Groups as Unions of Proper Subgroups

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1. INTRODUCTION. The following question has been asked many times (see, e.g., [**20**], [**2**]), and also appeared as a problem on the 1965 Willam Lowell Putnam Examination: *When is a group the union of two of its proper subgroups?*

The answer, as is now well known, is:

Theorem 1. *A group is* never *the union of two proper subgroups.*

The argument is quite simple, and makes an excellent problem for an undergraduate group theory class. Suppose G is the union of two proper subgroups A and B. Choose an element $a \in A$ that is not in B, and an element $b \in B$ that is not in A, and consider the product $ab \in G$. It cannot be in A, for then b would also have to be in A, a contradiction. Similarly, ab cannot be in B. We are forced to conclude that G cannot be the union of two proper subgroups.

There are several natural variants of the above question, however, that have positive and, in many cases, very pretty and surprising answers. In this article, we survey some of the many fascinating recent results in this direction and also present some new ones.

2. GROUPS AS UNIONS OF *n* PROPER SUBGROUPS. As no group is the union of two proper subgroups, the next natural question that arises is whether a group can be the union of *three* proper subgroups. (In fact, this was part (b) of the aforementioned Putnam problem!) This time the question has a positive answer, and in fact, one can classify all such groups. The complete answer was given by Scorza [**20**]:

Theorem 2 (Scorza). *A group is the union of three proper subgroups if and only if it has a quotient isomorphic to $C_2 \times C_2$, where C_2 denotes the cyclic group of order* 2.

Scorza's Theorem may be proven using arguments quite similar to those used for the case $n = 2$.

Proof. Suppose that $G = A \cup B \cup C$, where A, B, and C are proper subgroups of G, and let us partition G into seven parts as shown in Figure 1. By Theorem 1, S_A, S_B, and S_C are nonempty.

We claim that $S_{AB} = S_{BC} = S_{AC} = \phi$. To see this, let $a \in S_A$ and $x \in S_{BC}$. Then $ax \notin A$, for otherwise it would mean that $x \in A$, a contradiction. Also $ax \notin B \cup C$, because otherwise this would imply $a \in B$ or $a \in C$, again a contradiction. Therefore $S_{BC} = \phi$. Similarly, $S_{AB} = S_{AC} = \phi$. We conclude $G = S_A \cup S_B \cup S_C \cup S_{ABC}$.

Now note that if $a \in S_A$ and $b \in S_B$, then $ab \notin A \cup B$. Hence $ab \in S_C$, implying that $S_A S_B \subset S_C$ (and similarly, $S_C S_B \subset S_A$, etc.). Conversely, let $c \in S_C$, and for any $b \in S_B$, set $a = cb^{-1} \in S_C S_B \subset S_A$. Then $c = ab \in S_A S_B$, so that $S_C \subset S_A S_B$. We conclude that $S_A S_B = S_C$. The equalities $S_B S_C = S_A$, $S_C S_A = S_B$, and also $S_B S_A = S_C$, $S_C S_B = S_A$, and $S_A S_C = S_B$ follow likewise.

By nearly identical arguments, one shows that $S_A^2 = S_B^2 = S_C^2 = S_A S_B S_C = S_{ABC}$, and also, for any $a \in S_A$, we have $a \cdot S_{ABC} = S_{ABC} \cdot a = S_A$. Similarly, for any $b \in S_B$ and $c \in S_C$, we have $b \cdot S_{ABC} = S_{ABC} \cdot b = S_B$ and $c \cdot S_{ABC} = S_{ABC} \cdot c = S_C$.

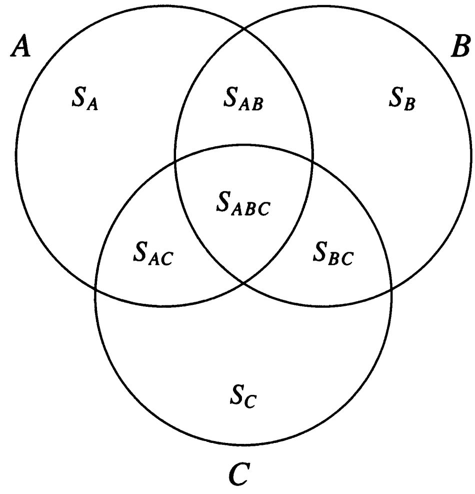

Figure 1.

