Proofs of some fairly technical results

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1 Comparison theorem for cohomology theories

Fill in later

2 Brown's representability theorem

In this section, we shall see that all reduced cohomology theories that satisfy the wedge sum (DV) axiom are representable functors, i.e. they are naturally isomorphic to the hom functor in the homotopy category hCW_*. In particular, for a given reduced cohomology theory h^*, we'll construct a sequence of spaces $\mathcal{Z}(n)$, which we'll call a spectrum, such that $h^n(X)$ is naturally isomorphic to $[X, \mathcal{Z}(n)]$.

2.1 Spectra and cohomology theories

Definition 2.1 (Spectrum)**.** A spectrum is a $\mathbb{Z}$ indexed sequence of pointed spaces $\mathcal{Z}(n)$ together with structure maps $\sigma_n : \Sigma\mathcal{Z}(n) \to \mathcal{Z}(n+1)$. If the adjoints of the structure maps, i.e. the maps $\widetilde{\sigma}_n : \mathcal{Z}(n) \to \Omega\mathcal{Z}(n+1)$ are homotopy equivalences, then the spectrum is called an Ω-spectrum.

Proposition 2.1. *Given a spectrum $\mathcal{Z}$, one can define the following functor.*

$$h^n(X; \mathcal{Z}) = \underset{k\to\infty}{\underrightarrow{\mathrm{colim}}} \left[S^k \wedge X, \mathcal{Z}(k+n)\right]$$

This is a contravariant functor which satisfies the homotopy invariance (H)*, suspension* (S)*, and exactness* (E) *axiom. Furthermore, if $\mathcal{Z}$ is an Ω-spectrum, the functor also satisfies the* (DV) *axiom. It is therefore a reduced cohomology theory.*

Proof. We'll deal with the axioms one at a time.

Homotopy invariance (H): This is obvious, because we are working in the homotopy category hCW_*. Any two continuous maps which are homotopy equivalent are the same morphism in this category, hence correspond to the same map to $\mathcal{Z}(k+n)$. And since the maps are same for all k, they also agree in the colimit.

Suspension (S): We need to show there is a natural isomorphism from $h^n(X)$ to $h^{n+1}(\Sigma X)$. To see this natural isomorphism, note that $h^n(X)$ is the colimit of the following diagram.

$$[X, \mathcal{Z}(n)] \to [\Sigma X, \mathcal{Z}(n+1)] \to [\Sigma^2 X, \mathcal{Z}(n+2)] \to \cdots$$

But for a diagram like this, the colimit won't change if we drop finitely many groups from the beginning of the diagram. Hence, $h^n(X)$ is also the colimit of the following diagram.

$$[\Sigma X, \mathcal{Z}(n+1)] \to [\Sigma^2 X, \mathcal{Z}(n+2)] \to [\Sigma^3 X, \mathcal{Z}(n+3)] \to \cdots$$

Similarly, $h^{n+1}(\Sigma X)$ is the colimit of the following diagram.

$$[\Sigma X, \mathcal{Z}(n+1)] \to [\Sigma^2 X, \mathcal{Z}(n+2)] \to [\Sigma^3 X, \mathcal{Z}(n+3)] \to \cdots$$

But now we notice that the groups are the same in the diagram for $h^n(X)$ and $h^{n+1}(\Sigma X)$. We draw isomorphisms between the corresponding groups, and that gives an natural isomorphism in the colimit.

Exactness (E): We need to show for any cofibration $i : A \hookrightarrow X$, the following sequence is exact.

$$h^n(A) \leftarrow h^n(X) \leftarrow h^n\left(X/A\right)$$

Using the cofibre sequence, we get that following sequences are exact for all k.

$$[\Sigma^k A, \mathcal{Z}(n+k)] \leftarrow [\Sigma^k X, \mathcal{Z}(n+k)] \leftarrow \left[\Sigma^k\left(X/A\right), \mathcal{Z}(n+k)\right]$$

And since filtered colimits of exact sequences are exact in AbGrp, the cohomology sequence is exact. (Should check this by hand in this particular case.)

Wedge sum (DV): Using the fact that suspension and loop space functors are adjoints, we have the following.

$$[\Sigma^k X, \mathcal{Z}(n+k)] \cong [X, \Omega^k \mathcal{Z}(n+k)]$$

This means that $h^n(X)$ is the colimit of the following diagram.

$$[X, \mathcal{Z}(n)] \to [X, \Omega\mathcal{Z}(n+1)] \to [X, \Omega^2\mathcal{Z}(n+2)] \to \cdots$$

If $\mathcal{Z}$ is an Ω-spectrum, then all the arrows in the above diagram are isomorphisms. That means $h^n(X) = [X, \mathcal{Z}(n)]$. But the functor $[\cdot, \mathcal{Z}(n)]$ satisfies (DV) axiom. That means h^* satisfies (DV) axiom.

□

A Definitions and notation

Definition A.1 (Suspension of a pointed space)**.** The suspension ΣX of a pointed space X is the smash product $S^1 \wedge X$.

Definition A.2 (Loop space of a pointed space)**.** The loop spaces ΩX of a pointed space X is the set of all pointed maps from S^1 to X with the compact-open topology.

B Some useful lemmas and theorems

Note: Although we state many of the lemmas here for TOP, they are also true for TOP_*, and the proof is similar.

Lemma B.1. *If $i : A \hookrightarrow X$ is a cofibration (in the category TOP), then the mapping cone $C(i)$ is homotopy equivalent to X/A.*

Proof. We will first construct the maps to and from $C(i)$ to X/A. The maps from $C(i)$ to X/A is the map that collapses the cone of A to a point corresponding to A in X/A. Now consider a map from $H : A \times I$ to $C(i)$, such that H contracts A to a point in $C(i)$, starting from the inclusion of A in X. Let the map j from X to $C(i)$ be the inclusion map. Since i is a cofibration, we can extend H with the initial condition j to a map $J : X \times I \to C(i)$. But $J(\cdot, 1)$ collapses A to a point. That means it factors through a X/A. This gives us a map k from X/A to $C(i)$.

The fact that these maps are homotopy inverses can be verified using the homotopy J. (Not sure of this. Verify later.) □

Lemma B.2. *In the category TOP, the following sequence is h-coexact.*

$$A \xrightarrow{f} B \xrightarrow{i} C(f)$$

That means for any space Z, the following sequence of abelian groups is exact.

$$[A, Z] \leftarrow [B, Z] \leftarrow [C(f), Z]$$

Proof. If an element $[c] \in [B, Z]$ goes to 0 in $[A, Z]$, that means $c \circ f : A \to Z$ is nullhomotopic, where c is a representative of $[c]$. But that means there is some function $d \in C(f)$ such that $c = d \circ i$. This shows the exactness of the sequence. □

Lemma B.3. *If K is a compact space, let A_i be a sequence of spaces where points are closed, and A is the colimit of the following diagram:*

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots$$

where all the embeddings are closed, then a map from K to A factors finitely through some A_i.

Proof. Let $J = f(K)$ be the compact image of K in A. For each set $A_i \setminus A_{i-1}$, pick an element c_i of J in the set, if J intersects $A_i \setminus A_{i-1}$. Since A_i's are closed, that means the subset c_i has the discrete topology. Furthermore, since points are closed, the set $\{\cup c_i\}$ is a closed subset of J, hence compact. And compact spaces with discrete topology are finite. That means only finitely many $A_i \setminus A_{i-1}$ intersect J. This means the map factors through at some finite stage. □

Theorem B.4 (Cofibre sequence)**.** *Put in result*