Partial summary of work done in summer of 2016

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July 3, 2016

1 Heating a disc

Laplace equation. Solution to Laplace equation with the appropriate boundary conditions leads naturally to Fourier series. Questions of convergence raised. Must be answered in pieces.

2 The fourier series

Rather than dealing with regular convergence, dealt with Abel summability of the fourier series. Poisson kernel.

3 Digression: All about kernels

Convolution operation. Kernels. Dirac sequences of kernels. Proved that $f * D_n$ converges uniformly to f.

4 Weaker notions of convergence

Master theorem showed that the fourier series is Abel summable. Now create similar kernels for finite fourier series (Dirichlet kernel) and averages of first n terms (Fejér kernel). Fejér kernels form a Dirac sequence, hence the fourier series is Cesàro summable.

5 Orthonormal basis for $C(T)$

Proved exponential polynomials dense in $C(T)$: two different proofs using Poisson and Fejér kernels (Fejér kernel gives explicit approximation). Two line proof using Stone-Weirstrass, and a much nastier proof using Weirstrass approximation.

6 Strengthening the conditions on f

Used density result to show $\lim\limits_{n\to\infty} \hat{f}(n) = 0$ for a continuous f (Riemann-Lebesgue lemma). Proved the principle of localisation. Then showed that if $\hat{f}(n)$ is $O\left(\frac{1}{n}\right)$, then the fourier series converges. Subsequently showed that if $f \in C^1(T)$, then $\hat{f}(n)$ is $o\left(\frac{1}{n}\right)$

7 Computing the zeta function for positive even integers

Used the result that $x^2 \in C^1(T)$, hence it's fourier series converges at $x = 0$ to get

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{8}$$

And a consequence of this result is that $\zeta(2) = \frac{\pi^2}{6}$. Similarly, by computing the fourier coefficients of x^{2k}, one can compute $\zeta(2k)$.

8 Alternative formula for $\zeta(s)$ where $s > 1$

An alternative formula for $\zeta(s)$ when $s > 1$ is given by

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}$$

where $\mathcal{P}$ is the set of prime numbers.

One can prove this using the fundamental theorem of arithmetic.

9 Proving Weirstrass approximation theorem from Fejér's theorem

From Fejér's theorem we got that trigonometric polynomials are dense in $C(T)$. It will then suffice to show that $\cos(n\theta)$ and $\sin(n\theta)$ can be approximated using polynomials on T. Consider the m^{th} Taylor polynomial for these functions and look at the remainder. For sufficiently large m, the remainder can be made smaller than a given $\varepsilon > 0$. This gives a polynomial approximation for the function and proves the proposition.

10 Showing $\bar{z}$ cannot be uniformly approximated by polynomials in z in a compact set in $\mathbb{C}$

Without loss of generality, assume the compact set in question contains the unit circle (if it doesn't, rescaling and translation should do the trick). Now assume some polynomial p ε approximates $\bar{z}$ where $\varepsilon < 0.5$. In that case

$$|zp(z) - z\bar{z}| < |z|\varepsilon$$

In particular, on the unit circle, the inequality reduces to

$$|zp(z) - 1| < \varepsilon$$

This would mean for all points x on the circle, the real part of $xp(x)$ lies between 0.5 and 1.5. But notice that if take $2(n!)$ equally spaced points on the circle, where n is the degree of $zp(z)$, then the sum of $zp(z)$ over those points is 0, which means the real part of $zp(z)$ on at least one of those points must be less than or equal to 0. We have a contradiction.

11 Poisson summation of the normal distribution

The task was to show

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2} + i\lambda x\right) dx = \exp\left(-\frac{\lambda^2}{2}\right)$$

Denote the value of the integral by $I(\lambda)$. The integral can be evaluated by first noting the imaginary part of the function inside the integral is odd; it goes to 0. Performing integration by parts on the real part of the function, one notices that $I(\lambda)$ satisfies the following differential equation:

$$\frac{dI}{d\lambda} = -\lambda I$$

This gives the required expression.

12 Hausdorff moment theorem

The Hausdorff moment theorem states that if for continuous function f and g and a compact interval I, the following equation holds:

$$\int_I x^n f(x) dx = \int_I x^n g(x) dx$$

for all non-negative integers n, then $f \equiv g$ on I.

This problem is equivalent to showing $k \equiv 0$ where $k = g - f$ which can be done by showing

$$\int_I k^2(x)dx = 0$$

Let p be an ε polynomial approximation of k. Then

$$\left| \int_I k^2(x)dx - \int_I p(x)k(x)dx \right| < |I|\varepsilon$$

But by the hypothesis, $\int_I p(x)k(x)dx = 0$ for all polynomials p. This completes the proof.