Therefore S_A, S_B, S_C, and S_{ABC} form the four (both left and right!) cosets of S_{ABC}. It follows that S_{ABC} is a normal subgroup of G, and G/S_{ABC} has a group structure that is isomorphic to the Klein four group $C_2 \times C_2$.

On the other hand, if G has a quotient G/N isomorphic to $C_2 \times C_2$ then G/N is the union of its three subgroups H_1, H_2, and H_3 of index 2. The inverse images of H_1, H_2, and H_3 under the surjective homomorphism $\phi : G \to G/N$ are then seen to be three proper subgroups of index 2 that cover G. This completes the proof. ■

Scorza's result was so elegant that people thought it ought to be generalized to numbers higher than three. In 1994, Cohn [**9**] performed a detailed analysis in this spirit and obtained some very interesting analogues for four, five, and six. Following Cohn's definition, let us write $\sigma(G) = n$ whenever G is the union of n proper subgroups, but is not the union of any smaller number of proper subgroups. Thus, for example, Scorza's result asserts that $\sigma(G) = 3$ if and only if G has a quotient isomorphic to $C_2 \times C_2$. Cohn's extensions for $\sigma(G) = 4$, 5, and 6 are then as follows:

Theorem 3 (Cohn). *Let G be a group. Then*

(a) $\sigma(G) = 4$ *if and only if* $\sigma(G) \neq 3$ *and G has a quotient isomorphic to the symmetric group* S_3 *or* $C_3 \times C_3$.

(b) $\sigma(G) = 5$ *if and only if* $\sigma(G) \notin \{3, 4\}$ *and G has a quotient isomorphic to the alternating group* A_4.

(c) $\sigma(G) = 6$ *if and only if* $\sigma(G) \notin \{3, 4, 5\}$ *and G has a quotient isomorphic to the dihedral group* D_5, $C_5 \times C_5$, *or W, where* $W = C_4 \ltimes C_5$ *is the group of order* 20 *having two generators a and b satisfying* $a^5 = b^4 = e$, $ba = a^2b$.

The next case, $\sigma(G) = 7$, gets considerably more unwieldy, involving extensive details and several pages of arguments. But, in the end, the answer is amusing and rather unexpected. Cohn [**9**] conjectured, and in 1997 Tomkinson [**18**] proved:

Theorem 4. (Tomkinson) *There is no group G such that* $\sigma(G) = 7$.

The proof of Tomkinson's theorem was indeed very complicated and required the analysis of numerous cases. Thus the idea of classifying groups that are the union of n proper subgroups for higher n would no doubt be a formidable task, since there appears to be no consistent pattern for small n that suggests a general answer.

However, we notice in all the above theorems that, to determine whether a group G is the union of n proper subgroups ($n \leq 7$), one need only determine whether G has a quotient isomorphic to one among a certain finite set of finite groups. The question thus arises as to whether a similar criterion might hold for any given n, however large n may be. It turns out that such a criterion does indeed exist, and we prove the following general finiteness result [**1**]:

Theorem 5. *For any positive integer n, there exists a unique minimal finite set $S(n)$ of finite groups such that $\sigma(G) = n$ if and only if $\sigma(G) \notin \{3, 4, \dots, n-1\}$ and G has a quotient isomorphic to some group $K \in S(n)$.*

For example, $S(1) = S(2) = \phi$, while $S(3) = \{C_2 \times C_2\}$, $S(4) = \{S_3, C_3 \times C_3\}$, $S(5) = \{A_4\}$, $S(6) = \{D_5, C_5 \times C_5, W\}$, and $S(7) = \phi$.

