

$e^{in\theta} = e_n(\theta) : \mathbb{T} \rightarrow \mathbb{R}$
 $\{e_n\}$ ON set in $L^2(\mathbb{T})$.

Q: Is it an ONB?

Equivalently, $\overline{\text{span}\{e_n | n \in \mathbb{Z}\}} = L^2(\mathbb{T})$.

Q': $e_n \in C(\mathbb{T})$ (Banach space with $\|\cdot\|_{\text{sup}}$)
 $\overline{\text{span}\{e_n | n \in \mathbb{Z}\}}$ is dense in $C(\mathbb{T})$?

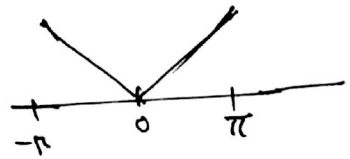
Yes to Q' \Rightarrow Yes to Q

($\because C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$ and $\|\cdot\|_{\text{sup}}$ convergence is stronger than $\|\cdot\|_L$ with $C(\mathbb{T})$)

Fejer: Yes to Q'

- 1) Read proof. (more explicit construction)
- 2) Give a proof using Stone-Weierstrass.

3) Find Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$



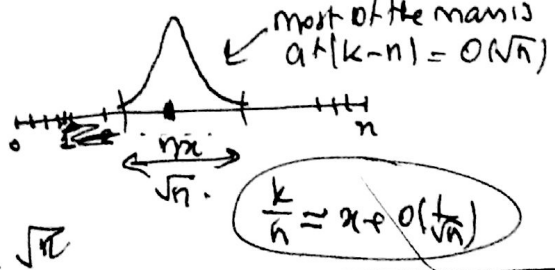
Weierstrass thm: Polynomials are dense $C[0,1]$.

Proof: $f \in C[0,1]$. $B_{f,n}(x) = \sum_{k=0}^n f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} \rightarrow \text{a polynomial. (Bernstein)}$

Claim: $\|B_{f,n} - f\|_{\text{sup}} \rightarrow 0$ as $n \rightarrow \infty$.

Idea: $p_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$ $0 \leq k \leq n$.

mean = nx
 $\text{std. dev} = \sqrt{nx(1-x)} < \sqrt{n}$



$H = H.s.$ $\langle \cdot, \cdot \rangle$ complete in the norm introduced assume separable.

$\Rightarrow \exists$ ONB $\{e_n | n=1,2,\dots\}$ means $\langle e_n, e_m \rangle = \delta_{nm}$ and $\overline{\text{span}\{e_n | n \geq 1\}} = H$.

$L^2(\mathbb{T})$: Consider $f: \mathbb{T} \rightarrow \mathbb{C}$ or f^2 is integrable.

$N = \{f \text{ of such } f\}$

\hookrightarrow vector space

$\hookrightarrow \langle f, g \rangle = \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi}$ inner product on N .

Problem: Not complete

Taken care of by going to Lebesgue integral.

Then we get $L^2(\mathbb{T}) = \{f: \mathbb{T} \rightarrow \mathbb{C} \mid \int_0^{2\pi} |f|^2 \frac{d\theta}{2\pi} < \infty\}$
 $\langle f, g \rangle = \int_0^{2\pi} f \overline{g} \frac{d\theta}{2\pi}$
 Quotient by $W = \{f \mid f=0 \text{ a.e.}\}$
 Hilbert space.

$H = H.s.$

W -subspace (f.d. or closed)

Given $v \in H \exists! w \in W$ that minimizes $\|v-w\|$.
 \hookrightarrow proj of v onto W .

If u_1, \dots, u_m is an ONB for W

then $w = \sum_{k=1}^m \langle v, u_k \rangle u_k$

$\hat{f}(n) = f \in \mathbb{T} \rightarrow \mathbb{R}$ integrable,

$\hat{f}(n) = \langle f, e_n \rangle$

$\downarrow = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$

Fourier coefficients.

