Notes on Homotopy Theory

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1 Categorical preliminaries

In this section, we'll define the categories we'll be dealing with in the rest of the notes. We'll also define some categorical constructions: in particular the *pushout* and the *pullback*.

1.1 Some important categories

- SET: This is the category of sets, where the objects are sets, and the morphisms between objects are set maps.
- TOP: This is the category of topological spaces, where the objects are topological spaces, and the maps are continuous maps between topological spaces.
- hTOP: This is the category with the objects being topological spaces, but the maps are homotopy classes of continuous maps, rather than being continuous maps themselves.
- TOP^0 : This is the category of pointed spaces, i.e. the objects are tuples of spaces and a basepoint in them, and morphisms are continuous maps that take basepoints to basepoints.
- $hTOP^0$: This is the homotopy category of pointed spaces, i.e. the objects are the same as in TOP^0 , but the maps are homotopy classes of maps between pointed spaces.
- TOP(2): This is the category of pairs of spaces. The objects here are (X,A), where $A \subset X$, and a morphism from (X,A) to (Y,B) is a continuous map $f:X \to Y$ such that $f(A) \subset B$.
- W(X,Y): Here, X and Y are two topological spaces. The objects of W(X,Y) are the continuous maps between X and Y, and the morphisms are homotopies between maps.
- TOP $_B$: Given a fixed topological space B, an object in the category TOP $_B$ is a topological space X along with a map $f:X\to B$. Given two objects $(X,f:X\to B)$ and $(Y,g:Y\to B)$, a morphism from the former to the latter is a continuous map h from X to Y such that the following diagram commutes.

$$X \xrightarrow{f} B$$

$$\downarrow g \uparrow$$

$$Y$$

This is the category of spaces over B.

- hTOP_B: This is the homotopy category of TOP_B , where the objects are the same, but the maps are quotiented out by homotopies.
- TOP^A : Given a fixed topological space A, an object in the category TOP^A is a space X along with a map $f:A\to X$. Given two objects $(X,f:A\to X)$ and $(Y,g:A\to Y)$,

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a morphism between these objects is a map $h:X\to Y$ such that the following diagram commutes.

$$\begin{array}{c}
A \xrightarrow{f} X \\
\downarrow g \\
Y
\end{array}$$

This is the category of spaces under A.

 $hTOP^A$: This is the homotopy category of TOP^A , described in a manner similar to $hTOP_B$.

1.2 Categorical constructions

1.2.1 Product

Definition 1.1. Given two objects A and B in a category C, their product is an object $A \times B$ along with maps $\pi_1: A \times B \to A$ and $\pi_2: A \times B \to B$ such that for any object F with maps $f_1: F \to A$ and $f_2: F \to B$, there exists a unique map from F to $A \times B$ making the following diagram commute.

$$A \stackrel{f_1}{\longleftarrow} A \times B \stackrel{f_2}{\longrightarrow} B$$

Products may not exist in all categories, but when they do, they are unique. They exist in SET and TOP, are the usual product.

1.2.2 Coproduct

Definition 1.2. In a category C, the coproduct of objects A and B is the object $A \coprod B$ along with maps $i_1: A \to A \coprod B$ and $i_2: B \to A \coprod B$ such that for any pair of maps $g_1: A \to G$ and $g_2: B \to G$, there exists a unique factorization via $A \coprod B$.

$$A \xrightarrow{i_1} A \coprod_{g_1} B \xleftarrow{i_2} B$$

$$\downarrow_{\exists !} g_2$$

$$G$$

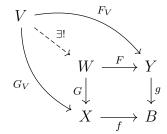
Coproducts exists in SET and TOP and are the disjoint union in these two categories. In TOP^0 , the coproduct is the wedge sum along the basepoint.

1.2.3 Pullback

Definition 1.3. In a category C, given two maps $f: X \to B$ and $g: Y \to B$, the pullback of f and g is the following diagram

$$\begin{array}{ccc} W & \xrightarrow{F} Y \\ G \downarrow & & \downarrow g \\ X & \xrightarrow{f} B \end{array}$$

along with the universal property that for any V with maps F_V and G_V to X and Y, F_V and G_V factor uniquely through W.



In TOP, the pullback exists, and is given by the following subspace.

$$W = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

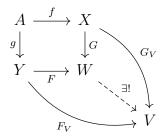
Alternatively, a pullback can be shown to be the product in the category TOP_B .

1.2.4 Pushout

Definition 1.4. A pushout is the dual notion to a pullback. Given a category C, and maps $f: A \to X$ and $g: A \to Y$, the pushout of f and g is the following diagram.

$$\begin{array}{ccc}
A & \xrightarrow{f} & X \\
g \downarrow & & \downarrow G \\
Y & \xrightarrow{F} & W
\end{array}$$

 ${\it W}$ must also satisfy the following universal property.



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In TOP, the pushout W is the following space.

$$W = \frac{(X \coprod Y)}{f(a) \sim g(a)}$$

Alternatively, a pushout can be seen as a coproduct in the category TOP^A .

2 Homotopical Constructions

In this section, we'll cover the construction of the essential spaces in homotopy theory: the mapping cylinder, cones, suspensions, and loop spaces; we'll cover the pointed and unpointed versions of these, and prove their homotopy equivalence whenever applicable.

2.1 Mapping cylinder

Definition 2.1. Given a map $f: X \to Y$, the mapping cylinder Z(f) is constructed via the following pushout.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_1^X \Big\downarrow & & \Big\downarrow J \\ X \times I & \xrightarrow{a} & Z(f) \end{array}$$

Topologically, the mapping cylinder is the disjoint union of $X \times I$ and Y quotiented with the relation $(x,1) \sim f(x)$.

Insert picture later