Contents

1 Convergence results for fourier series

1.1 A historical perspective

Fourier series made their first appearance in mathematics through the way of physics, as is the norm with a lot of classical analysis. In particular, they turned up in the analysis of the following differential equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{1}$$

on the unit disc, with the following boundary conditions:

$$u(1, \theta) = f(\theta)$$

where f is an arbitrary continous function on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$.

Equation 1 is what is called the Laplace equation, which describes the temperature u on a conducting surface at steady state. This naturally raises the question: What would the temperature profile on a disk look like at steady state if the boundary was maintained at some temperature $f(\theta)$. That is precisely what the described differential equation along with the boundary conditions tries to answer.

Once one makes the simplifying assumption that the solution is separable, i.e.

$$u(r, \theta) = a(r) \cdot b(\theta)$$

equation 1 is easy to solve, and one sees that it has solutions of the form:

$$u_n(r, \theta) = r^n e^{in\theta}$$

for all integers n. And finite linear combinations of the solutions u_n are also solutions to the differential equation 1. But there is one problem. Not every continuous function f on $[-\pi, \pi]$ is a finite linear combination of $e^{in\theta}$, e.g. $|x|$. Leaving that problem aside for now, one notes that if f is of the form

$$f(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$

then a_n is can be determined in the following manner:

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \tag{2}$$

Here a_n is called the n^{th} fourier coefficient of f. Since we assumed that f is a finite linear combination of $e^{-in\theta}$, that means it has only finitely many non-zero coefficients. Contrariwise, continuous functions that are not finte linear combinations of $e^{in\theta}$ must have infinitely many non-zero fourier coefficients. Could it perhaps be that f is an *infinite* linear combination of $e^{in\theta}$? To be more precise, does the following sequence converge to f (in an appropriate sense) as N goes to ∞?

$$S_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$

where a_n is defined as in equation 2.

The best we can hope for is for the sequence S_N to converge uniformly to the function f, but as we shall see, that is not actually true, i.e. there are continuous functions whose fourier series diverges for some point in the domain. The next step would be either weakening the mode of convergence, i.e. instead of expecting convergence in the L^∞ norm, one could expect convergence in weaker norms such as L^1 or L^2, or even Cesàro convergence. Another possible way to go about would be to strengthen the conditions on the function, i.e. forcing the function to be C^1, or even absolutely continuous ensures that the sequence S_N converges uniformly to f.

1.2 Fourier series of continuous functions are Cesàro summable

Before we go on to show that fourier series converge under weaker notions of convergence, we'll develop a technique that can be used more generally to show certain sequences of functions converge uniformly. Furthermore, in all the sections that are to follow, we'll be working with continuous complex valued functions on the circle T, which is defined as the space $\mathbb{R}$ quotiented with the equivalence relation $\sim$ defined as

$$a \sim b \iff (a-b) = 2n\pi,\ n \in \mathbb{Z}$$

1.2.1 Convolution

For two continuous functions f and g on T, we can define a binary operation $*$:

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-t)g(t)dt$$

It follows that the operation is commutative (substitution of variables), associative (again substitution of variable), and distributive (integration is a linear operator). It also follows that $f * g$ is continuous since f is uniformly continuous.

It's not too hard to show that this operation has no identity element.

Proposition 1.1. *There exists no continuous function g on T such that for all continuous functions f*

$$f * g = f$$

Proof. We will prove the result by showing that if such a g existed, then $g(x) = 0$ for $x \neq 0$. But that would mean g is non-zero on a set of measure 0, hence $\int_T f(x-t)g(t)dt = 0$ for all f, which means g is not the identity, hence a contradiction.

Assume we have a continuous function g such that for all continuous f, $f * g = f$. Pick any non-zero x_0. We claim that $g(x_0)$ must be 0. If it's not 0, then without loss of generality, let $g(x_0) = \varepsilon > 0$. That means there exists a δ_0 such that for all $x \in (x_0 - \delta_0, x + \delta_0)$, $g(x) > \frac{\varepsilon}{2}$. Let $\delta = \min(\delta_0, |x_0|)$.

