Partial summary of work done in summer of 2016

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1 Heating a disc

Laplace equation. Solution to Laplace equation with the appropriate boundary conditions leads naturally to Fourier series. Questions of convergence raised. Must be answered in pieces.

2 The fourier series

Rather than dealing with regular convergence, dealt with Abel summability of the fourier series. Poisson kernel.

3 Digression: All about kernels

Convolution operation. Kernels. Dirac sequences of kernels. Proved that $f * D_n$ converges uniformly to f.

4 Weaker notions of convergence

Master theorem showed that the fourier series is Abel summable. Now create similar kernels for finite fourier series (Dirichlet kernel) and averages of first n terms (Fejér kernel). Fejér kernels form a Dirac sequence, hence the fourier series is Cesàro summable.

5 Orthonormal basis for $C(T)$

Proved exponential polynomials dense in $C(T)$: two different proofs using Poisson and Fejér kernels (Fejér kernel gives explicit approximation). Two line proof using Stone-Weirstrass, and a much nastier proof using Weirstrass approximation.

6 Strengthening the conditions on f

Used density result to show $\lim_{n\to\infty} \hat{f}(n) = 0$ for a continuous f (Riemann-Lebesgue lemma). Proved the principle of localisation. Then showed that if $\hat{f}(n)$ is $O\left(\frac{1}{n}\right)$, then the fourier series converges. Subsequently showed that if $f \in C^1(T)$, then $\hat{f}(n)$ is $o\left(\frac{1}{n}\right)$

7 Computing the zeta function for positive even integers

Used the result that $x^2 \in C^1(T)$, hence it's fourier series converges at $x = 0$ to get

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = \frac{\pi^2}{8}$$

And a consequence of this result is that $\zeta(2) = \frac{\pi^2}{6}$. Similarly, by computing the fourier coefficients of x^{2k}, one can compute $\zeta(2k)$.

8 Alternative formula for $\zeta(s)$ where $s > 1$

An alternative formula for $\zeta(s)$ when $s > 1$ is given by

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}$$

where $\mathcal{P}$ is the set of prime numbers.

One can prove this using the fundamental theorem of arithmetic.

9 Proving Weirstrass approximation theorem from Fejér's theorem

From Fejér's theorem we got that trigonometric polynomials are dense in $C(T)$. It will then suffice to show that $\cos(n\theta)$ and $\sin(n\theta)$ can be approximated using polynomials on T. Consider the m^{th} Taylor polynomial for these functions and look at the remainder. For sufficiently large m, the remainder can be made smaller than a given $\varepsilon > 0$. This gives a polynomial approximation for the function and proves the proposition.

10 Showing $\bar{z}$ cannot be uniformly approximated by polynomials in z in a compact set in $\mathbb{C}$

Without loss of generality, assume the compact set in question contains the unit circle (if it doesn't, rescaling and translation should do the trick). Now assume some polynomial p ε approximates $\bar{z}$ where $\varepsilon < 0.5$. In that case

$$|zp(z) - z\bar{z}| < |z|\varepsilon$$

In particular, on the unit circle, the inequality reduces to

$$|zp(z) - 1| < \varepsilon$$

This would mean for all points x on the circle, the real part of $xp(x)$ lies between 0.5 and 1.5. But notice that if take $2(n!)$ equally spaced points on the circle, where n is the degree of $zp(z)$, then the sum of $zp(z)$ over those points is 0, which means the real part of $zp(z)$ on at least one of those points must be less than or equal to 0. We have a contradiction.

11 Poisson summation of the normal distribution

The task was to show

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2} + i\lambda x\right) dx = \exp\left(-\frac{\lambda^2}{2}\right)$$

Denote the value of the integral by $I(\lambda)$. The integral can be evaluated by first noting the imaginary part of the function inside the integral is odd; it goes to 0. Performing integration by parts on the real part of the function, one notices that $I(\lambda)$ satisfies the following differential equation:

$$\frac{dI}{d\lambda} = -\lambda I$$

This gives the required expression.

12 Hausdorff moment theorem

The Hausdorff moment theorem states that if for continuous function f and g and a compact interval I, the following equation holds:

$$\int_I x^n f(x)dx = \int_I x^n g(x)dx$$

for all non-negative integers n, then $f \equiv g$ on I.

This problem is equivalent to showing $k \equiv 0$ where $k = g - f$ which can be done by showing

$$\int_I k^2(x)dx = 0$$

Let p be an ε polynomial approximation of k. Then

$$\left| \int_I k^2(x)dx - \int_I p(x)k(x)dx \right| < |I|\varepsilon$$

But by the hypothesis, $\displaystyle\int_I p(x)k(x)dx = 0$ for all polynomials p. This completes the proof.