To prove Theorem 5, we require a result of Neumann [**15**]. Let us say that a group G is an *irredundant* union $\cup_{i=1}^{n} A_i$ of subsets $A_i \subset G$ if $G = \cup_{i=1}^{n} A_i$ but no A_j is contained in the union $\cup_{i \neq j} A_i$ of the others. Then Neumann's Theorem states that, if a group G is an irredundant union of subgroups A_i for $i = 1, 2, \dots, n$, then the index $[G : \cap_{i=1}^{n} A_i]$ of their intersection is bounded by a finite constant $f(n)$ depending only on n. This implies, in particular, that all A_i must have finite index in G. In Theorem 6, we will present a refined version of Neumann's result due to Tomkinson [**19**].

We are now ready to prove Theorem 5.

Proof of Theorem 5. Let $S(n)$ be the set of all groups K, up to isomorphism, for which $\sigma(K) = n$ but no nontrivial quotient L of K satisfies $\sigma(L) = n$. We show that $S(n)$ satisfies all properties we require of it.

First, let G be any group with $\sigma(G) = n$, and write $G = \cup_{i=1}^{n} A_i$ for some proper subgroups A_i of G. Set $H = \cap_{i=1}^{n} A_i$. Then, by Neumann's Theorem, there exists a finite constant $f(n)$, depending only on n, such that the index of H in G is at most $f(n)$. Let $M = \operatorname{core}(H)$ denote the intersection of the conjugates of H. Then evidently $\sigma(G/M) = n$ too, for $G/M = \cup_{i=1}^{n}(A_i/M)$, and if G/M were the union of fewer than n proper subgroups B_j/M, then we would have $G = \cup_j B_j$, a contradiction.

It follows now that G/M is isomorphic to a subgroup of the symmetric group $S_{[G:H]}$, since the action of G by left multiplication on the left cosets of H yields a homomorphism from G to the group $\mathrm{Perm}(G/H)$ of permutations of the set G/H. The kernel of this homomorphism is seen to be M. Thus

$$|G/M| \leq |S_{[G:H]}| \leq |S_{f(n)}| = f(n)!. \tag{1}$$

Let $N \supseteq M$ be a normal subgroup of G that is maximal with respect to the property that $\sigma(G/N) = n$ (such a maximal N exists because the index of M in G is finite). Then clearly $G/N \in S(n)$, which shows that any group G with $\sigma(G) = n$ must have a quotient in $S(n)$ of order at most $f(n)!$.

In particular, if G is a group in $S(n)$, then the corresponding normal subgroup M must equal $\{1\}$, so by (1) we have $|G| \leq f(n)!$. Thus any element of $S(n)$ must have order at most $f(n)!$. Since there are only finitely many groups G of order at most $f(n)!$, and since only a subset of these can satisfy $\sigma(G) = n$, our set $S(n)$ must be a finite set of finite groups.

Thus we have shown that if G is any group with $\sigma(G) = n$, then it must have a quotient isomorphic to some group K in the finite set $S(n)$ of finite groups. Conversely, suppose G is a group such that $\sigma(G) \notin \{3, 4, \dots, n-1\}$ and G has a quotient isomorphic to some group $K \in S(n)$. Then since K is covered by n proper subgroups, the inverse images of these subgroups, under the projection $\pi : G \to K$, form a covering of G by n proper subgroups; it follows that $\sigma(G) = n$.

It remains only to prove the minimality and uniqueness of the set $S(n)$. To this end, suppose S is any other set of groups such that, for all groups G, we have:

$$\begin{array}{l}\sigma(G) = n \text{ if and only if } \sigma(G) \notin \{3, 4, \dots, n-1\} \text{ and} \\ G \text{ has a quotient isomorphic to some group } K \in S.\end{array} \tag{2}$$

Assume, furthermore, that there exists a group $H \in S(n)$ not contained in S. Since $\sigma(H) = n$, the group H must have a quotient isomorphic to some $K \in S$. As $H \notin S$, this quotient of H must be nontrivial. Moreover, since $H \in S(n)$, no nontrivial quotient L of H can satisfy $\sigma(L) \leq n$. (If $\sigma(L) < n$, then any covering of L by fewer than n proper subgroups would lift to such a covering of H; and if $\sigma(L) = n$, then the group H cannot be a member of $S(n)$ by the definition of $S(n)$.) It follows that $\sigma(K) > n$.