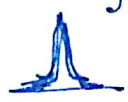
$f(x) = |x|$ on $[-\pi, \pi]$
 $\hat{f}(0) = \pi^2$ $\hat{f}(n) = \frac{4}{n^2}$ n odd.
 $a_n = 0$ $b_n = 0$
 $\hat{f}(n) = \begin{cases} 0 & \text{if } n \text{ even } n \neq 0 \\ -\frac{4}{n^2} & \text{if } n \text{ is odd.} \end{cases}$

$\hat{f}(n) = \langle f, e_n \rangle$ $e_n = \cos(n \cdot) + i \sin(n \cdot)$
 $= a_n + i b_n$ $a_n = \langle f, \cos(n \cdot) \rangle$
 $b_n = \langle f, \sin(n \cdot) \rangle$

n even n odd
 $a_n = 0$ $a_n = -\frac{4}{n^2}$
 $b_n = 0$ $b_n = 0$

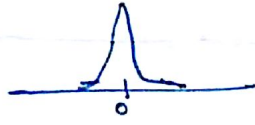
$S_n f = (f * D_n)$
 $= \sum_{k=-n}^n \hat{f}(k) e_k(\cdot)$
 $S_n f \rightarrow f$ unit if $\hat{f}(k) = O(1/k)$
 $\hat{a}_n f = \frac{1}{n+1} (\hat{a}_0 f + \dots + \hat{a}_n f)$
 $\rightarrow f$ unit if $\hat{f}(k) = O(1/k)$

$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e_k(x)$ p.w.
 $x=0. \quad 0 = \pi^2 - \sum_{k=1}^{\infty} \frac{8}{k^2}$ $\frac{9(2k)}{?}$

Wienerstrass: $g_n \in f * [\cos(x) - x^2]$ \checkmark $\int f(t) P_n(t) dt$
 $g_n \xrightarrow{unit} f$ 



Another kernel: Heat kernel: $p_t(x) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$ $N(0,t)$ density.

Approx. identity as $t \rightarrow 0$ 

For nice f , $(f * p_t) \rightarrow f$ unit as $t \rightarrow 0$ on compact.
 (Opt. supported).

$f * p_t$

Exer: Deduce Wienerstrass from Fejer thm.

$f(n)$

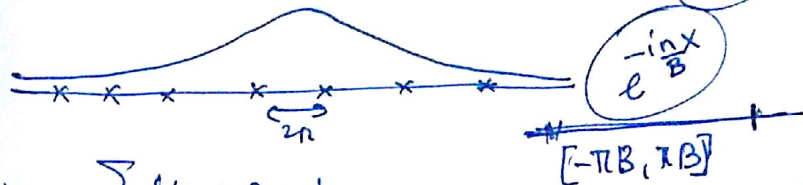
Q: In \mathbb{C} -plane can you approximate C^1 function on a compact set (eg. disk/square) uniformly by polynomials?

Ans. Yes: If polynomials of (x,y) . $\sum_{i,j=0}^n a_{ij} x^i y^j$

No: If polynomials of $\bar{z} = x + iy$. $\sum_{k=0}^n c_k \bar{z}^k$

Exer: \rightarrow Gg: \bar{z} cannot be approximated by such polynomials.

Let $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} : \mathbb{R} \rightarrow \mathbb{R}$.



can let $f_p(x) : [-\pi, \pi] \rightarrow \mathbb{R}$ $f_p(x) = \sum_{n \in \mathbb{Z}} f(x + 2\pi n)$

$f(x) \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{-inx}$

Can find $\hat{f}_p(m) = \int_{-\pi}^{\pi} f_p(x) e^{-imx} dx = \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} f(x + 2\pi n) e^{-imx} dx$
 $= \sum_{n \in \mathbb{Z}} \int_{(n-1)\pi}^{(n+1)\pi} f(y) e^{-imy} dy = \int_{\mathbb{R}} f(t) e^{-imt} dt$
check $\frac{1}{\sqrt{2\pi}} e^{-m^2/2}$

Exer: For any $\lambda \in \mathbb{R}$ $\int_{\mathbb{R}} f(t) e^{i\lambda t} dt = e^{-\lambda^2/2}$

