$e^{in\theta} = e_n(\theta) : \underset{[0,2\pi)}{\overset{}{\mathbb{T}}} \to \mathbb{R}$

$\{e_n\}_{n\in\mathbb{Z}}$ ON set in $L^2(\mathbb{T})$.

Q: Is it an ONB?

Equivalently, $\overline{\text{span}\{e_n \mid n\in\mathbb{Z}\}} = L^2(\mathbb{T})$.

Q': $e_n \in C(\mathbb{T})$ → (Banach) space with $\|\cdot\|_{sup}$.

trig. polynomials: $\text{span}\{e_n \mid n\in\mathbb{Z}\}$ is dense in $C(\mathbb{T})$?

Yes to Q' ⇒ Yes to Q

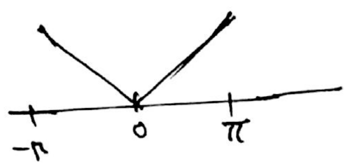
(∵ $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$ and $\|\cdot\|_{sup}$ convergence in stronger than $\|\cdot\|_{L^2}$ with $C(\mathbb{T})$)

$\boxed{\text{Féjer: Yes to Q'}}$

1) Read proof. (more explicit construction)
2) Give a proof using Stone-Wierstrass.

3) Find Fourier coefficients for ~~f(θ)~~ =

$f(\theta) = |\theta| \quad \theta \in [-\pi, \pi]$



H - H.S. $\langle \cdot , \cdot \rangle$ complete in the norm introduced assume separable.

⇒ ∃ ONB $\{v_n \mid n = 1, 2, \dots\}$ means $\langle v_n, v_m \rangle = \delta_{nm}$ and $\overline{\text{span}\{v_n \mid n \geq 1\}} = H$.

$L^2(\mathbb{T})$: Consider $f : \mathbb{T} \to \mathbb{C}$ st f^2 is integrable.

V = set of such f

↳ vector space ✓

↳ $\langle f, g \rangle = \int_0^{2\pi} f(\theta)\overline{g(\theta)} \frac{d\theta}{2\pi}$ inner product on V.

Problem: Not complete

Taken care of by going to Lebesgue integral.

Then we get $L^2(\mathbb{T}) = \{ f : \mathbb{T} \to \mathbb{C} \mid \int_0^{2\pi} f^2 \frac{d\theta}{2\pi} < \infty \} / \sim$ ↳ Leb. int.

$\langle f, g \rangle = \int_0^{2\pi} f\bar{g} \frac{d\theta}{2\pi}$ ↙ O Quotient by $W = \{ f \mid f = 0 \text{ a.e.} \}$

Hilbert space.

H. - H.S. ~~$v_1, \dots, v_m$~~

W - subspace (f.d. or closed)

Given $v \in H$ ∃! $w \in W$ that minimizes $\|v - w\|$.

↳ proj of v onto W.

If $u_1, \dots, u_m$ is an ONB for W

then $w = \sum_{k=1}^{m} \langle v, u_k \rangle u_k$

$\hat{f}(n) =$ $f : \mathbb{T} \to \mathbb{R}$ integrable,

$\hat{f}(n) = \langle f, e_n \rangle$

$= \int_0^{2\pi} f(\theta) e^{-in\theta} \frac{d\theta}{2\pi}$

↓

Fourier coefficients.

Weierstrass thm: Polynomials are dense $C[0,1]$.

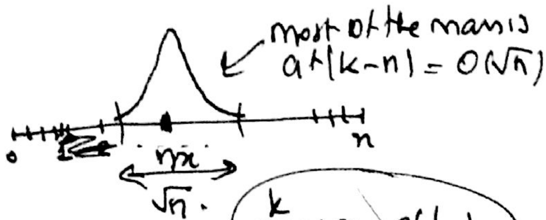
Proof: $f \in C[0,1]$. $B_{f,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$ → a polynomial. (Bernstein)

Claim: $\|B_{f,n} - f\|_{sup} \to 0$ as $n \to \infty$.

Idea: $p_{n,x}(k) = \binom{n}{k} x^k (1-x)^{n-k} \quad 0 \leq k \leq n$.

mean = nx

~~var~~ Std. dev = $\sqrt{nx(1-x)} < \sqrt{n}$



$\frac{k}{n} \simeq x + O\left(\frac{1}{\sqrt{n}}\right)$

$f(x) = |x|$ on $[-\pi, \pi]$

$\hat{f}(n) = \langle f, e_n \rangle$ $\quad e_n = \cos(n\cdot) + i\sin(n\cdot)$

$\hat{f}(0) = \pi^2$ $\quad \underbrace{\cancel{\hat{f}(n)}}_{a_n} = -\frac{4}{n^2}$ $\cancel{n \text{ odd}}$. $\quad = a_n + i b_n$. $\quad a_n = \langle f, \cos(n\cdot)\rangle$, $\quad b_n = \langle f, \sin(n\cdot)\rangle$

$b_n = 0$

n even	n odd
$a_n = 0$	$a_n = -\frac{4}{n^2}$
$b_n = 0$	$b_n = 0$.

$$\hat{f}(n) = \begin{cases} 0 & \text{if } n \text{ even } n \neq 0 \\ -4/n^2 & \text{if } n \text{ is odd.} \end{cases}$$

$S_n f = (f * D_n) = \sum_{k=-n}^{n} \hat{f}(k) e_k(\cdot)$

$S_n f \to f$ unif if $\hat{f}(n) = O(1/n)$

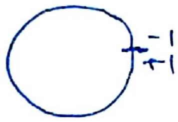
$\sigma_n f = \frac{1}{n+1}(S_0 f + \dots + S_n f)$ $\to f$ unif $\forall f$ cts.