Now the group K satisfies $\sigma(K) \notin \{3, 4, \dots, n-1\}$ and K has a quotient (namely itself) in S. By property (2) of S, we conclude that $\sigma(K) = n$, a contradiction. Thus $H \in S$ after all, and $S(n) \subset S$. Therefore $S(n)$ is the unique minimal set S (with respect to inclusion) satisfying (2), and this completes the proof. ■

In theory, our proof of Theorem 5 gives a finite (albeit very inefficient!) algorithm for determining $S(n)$ for any given n. One looks at all groups G having order less than or equal to $f(n)!$, and determines which of these groups satisfy $\sigma(G) = n$ and do not have a quotient with the same property.

It is an interesting question what the optimal value of $f(n)$ is for each n. Neumann **[15]** gave general upper bounds for $f(n)$, which were later refined by Tomkinson **[19]** using a very elegant argument:

Theorem 6 (Tomkinson). *Let $G = \cup_{i=1}^{n} x_i A_i$ be an irredundant union of* cosets *of subgroups $A_i \subset G$ $(1 \leq i \leq n)$. Then $[G : \cap_{i=1}^{n} A_i] \leq n!$.*

Proof. Let $H = \cap_{i=1}^{n} A_i$. We first prove that, for $0 \leq k \leq n-1$, we have

$$[A_{i_1} \cap \cdots \cap A_{i_{n-k}} : H] \leq k!, \tag{3}$$

where $i_1, i_2, \dots, i_n$ is any permutation of $\{1, 2, \dots, n\}$. We proceed by induction. For $k = 0$, the inequality (3) is clearly true. Assume that (3) is true for $k-1$, i.e., the index of H in the intersection of any $n-k+1$ of the A_i's is at most $(k-1)!$. Then to prove (3) for k, let $x \in G - (x_{i_1} A_{i_1} \cup \cdots \cup x_{i_{n-k}} A_{i_{n-k}})$. Set $Y = \cap_{r=1}^{n-k} A_{i_r}$. Then the coset xY is disjoint from $x_{i_1} A_{i_1} \cup \cdots \cup x_{i_{n-k}} A_{i_{n-k}}$, so that

$$xY \subset x_{i_{n-k+1}} A_{i_{n-k+1}} \cup \cdots \cup x_{i_n} A_{i_n}$$

implying that

$$Y \subset x^{-1} x_{i_{n-k+1}} A_{i_{n-k+1}} \cup \cdots \cup x^{-1} x_{i_n} A_{i_n},$$

and thus

$$Y = (x^{-1}x_{i_{n-k+1}}A_{i_{n-k+1}} \cap Y) \cup \cdots \cup (x^{-1}x_{i_n}A_{i_n} \cap Y). \tag{4}$$

Now $x^{-1}x_{i_{n-k+r}}A_{i_{n-k+r}} \cap Y$, for $r = 1, 2, \ldots, k$, is either empty or is a coset of $A_{i_{n-k+r}} \cap Y$. By (4), Y is the union of at most k such cosets, and by the induction hypothesis, each of these k cosets is the union of at most $(k-1)!$ cosets of H. Thus $[Y : H] \leq k \cdot (k-1)!$, proving (3) for $k = 0, \ldots, n-1$.

