$e^{in\theta} = e_n(\theta) : \underset{[0,2\pi)}{\mathbb{T}} \to \mathbb{R}$

$\{e_n\}_{n\in\mathbb{Z}}$ ON set in $L^2(\mathbb{T})$.

Q: Is it an ONB?

Equivalently, $\overline{\text{span}}\{e_n | n\in\mathbb{Z}\} = L^2(\mathbb{T})$.

Q': $e_n \in C(\mathbb{T})$ → (Banach) space with $\|\cdot\|_{sup}$.

trig. polynomials: $\text{span}\{e_n | n\in\mathbb{Z}\}$ is dense in $C(\mathbb{T})$?

Yes to Q' $\Rightarrow$ Yes to Q

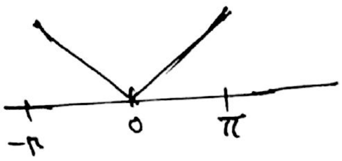
($\because$ $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$ and $\|\cdot\|_{sup}$ convergence in stronger than $\|\cdot\|_{L^2}$ with $C(\mathbb{T})$)

$\boxed{\text{Féjer: Yes to Q'}}$

1) Read proof. (more explicit construction)
2) Give a proof using Stone-Wierstrass.

---

3) Find Fourier coefficients for ~~f(θ)~~ $f(\theta) = |\theta|$ $\quad \theta \in [-\pi, \pi)$



H - H.S. $\langle \cdot, \cdot \rangle$ complete in the norm introduced
assume separable.

$\Rightarrow \exists$ ONB $\{v_n | n=1,2,\dots\}$ means $\langle v_n, v_m\rangle = \delta_{nm}$
and $\overline{\text{span}}\{v_n | n\geq 1\} = H$.

$L^2(\mathbb{T})$: Consider $f: \mathbb{T} \to \mathbb{C}$ s.t. f^2 is integrable.

V = set of such f

↳ vector space ✓

↳ $\langle f, g\rangle = \int_0^{2\pi} f(\theta)\overline{g(\theta)}\,\frac{d\theta}{2\pi}$ inner product on V.

Problem: Not complete

Taken care of by going to Lebesgue integral.

Then we get $L^2(\mathbb{T}) = \{f: \mathbb{T}\to\mathbb{C} \mid \int_0^{2\pi} f^2 \frac{d\theta}{2\pi} < \infty\}/\sim$
↳ Leb. int.

$\langle f, g\rangle = \int_0^{2\pi} f\bar{g}\,\frac{d\theta}{2\pi}$ $\quad$ O $\quad$ Quotient by $W = \{f \mid f = 0 \text{ a.e.}\}$

Hilbert space.

H. - H.S. ~~$v_1 \dots v_m$~~

W - subspace (f.d. or closed)

Given $v \in H$ $\exists!$ $w \in W$ that minimizes $\|v - w\|$.
↳ proj of v onto W.

If $u_1, \dots, u_m$ is an ONB for W

then $w = \sum_{k=1}^{m} \langle v, u_k\rangle u_k$

$f: \mathbb{T} \to \mathbb{R}$ integrable,

$\hat{f}(n) = \langle f, e_n\rangle$

$= \int_0^{2\pi} f(\theta) e^{-in\theta}\,\frac{d\theta}{2\pi}$

↓

Fourier coefficients.

Wierstrass thm: Polynomials are dense $C[0,1]$.

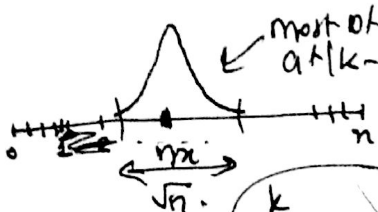
Proof: $f \in C[0,1]$. $\quad B_{f,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\underbrace{\binom{n}{k} x^k (1-x)^{n-k}}$ → a polynomial. (Bernstein)

Claim: $\|B_{f,n} - f\|_{sup} \longrightarrow 0$ as $n \to \infty$.

Idea: $p_{n,x}(k) = \binom{n}{k} x^k (1-x)^{n-k}$ $\quad 0 \leq k \leq n$.

mean $= nx$

std. dev $= \sqrt{nx(1-x)} < \sqrt{n}$



most of the mass is at $|k-n| = O(\sqrt{n})$

$\frac{k}{n} \simeq x + O\left(\frac{1}{\sqrt{n}}\right)$

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$f(x) = |x|$ on $[-\pi, \pi]$

$\hat{f}(0) = \pi^2$ $\quad \underset{a_n}{\hat{f}(n)} = -\frac{4}{n^2}$ n odd.

