# Notes on Homotopy Theory

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# 1 Categorical preliminaries

In this section, we'll define the categories we'll be dealing with in the rest of the notes. We'll also define some categorical constructions: in particular the *pushout* and the *pullback*.

## 1.1 Some important categories

$\mathrm{SET}$: This is the category of sets, where the objects are sets, and the morphisms between objects are set maps.

$\mathrm{TOP}$: This is the category of topological spaces, where the objects are topological spaces, and the maps are continuous maps between topological spaces.

$\mathrm{hTOP}$: This is the category with the objects being topological spaces, but the maps are homotopy classes of continuous maps, rather than being continuous maps themselves.

$\mathrm{TOP}^0$: This is the category of pointed spaces, i.e. the objects are tuples of spaces and a basepoint in them, and morphisms are continuous maps that take basepoints to basepoints.

$\mathrm{hTOP}^0$: This is the homotopy category of pointed spaces, i.e. the objects are the same as in $\mathrm{TOP}^0$, but the maps are homotopy classes of maps between pointed spaces.

$\mathrm{TOP}(2)$: This is the category of pairs of spaces. The objects here are $(X, A)$, where $A \subset X$, and a morphism from $(X, A)$ to $(Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subset B$.

$W(X, Y)$: Here, $X$ and $Y$ are two topological spaces. The objects of $W(X, Y)$ are the continuous maps between $X$ and $Y$, and the morphisms are homotopies between maps.

$\mathrm{TOP}_B$: Given a fixed topological space $B$, an object in the category $\mathrm{TOP}_B$ is a topological space $X$ along with a map $f : X \to B$. Given two objects $(X, f : X \to B)$ and $(Y, g : Y \to B)$, a morphism from the former to the latter is a continuous map $h$ from $X$ to $Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & B \\ & Y \arrow[u, "g"'] \end{tikzcd}$$

This is the *category of spaces over* $B$.

$\mathrm{hTOP}_B$: This is the homotopy category of $\mathrm{TOP}_B$, where the objects are the same, but the maps are quotiented out by homotopies.

$\mathrm{TOP}^A$: Given a fixed topological space $A$, an object in the category $\mathrm{TOP}^A$ is a space $X$ along with a map $f : A \to X$. Given two objects $(X, f : A \to X)$ and $(Y, g : A \to Y)$, a morphism between these objects is a map $h : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[dl, "h"] \\ Y & \end{tikzcd}$$

This is the *category of spaces under $A$*.

**$\mathrm{hTOP}^A$:** This is the homotopy category of $\mathrm{TOP}^A$, described in a manner similar to $\mathrm{hTOP}_B$.

## 1.2 Categorical constructions

### 1.2.1 Product

**Definition 1.1.** Given two objects $A$ and $B$ in a category $\mathcal{C}$, their product is an object $A \times B$ along with maps $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for any object $F$ with maps $f_1 : F \to A$ and $f_2 : F \to B$, there exists a unique map from $F$ to $A \times B$ making the following diagram commute.

$$\begin{tikzcd} & F \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Products may not exist in all categories, but when they do, they are unique. They exist in $\mathrm{SET}$ and $\mathrm{TOP}$, are the usual product.

### 1.2.2 Coproduct

**Definition 1.2.** In a category $\mathcal{C}$, the coproduct of objects $A$ and $B$ is the object $A \coprod B$ along with maps $i_1 : A \to A \coprod B$ and $i_2 : B \to A \coprod B$ such that for any pair of maps $g_1 : A \to G$ and $g_2 : B \to G$, there exists a unique factorization via $A \coprod B$.

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[dr, "g_1"'] & A \coprod B \arrow[d, dashed, "\exists!"] & B \arrow[l, "i_2"'] \arrow[dl, "g_2"] \\ & G & \end{tikzcd}$$

Coproducts exists in $\mathrm{SET}$ and $\mathrm{TOP}$ and are the disjoint union in these two categories. In $\mathrm{TOP}^0$, the coproduct is the wedge sum along the basepoint.

### 1.2.3 Pullback

**Definition 1.3.** In a category $\mathcal{C}$, given two maps $f : X \to B$ and $g : Y \to B$, the pullback of $f$ and $g$ is the following diagram

$$\begin{tikzcd} W \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

along with the universal property that for any $V$ with maps $F_V$ and $G_V$ to $X$ and $Y$, $F_V$ and $G_V$ factor uniquely through $W$.

$$\begin{tikzcd} V \arrow[drr, bend left, "F_V"] \arrow[ddr, bend right, "G_V"'] \arrow[dr, dashed, "\exists!"] & & \\ & W \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & B \end{tikzcd}$$

In TOP, the pullback exists, and is given by the following subspace.

$$W = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

Alternatively, a pullback can be shown to be the product in the category $\mathrm{TOP}_B$.

### 1.2.4 Pushout

**Definition 1.4.** A pushout is the dual notion to a pullback. Given a category $\mathcal{C}$, and maps $f : A \to X$ and $g : A \to Y$, the pushout of $f$ and $g$ is the following diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "G"] \\ Y \arrow[r, "F"'] & W \end{tikzcd}$$

$W$ must also satisfy the following universal property.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "G"] \arrow[ddr, bend left, "G_V"] & \\ Y \arrow[r, "F"'] \arrow[drr, bend right, "F_V"'] & W \arrow[dr, dashed, "\exists!"] & \\ & & V \end{tikzcd}$$

In TOP, the pushout $W$ is the following space.

$$W = \frac{(X \coprod Y)}{f(a) \sim g(a)}$$

Alternatively, a pushout can be seen as a coproduct in the category $\mathrm{TOP}^A$.

# 2 Homotopical Constructions

In this section, we'll cover the construction of the essential spaces in homotopy theory: the mapping cylinder, cones, suspensions, and loop spaces; we'll cover the pointed and unpointed versions of these, and prove their homotopy equivalence whenever applicable.

## 2.1 Mapping cylinder

**Definition 2.1.** Given a map $f : X \to Y$, the mapping cylinder $Z(f)$ is constructed via the following pushout.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_1^X}\downarrow & & \downarrow{\scriptstyle J} \\ X \times I & \xrightarrow[a]{} & Z(f) \end{array}$$

Topologically, the mapping cylinder is the disjoint union of $X \times I$ and $Y$ quotiented with the relation $(x, 1) \sim f(x)$.

Insert picture later