Finally, each A_i is the union of at most $(n-1)!$ cosets of H by (3), and hence $G = \cup_{i=1}^{n} x_i A_i$ is the union of at most $n!$ cosets of H. This proves the theorem. ∎

Using Theorem 6, we obtain an interesting upper bound on the size of the groups that occur in $S(n)$:

Corollary 1. *If $K \in S(n)$, then $|K| \leq n!!$.*

In **[19]**, using some more involved arguments, Tomkinson was able to improve the bound in Theorem 6 slightly to $f(n) \leq \max\{(n-1)^2, (n-2)^3\} \cdot (n-3)!$. This yields the corresponding improvement in Corollary 1. Of course, these various bounds for $f(n)$, and the corresponding bounds on the size of elements of $S(n)$, are likely far from optimal. For small values of n, optimal values for $f(n)$ have been pursued by Bryce, Fedri, and Serena **[6]**, who show that for $n = 5$ the best value for $f(n)$ is 16. For $n > 5$, however, the question remains open.

It would be nice to have some general results regarding the behavior of $\sigma(G)$ for various groups G. For solvable G, a complete method for computing $\sigma(G)$ was given by Tomkinson, who showed that for such G we have $\sigma(G) = p^a + 1$, where p^a is the "order of the smallest chief factor of G having at least two complements in G." For details on this beautiful result, see Tomkinson's paper **[18]**.

Important in the proofs of Cohn's and Tomkinson's Theorems 3 and 4 was understanding the values of $\sigma(S_n)$ and $\sigma(A_n)$ for various small values of n. In the course of their proofs, they showed

$$\begin{array}{lcl} \sigma(S_1) = \ * & & \sigma(A_1) = \ * \\ \sigma(S_2) = \ * & & \sigma(A_2) = \ * \\ \sigma(S_3) = \ 4 & \text{and} & \sigma(A_3) = \ * \\ \sigma(S_4) = \ 4 & & \sigma(A_4) = \ 5 \\ \sigma(S_5) = 16 & & \sigma(A_5) = 10 \\ \sigma(S_n) > \ 7 & & \sigma(A_n) > \ 7 \qquad (n \geq 6) \end{array}$$

where we have used the notation $\sigma(G) = *$ for G cyclic, i.e., for a group G that is not the union of proper subgroups. The values of $\sigma(S_n)$ and $\sigma(A_n)$ are not completely known for $n \geq 6$, and it would be interesting to understand how fast they grow, or at least to have good upper and lower bounds. There have been some exciting recent advances in this direction by Maróti **[14]**, who shows $\sigma(S_n) = 2^{n-1}$ for $n > 1$ odd and $n \neq 9$, and $\sigma(A_n) = 2^{n-2}$ for $n > 2$ and $n \equiv 2 \pmod 4$. For the remaining values of n, Maróti obtains upper bounds for $\sigma(S_n)$ and lower bounds for $\sigma(A_n)$, although it is unknown whether these bounds are sharp.

The behavior of $\sigma(G)$ for other finite simple groups G has also been the subject of recent study; see, e.g., the works of Bryce-Fedri-Serena **[7]**, Lucido **[13]**, Maróti **[14]**, and Holmes **[11]**.

3. GROUPS AS UNIONS OF PROPER NORMAL SUBGROUPS. As mentioned in the previous section, the situation gets increasingly complicated as we require a group G to be the union of larger numbers of subgroups. Perhaps this indicates that we have not been asking quite the right question in generalizing Scorza's result!

Notice that our proof of Scorza's theorem implies:

Theorem 7. *A group that is the union of three proper subgroups is actually the union of three proper normal subgroups.*

Thus we may add the word *normal* to Scorza's result, and it remains true! This suggests that we might ask: *when is a group the union of proper normal subgroups?*

Following **[3]**, we may call a group that is the union of its proper normal subgroups *anti-simple*, because such a group is quite the opposite of simple. Indeed, in an anti-simple group G, not only do there exist nontrivial proper normal subgroups, but every element of G is contained in one. As in **[3]**, let us write $\eta(G) = n$ if G is the union of n proper normal subgroups but is not the union of fewer than n proper normal subgroups.