$b_n = 0$

$\hat{f}(n) = \langle f, e_n \rangle$ $\quad e_n = \cos(n\cdot) + i\sin(n\cdot)$

$= a_n + i b_n$. $\quad a_n = \langle f, \cos(n\cdot) \rangle$

$b_n = \langle f, \sin(n\cdot) \rangle$

n even	n odd
$a_n = 0$	$a_n = -\frac{4}{n^2}$
$b_n = 0$	$b_n = 0$.

$$\hat{f}(n) = \begin{cases} 0 & \text{if } n \text{ even } n \neq 0 \\ -4/n^2 & \text{if } n \text{ is odd.} \end{cases}$$

$S_n f = (f * D_n)$
$= \sum_{k=-n}^{n} \hat{f}(k) e_k(\cdot)$

$S_n f \to f$ unif if $\hat{f}(n) = O(1/n)$

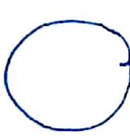
$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e_k(x) \quad \text{p.w.}$$

$x = 0$. $\quad 0 = \pi^2 - \sum_{k=1}^{\infty} \frac{8}{k^2}$ $\quad$ $\underline{\underline{a(2k)}}$?..

$\sigma_n f = \frac{1}{n+1}(S_0 f + \dots + S_n f)$
$\to f$ unif $\forall f$ cts.

<u>Wierstrass:</u> $g_n \in f * [c_n(1-x^2)^n]$ ✓. $\quad$ ~~$\int f(t)\, P(x-t)\, dt$~~

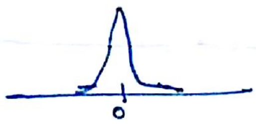
$g_n \xrightarrow{\text{unif}} f$



$t = -1$
$t = 1$

<u>Another kernel:</u> Heat kernel: $\quad p_t(x) = \dfrac{e^{-x^2/2t}}{\sqrt{2\pi t}}$ $\quad$ $N(0,t)$ density.

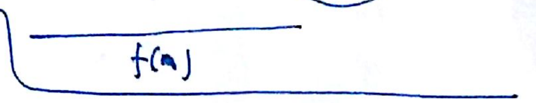
Approx. identity as $t \to 0$ $\quad$ analytic



For nice f, $(f * p_t) \to f$ unif as $t \to 0$
on compacts.

(cpt supported).

$f * p_t$



$f(a)$

<u>Exer:</u> Deduce Weierstrass' thm from Fejer's thm.

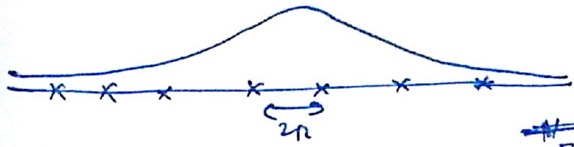
Q: In $\mathbb{C}$-plane can you approximate cts fns on a compact set (e.g. disk / square) uniformly by polynomials?

Ans. Yes: If polynomials of ~~(x, y)~~ (x, y). $\quad \sum_{i,j=0}^{n} a_{ij} x^i y^j$

No: If polynomials of $\underline{\underline{z = x+iy}}$. $\quad \sum_{k=0}^{n} c_k z^k$

<u>Exer:</u> $\longrightarrow$ Eg: $\bar{z}$ cannot be approximated by $\exists$ such polynomials.

Let $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} : \mathbb{R} \to \mathbb{R}$.



$n\in\mathbb{Z}$

$e^{\frac{-inx}{B}}$

$[-\pi B, \pi B]$

Can let $f_p(x)$ $f_p : [-\pi,\pi] \to \mathbb{R}$ $\quad f_p(x) = \sum_{n\in\mathbb{Z}} f(x+2\pi n)$

$f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{n}{B}\right) e^{-\frac{inx}{B}}$

Cts fn!!

$$\hat{f}_p(m) = \int_{-\pi}^{\pi} f_p(x)\, e^{-imx}\,dx = \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} f(x+2\pi n)\, e^{-imx}\,dx$$

$$= \sum_{n\in\mathbb{Z}} \int_{(2n-1)\pi}^{(2n+1)\pi} f(y)\, e^{-imy}\,dy = \int_{\mathbb{R}} f(t)\, e^{-imt}\,dt$$

