ $\quad x, y \geq 0$ $\quad X, Y$ uavh.

$W = X + Y$. Finn $f_W(w)$

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx$$

$$= \int_0^w xe^{-x} e^{x-w}\, dx = \int_0^w xe^{-w}\, dx = \left[\frac{x^2}{2} e^{-w}\right]_0^w$$

$$\underline{\underline{= \frac{w^2}{2} e^{-w}}}$$

11) X, Y uavh. $f_X(x) = xe^{-x}$ $\quad f_Y(y) = e^{-y}$. Finn $f_W(w)$ når $W = \frac{Y}{X}$

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx)\, dx$$

$$= \int_0^{\infty} x \cdot xe^{-x} \cdot e^{-wx}\, dx$$

$$= \int_0^{\infty} x^2 e^{-x(1+w)}\, dx \qquad u = x(1+w) \qquad \begin{matrix} du = 1+w \\ dx = (1+w)^{-1}\, du \end{matrix}$$

$$\int \frac{u^2 e^{-u}}{(1+w)^3}\, du = \frac{1}{(1+w)^3} \int u^2 e^{-u}\, du = \frac{1}{(1+w)^3}\left[-e^{-u} u^2 + 2(-e^{-u} u - e^{-u})\right]$$

$$= \frac{1}{(1+w)^3}\left[-e^{-x(1+w)} x^2 (1+w)^2 + 2\left(-e^{-x(1+w)} x(1+w) - e^{-x(1+w)}\right)\right]_0^{\infty}$$

$$= \frac{1}{(1+w)^3} \cdot 2 \qquad \underline{\underline{= \frac{2}{(1+w)^3}}}$$

3.9

1) r chips med tilbakelegging fra n chips merket $1 \to n$. V = sum. Finn $E(V)$

La $V = V_1 + \dots + V_r$, der V_i = nummer på chip i, $i \in \{1, \dots, r\}$

$$p_{X_i}(x) = \frac{1}{n} \quad \forall\, x \in \{1, \dots, n\}$$

$$E(V_i) = \frac{1}{n} \sum_{k=0}^{n} k = \frac{1+2+\dots+n}{n} = \underline{\underline{\frac{n+1}{2}}}$$

$$E(V) = E(V_1) + E(V_2) + \dots + E(V_r) \underline{\underline{= \frac{r}{2}(n+1)}}$$

ØVING 8 side 2 Andreas B. Berg

3.9

11) X = punkt, $x \in [0,1]$ $\quad Y$ = punkt, $y \in [0,1]$

La V = areal av trekant med hjørne $(X,0), (0,Y), (0,0)$. Hva er $E(V)$?

$V = \frac{1}{2}XY$, så $E(V) = \frac{1}{2}E(X)E(Y)$ $\quad$ [X, Y uavh, t. 3.9.3]

Alle pkt. like sannsynlig gir $f_X(x) = 1 = f_Y(y)$, $\quad x, y \in [0,1]$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \left[\tfrac{1}{2}y^2\right]_0^1 = \tfrac{1}{2}$$

$$\Rightarrow E(V) = \tfrac{1}{2}E(X)E(Y) = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} = \underline{\underline{\tfrac{1}{8}}}$$

13) To terningkast. X = første terning. Y = høyeste terning. Finn $Cov(X, Y)$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$p_X(x) = \frac{1}{6} \quad \forall\, x \in \{1, \dots, 6\}$ $\qquad E(X) = \frac{1}{6}\sum_{i=1}^{6} i = 3{,}5$

y	1	2	3	4	5	6
$p_Y(y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$E(Y) = \frac{1}{36} + \frac{3\cdot 2}{36} + \frac{5\cdot 3}{36} + \frac{7\cdot 4}{36} + \frac{9\cdot 5}{36} + \frac{11\cdot 6}{36} = \frac{161}{36}$$

$y \backslash x$	1	2	3	4	5	6
1	$\frac{1}{6}$	0	0	0	0	0
2	$\frac{1}{6}$	$\frac{2}{6}$	0	0	0	0
3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{6}$	0	0	0
4	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{4}{6}$	0	0
5	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	0
6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