Then the answer to this new question, regarding when $\eta(G) = n$, yields another pretty generalization of Scorza's Theorem. In **[3]**, we proved:

Theorem 8. *Suppose a group G is the union of its proper normal subgroups. Then $\eta(G) = p + 1$, where p is the smallest prime such that G has a quotient isomorphic to $C_p \times C_p$, if such a prime p exists; and $\eta(G) = \infty$ otherwise.*

For finite groups, Theorem 8 in particular gives a complete classification of anti-simple groups:

Corollary 2. *A finite group is the union of proper normal subgroups if and only if it has a quotient isomorphic to $C_p \times C_p$ for some prime p.*

As in Scorza's result, one direction of Corollary 2 is easy to see: $C_p \times C_p$ is the union of its normal subgroups $H_1, H_2, \dots, H_{p+1}$ of index p, and if $\phi : G \to C_p \times C_p$ is a surjective homomorphism, then G will be the union of the proper normal subgroups $\phi^{-1}(H_1), \phi^{-1}(H_2), \dots, \phi^{-1}(H_{p+1})$.

The reverse direction is a good deal more involved, and hence we omit the proof here. (Readers are referred to **[3]** for details on the proofs of Theorem 8 and Corollary 2.)

We note that Corollary 2 also applies to infinite groups G, provided we require our group G to be the union of *finitely many* normal subgroups. Indeed, suppose G is an irredundant union $\cup_{i=1}^{n} H_i$ of finitely many normal subgroups. Then, by Neumann's Theorem, $H = \cap_{i=1}^{n} H_i$ has finite index in G, implying that the finite group G/H is the union of the proper normal subgroups H_i/H. By Theorem 8, G/H has a quotient isomorphic to $C_p \times C_p$ for some prime p. This quotient is then also a quotient of G, proving that G has a quotient isomorphic to $C_p \times C_p$. The converse, of course, follows just as in the case of finite G.

If, however, we drop the condition that G be a finite union, then Corollary 2 does not hold anymore. For example, for each $i = 1, 2, 3, \dots$, let G_i be a nonabelian simple group, and define $G = \oplus_{i=1}^{\infty} G_i$. For each $j = 1, 2, 3, \dots$, let H_j denote the (normal) subgroup $\oplus_{i=1}^{j} G_i$ of G. Then clearly $G = \cup_{j=1}^{\infty} H_j$, and hence is anti-simple; however, it may be seen that the commutator subgroup of G must be G itself, and so G cannot have a quotient isomorphic to $C_p \times C_p$ or any nontrivial abelian group. Therefore, we invite the reader to ponder the question:

Question. What is the classification of anti-simple *infinite* groups?

An elegant solution to the above problem would be a most welcome final touch to the complete classification of anti-simple groups.

4. GROUPS AS CONJUGATE UNIONS OF PROPER SUBGROUPS. Another interesting question one may ask, which has applications to Galois theory, is: when can a group be covered by the conjugates of n proper subgroups? More precisely, we say that G is the *conjugate union* of its subgroups $A_1, A_2, \ldots, A_n$ if it is the union of the conjugates of the subgroups A_i for $i = 1, \ldots, n$. Equivalently, a group is the conjugate union of $A_1, \ldots, A_n$ if the union $\cup_{i=1}^n A_i$ intersects every conjugacy class of G.

The first question that arises in this context is whether a group G can be the conjugate union of one subgroup. In other words: is it possible for a proper subgroup of G to contain at least one element from every conjugacy class of G? For infinite groups, it is possible for a group to be the conjugate union of one subgroup. For example, the group $\mathrm{GL}_n(\mathbb{C})$ is the conjugate union of the subgroup of upper triangular matrices, because every $n \times n$ matrix is conjugate to an upper triangular matrix. More generally, in any connected linear algebraic group over an algebraically closed field, all Borel subgroups B are conjugate to each other and together they cover G. However, such a scenario is not possible for finite groups:

Theorem 9. *A finite group cannot be the conjugate union of one proper subgroup.*

