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When Is a Group the Union of Proper Normal Subgroups?

Mira Bhargava

It is well known, and has been rediscovered many times, that a group G cannot be the union of two proper subgroups; see [**1**], [**2**], [**4**], or [**6**]. It is possible, however, for a group to be the union of three proper subgroups. A simple description of all groups that can be expressed as a union of three proper subgroups seems first to have been discovered by Scorza [**6**], who showed:

Theorem 1. *A group G is expressible as the union of three proper subgroups if and only if it has a quotient that is isomorphic to the Klein four-group $C_2 \times C_2$.*

One direction of the theorem is, of course, easy to see: $C_2 \times C_2$ is the union of its three subgroups H_1, H_2, H_3 of index 2, and if one has a surjective homomorphism $\phi : G \to C_2 \times C_2$, then G is the union of its proper subgroups $\phi^{-1}(H_1)$, $\phi^{-1}(H_2)$, $\phi^{-1}(H_3)$.

As indicated by the works of Cohn [**3**] and Tomkinson [**8**], the situation becomes increasingly complicated as we require G to be the union of larger numbers of subgroups. On the other hand, Theorem 1 implies that if a group is expressible as the union of three proper subgroups, then it must actually be a union of three proper *normal* subgroups. Thus it seemed natural to ask the question posed in the title.

Groups that are the union of their proper normal subgroups may be called "anti-simple", as they are in a sense as far from simple as can be: in an anti-simple group G, not only do there exist nontrivial proper normal subgroups, but every element of G is contained in one. The problem of determining all finite simple groups was a huge collaborative effort that took nearly a century to resolve. Fortunately, the problem of classifying anti-simple groups is not quite so difficult. In fact, the answer to our question, at least in the case of finite groups, turns out to be remarkably similar to Theorem 1.

Theorem 2. *A finite group G is the union of its proper normal subgroups if and only if it has a quotient that is isomorphic to $C_p \times C_p$ for some prime p.*

Proof. Clearly, $C_p \times C_p$ is the union of its normal subgroups $H_1, H_2, \ldots, H_{p+1}$ of index p, and if $\phi : G \to C_p \times C_p$ is a surjective homomorphism, then G is the union of the proper normal subgroups $\phi^{-1}(H_1), \phi^{-1}(H_2), \ldots, \phi^{-1}(H_{p+1})$.

Conversely, suppose $G = \cup_{i=1}^n H_i$, where the H_i are proper normal subgroups of G. We may assume that each H_i is a *maximal* normal subgroup, i.e., each is contained in no other proper normal subgroup. For convenience, we also assume that the union $\cup_{i=1}^n H_i$ is *irredundant*, i.e., no subgroup H_j is contained in the union of the others, $\cup_{i \neq j} H_i$.

Let $N = \cap_{i=1}^n H_i$, and let $\psi : G \to G/N$ be the natural homomorphism. Then $G/N = \cup_{i=1}^n \psi(H_i)$ is also a union of proper normal subgroups; moreover, if G/N has a quotient that is isomorphic to $C_p \times C_p$, then so does G. Hence to prove the theorem, we may replace G by G/N, H_i by $\psi(H_i)$, and so we may assume that $\cap_{i=1}^n H_i = \{e\}$.

Let $I \subset \{1, 2, \ldots, n\}$ be a maximal index set such that $\cap_{i \in I} H_i \neq \{e\}$. Let $k = |I|$. Relabeling if necessary, we may assume that $I = \{1, 2, \ldots, k\}$.

We claim that $k < n - 1$. For if $k = n - 1$, then $H_1 \cap \cdots \cap H_{n-1}$ contains an element $a \neq e$. Since $\cap_{i=1}^n H_i = \{e\}$, it follows that $a \notin H_n$, and since $\cup_{i=1}^n H_i$ is irredundant, there exists some $b \in H_n$ such that $b \notin H_i$ for all $i < n$. Consider the product $ab \in G = \cup_{i=1}^n H_i$. If $ab \in H_n$, then $a \in H_n$, a contradiction, and if $ab \in H_i$ for some $i < n$, then $b \in H_i$, also a contradiction. Hence a cannot exist, so $H_1 \cap \cdots \cap H_{n-1}$ is trivial and $k < n - 1$, as claimed.

Let $T = H_1 \cap \cdots \cap H_k$. Then T is a nontrivial normal subgroup of G, with $T \cap H_i = \{e\}$ for all $i \in \{k + 1, \ldots, n\}$. Let us pick any ℓ, m with $k + 1 \leq \ell, m \leq n$. Then TH_ℓ and TH_m are normal subgroups of G that strictly contain H_ℓ and H_m, respectively. As the latter are maximal normal subgroups, we must have $G = TH_\ell = TH_m$, and thus $G \cong T \times H_\ell \cong T \times H_m$.

