THE WEYL INEQUALITY FOR QUADRATIC POLYNOMIALS

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A *Weyl sum* is an exponential sums of the form

$$S = \sum_{n=1}^{N} e^{2\pi i P(n)} \tag{1}$$

where $P(x)$ is a polynomial with real coefficients. The purpose of this note is to derive Weyl's estimates for such sums in the special case when $P(x) = \alpha x^2 + \beta x + \gamma$.

We closely follow the fine treatments in [1] and [2].

Theorem 1 (The Weyl inequality for quadratic polynomials)**.** *Let $a \in \mathbf{Z}$ and $q \in \mathbf{N}$ with $(a,q) = 1$, and $N \in \mathbf{N}$ with $N \geq 2$. If $\alpha \in \mathbf{R}$ with $|\alpha - a/q| \leq q^{-2}$, then*

$$\Big|\sum_{n=1}^{N} e^{2\pi i(\alpha n^2 + \beta n + \gamma)}\Big| \leq 20 N \log N (1/q + 1/N + q/N^2)^{1/2}.$$

We remark that this gives a non-trivial estimate whenever $N^\eta \leq q \leq N^{2-\varepsilon}$ for some $0 < \eta, \varepsilon < 1$.

We begin with the following elementary lemma.

Lemma 2. *Let $\alpha \in \mathbf{R}$. Then for all $N \in \mathbf{N}$,*

$$\Big|\sum_{n=1}^{N} e^{2\pi i(\alpha n + \beta)}\Big| \leq \min\Big\{N, \frac{1}{2\|\alpha\|}\Big\}$$

where $\|\alpha\|$ is the distance from α to the nearest integer.

Proof. The constant β has no effect on the inequality whatsoever. If $\alpha = 0 \mod 1$, then the sum is N. If $\alpha \neq 0 \mod 1$, then

$$\Big|\sum_{n=1}^{N} e^{2\pi i \alpha n}\Big| \leq \frac{|1 - e^{2\pi i \alpha N}|}{|1 - e^{2\pi i \alpha}|} \leq \frac{|\sin \pi \alpha N|}{|\sin \pi \alpha|} \leq \frac{1}{2\|\alpha\|}. \qquad \square$$

The method of *Weyl differencing* allows us to treat higher degree polynomials, the idea is simply to *square-out* the Weyl sum (1);

$$\begin{aligned}
|S|^2 &= \sum_{n=1}^{N} \sum_{m=1}^{N} e^{2\pi i [P(m) - P(n)]} \\
&= \sum_{n=1}^{N} \sum_{h=1-n}^{N-n} e^{2\pi i [P(n+h) - P(n)]} \\
&= N + \sum_{h=1}^{N-1} \sum_{n=1}^{N-h} e^{2\pi i [P(n+h) - P(n)]} + \sum_{h=1-N}^{-1} \sum_{n=1-h}^{N} e^{2\pi i [P(n+h) - P(n)]} \\
&= N + 2\,\mathrm{Re} \sum_{h=1}^{N-1} \sum_{n=1}^{N-h} e^{2\pi i [P(n+h) - P(n)]} \\
&\leq N + 2 \sum_{h=1}^{N-1} \Big| \sum_{n=1}^{N-h} e^{2\pi i [P(n+h) - P(n)]} \Big|.
\end{aligned}$$

Since $P(x+h)-P(x)$ is a polynomial of degree one less than that of $P(x)$, the possibility of inducting on the degree of P arises.

In Theorem 1 we are only considering Weyl sums with $P(x)=\alpha x^2+\beta x+\gamma$ so in this case the difference $P(x+h)-P(x)=2\alpha hx+\alpha h^2+\beta h$, and it follows from *Weyl differencing* and Lemma 2 that

$$\begin{aligned}|S|^2 &\leq N+2\sum_{h=1}^{N-1}\Big|\sum_{n=1}^{N-h}e^{2\pi i(2\alpha h)n}\Big|\\ &\leq N+2\sum_{h=1}^{N-1}\min\Big\{N-h,\frac{1}{\|2\alpha h\|}\Big\}\\ &\leq N+2\sum_{h=1}^{2N}\min\Big\{N,\frac{1}{\|\alpha h\|}\Big\}.\end{aligned}$$

Theorem 1 therefore follows immediately from the following proposition (with $H=2N$).

Proposition 3. *Let $a\in\mathbf{Z}$ and $q\in\mathbf{N}$ with $(a,q)=1$, $N\in\mathbf{N}$ with $N\geq 2$, and $H\in\mathbf{N}$. If $\alpha\in\mathbf{R}$ with $|\alpha-a/q|\leq q^{-2}$, then*

$$\sum_{h=1}^{H}\min\Big\{N,\frac{1}{\|\alpha h\|}\Big\}\leq 24\log N(N+q+H+HN/q).$$

The proof of this proposition follows from the lemma below together with the key observation that if $0<|h_2-h_1|\leq q/2$, then $\|\alpha h_2-\alpha h_1\|\geq 1/2q$.

Lemma 4. *Let $L,M,N\in\mathbf{N}$ with $N\geq 2$ and $L\leq M$. If $\alpha_1,\dots,\alpha_L\in\mathbf{R}$ with $\|\alpha_\ell-\alpha_{\ell'}\|\geq M^{-1}$ whenever $\ell\neq\ell'$, then*

$$\sum_{\ell=1}^{L}\min\Big\{N,\frac{1}{\|\alpha_\ell\|}\Big\}\leq 6(N+M)\log N.$$

Proof of Proposition 3. We first note that if $0<|h_2-h_1|\leq q/2$, then

$$\|\alpha h_2-\alpha h_1\|\geq\|(h_2-h_1)a/q\|-|h_2-h_1|/q^2\geq 1/q-1/2q=1/2q$$

since $(h_2-h_1)a\neq 0 \mod q$. It then follows from Lemma 4 that

$$\sum_{h=1}^{H}\min\Big\{N,\frac{1}{\|\alpha h\|}\Big\}\leq\sum_{k=0}^{\lfloor 2H/q\rfloor}\sum_{h=k\lfloor q/2\rfloor+1}^{(k+1)\lfloor q/2\rfloor}\min\Big\{N,\frac{1}{\|\alpha h\|}\Big\}\leq 6(1+2H/q)(N+2q)\log N.\qquad\square$$

Proof of Lemma 4. Without loss of generality we may assume that each $\alpha_\ell\in[-1/2,1/2]$ and that

$$S^+=\sum_{\substack{1\leq\ell\leq L\\ \alpha_\ell\geq 0}}\min\Big\{N,\frac{1}{\|\alpha_\ell\|}\Big\}\geq\frac{1}{2}\sum_{\ell=1}^{L}\min\Big\{N,\frac{1}{\|\alpha_\ell\|}\Big\}.$$

Relabeling the non-negative α_ℓ as $0\leq\alpha_1<\alpha_2<\cdots<\alpha_K$ and noting that $\alpha_k\geq(k-1)/M$ for $k=1,\dots,K$, we see that

$$S^+\leq\sum_{k=0}^{K-1}\min\Big\{N,\frac{M}{k}\Big\}=\sum_{k=0}^{\lfloor M/N\rfloor}N+\sum_{M/N<k<K}\frac{M}{k}\leq(N+M)+2M\log N.\qquad\square$$

References

[1] W. T. Gowers, *Additive and Combinatorial Number Theory*, www.dpmms.cam.ac.uk/∼wtg10/addnoth.notes.dvi.

[2] H. L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conference Series in Mathematics, 84.