Proof. Suppose G is the conjugate union of the proper subgroup A, i.e., $G = A^{(1)} \cup \cdots \cup A^{(r)}$ where the $A^{(i)}$ $(i = 1, \ldots, r)$ denote the conjugates of A. Clearly, we may assume $r > 1$. Let $N \supset A$ denote the normalizer of A. Then $r = |G/N| \leq |G/A|$. Now $A^{(1)}, \ldots, A^{(r)}$, being subgroups, all contain the identity element of G; therefore, the total number of elements in $A^{(1)} \cup \cdots \cup A^{(r)}$ is at most

$$\begin{aligned}|A^{(1)}| + \cdots + |A^{(r)}| - (r-1) = r \cdot |A| - (r-1) &\leq |G/A| \cdot |A| - (r-1)\\ &= |G| - (r-1).\end{aligned}$$

Thus $\cup_{i=1}^r A^{(i)}$ cannot be all of G. ■

The proof shows that, if A is a subgroup of small index r in a group G, with normalizer $N_G(A) = A$ and $A^{(i)} \cap A^{(j)} = \{1\}$ for all distinct i and j, then $\cup A^{(i)}$ will cover all of G except for $r - 1$ elements. Thus, in a sense, $\cup_{i=1}^r A^{(i)}$ just "barely misses" covering all of G in such a case. Hence it initially seems clear that it should not be difficult to write various groups as the conjugate union of *two* proper subgroups. For example, the symmetric group S_3 is the conjugate union of two of its subgroups having index 2 and 3 respectively. However, it remains an open question to determine which groups are the conjugate union of two proper subgroups!

We write $\xi(G) = n$ if G is the conjugate union of n proper subgroups, but is not the conjugate union of fewer than n proper subgroups. We establish below some necessary and sufficient conditions for $\xi(G) = 2$. First, we have:

Theorem 10. *Suppose a finite group G is the conjugate union of two proper subgroups. Then G possesses at least one maximal subgroup that is not normal.*

Proof. Suppose G is the conjugate union of the proper subgroups A and B. Without loss of generality, we may assume A and B are maximal, for otherwise they could be replaced by any subgroups properly containing them. If both A and B are normal, then G would be the union of A and B, contradicting Theorem 1. Hence at least one of A or B must be nonnormal, yielding the desired conclusion. ∎

Theorem 10 immediately rules out a number of possibilities for G if we are to have $\xi(G) = 2$. Obviously, G cannot be abelian. In addition, one checks for example that the dihedral group D_4 of order 8 has the property that every maximal subgroup is of index 2 and hence normal. Therefore, we must have $\xi(D_4) > 2$. In fact, one can show $\xi(D_4) = 3$, leading to the interesting equality $\sigma(D_4) = \eta(D_4) = \xi(D_4) = 3$. More generally, a finite group in which every maximal subgroup is normal is called a *nilpotent* group (see **[10]** for other, more standard definitions). Thus Theorem 10 implies that if G is nilpotent, then $\xi(G) > 2$.

In the case of finite solvable groups G, the question of whether G is the conjugate union of two proper subgroups is not entirely unrelated to the question we considered in the previous section, namely, whether G can be expressed as the union of normal subgroups! As pointed out in the work of Jamali and Mousavi **[12]**, the techniques of Cohn and Tomkinson in their work on solvable groups heavily involve a very subtle interplay between unions of normal subgroups and conjugate unions of two subgroups. In particular, Jamali and Mousavi derive the following theorem:

Theorem 11 (Jamali-Mousavi). *Let G be a finite noncyclic solvable group. Then G satisfies at least one of the following conditions:* (i) *G is the union of proper normal subgroups, or* (ii) *G is the conjugate union of two proper subgroups.*

Their techniques, which follow the methods of Cohn and Tomkinson, do not naturally enable one to separate the two conditions, however!