In particular, T commutes with both H_ℓ and H_m, so T commutes with $G = H_\ell H_m$, which implies that T is abelian. Moreover, $T \cong G/H_\ell \cong G/H_m$, and since H_ℓ and H_m are maximal normal subgroups, we see that T is simple. It follows that $T \cong C_p$ for some prime p.

Finally, since $G/H_\ell \cong G/H_m \cong C_p$, we have $G/(H_\ell \cap H_m) \cong C_p \times C_p$. ■

Unlike Theorem 1, however, Theorem 2 does not hold for infinite groups. For example, let G be the additive group of rationals; then G is clearly the union of proper normal subgroups, but has no quotient of the form $C_p \times C_p$. One might then suspect that a group G (finite or infinite) is the union of proper normal subgroups if and only if it has some noncyclic abelian quotient (finite or infinite). But even this is not true. Indeed, let $G = \oplus_{i=0}^\infty A_5$ be the infinite direct sum of the alternating group A_5 with itself. Then G is the union of its normal subgroups $H_n = \oplus_{i=0}^n A_5$. However, G has no (nontrivial) abelian quotient, since the commutator subgroup of G is easily seen to be G itself. Is there a modification of Theorem 2 that holds for infinite groups?

In case we are interested in groups that are unions of *finitely many* proper normal subgroups, the answer then turns out to be the same, as we show in the following strengthening of Theorem 2.

Theorem 3. *A group G (finite or infinite) is the union of finitely many proper normal subgroups if and only if it has a quotient that is isomorphic to $C_p \times C_p$ for some prime p.*

Proof. Let G be the union of proper normal subgroups $H_1, \ldots, H_n$. Then by an argument of Sonn [**7**, proof of Theorem 1], there exists a finite group G' and a surjective homomorphism $\phi : G \to G'$ such that G' is the union of proper normal subgroups $\phi(H_1), \ldots, \phi(H_n)$. We may therefore apply Theorem 2 to obtain a quotient of G' of the form $C_p \times C_p$; the latter is then also a quotient of G, as desired. ■

Finally, it is interesting to ask what is the smallest number, call it $\eta(G)$, such that G can be expressed as the union of $\eta(G)$ proper normal subgroups. This is addressed in Theorem 4.

Theorem 4. *Suppose G is anti-simple. Let p_0 be the smallest prime such that G has a quotient that is isomorphic to $C_{p_0} \times C_{p_0}$, if such a quotient exists; otherwise let $p_0 = \infty$. Then $\eta(G) = p_0 + 1$.*

Proof. If G has a quotient that is isomorphic to $C_{p_0} \times C_{p_0}$, where p_0 is minimal, then we have seen that G is the union of $p_0 + 1$ proper normal subgroups. Hence $\eta(G) \leq p_0 + 1$.

To see that $\eta(G) \geq p_0 + 1$, we proceed as in the proof of Theorem 2. Let $G = \cup_{i=1}^n H_i$ be as before, with T, k, p, etc. defined in the identical manner. We wish to show that $n \geq p_0 + 1$.

Let x be an element of H_m that is not contained in H_i for any $i \in \{1, \ldots, k\}$. Then the coset xT must be contained in $\cup_{r=k+1}^n H_r$. We claim that each H_r with $k + 1 \leq r \leq n$ contains at most one element of xT; for if H_r were to contain two distinct elements $xt_1, xt_2 \in xT$, then $t_1 t_2^{-1}$ would be in $T \cap H_r = \{e\}$, a contradiction. Therefore, since $xT \subset \cup_{r=k+1}^n H_r$, we must have $n - k \geq |T| = p$. As $k \geq 1$, we have $n \geq p + 1 \geq p_0 + 1$, which is the desired conclusion. ■

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A Short Proof of Hall's Theorem on SDRs

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Let $\mathbf{X} = [X_1, \ldots, X_n]$ be a finite sequence of finite sets. A *system of distinct representatives* for $\mathbf{X}$ (or, for short, an SDR for X) is a sequence $(x_1, \ldots, x_n)$ of distinct elements such that $x_i \in X_i$ for $1 \leq i \leq n$. The element x_i of X_i is called the *representative* of X_i. Let $|S|$ denote the cardinality of the set S. Using an idea of Rizzi [**1**], we give the following short proof of Philip Hall's theorem on SDRs.