ØVING 7 side 1 Andreas B. Berg

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3). Test om T diagonaliserbar, finn basis β så $[T]_\beta^\beta$ diagonal

a) $V = P_3(R)$, $T(f(x)) = f'(x) + f''(x)$

$T(a + bx + cx^2 + dx^3)$

$= 0 + b + 2cx + 3dx^2 + 2c + 6dx = (b+2c) + (2c+6d)x + 3dx^2$

$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b+2c \\ 2c+6d \\ 3d \\ 0 \end{pmatrix}$ $T(1) = 0$ $T(x) = 1$ $T(x^2) = 2x+2$ $T(x^3) = 3x^2+6x$

La $\gamma = \{1, x, x^2, x^3\}$ basis for $P_3(R)$

$[T]_\gamma^\gamma = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\det(T - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 2 & 0 \\ & -\lambda & 2 & 6 \\ & & -\lambda & 3 \\ 0 & & & -\lambda \end{pmatrix} = \lambda^4$

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 0$

$E_0 = N(T - 0I) = N(T) \Rightarrow$ basis $\{1\}$

$\Rightarrow \dim E_0 = 1 < 4 \Rightarrow$ T ikke diagonaliserbar

b) $V = P_2(R)$ $T(ax^2 + bx + c) = cx^2 + bx + a$

$\gamma = \{x^2, x, 1\}$ basis for $P_2(R)$ $T(x^2) = 1$ $T(x) = x$ $T(1) = x^2$

$[T]_\gamma^\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$\det(T - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = (1-\lambda) \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$

$= (1-\lambda)(\lambda^2 - 1) = (1-\lambda)(\lambda+1)(\lambda-1) = 0$

$\Downarrow$

$\lambda_1 = \lambda_2 = 1$ $\lambda_3 = -1$

$T - 1I$ $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\dim E_1 = \dim N(T - 1I) = 3 - \text{rank}(T - I) = 2 \Rightarrow$ T diagonaliserbar

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3) b) Skal finne basis β så $[T]_\beta^\beta$ diagonal

Basis for $E_1 = N(T-I)$:

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & a \\ 0 & 0 & 0 & b \\ 1 & 0 & -1 & c \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} -1 & 0 & 1 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c+a \end{array}\right)$$

Basis: $\vec{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Basis for $E_{-1} = N(T+I)$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 2 & 0 & b \\ 1 & 0 & 1 & c \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 2 & 0 & b \\ 0 & 0 & 0 & c-a \end{array}\right) \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\beta_1 = ([\vec{u}]_\gamma)^{-1} = x$ $\quad \beta_2 = ([\vec{v}]_\gamma)^{-1} = x^2+1$ $\quad \beta_3 = ([\vec{w}]_\gamma)^{-1} = x^2-1$

$$\underline{\underline{[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ for basis } \beta = \{x, x^2+1, x^2-1\}}}$$

c) $V = \mathbb{R}^3$ $\quad T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}$

γ = basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$ ordnet.

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ $\quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$$\rightsquigarrow [T]_\gamma^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = A$$

Ser at A med liten endring i rekkefølgen på basisen er diagonal:

<u>La $\beta = \{(0,1,0), (1,0,0), (0,0,1)\}$</u> ordn. basis for $\mathbb{R}^3$

$$\underline{\underline{\Rightarrow [T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ diagonal matrise.}}}$$

Merk: Ser fasiten mener denne ikke er diagonaliserbar. Hvor har jeg gjort feil?

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3) d) $V = P_2(\mathbb{R}) \quad T(f(x)) = f(0) + f(1)(x + x^2)$

$$T(ax^2 + bx + c) = c + (a+b)(x+x^2) = (a+b+c)x^2 + (a+b+c)x + c$$

$\gamma = \{x^2, x, 1\}$ ordn. basis for $P_2(\mathbb{R})$

$$T(x^2) = x^2 + x \qquad T(x) = x^2 + x \qquad T(1) = x^2 + x + 1$$

$$[T]_\gamma^\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = A$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(\cancel{1} - 2\lambda + \lambda^2 - \cancel{1}) = \lambda(1-\lambda)(\lambda - 2)$$

$$\Rightarrow \lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 2 \quad \Rightarrow \underline{\underline{T \text{ diagonaliserbar}}}$$

$\underline{\lambda_1 = 0:}$

$(A - 0I) = A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Finner basis for $E_0 = N(A)$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 1 & 0 & b \\ 1 & 1 & 1 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 0 & 0 & b-a \\ 1 & 1 & 1 & c \end{array}\right) \Rightarrow \vec{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\beta_1 = ([\vec{u}]_\gamma)^{-1} = x^2 - x$$

$\underline{\lambda_2 = \phi:}$

$(A - I) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Basis: $\left(\begin{array}{ccc|c} 0 & 1 & 0 & a \\ 1 & 0 & 0 & b \\ 1 & 1 & 0 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 0 & 0 & c-b-a \end{array}\right)$

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \beta_2 = ([\vec{v}]_\gamma)^{-1} = x^2 + x - 1$$

$\underline{\lambda_3 = 2:}$

$(A - 2I) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. Basis E_2:

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & a \\ 1 & -1 & 0 & b \\ 1 & 1 & -1 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & a \\ 0 & 0 & 0 & b+a \\ 1 & 1 & -1 & c \end{array}\right) \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_3 = ([\vec{w}]_\gamma)^{-1} = x^2 + x$$

$$\underline{\underline{[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ for } \beta = \{x^2 - x,\ x^2 + x - 1,\ x^2 + x\}}}$$

ordnet basis.