Conclusion: (Assuming unit p.w. conv.) $\sum_{n \in \mathbb{Z}} \frac{e^{-(x+2\pi n)^2/2}}{\sqrt{2\pi}} = \sum_{m \in \mathbb{Z}} e^{-m^2/2} e^{imx}$

$x=0: \sum_{n \in \mathbb{Z}} e^{-2\pi^2 n^2} = \sqrt{2\pi} \sum_{m \in \mathbb{Z}} e^{-m^2/2}$

Exer: Start with $f_t(x) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$ and get $\sum_{n \in \mathbb{Z}} e^{-4\pi^2 n^2/t} = (2\pi t) \sum_{m \in \mathbb{Z}} e^{-m^2/t}$

$\int_{-\pi}^{\pi} e^{imx} \frac{dx}{2\pi} = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$

$S \subseteq \mathbb{Z}$ finite. 3-term AP in S means $x, y, z \in S$ s.t. $x+z=2y=0$.

3-term APs = $\sum_{(x,y,z) \in S^3} \int_{-\pi}^{\pi} e^{i(x+z-2y)\theta} \frac{d\theta}{2\pi}$
 $= \int_{-\pi}^{\pi} \left(\sum_{x \in S} e^{ix\theta} \right) \left(\sum_{z \in S} e^{iz\theta} \right) \left(\sum_{y \in S} e^{-iy\theta} \right) \frac{d\theta}{2\pi}$

Analysis of exponential sum.

Will come to later (Poisson summation)

Plan

- Read book 1. → Bhattach
- Key's equidistribution thm
- Roth's thm: A subset of \mathbb{N} with positive density has a 3-term AP.
- Construction of expander graphs.
- $\sum_{n \in \mathbb{Z}} a_n \frac{e^{inx}}{i^n} \rightarrow$ how dense diff. chn (w.p. 1)

Ex 1
 $G_n = \mathbb{Z}/n\mathbb{Z}$ = multiplication group of n^{th} roots of 1.

$\{0, 1, \dots, n-1\}$
 $+ (\text{mod } n)$ → Define for $0 \leq l \leq n-1$.

$\chi_l: G_n \rightarrow \mathbb{C}$
 $\chi_l(k) = e^{2\pi i k l / n} \quad 0 \leq k \leq n-1$

$G_n = \{0, 1, \dots, n-1\}$

$\ell^2(G_n) = \{f: G_n \rightarrow \mathbb{C}\}$
 $\langle f_1, f_2 \rangle = \frac{1}{n} \sum_{k=0}^{n-1} f_1(k) \overline{f_2(k)}$

$(\ell^2(G_n) \cong \mathbb{C}^n)$

Exer 1) χ_l is a homomorphism from G_n into $(\mathbb{C}^\times, \cdot)$.

2) $\{\chi_l \mid 0 \leq l \leq n-1\}$ form an ONB for $\ell^2(G_n)$.

3) If $f: G_n \rightarrow \mathbb{C}$, then $f = \sum_{l=0}^{n-1} \hat{f}(l) \chi_l$ where $\hat{f}(l) = \langle f, \chi_l \rangle$

4) $\frac{1}{n} \sum_{k=0}^{n-1} |f(k)|^2 = \sum_{l=0}^{n-1} |\hat{f}(l)|^2$

$= \frac{1}{n} \sum_{k=0}^{n-1} f(k) \overline{\chi_l(k)}$

$= \frac{1}{n} \sum_{k=0}^{n-1} f(k) e^{-2\pi i k l / n}$

Ex 2: $G_n = \mathbb{Z}/n\mathbb{Z} \cong G_n = \{1, i\}^n \quad ((\mathbb{Z}/2\mathbb{Z})^n)$.

Exer

Find all homomorphisms of G_n into \mathbb{C}^\times .

$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n)$
 $= (x_1 y_1, x_2 y_2, \dots, x_n y_n)$

Later: These homomorphisms will form an ONB for $\ell^2(G_n) = \{f: G_n \rightarrow \mathbb{C}\}$

"Representation theory of finite groups"

J. P. Serre

(works with finite abelian groups)

$\langle f_1, f_2 \rangle = \frac{1}{|G_n|} \sum_{g \in G_n} f_1(g) \overline{f_2(g)}$