$p_{Y|X}(y) = 6\,p_{XY}(x,y)$

$$p_{XY}(x,y) = p_{Y|X}(y)\cdot p_X(x) = \tfrac{1}{6}\,p_{Y|X}(y)$$

$$E(XY) = \sum_{x=1}^{6}\sum_{y=1}^{6} xy\,p_{XY}(x,y) = \sum_{x=1}^{6}\sum_{y=x}^{6} xy\,\tfrac{1}{6}\,p_{Y|X}(y)$$

$$= \underbrace{\frac{1+2+3+4+5+6}{36}}_{x=1} + \underbrace{\frac{8+6+8+10+12}{36}}_{x=2} + \underbrace{\frac{27+12+15+18}{36}}_{x=3} + \underbrace{\frac{64+20+24}{36}}_{x=4} + \underbrace{\frac{125+30}{36}}_{x=5} + \underbrace{6}_{x=6}$$

$$= \frac{154}{9}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{154}{9} - \frac{7}{2}\cdot\frac{161}{36} = \underline{\underline{\frac{35}{24}}} \quad \left(= \frac{105}{72}\right)$$

 Andreas B. Berg

3.10

1) Ventetid i min fordelt jevnt på $(0,10)$. Hva er sanns. for at nest lengste av 4 ventetider < 5 min?

Ser at $f_Y(y) = \frac{1}{10}$

$F_Y(y) = \int_0^y f_Y(t)\,dt = \frac{1}{10}y$

La $Y_{(1)}, \dots, Y_{(4)}$ være ventetidene, hvor $Y_{(1)} < Y_{(2)} < Y_{(3)} < Y_{(4)}$.

Ser på $Y_{(3)}$. Vil finne $F_{Y_{(3)}}(5) = P(Y_{(2)} \leq 5)$

$$f_{Y(3)}(y) = \frac{4!}{1!\,2!}\left(F_Y(y)\right)^2\left(1 - F_Y(y)\right)^1 f_Y(y)$$

$$= 4 \cdot 3 \cdot \frac{1}{100}y^2 \cdot \left(1 - \frac{1}{10}y\right) \cdot \frac{1}{10}$$

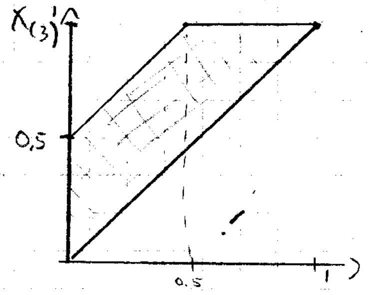
$$= \frac{12y^2}{1000} \cdot \left(1 - \frac{1}{10}y\right) = \frac{12y^2}{1000} - \frac{12y^3}{10000}$$

$$F_{Y(2)}(5) = \int_0^5 f_{Y(2)}(t)\,dt = \int_0^5 \frac{12t^2}{1000} - \frac{12t^3}{10000}\,dt$$

$$= \left[\frac{t^3}{250} - \frac{3t^4}{10000}\right]_0^5 = \frac{1}{2} - \frac{3}{16} = \underline{\underline{\frac{5}{16}}}$$

15) Tre punkt, X_1, X_2, X_3 på $[0,1]$. Hva er $P(\text{alle } X \text{ innenfor } 0.5)$?

La $X_{(1)} \leq X_{(2)} \leq X_{(3)}$



La $R = \text{range} = X_{(3)} - X_{(1)}$

$f_Y(y) = 1, \quad y \in [0,1]$

$F_Y(y) = y \quad y \in [0,1]$

$$f_{X_{(1)}X_{(3)}}(u, v) = \frac{3!}{0!\,1!\,0!}\, u^0 (v-u)^1 (1-v)^0 \cdot 1 \cdot 1 = 6(v-u)$$

$$0 \leq u < v \leq 1$$

$$P(R \leq 0.5) = F_R(0.5) = \int_0^{\frac{1}{2}} \int_u^{u+\frac{1}{2}} 6(v-u)\,dv\,du + \int_{0.5}^1 \int_u^1 6(v-u)\,dv\,du$$

$$= \int_0^{\frac{1}{2}} \left[3v^2 - 6vu\right]_u^{u+\frac{1}{2}} du + \int_{\frac{1}{2}}^1 \left[3v^2 - 6vu\right]_u^1 du$$