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12) T lin. op., inversibel, $T: V \to V$, V-endelig dim.

a) E_λ egenrom i T, $E_{\lambda^{-1}}$ eg.rom i T^{-1}

La v vektor i E_λ

$$\Rightarrow T(v) = \lambda v \Rightarrow T^{-1}(T(v)) = T^{-1}(\lambda v) \Rightarrow v = \lambda T^{-1}(v)$$

$$\Rightarrow \lambda^{-1} v = T^{-1}(v) \Rightarrow \underline{v \in E_{\lambda^{-1}}}$$

La $u \in E_{\lambda^{-1}}$

$$\Rightarrow T^{-1}(u) = \lambda^{-1} u \Rightarrow u = T(\lambda^{-1} u) = \lambda^{-1} T(u)$$

$$\Rightarrow \lambda u = T(u) \Rightarrow \underline{u \in E_\lambda}$$

$$\Rightarrow \underline{\underline{E_\lambda = E_{\lambda^{-1}}}}$$

b) Hvis T diagonaliserbar, finnes en basis av egenvektorer i E_λ. Siden $E_{\lambda^{-1}} = E_\lambda$ finnes samme basis av egenvektorer i $E_{\lambda^{-1}}$, så T^{-1} har basis av eg. vektorer $\Rightarrow$ $\underline{\underline{T^{-1} \text{ diagonaliserbar}}}$

14) Finn gen. løsning:

a) $x' = x + y$, $y' = 3x - y$

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{v'} = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v}$$

Diagonaliserer A:

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} = -1 + \lambda^2 - 3 = -(4 - \lambda^2)$$

$$= -(2+\lambda)(2-\lambda) \Rightarrow \lambda_1 = 2, \ \lambda_2 = -2$$

$\underline{\lambda_1 = 2:}$

$$A - 2I = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} -1 & 3 & a \\ 1 & -3 & b \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 3 & a \\ 0 & 0 & b+a \end{array}\right)$$

$$\Rightarrow E_2 \text{ basis } \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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14) a) $\lambda_2 = -2$:

$$(A+2I) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 3 & 3 & a \\ 1 & 1 & b \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 0 & a-3b \\ 1 & 1 & b \end{array}\right)$$

$$\Rightarrow E_{-2} \text{ basis } \vec{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\Rightarrow A = Q D Q^{-1} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} Q^{-1}$$

$$Av = v' \Rightarrow Q D Q^{-1} v = v'$$

$$D Q^{-1} v = (Q^{-1} v)'$$

La $z = Q^{-1} v$

$$D z = z'$$

$$\Rightarrow z = Q^{-1} v = \begin{bmatrix} e^{-2t} \\ e^{2t} \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = Q \begin{bmatrix} C_1 e^{-2t} \\ C_2 e^{2t} \end{bmatrix}$$

$$\underline{\underline{x = C_1 e^{-2t} + C_2 e^{2t} \quad y = -3C_1 e^{-2t} + C_2 e^{2t}}} \quad C_1, C_2 \in \mathbb{R}$$

b) $x_1' = 8x_1 + 10x_2$
$x_2' = -5x_1 - 7x_2$

$$v' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ -5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Av$$

Diagonaliserer A:

$$\det(A - \lambda I) = \det\begin{pmatrix} 8-\lambda & 10 \\ -5 & -7-\lambda \end{pmatrix} = -56 - \lambda + \lambda^2 + 50 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$$

$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = -2$$

$\underline{\lambda_1 = 3}$

$$A - 3I = \begin{pmatrix} 5 & 10 \\ -5 & -10 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 5 & -5 & a \\ 10 & -10 & b \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & -5 & a \\ 0 & 0 & b-2a \end{array}\right)$$

$$\Rightarrow E_3 \text{ basis } \vec{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = -2}$

$$A + 2I = \begin{pmatrix} 10 & 10 \\ -5 & -5 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 10 & -5 & a \\ 10 & -5 & b \end{array}\right) \sim \left(\begin{array}{cc|c} 10 & -5 & a \\ 0 & 0 & b-a \end{array}\right)$$

$$\Rightarrow E_{-2} \text{ basis } \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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14) b) $A = QDQ^{-1}$, $Q = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

$$Av = QDQ^{-1}v = v'$$

$$DQ^{-1}v = (Q^{-1}v')'$$

$$Q^{-1}v = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-2t} \end{bmatrix}$$

$$v = Q\begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-2t} \end{bmatrix}$$

$$\underline{\underline{x_1 = 2c_1e^{3t} + c_2e^{-2t} \qquad x_2 = -c_1e^{3t} - c_2e^{-2t} \qquad c_1, c_2 \in \mathbb{R}}}$$

c) $v' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Av$

A = øvre triangulær matrise $\Rightarrow \lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$

$\underline{\lambda_1 = \lambda_2 = 1:}$

$$A - I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\dim E_1 = \dim N(A - I) = 3 - \text{rang}(A - I) = \underline{2}$. Basis:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 1 & 1 & 1 & c \end{array}\right) \Rightarrow \vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ basis } E_1$$

$\underline{\lambda_3 = 2}$

$$A - 2I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 1 & 1 & 0 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & 0 & c+a+b \end{array}\right) \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ basis } E_2$$

$A = QDQ^{-1}$, $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$QDQ^{-1}v = v' \Rightarrow DQ^{-1}v = (Q^{-1}v)'$$

Vet fra (a) og (b): $v = Q\begin{bmatrix} c_1 e^{t} \\ c_2 e^{t} \\ c_3 e^{2t} \end{bmatrix}$

$$\underline{\underline{x_1 = c_1e^{t} + c_3e^{2t} \qquad x_2 = c_2e^{t} + c_3e^{2t} \qquad x_3 = c_3e^{2t}, \quad c_i \in \mathbb{R} \ \forall i}}$$