Check $\quad e^{-m^2/2}$

Exer: For any $\lambda\in\mathbb{R}$ $\int_{\mathbb{R}} f(t)\, e^{-i\lambda t}\,dt = e^{-\lambda^2/2}$

Conclusion: (Assuming unif p.w. conv.) $\quad \sum_{n\in\mathbb{Z}} \frac{e^{-(x+2\pi n)^2/2}}{\sqrt{2\pi}} = \sum_{m\in\mathbb{Z}}' e^{-m^2/2}\, e^{imx}$

$x=0$: $\quad \sum_{n\in\mathbb{Z}} e^{-2\pi^2 n^2} = \sqrt{2\pi} \sum_{m\in\mathbb{Z}} e^{-m^2/2}$

Exer: Start with $f_t(x) = \frac{e^{-x^2/2t}}{\sqrt{4\pi t}}$ and get $\sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2/t} = (2\pi t) \sum_{m\in\mathbb{Z}} e^{-m^2 t}$ $\forall t$

↳ will come to later (Poisson summation)

$$\int_{-\pi}^{\pi} e^{imx}\,\frac{dx}{2\pi} = \begin{cases} 0 & \text{if } m\neq 0 \\ 1 & \text{if } m=0 \end{cases}$$

$S\subseteq\mathbb{Z}$ finite. 3-term APs in S mean $x,y,z\in S$ st $x+z-2y=0$.

$$\#\text{ 3-term APs} = \sum_{(x,y,z)\in S} \int_{-\pi}^{\pi} e^{i(x+z-2y)\theta}\,\frac{d\theta}{2\pi}$$

$$= \int_{-\pi}^{\pi} \Big(\sum_{x\in S} e^{ix\theta}\Big)\Big(\sum_{z\in S} e^{iz\theta}\Big)\Big(\sum_{y\in S} e^{-i2y\theta}\Big)\,\frac{d\theta}{2\pi}$$

↓

Analysis of exponential sums.

Plan

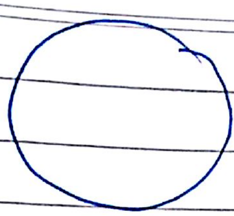
- Read basics → Bhatia
- Weyl's equidistribution thm
- Roth's thm: A subset of $\mathbb{N}$ with positive density has a 3-term AP.
- Construction of expander graphs.
- $\sum_{n\in\mathbb{Z}\setminus\{0\}} a_n \frac{e^{in\theta}}{in}$ → nowhere diff cts fn (w.p. 1)
  → $X_n + iY_n$, X_n, Y_n iid $N(0,1)$

Eg 1

$G_n = \mathbb{Z}/(n)$ = multiplication group of n^{th} roots of 1.

$\{0,1,\dots,n-1\}$

$+ \pmod{n}$ · → Define for $0 \le \ell \le n-1$.



$$\chi_\ell : G_n \to \mathbb{C}$$

$$\chi_\ell(k) = e^{2\pi i k\ell/n} \quad 0 \le k \le n-1.$$

$G_n = \{0,1,\dots,n-1\}$

$\ell^2(G_n) = \{f : G_n \to \mathbb{C}\}$

$\langle f_1, f_2\rangle = \frac{1}{n}\sum_{k=0}^{n-1} f_1(k)\overline{f_2(k)}$

$(\ell^2(G_n) \cong \mathbb{C}^n)$

Exer

1) χ_ℓ is a homomorphism from G_n into $(\mathbb{C}^\times, \cdot)$.

2) $\{\chi_\ell \mid 0 \le \ell \le n-1\}$ form an ONB for $\ell^2(G_n)$.

3) If $f : G_n \to \mathbb{C}$, then $f = \sum_{\ell=0}^{n-1} \hat{f}(\ell)\chi_\ell$ where $\hat{f}(\ell) = \langle f, \chi_\ell\rangle$

$$= \frac{1}{n}\sum_{k=0}^{n-1} f(k)\overline{\chi_\ell(k)}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} f(k)\, e^{-2\pi i k\ell/n}$$

4) $\frac{1}{n}\sum_{k=0}^{n-1}|f(k)|^2 = \sum_{\ell=0}^{n-1}|\hat{f}(\ell)|^2$

Eg 2: $G_n = \{-1,1\}^n \quad ((\mathbb{Z}/(2))^n)$.