13 Filler

Fill missing stuff here

14 Roth's Theorem

In very loose terms, Roth's theorem states that for a given number $0 < \delta \leq 1$, also called density, there exists a natural number N such that any subset A of $[N]$[1] which has cardinality more than or equal to δN contains a three term AP.

Clearly, if the density is 1, then the statement is quite obvious. What the following proof does is given a set $[N]$ and its subset A, it tries to find a large enough subset N' of $[N]$ such that N' is also an AP, and the density of $A \cap N'$ in N' is more than δ by some fixed multiplicative factor. Once we have this capability, we iterate until the density of A in some subprogression exceeds 1, in which case we're done.

What we do first is identify the set $[N]$ with the group $\mathbb{Z}/N\mathbb{Z}$ in the natural way. This way, we can also identify the subset A as the subset of the group. Consider the function $\mathbf{1}_A$ from $\mathbb{Z}/N\mathbb{Z}$ to $\mathbb{C}$ which is 1 is the argument is in A, otherwise 0. Now we can do some fourier analysis since we have an L_2 function.

Consider all the elements x, y, z of A such that $x + z = 2y$. Clearly, these form a 3-AP in $\mathbb{Z}/N\mathbb{Z}$. These $\mathbb{Z}/N\mathbb{Z}$ progressions need not be $\mathbb{Z}$ progressions. But let's find a way of counting these $\mathbb{Z}/N\mathbb{Z}$ progressions nevertheless.

Consider the following function:

$$f(x, y, z) = \frac{1}{N} \sum_{k=0}^{N-1} \exp\left(-\frac{2\pi i}{N}(x - 2y + z)k\right)$$

Clearly, this function will be 1 if (x, y, z) form a $\mathbb{Z}/N\mathbb{Z}$ progression, otherwise it will be 0. Hence the number of such progressions is given by:

$$\begin{aligned} S_0 &= \sum_{x,y,z \in A} f(x, y, z) \\ &= \sum_{x,y,z \in A} \frac{1}{N} \sum_{k=0}^{N-1} \exp\left(-\frac{2\pi i}{N}(x - 2y + z)k\right) \\ &= \sum_{x,y,z \in [N]} \mathbf{1}_A(x)\mathbf{1}_A(y)\mathbf{1}_A(z) \frac{1}{N} \sum_{k=0}^{N-1} \exp\left(-\frac{2\pi i}{N}(x - 2y + z)k\right) \end{aligned}$$

Since the sums are finite, with appropriate rearrangement, we get:

$$S_0 = \frac{1}{N} \sum_{k=0}^{N-1} \widehat{\mathbf{1}_A}(k)^2 \widehat{\mathbf{1}_A}(-2k)$$

[1] $[N]$ is shorthand for the set $\{0, 1, \ldots, N - 1\}$.

where $\widehat{f}(k)$ is the k^{th} fourier coefficient of f, which is defined by:

$$\widehat{f}(k) = \sum_{x=0}^{N-1} f(x) \exp\left(-\frac{2\pi i}{N} kx\right)$$

It is then obvious that $\widehat{\mathbf{1}_A}(0) = \delta N$. The original sum then becomes:

$$S_0 = \delta^3 N^2 + \frac{1}{N} \sum_{k=1}^{N-1} \widehat{\mathbf{1}_A}(k)^2 \widehat{\mathbf{1}_A}(-2k)$$

Without loss of generality, assume that N is odd so that $-2k$ is never 0, as k goes from 1 to $N-1$. If the $\widehat{\mathbf{1}_A}(k)$ are bounded by a small enough number, then perhaps we can say something about the number of $\mathbb{Z}/N\mathbb{Z}$ progressions in A. Notice that the sum S_0 also counts the trivial progressions. Hence the number of non-trivial progressions is $S_0 - \delta N$.

Clearly, the bound on $\widehat{\mathbf{1}_A}(k)$ depends upon $\mathbf{1}_A$, which in turn depends upon the set A. We call a set ε-uniform if $\widehat{\mathbf{1}_A}(k) \leq \varepsilon N$ for all non-zero k. Also, define M_A to be the set $\left[\frac{N}{3}, \frac{2N}{3}\right) \cap A$. Clearly, if x and y belong to M_A, and x, y, z form a $\mathbb{Z}/N\mathbb{Z}$ progression, then they also form a $\mathbb{Z}$ progression.

