Contents

1 Convergence results for fourier series

1.1 A historical perspective

Fourier series made their first appearance in mathematics through the way of physics, as is the norm with a lot of classical analysis. In particular, they turned up in the analysis of the following differential equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{1}$$

on the unit disc, with the following boundary conditions:

$$u(1, \theta) = f(\theta)$$

where f is an arbitrary continous function on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$.

Equation 1 is what is called the Laplace equation, which describes the temperature u on a conducting surface at steady state. This naturally raises the question: What would the temperature profile on a disk look like at steady state if the boundary was maintained at some temperature $f(\theta)$. That is precisely what the described differential equation along with the boundary conditions tries to answer.

Once one makes the simplifying assumption that the solution is separable, i.e.

$$u(r, \theta) = a(r) \cdot b(\theta)$$

equation 1 is easy to solve, and one sees that it has solutions of the form:

$$u_n(r, \theta) = r^n e^{in\theta}$$

for all integers n. And finite linear combinations of the solutions u_n are also solutions to the differential equation 1. But there is one problem. Not every continuous function f on $[-\pi, \pi]$ is a finite linear combination of $e^{in\theta}$, e.g. $|x|$. Leaving that problem aside for now, one notes that if f is of the form

$$f(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$

then a_n is can be determined in the following manner:

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \tag{2}$$

Here a_n is called the n^{th} fourier coefficient of f. Since we assumed that f is a finite linear combination of $e^{-in\theta}$, that means it has only finitely many non-zero coefficients. Contrariwise, continuous functions that are not finte linear combinations of $e^{in\theta}$ must have infinitely many non-zero fourier coefficients. Could it perhaps be that f is an *infinite* linear combination of $e^{in\theta}$? To be more precise, does the following sequence converge to f (in an appropriate sense) as N goes to ∞?

$$S_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$

where a_n is defined as in equation 2.

The best we can hope for is for the sequence S_N to converge uniformly to the function f, but as we shall see, that is not actually true, i.e. there are continuous functions whose fourier series diverges for some point in the domain. The next step would be either weakening the mode of convergence, i.e. instead of expecting convergence in the L^∞ norm, one could expect convergence in weaker norms such as L^1 or L^2, or even Abel or Cesàro convergence. Another possible way to go about would be to strengthen the conditions on the function, i.e. forcing the function to be C^1, or even absolutely continuous ensures that the sequence S_N converges uniformly to f.

1.2 Fourier series of continuous functions are Abel and Cesàro summable

Before we go on to show that fourier series converge under weaker notions of convergence, we'll develop a technique that can be used more generally to show certain sequences of functions converge uniformly. Furthermore, in all the sections that are to follow, we'll be working with continuous complex valued functions on the circle T, which is defined as the space $\mathbb{R}$ quotiented with the equivalence relation $\sim$ defined as

$$a \sim b \iff (a-b) = 2n\pi,\ n \in \mathbb{Z}$$

1.2.1 Convolution

For two continuous functions f and g on T, we can define a binary operation $*$:

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-t)g(t)dt$$

It follows that the operation is commutative (substitution of variables), associative (again substitution of variable), and distributive (integration is a linear operator). It also follows that $f * g$ is continuous since f is uniformly continuous.

It's not too hard to show that this operation has no identity element.

Proposition 1.1. *There exists no continuous function g on T such that for all continuous functions f*

$$f * g = f$$

Proof. We will prove the result by showing that if such a g existed, then $g(x) = 0$ for $x \neq 0$. But that would mean g is non-zero on a set of measure 0, hence $\int_T f(x-t)g(t)dt = 0$ for all f, which means g is not the identity, hence a contradiction.

Assume we have a continuous function g such that for all continuous f, $f * g = f$. Pick any non-zero x_0. We claim that $g(x_0)$ must be 0. If it's not 0, then without loss of generality, let $g(x_0) = \varepsilon > 0$. That means there exists a δ_0 such that for all $x \in (x_0 - \delta_0, x + \delta_0)$, $g(x) > \frac{\varepsilon}{2}$. Let $\delta = \min(\delta_0, |x_0|)$.