Now define a continuous function f in the following manner:

$$f(-x) = \begin{cases} 1 & x \in (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}) \\ \frac{2(x-(x_0-\delta))}{\delta} & x \in \left(x_0 - \delta, x_0 - \frac{\delta}{2}\right] \\ \frac{2((x_0+\delta)-x)}{\delta} & x \in \left[x_0 + \frac{\delta}{2}, x_0 + \delta\right) \\ 0 & \text{otherwise} \end{cases}$$

It's clear that $f(0) = 0$.

$$\begin{aligned}
f(0) &= (f * g)(0) \\
&= \int_{-\pi}^{\pi} f(-t)g(t)dt \\
&\geq \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f(-t)g(t)dt \\
&\geq \frac{\varepsilon\delta}{2} \\
&> 0
\end{aligned}$$

But $f(0) = 0$, and $0 \not> 0$. We have a contradiction, which means $g(x) = 0$ for all $x \neq 0$. This completes the proof. □

1.2.2 Dirac sequences

We saw that there is no continuous function which acts as an identity for $*$ operation. However, there do exist sequences of functions, such that the sequence $\{f*g_n\}$ converges uniformly to f as n goes to infinity. Such a sequence $\{g_n\}$ is called a Dirac sequence. The formal definition of a Dirac sequence is the following:

Definition 1.1. A sequence of a continuous functions $\{g_n\}$ is called a Dirac sequence if it satisfies the following conditions:

1. $g_n(x) \geq 0$ for all $n \in \mathbb{N}$ and all $x \in T$.
2. $g_n(x) = g_n(-x)$ for all $n \in \mathbb{N}$ and all $x \in T$.
3. $\displaystyle\int_{-\pi}^{\pi} g_n(t)dt = 1$ for all $n \in \mathbb{N}$.
4. For all $\varepsilon > 0$ and $\delta > 0$, there exists an N such that for all $n > N$,
$$\int_{-\pi}^{-\delta} g_n(t)dt + \int_{\delta}^{\pi} g_n(t)dt < \varepsilon$$

Following from the definition, we get this very useful theorem:

Theorem 1.2. *If $\{g_n\}$ is a Dirac sequence, then for all continuous functions f, the sequence $\{f * g_n\}$ converges uniformly to f.*

Proof. Pick any $\varepsilon > 0$. We need to show there exists an N such that for all $n > N$, $|(f * g_n) - f|_\infty < \varepsilon$. Pick a particular continuous function f. Let M be the maximum of $|f|$ on T. Pick $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\varepsilon_1\varepsilon_2 + \varepsilon_2 + 2M\varepsilon_1 < \varepsilon \tag{3}$$

Since f is uniformly continuous on T, pick a δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon_2$.

Now pick an N such that for all $n > N$

$$\int_{-\pi}^{-\delta} g_n(t)dt + \int_{\delta}^{\pi} g_n(t)dt < \varepsilon_1$$

This would imply for all $n > N$

$$\int_{-\delta}^{\delta} g_n(t)dt > 1 - \varepsilon_1$$

Now consider $(f * g_n)(x_0)$ for some $x \in T$.

$$\begin{aligned}
(f * g_n)(x_0) &= \int_{-\pi}^{\pi} f(x_0 - t)g_n(t)dt \\
&= \left(\int_{-\pi}^{-\delta} f(x_0 - t)g_n(t)dt + \int_{\delta}^{\pi} f(x_0 - t)g_n(t)dt\right) + \left(\int_{-\delta}^{\delta} f(x_0 - t)g_n(t)dt\right)
\end{aligned}$$

Let's analyze the two terms separately. Since $-M \leq f(x_0 - t) \leq M$, we can bound the first term as

$$-M\varepsilon_1 \leq \int_{-\pi}^{-\delta} f(x_0 - t)g_n(t)dt + \int_{\delta}^{\pi} f(x_0 - t)g_n(t)dt \leq M\varepsilon_1$$

Similarly, for all x in the interval $(x_0 - \delta, x_0 + \delta)$

$$f(x_0) - \varepsilon_2 \leq f(x) \leq f(x_0) + \varepsilon_2$$

This lets us bound the second term in the following manner:

$$(1 - \varepsilon_1)(f(x_0) - \varepsilon_2) \leq \int_{-\delta}^{\delta} f(x_0 - t)g_n(t)dt \leq f(x_0) + \varepsilon_2$$

This gives us a complete bound on $(f * g_n)(x_0)$.

$$-M\varepsilon_1 + (1 - \varepsilon_1)(f(x_0) - \varepsilon_2) \leq (f * g_n)(x_0) \leq M\varepsilon_1 + (f(x_0) + \varepsilon_2)$$