Our Theorem 8 and its corollary do separate out condition (i). Thus we may give various sufficient conditions for a finite solvable group to be the conjugate union of two proper subgroups. For example, combining Corollary 2 and Theorem 11, we easily obtain:

Corollary 3. *Let G be a finite solvable group. Suppose G possesses a quotient K such that:* (1) *K is not cyclic, and* (2) *K does not have a quotient isomomorphic to $C_p \times C_p$ for any prime p. Then G is the conjugate union of two proper subgroups.*

Proof. Since G is solvable, its quotient K is solvable. Also, by hypotheses (1) and (2), K is noncyclic and does not have a quotient isomorphic to $C_p \times C_p$ for any prime p. Corollary 2 and Theorem 11 thus imply that K is the conjugate union of two proper subgroups A and B. If $\pi : G \to K$ is the natural projection, then the inverse images of the conjugates of A and B cover G. Since the inverse images of the conjugates of A and B are the conjugates of the inverse images of A and B respectively, we conclude that G is the conjugate union of its proper subgroups $\pi^{-1}(A)$ and $\pi^{-1}(B)$. ∎

As an example, let us consider the dihedral group D_n of order $2n$. If $n = 2^m$ is a power of 2, then it is an easy matter to check that D_n is nilpotent, so that $\xi(D_{2^m}) > 2$. On the other hand, since any dihedral group D_{2^m} $(m > 1)$ has a quotient isomorphic to D_4 and $\xi(D_4) = 3$, we see that $\xi(D_{2^m}) = 3$ for all $m > 1$. Now let $n = 2^m r$, where $r > 1$ is odd. Then $D_{2^m r}$ has a quotient isomorphic to D_r. The group D_r is not cyclic and has no quotient isomorphic to $C_p \times C_p$ for any prime p (including $p = 2$!). By Corollary 3, $\xi(D_{2^m r}) = 2$. In summary, $\xi(D_n) = 3$ or 2 in accordance with whether n is a power of 2 or not.

The general answer to the question of the value of $\xi(G)$ for solvable G would require a more complete separation between conditions (i) and (ii) in Theorem 11, which seems not to be a trivial problem. It forms a natural direction for further investigation.

The situation for general finite groups G, which are not necessarily solvable, remains completely open. Is it true that a nonsolvable group is always the conjugate union of two subgroups? It would be interesting to investigate this question for families of finite simple groups.

Finally, as we mentioned at the start of this section, the notion of conjugate union is closely related to Galois-theoretic questions concerning the factorization of integer polynomials modulo p. We hope to discuss these applications in a future paper.

5. FURTHER READING. There is a wide range of problems concerning coverings of groups by subgroups. We have considered here coverings by proper subgroups, proper normal subgroups, and conjugates of proper subgroups. Other interesting types of coverings, such as Hughes coverings and abelian coverings, have been extensively pursued by Bryce-Fedri-Serena [**7**], Baer (unpublished; see Robinson [**16**, Chapter 4]) and others. Coverings by nilpotent subgroups have been investigated in the work of Brodie and Kappe [**5**]. A related problem concerns counting how many coverings there are of a given type. In this direction, for example, Brodie [**4**] has classified those groups admitting exactly one covering by proper subgroups. It is evident that there are many exciting open questions in this area that remain to be studied, and readers interested in these and further topics are referred to the bibliography below.

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Mathematics Is . . .

"Pure Mathematics is the class of all propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants."

Bertrand Russell, *Principles of Mathematics*,
W. W. Norton, New York, 1996, p. 3.

"Mathematics is the manhood of logic."

Bertrand Russell, *Introduction to Mathematical Philosophy*,
Dover, New York, 1993, p. 194.

"Pure mathematics is one of the highest forms of art; it has a sublimity quite special to itself, and an immense dignity derived from the fact that its world is exempt from change and time."

Bertrand Russell, letter to Helen Thomas, Dec. 30, 1901, in
The Selected Letters of Bertrand Russell: The Private Years, 1884–1914, N. Griffin, ed., Routledge, London, 2002, p. 218.

—Submitted by Carl C. Gaither, Killeen, TX