$(x_1,\dots,x_n)\cdot(y_1,\dots,y_n) = (x_1y_1, x_2y_2, \dots, x_ny_n)$

Exer

· Find all homomorphisms of G_n into $\mathbb{C}$?

Later: These homomorphisms will form an ONB for $\ell^2(G_n) = \{f : G_n \to \mathbb{C}\}$

$$\langle f_1, f_2\rangle = \frac{1}{|G_n|}\sum f_1(g)\overline{f_2(g)}$$

"Representation theory of finite groups"

· J. P. Serre

(we stay with finite abelian groups)

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$X_1, X_2, \ldots$ iid mean 0, Var 1.

$S_n = X_1 + \cdots + X_n$.

F_n = CDF of $\frac{S_n}{\sqrt{n}}$.

CLT: $F_n(t) \longrightarrow \Phi(t)$ p.w.

$\Uparrow$

Levy: $E[e^{i\lambda \frac{S_n}{\sqrt{n}}}] \longrightarrow E[e^{i\lambda Z}]$ $\forall \lambda \in \mathbb{R}$

where $Z \sim N(0,1)$ r.v. $(e^{-\lambda^2/2})$

$Y \sim$ r.v. with density g

$$\underbrace{E[e^{i\lambda Y}]} = \int_{-\infty}^{\infty} e^{i\lambda x} g(x)\,dx \rightarrow \text{Fourier transform of } g$$

$$E[e^{i\lambda \frac{S_n}{\sqrt{n}}}] = E\left[\prod_{k=1}^{n} e^{i\lambda X_k/\sqrt{n}}\right] = \prod_{k=1}^{n} E[e^{i\lambda X_k/\sqrt{n}}]$$

$$= \Big(\underbrace{E[e^{i\lambda X_1/\sqrt{n}}]}\Big)^n$$

....

$f: \mathbb{R} \to \mathbb{R}$ - integrable $\quad \hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-i\lambda x}\,dx \quad \langle f, g_\lambda \rangle$

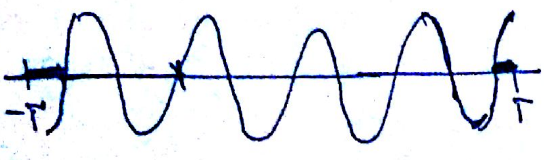
$f: \mathbb{R} \to \mathbb{R}$.

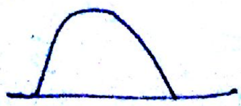
$e_\lambda(x) = e^{i\lambda x} \qquad \int e_{\lambda_1} \bar{e}_{\lambda_2}$ makes no sense?

$$\frac{1}{2T}\int_{-T}^{T} e_{\lambda_1}\bar{e}_{\lambda_2} = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 \\ O\left(\frac{1}{T}\right) \text{ as } T \to \infty, & \text{if } \lambda_1 \neq \lambda_2 \end{cases}$$

$$f(x) = \int_{\mathbb{R}} \hat{f}(\lambda) e^{i\lambda x} \frac{d\lambda}{2\pi}$$

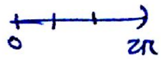
$e_{\lambda_1 - \lambda_2}$





Weyl's equidistribution: $\alpha \notin \mathbb{Q}[0,1]$

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{k\alpha \pmod 1}$$



F_n = i.th CDF $\quad F_n(t) = \mu_n[0,t]$

Weyl's equidistribution: $F_n(t) \to t \ \forall t$.

$\hookrightarrow$ CDF of unif$[0,1]$.

$\Uparrow$

$\boxed{\begin{matrix} X_n \sim \mu_n. \\ X \sim \text{unif}[0,1] \end{matrix}}$ $E[e^{ikX_n}] \to E[e^{ikX}]$ $\forall k \in \mathbb{Z}$. $\begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$

$Y \sim$ r.v. with density f on $[0, 2\pi)$

$$E[e^{ikY}] = \int_0^{2\pi} f(x) e^{ikx}\,dx = \hat{f}(k).$$

Exer: F_n, F are CDFs on $\mathbb{R}$ (on $[0,2\pi]$)

with $F_n(0) = 0 = F(0)$

$F_n(2\pi) = 1 = F(2\pi)$

$F_n(t) \to F(t) \ \forall t$

$\Leftrightarrow \quad E[e^{ikY_n}] \to E[e^{ikY}]$

$\forall k \in \mathbb{Z}$

where $Y_n \sim F_n$, $Y \sim F$

(assume F_n, F have densities)