Using the triangle inequality and the fact that $|f(x_0)| < M$, we get

$$f(x_0) - (\varepsilon_1\varepsilon_2 + \varepsilon_2 + 2M\varepsilon_1) \leq (f * g_n)(x_0) \leq f(x_0) + (\varepsilon_1\varepsilon_2 + \varepsilon_2 + 2M\varepsilon_1)$$

But from inequality 3, we get

$$f(x_0) - \varepsilon \leq (f * g_n)(x_0) \leq f(x_0) + \varepsilon$$

This shows that $\{f * g_n\}$ converges uniformly to f for all continuous f. $\square$

This is a useful result because this will let us deal with the question of convergence of the partial fourier series of some continuous function f. One can write the n^{th} partial fourier series as the convolution of f with some function g_n, called a kernel, and if one shows that the kernels g_n form a Dirac sequence, then the fourier series also converges.

1.2.3 Cesàro summability

We mentioned that for general continuous functions, their fourier series need not converge uniformly, or even pointwise, to the function; weakening the notion of convergence lets the conjecture go through. A weaker notion of convergence for infinite sums is the notion of Cesàro summability.

Definition 1.2. A sequence $\{x_n\}$ is said to be Cesàro summable if the following sequence $\{\sigma_n\}$ converges:

$$\sigma_n = \frac{\sum_{k=1}^{n} s_k}{n}$$

where s_k is the k^{th} partial sum of the sequence $\{x_n\}$. If the sequence $\{\sigma_n\}$ converges to L, then L is called the Cesàro sum of $\{x_n\}$.

It's easy to see that if the series $\displaystyle\sum_{k=1}^{n} x_n$ converges, then the Cesàro sums also converge to the same limit. To show it is a strictly weaker notion of convergence, consider the sequence $x_n = (-1)^n$. Clearly, the partial sums of x_n do not converge, but the Cesàro sums do converge to $\frac{1}{2}$.

1.2.4 The Fejér kernel

As outlined in a previous subsection, one would like to find functions g_N such that

$$(f * g_N)(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$

where a_n is the n^{th} fourier coefficient of f. Rewriting the above sum, we get

$$\sum_{n=-N}^{N} a_n e^{in\theta} = \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \right) e^{in\theta}$$

Since this is a finite sum, we can exchange the sum and the integral to get

$$\sum_{n=-N}^{N} a_n e^{in\theta} = \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(\theta-t)} \right) f(t) dt$$

Summing up the geometric series, we get

$$\frac{1}{2\pi} \sum_{n=-N}^{N} e^{int} = \frac{1}{2\pi} \frac{\sin\left(Nt + \frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}$$

The function g_N we wanted was is this function

$$g_N(t) = \frac{1}{2\pi} \frac{\sin\left(Nt + \frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}$$

g_N certainly is even, and its integral over T is 1, but it is not non-negative everywhere, hence it fails to form a Dirac sequence. Let's look at the Cesàro partial sums instead, and concentrates on the kernel corresponding to those, which are called Fejér kernels. Clearly, the kernel corresponding to the N^{th} Cesàro sum would be the following:

$$F_N = \frac{\sum_{n=1}^{N} g_n}{N}$$

Summing up the geometric progression once again, we get the following closed form expression for F_N:

$$\begin{aligned} F_N(t) &= \frac{1}{2N\pi} \frac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)} \\ &= \frac{1}{2N\pi} \frac{1-\cos(nt)}{1-\cos(t)} \end{aligned}$$

It follows that F_N is even and its integral over T is 1. Furthermore, it is non-negative, and finally we have the following inequality for $0 < \delta < |t| \leq \pi$

$$\begin{aligned} \frac{1}{2N\pi} \frac{1-\cos(nt)}{1-\cos(t)} &\leq \frac{1}{2N\pi} \frac{2}{1-\cos(t)} \\ &\leq \frac{1}{N\pi} \frac{1}{1-\cos(\delta)} \end{aligned}$$

This shows that $\{F_N\}$ satisfies all the conditions required to be a Dirac sequence. This easily leads to the following result.

Theorem 1.3 (Fejér's Theorem)**.** *The fourier series of a continuous function is Cesàro summable, and the partial Cesàro sums converge uniformly to f.*

Proof. The Fejér kernels form a Dirac sequence. Now use the result of theorem 1.2. $\square$

2 Weyl's equidistribution theorem

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3 Roth's theorem

3.1 Doing fourier analysis on finite cyclic groups

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