Notes on Homotopy Theory

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Contents

1 Categorical preliminaries

In this section, we'll define the categories we'll be dealing with in the rest of the notes. We'll also define some categorical constructions: in particular the *pushout* and the *pullback*.

1.1 Some important categories

SET**:** This is the category of sets, where the objects are sets, and the morphisms between objects are set maps.

TOP**:** This is the category of topological spaces, where the objects are topological spaces, and the maps are continuous maps between topological spaces.

hTOP**:** This is the category with the objects being topological spaces, but the maps are homotopy classes of continuous maps, rather than being continuous maps themselves.

TOP^0**:** This is the category of pointed spaces, i.e. the objects are tuples of spaces and a basepoint in them, and morphisms are continuous maps that take basepoints to basepoints.

hTOP^0**:** This is the homotopy category of pointed spaces, i.e. the objects are the same as in TOP^0, but the maps are homotopy classes of maps between pointed spaces.

$\mathrm{TOP}(2)$**:** This is the category of pairs of spaces. The objects here are (X, A), where $A \subset X$, and a morphism from (X, A) to (Y, B) is a continuous map $f : X \to Y$ such that $f(A) \subset B$.

$W(X, Y)$**:** Here, X and Y are two topological spaces. The objects of $W(X, Y)$ are the continuous maps between X and Y, and the morphisms are homotopies between maps.

TOP_B**:** Given a fixed topological space B, an object in the category TOP_B is a topological space X along with a map $f : X \to B$. Given two objects $(X, f : X \to B)$ and $(Y, g : Y \to B)$, a morphism from the former to the latter is a continuous map h from X to Y such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & B \\ & Y \arrow[u, "g"'] \end{tikzcd}$$

This is the *category of spaces over* B.

hTOP_B**:** This is the homotopy category of TOP_B, where the objects are the same, but the maps are quotiented out by homotopies.

TOP^A**:** Given a fixed topological space A, an object in the category TOP^A is a space X along with a map $f : A \to X$. Given two objects $(X, f : A \to X)$ and $(Y, g : A \to Y)$, a morphism between these objects is a map $h : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[dl, "h"] \\ Y & \end{tikzcd}$$

This is the *category of spaces under A*.

hTOP^A**:** This is the homotopy category of TOP^A, described in a manner similar to hTOP_B.

1.2 Categorical constructions

1.2.1 Product

Definition 1.1. Given two objects A and B in a category $\mathcal{C}$, their product is an object $A \times B$ along with maps $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for any object F with maps $f_1 : F \to A$ and $f_2 : F \to B$, there exists a unique map from F to $A \times B$ making the following diagram commute.

$$\begin{tikzcd} & F \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Products may not exist in all categories, but when they do, they are unique. They exist in SET and TOP, are the usual cartesian product.

1.2.2 Coproduct

Definition 1.2. In a category $\mathcal{C}$, the coproduct of objects A and B is the object $A \coprod B$ along with maps $i_1 : A \to A \coprod B$ and $i_2 : B \to A \coprod B$ such that for any pair of maps $g_1 : A \to G$ and $g_2 : B \to G$, there exists a unique factorization via $A \coprod B$.

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[dr, "g_1"'] & A \coprod B \arrow[d, dashed, "\exists!"] & B \arrow[l, "i_2"'] \arrow[dl, "g_2"] \\ & G & \end{tikzcd}$$

Coproducts exists in SET and TOP and are the disjoint union in these two categories. In TOP^0, the coproduct is the wedge sum along the basepoint.

1.2.3 Pullback

Definition 1.3. In a category $\mathcal{C}$, given two maps $f : X \to B$ and $g : Y \to B$, the pullback of f and g is the following diagram

$$\begin{tikzcd}
W \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & B
\end{tikzcd}$$

along with the universal property that for any V with maps F_V and G_V to X and Y, F_V and G_V factor uniquely through W.

$$\begin{tikzcd}
V \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "F_V"] \arrow[ddr, bend right, "G_V"'] & & \\
& W \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & B
\end{tikzcd}$$

In TOP, the pullback exists, and is given by the following subspace.

$$W = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

Alternatively, a pullback can be shown to be the product in the category TOP_B.

1.2.4 Pushout

Definition 1.4. A pushout is the dual notion to a pullback. Given a category $\mathcal{C}$, and maps $f : A \to X$ and $g : A \to Y$, the pushout of f and g is the following diagram.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "G"] \\
Y \arrow[r, "F"'] & W
\end{tikzcd}$$

W must also satisfy the following universal property.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "G"] \arrow[ddr, bend left, "G_V"] & \\
Y \arrow[r, "F"'] \arrow[drr, bend right, "F_V"'] & W \arrow[dr, dashed, "\exists!"] & \\
& & V
\end{tikzcd}$$

In TOP, the pushout W is the following space.

$$W = \frac{(X \coprod Y)}{f(a) \sim g(a)}$$

Alternatively, a pushout can be seen as a coproduct in the category TOP^A.

2 Homotopical Constructions

In this section, we'll cover the construction of the essential groups and spaces in homotopy theory: the homotopy groupoid, mapping cylinder, cones, suspensions, and loop spaces.