15) $P(R \le 0{,}5) = \int_0^{\frac{1}{2}} 3(u+\tfrac{1}{2})^2 - 6(u+\tfrac{1}{2})u - 3u^2 + 6u^2 \, du$

$$+ \int_{\frac{1}{2}}^{1} 3 - 6u - 3u^2 + 6u^2 \, du$$

$$= \int_0^{\frac{1}{2}} 3u^2 + 3u + \frac{3}{4} - 6u^2 - 3u + 3u^2 \, du$$

$$+ \int_{\frac{1}{2}}^{1} 3 - 6u + 3u^2 \, du$$

$$= \left[\frac{3}{4}u\right]_0^{\frac{1}{2}} + \left[3u - 3u^2 + u^3\right]_{\frac{1}{2}}^{1}$$

$$= \frac{3}{8} + \cancel{3} - 3 + 1 - \frac{3}{2} + \frac{3}{4} - \frac{1}{8} = \underline{\underline{\frac{1}{2}}}$$

ARK

1) a) $Y = g(X)$

$$f_X(x)\,dx = f_Y(y)\,dy$$

$$\Downarrow$$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \quad \text{hvor } y = g(x)$$

$$\underline{= f_X(g^{-1}(y))\,\frac{d}{dy}(g^{-1}(y))} \quad \text{når } g \text{ økende} \Rightarrow \frac{d}{dy}g^{-1}(y) > 0$$

b) Når g avtagende $\Rightarrow \frac{d}{dy}g^{-1}(y) < 0 \Rightarrow \left|\frac{d}{dy}g^{-1}(y)\right| = -\frac{d}{dy}g^{-1}(y)$

$$\Rightarrow f_Y(y) = -f_X(g^{-1}(y))\,\frac{d}{dy}g^{-1}(y)$$

c) Se svar i (b) $\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$

evt a) $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}P(Y \le y) = \frac{d}{dy}P(g(X) \le y)$

$$= \frac{d}{dy}P(X \le g^{-1}(y)) = \frac{d}{dy}F_X(g^{-1}(y))$$

$$\underline{= f_X(g^{-1}(y))\,\frac{d}{dy}g^{-1}(y)} \quad [\text{kjerneregel}]$$

 Andreas B. Berg

2) U, V uavh. $W = e^{U+V}$

La $Z = U + V$. $W = e^{Z} = g(z) \Rightarrow g^{-1}(w) = \ln z$

$$f_W(w) = f_Z(\ln w)\frac{1}{w}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_U(u) f_V(z-u)\,du$$

$$\Rightarrow \underline{f_W(w) = \int_{-\infty}^{\infty} \frac{f_U(u) f_V(\ln(w) - u)}{w}\,du}$$

$$f_W(w) = \frac{d}{dw} F_W(w) = \frac{d}{dw} P(W \leq w) = \frac{d}{dw} P(e^{U+V} \leq w)$$

$$e^{U+V} \leq w$$

$$Z \leq \ln w$$

$$= \frac{d}{dw} F_Z(\ln w) = f_Z(\ln w)\frac{1}{w} = \underline{\int_{-\infty}^{\infty} \frac{f_U(u) f_V(\ln(w) - u)}{w}\,du}$$

3) $E(X) = \mu_X$

$$Var(aX + b) = \int_{-\infty}^{\infty} (aX + b - E(aX+b))^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (aX + b - (E(aX) + b))^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (aX - aE(X))^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (a^2X^2 - 2a^2 X E(X) + a^2E(X)^2) f_X(x)\,dx$$

$$= a^2 \int_{-\infty}^{\infty} (x^2 - 2xE(x) + E(x)^2) f_X(x)\,dx$$

$$= a^2 \int_{-\infty}^{\infty} (x - E(x))^2 f_X(x)\,dx$$

$$\underline{\underline{= a^2\,Var(X)}}$$