$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e_k(x)$ ptw.

$x = 0$. $\quad 0 = \pi^2 - \sum_{k=1}^{\infty} \frac{8}{\cancel{k^2}}$ $\quad$ $\underline{\underline{q(2k)}}$?..

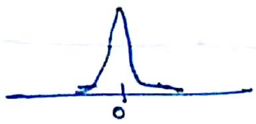
__Weierstrass:__ $g_n \in f * [c_n(1-x^2)^n]$ ✓. $\cancel{\int f(t) P(x-t) dt}$

$g_n \xrightarrow{\text{unif}} f$



$t = 1$, $t = -1$

__Another kernel:__ Heat kernel: $p_t(x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$ $\quad N(0,t)$ density.

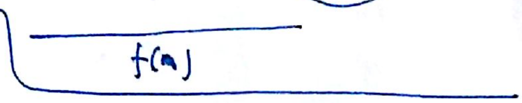


Approx. identity as $t \to 0$ $\quad$ analytic

For nice f, $(f * p_t) \to f$ unif as $t \to 0$ on compacts.

(cpt supported).

$f * p_t$



$f(a)$

__Exer:__ Deduce Weierstrass' thm from Fejer's thm.

Q: In $\mathbb{C}$-plane can you approximate cts fns on a compact set (eg. disk / square) uniformly by polynomials?

Ans. Yes: If polynomials of $\cancel{(x,y)}$ (x,y). $\quad \sum_{i,j=0}^{n} a_{ij} x^i y^j$

No: If polynomials of $z = x+iy$. $\quad \sum_{k=0}^{n} c_k z^k$

__Exer:__ $\longrightarrow$ Eg: $\bar{z}$ cannot be approximated by such polynomials.

$F(x) = |x|$

$n \in \mathbb{Z}$

Let $f(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}} : \mathbb{R} \to \mathbb{R}$.

$e^{-\frac{inx}{B}}$

$[-\pi B, \pi B]$

2π

Can let $f_p : [-\pi, \pi] \to \mathbb{R}$ $\quad f_p(x) = \sum_{n \in \mathbb{Z}} f(x + 2\pi n)$

Cts fn!

$f(x) \sim \sum_{n \in \mathbb{Z}} \hat{f}\left(\frac{n}{B}\right) e^{-\frac{inx}{B}}$

$$\hat{f}_p(m) = \int_{-\pi}^{\pi} f_p(x) e^{-imx} dx = \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} f(x + 2\pi n) e^{-imx} dx$$

$$= \sum_{n \in \mathbb{Z}} \int_{(2n-1)\pi}^{(2n+1)\pi} f(y) e^{-imy} dy = \int_{\mathbb{R}} f(t) e^{-imt} dt$$

Check $\cancel{e^{-x^2/2}}$ $e^{-m^2/2}$

Exer: For any $\lambda \in \mathbb{R}$ $\int_{\mathbb{R}} f(t) e^{-i\lambda t} dt = e^{-\lambda^2/2}$

Conclusion: (Assuming unif p.w. conv.). $\sum_{n \in \mathbb{Z}} \dfrac{e^{-(x+2\pi n)^2/2}}{\sqrt{2\pi}} = \sum_{m \in \mathbb{Z}} e^{-m^2/2} e^{imx}$

$x = 0$: $\sum_{n \in \mathbb{Z}} e^{-2\pi^2 n^2} = \sqrt{2\pi} \sum_{m \in \mathbb{Z}} e^{-m^2/2}$

Exer: Start with $f_t(x) = \dfrac{e^{-x^2/2t}}{\sqrt{4\pi t}}$ and get $\sum_{n \in \mathbb{Z}} e^{-4\pi^2 n^2/t} = (2\pi t) \sum_{m \in \mathbb{Z}} e^{-m^2 \cancel{t}} \quad \forall t$

→ Will come to later (Poisson summation)

$$\int_{-\pi}^{\pi} e^{imx} \frac{dx}{2\pi} = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$$

$S \subseteq \mathbb{Z}$ finite. 3-term APs in S mean $x, y, z \in S$ st $x + z - 2y = 0$.

$$\#\text{ 3-term APs} = \sum_{(x,y,z) \in S} \int_{-\pi}^{\pi} e^{i(x+z-2y)\theta} \frac{d\theta}{2\pi}$$

$$= \int_{-\pi}^{\pi} \left(\sum_{x \in S} e^{ix\theta}\right) \left(\sum_{z \in S} e^{iz\theta}\right) \left(\sum_{y \in S} e^{-i2y\theta}\right) \frac{d\theta}{2\pi}$$

↓

$\boxed{\text{Analysis of exponential sums.}}$

Plan

- Read basics → Bhatia
- Weyl's equidistribution thm
- Roth's thm: A subset of $\mathbb{N}$ with positive density has a 3-term AP.
- Construction of expander graphs.
- $\sum_{n \in \mathbb{Z} \setminus \{0\}} a_n \dfrac{e^{in\theta}}{in}$ → nowhere diff cts fn (w.p. 1) → $X_n + iY_n$, X_n, Y_n iid $N(0,1)$