2.1 Homotopy groupoid

Definition 2.1. Let X and Y be topological spaces. The category $\Pi(X, Y)$ has its objects as maps from X to Y, and its morphisms are homotopies between maps quotiented by the following relation. Two homotopies between maps f and g, $\mathcal{P}$ and $\mathcal{Q}$ are the same morphism if there is a homotopy $\mathcal{M}$ from $\mathcal{P}$ to $\mathcal{Q}$ relative to [1] $X \times \partial I$.

The quotienting gives the collection of morphisms a groupoid structure. In particular, associativity only works out because of the quotienting. The fundamental groupoid is a special case of the homotopy groupoid $\Pi(X, Y)$, when X is just a point. Similarly, we can describe the pointed version of the homotopy groupoid, which we denote by $\Pi^0(X, Y)$ for pointed spaces X and Y.

2.2 Mapping cylinder

Definition 2.2. Given a map $f : X \to Y$, the mapping cylinder $Z(f)$ is constructed via the following pushout.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_1^X"'] & Y \arrow[d, "J"] \\ X \times I \arrow[r, "a"'] & Z(f) \end{tikzcd}$$

Topologically, the mapping cylinder is the disjoint union of $X \times I$ and Y quotiented with the relation $(x, 1) \sim f(x)$.

We construct some more maps.

$$\begin{aligned} q &: Z(f) \to Y \\ q(x, t) &:= f(x) \\ q(y) &:= y \end{aligned}$$

$$\begin{aligned} j &: X \to Z(f) \\ j(x) &:= (x, 0) \end{aligned}$$

[1] A homotopy relative to a subspace is a homotopy that is constant on that subspace.

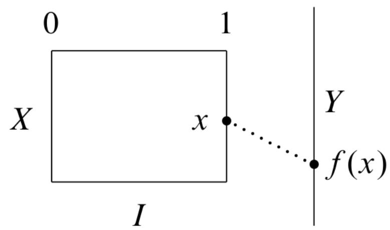

Figure 1: The mapping cylinder (Temporary. Put citation.)

We now have the following relations.

$$qj = f$$
$$qJ = \mathrm{id}_Y$$

We can also see the map Jq is homotopic to $\mathrm{id}_{Z(f)}$ relative to the Y subspace. This means $Z(f)$ is homotopy equivalent to Y and q and J are the homotopy equivalence. Note that j is a closed embedding. We have thus decomposed f into a closed embedding j, and a homotopy equivalence q.

2.3 Suspension

Definition 2.3. content...

3 Generalized (Co)homology Theories

3.1 Correspondence between unreduced and reduced homology theory

Definition 3.1 (Unreduced homology)**.** An unreduced homology theory is a functor h_* from the category hCW^2 to the category of $\mathbb{Z}$ graded abelian groups $\mathbb{Z} - \mathrm{Ab}$ such that h_* along with the following natural transformations.

$$\partial_* : h_* \to h_{*-1} \circ I$$

Here, I sends (X, A) to $(A, \varnothing)$. The funtor must satisfy the following axioms.

Homotopy equivalence (H): If two spaces in hCW^2 are homotopy equivalent via a map of pairs f, then $h_*(f)$ is a natural isomorphism.

Exactness (E): We have the following long exact sequence.

$$\cdots \longrightarrow h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X, A) \xrightarrow{\partial_*} h_{n-1}(A) \longrightarrow \cdots$$

Excision (A): For subcomplexes A and B, the map $i : (A, A \cap B) \hookrightarrow (A \cup B, B)$ induces an isomorphism from $h_*(A, A \cap B) \to h_*(A \cup B, B)$.

Direct union (DV): For any indexing set Ω, and a space $X = \coprod_{\alpha\in\Omega} X_\alpha$, the induced maps $h_*(X_\alpha) \to h_*(X)$ induce the following isomorphism.

$$\bigoplus_{\alpha\in\Omega} h_*(X_\alpha) \to h_*(X)$$

Definition 3.2 (Reduced homology)**.** A reduced homology theory is a functor $\widetilde{h}_*$ from hCW^0 to $\mathbb{Z}$ graded abelian groups along with the following natural transformation s_*.

$$\widetilde{h}_*(X) \to \widetilde{h}_{*+1}(SX)$$

This functor satisfies the following axioms.

Homotopy invariance (H): A pointed homotopy between spaces induces isomorphisms in the homology groups.

Suspension (S): The map s_* is a natural isomorphism.

Exactness (E): If $i : A \hookrightarrow X$ is a cofibration, then the following sequence is exact.

$$\widetilde{h}_*(A) \to \widetilde{h}_*(X) \to \widetilde{h}_*\left(X/A\right)$$

Direct union (DV): If $X = \bigvee_\alpha X_\alpha$, then the inclusions induce the following isomorphism.

$$\bigoplus_{\alpha} \widetilde{h}_*(X_\alpha) \to \widetilde{h}_*(X)$$

Unreduced and reduced cohomology are defined in a similar manner, by using contravariant functors and reversing all the arrows.