Now assume A is ε-uniform for $\varepsilon < \frac{\delta^2}{8}$ and if $|M_A| \geq \frac{\delta N}{4}$, then the number of $\mathbb{Z}$ progressions S is atleast $\frac{\delta^3 N^2}{32}$. Doing the same thing as we did for computing S_0, we get:

$$\begin{aligned} S &\geq \frac{1}{N} \sum_{k=0}^{N-1} \widehat{\mathbf{1}_{M_A}}(k) \widehat{\mathbf{1}_A}(k) \widehat{\mathbf{1}_{M_A}}(-2k) \\ &= \delta |M_A|^2 + \frac{1}{N} \sum_{k=1}^{N-1} \widehat{\mathbf{1}_{M_A}}(k) \widehat{\mathbf{1}_A}(k) \widehat{\mathbf{1}_{M_A}}(-2k) \end{aligned}$$

Let's try to bound the second term in the sum above.

$$\begin{aligned} \left| \sum_{k=1}^{N-1} \widehat{\mathbf{1}_{M_A}}(k) \widehat{\mathbf{1}_A}(k) \widehat{\mathbf{1}_{M_A}}(-2k) \right| &\leq \varepsilon N \sum_{k=1}^{N-1} \left| \widehat{\mathbf{1}_{M_A}}(k) \widehat{\mathbf{1}_{M_A}}(-2k) \right| \\ &\leq \varepsilon N \left(\sum_{k=1}^{N-1} \left| \widehat{\mathbf{1}_{M_A}}(k) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N-1} \left| \widehat{\mathbf{1}_{M_A}}(-2k) \right|^2 \right)^{\frac{1}{2}} && \text{(Cauchy-Schwarz inequality)} \\ &\leq \varepsilon N \sum_{k=0}^{N-1} \left| \widehat{\mathbf{1}_{M_A}}(k) \right|^2 \\ &= \varepsilon N^2 \sum_{k=0}^{N-1} |\mathbf{1}_{M_A}(k)|^2 && \text{(Plancherel's equality)} \\ &= \varepsilon N^2 |M_A| \end{aligned}$$

Now we use the inequalities we took for our hypotheses, and we get:

$$S \geq \delta |M_A|^2 - \varepsilon N |M_A| \geq \frac{\delta^3 N^2}{32}$$

However, note that S also counts the trivial progressions. There are exactly δN trivial progressions. Hence if $\frac{\delta^3 N^2}{32} - \delta N > 0$, only then will the set contain a 3-AP. That means if $N > \frac{32}{\delta^2}$, then the set contains a three term AP.

We have shown that if $N > \frac{32}{\delta^2}$, A is ε-uniform for $\varepsilon < \frac{\delta^2}{8}$, and $|M_A| \geq \frac{\delta N}{4}$, then A contains a three term $\mathbb{Z}$ progression.

On the contrary, assume that A does not contain a 3-AP. Then one of the above conditions must be unsatisfied. We can ignore the first condition, since we can make N as large as we want, given a fixed δ. That leaves the other two conditions. We'll show if either of those conditions are unsatisfied, then $[N]$ contains a sub-progression P, such that the $|P| \geq \frac{\delta^2 \sqrt{N}}{256}$ such that the density of $A \cap P$ is more than $\delta + \frac{\delta^2}{64}$.

If $|M_A| < \frac{\delta N}{4}$, then consider the sets L_A and R_A defined as follows:

$$L_A = A \cap \left[0, \frac{N}{3}\right)$$
$$R_A = A \cap \left[\frac{2N}{3}, N\right)$$

The density of A in one of these sets will be greater than or equal to $\frac{9\delta}{8}$, and these sets are subprogressions of $[N]$ of length $\frac{N}{3}$. This proves the theorem if $|M_A| < \frac{\delta N}{4}$.

Now consider what happens if the second condition is not satisfied, i.e. if $|\widehat{\mathbf{1}_A}(r)| \geq \varepsilon N$, where $\varepsilon = \frac{\delta^2}{8}$ for some r. We'll need the following lemma to progress:

Lemma 14.1. *If $|\widehat{\mathbf{1}_A}(r)| \geq \varepsilon N$, then there exists a non-overlapping $\mathbb{Z}/N\mathbb{Z}$ progression B such that $|B| > \frac{\sqrt{N}}{4}$ such that the density of $A \cap B$ in B is $\delta + \frac{\varepsilon}{4}$.*

Proof. It's rather long, and I don't feel like writing it. It uses a bit of fourier analysis though. I'll write it formally soon. □

That does it, doesn't it?