Now define a continuous function f in the following manner:

$$f(-x) = \begin{cases} 1 & x \in (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}) \\ \frac{2(x-(x_0-\delta))}{\delta} & x \in \left(x_0 - \delta, x_0 - \frac{\delta}{2}\right] \\ \frac{2((x_0+\delta)-x)}{\delta} & x \in \left[x_0 + \frac{\delta}{2}, x_0 + \delta\right) \\ 0 & \text{otherwise} \end{cases}$$

It's clear that $f(0) = 0$.

$$\begin{aligned} f(0) &= (f * g)(0) \\ &= \int_{-\pi}^{\pi} f(-t)g(t)dt \\ &\geq \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f(-t)g(t)dt \\ &\geq \frac{\varepsilon\delta}{2} \\ &> 0 \end{aligned}$$

But $f(0) = 0$, and $0 \not> 0$. We have a contradiction, which means $g(x) = 0$ for all $x \neq 0$. This completes the proof. □

1.2.2 Dirac sequences

We saw that there is no continuous function which acts as an identity for $*$ operation. However, there do exist sequences of functions, such that the sequence $\{f * g_n\}$ converges uniformly to f as n goes to infinity. Such a sequence $\{g_n\}$ is called a Dirac sequence. The formal definition of a Dirac sequence is the following:

Definition 1.1. A sequence of a continuous functions $\{g_n\}$ is called a Dirac sequence if it satisfies the following conditions:

1. $g_n(x) \geq 0$ for all $n \in \mathbb{N}$ and all $x \in T$.
2. $g_n(x) = g_n(-x)$ for all $n \in \mathbb{N}$ and all $x \in T$.
3. $\displaystyle\int_{-\pi}^{\pi} g_n(t)dt = 1$ for all $n \in \mathbb{N}$.
4. For all $\varepsilon > 0$ and $\delta > 0$, there exists an N such that for all $n > N$,

$$\int_{-\pi}^{-\delta} g_n(t)dt + \int_{\delta}^{\pi} g_n(t)dt < \varepsilon$$

Following from the definition, we get this very useful theorem:

Theorem 1.2. *If $\{g_n\}$ is a Dirac sequence, then for all continuous functions f, the sequence $\{f * g_n\}$ converges uniformly to f.*

Proof. Pick any $\varepsilon > 0$. We need to show there exists an N such that for all $n > N$, $|(f * g_n) - f|_\infty < \varepsilon$. Pick a particular continuous function f. Let M be the maximum of $|f|$ on T. Pick $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\varepsilon_1\varepsilon_2 + \varepsilon_2 + 2M\varepsilon_1 < \varepsilon \tag{3}$$

Since f is uniformly continuous on T, pick a δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon_2$.

Now pick an N such that for all $n > N$

$$\int_{-\pi}^{-\delta} g_n(t)dt + \int_{\delta}^{\pi} g_n(t)dt < \varepsilon_1$$

This would imply for all $n > N$

$$\int_{-\delta}^{\delta} g_n(t)dt > 1 - \varepsilon_1$$

Now consider $(f * g_n)(x_0)$ for some $x \in T$.

$$\begin{aligned}(f * g_n)(x_0) &= \int_{-\pi}^{\pi} f(x_0 - t)g_n(t)dt \\ &= \left(\int_{-\pi}^{-\delta} f(x_0 - t)g_n(t)dt + \int_{\delta}^{\pi} f(x_0 - t)g_n(t)dt\right) + \int_{-\delta}^{\delta} f(x_0 - t)g_n(t)dt\end{aligned}$$

Let's analyze the two terms separately. Since $-M \leq f(x_0 - t) \leq M$, we can bound the first term as

$$-M\varepsilon_1 \leq \int_{-\pi}^{-\delta} f(x_0 - t)g_n(t)dt + \int_{\delta}^{\pi} f(x_0 - t)g_n(t)dt \leq M\varepsilon_1$$

Similarly, for all x in the interval $(x_0 - \delta, x_0 + \delta)$

$$f(x_0) - \varepsilon_2 \leq f(x) \leq f(x_0) + \varepsilon_2$$

This lets us bound the second term in the following manner:

$$(1 - \varepsilon_1)(f(x_0) - \varepsilon_2) \leq \int_{-\delta}^{\delta} f(x_0 - t)g_n(t)dt \leq f(x_0) + \varepsilon_2$$

This gives us a complete bound on $(f * g_n)(x_0)$.

$$-M\varepsilon_1 + (1 - \varepsilon_1)(f(x_0) - \varepsilon_2) \leq (f * g_n)(x_0) \leq M\varepsilon_1 + (f(x_0) + \varepsilon_2)$$

Using the triangle inequality and the fact that $|f(x_0)| < M$, we get

$$f(x_0) - (\varepsilon_1\varepsilon_2 + \varepsilon_2 + 2M\varepsilon_1) \leq (f * g_n)(x_0) \leq f(x_0) + (\varepsilon_1\varepsilon_2 + \varepsilon_2 + 2M\varepsilon_1)$$

But from inequality 3, we get

$$f(x_0) - \varepsilon \leq (f * g_n)(x_0) \leq f(x_0) + \varepsilon$$

This shows that $\{f * g_n\}$ converges uniformly to f for all continuous f. $\square$

2 Weyl's equidistribution theorem

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3 Roth's theorem

3.1 Doing fourier analysis on finite cyclic groups

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