

# Notes on Homotopy Theory

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# 1 Categorical preliminaries

In this section, we'll define the categories we'll be dealing with in the rest of the notes. We'll also define some categorical constructions: in particular the *pushout* and the *pullback*.

## 1.1 Some important categories

**SET:** This is the category of sets, where the objects are sets, and the morphisms between objects are set maps.

**TOP:** This is the category of topological spaces, where the objects are topological spaces, and the maps are continuous maps between topological spaces.

**hTOP:** This is the category with the objects being topological spaces, but the maps are homotopy classes of continuous maps, rather than being continuous maps themselves.

**TOP<sup>0</sup>:** This is the category of pointed spaces, i.e. the objects are tuples of spaces and a basepoint in them, and morphisms are continuous maps that take basepoints to basepoints.

**hTOP<sup>0</sup>:** This is the homotopy category of pointed spaces, i.e. the objects are the same as in TOP<sup>0</sup>, but the maps are homotopy classes of maps between pointed spaces.

**TOP(2):** This is the category of pairs of spaces. The objects here are  $(X, A)$ , where  $A \subset X$ , and a morphism from  $(X, A)$  to  $(Y, B)$  is a continuous map  $f : X \rightarrow Y$  such that  $f(A) \subset B$ .

**$W(X, Y)$ :** Here,  $X$  and  $Y$  are two topological spaces. The objects of  $W(X, Y)$  are the continuous maps between  $X$  and  $Y$ , and the morphisms are homotopies between maps.

**TOP<sub>B</sub>:** Given a fixed topological space  $B$ , an object in the category TOP<sub>B</sub> is a topological space  $X$  along with a map  $f : X \rightarrow B$ . Given two objects  $(X, f : X \rightarrow B)$  and  $(Y, g : Y \rightarrow B)$ , a morphism from the former to the latter is a continuous map  $h$  from  $X$  to  $Y$  such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ & \searrow h & \uparrow g \\ & & Y \end{array}$$

This is the *category of spaces over B*.

**hTOP<sub>B</sub>:** This is the homotopy category of TOP<sub>B</sub>, where the objects are the same, but the maps are quotiented out by homotopies.

**TOP<sup>A</sup>:** Given a fixed topological space  $A$ , an object in the category TOP<sup>A</sup> is a space  $X$  along with a map  $f : A \rightarrow X$ . Given two objects  $(X, f : A \rightarrow X)$  and  $(Y, g : A \rightarrow Y)$ ,

a morphism between these objects is a map  $h : X \rightarrow Y$  such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ g \downarrow & \swarrow h & \\ Y & & \end{array}$$

This is the *category of spaces under A*.

$\mathbf{hTOP}^A$ : This is the homotopy category of  $\mathbf{TOP}^A$ , described in a manner similar to  $\mathbf{hTOP}_B$ .

## 1.2 Categorical constructions

### 1.2.1 Product

**Definition 1.1.** Given two objects  $A$  and  $B$  in a category  $\mathcal{C}$ , their product is an object  $A \times B$  along with maps  $\pi_1 : A \times B \rightarrow A$  and  $\pi_2 : A \times B \rightarrow B$  such that for any object  $F$  with maps  $f_1 : F \rightarrow A$  and  $f_2 : F \rightarrow B$ , there exists a unique map from  $F$  to  $A \times B$  making the following diagram commute.

$$\begin{array}{ccccc} & & F & & \\ & f_1 \swarrow & \downarrow \exists! & \searrow f_2 & \\ A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \end{array}$$

Products may not exist in all categories, but when they do, they are unique. They exist in SET and TOP, are the usual cartesian product.

### 1.2.2 Coproduct

**Definition 1.2.** In a category  $\mathcal{C}$ , the coproduct of objects  $A$  and  $B$  is the object  $A \coprod B$  along with maps  $i_1 : A \rightarrow A \coprod B$  and  $i_2 : B \rightarrow A \coprod B$  such that for any pair of maps  $g_1 : A \rightarrow G$  and  $g_2 : B \rightarrow G$ , there exists a unique factorization via  $A \coprod B$ .

$$\begin{array}{ccccc} A & \xrightarrow{i_1} & A \coprod B & \xleftarrow{i_2} & B \\ & \searrow g_1 & \downarrow \exists! & \swarrow g_2 & \\ & & G & & \end{array}$$

Coproducts exist in SET and TOP and are the disjoint union in these two categories. In  $\mathbf{TOP}^0$ , the coproduct is the wedge sum along the basepoint.

### 1.2.3 Pullback

**Definition 1.3.** In a category  $\mathcal{C}$ , given two maps  $f : X \rightarrow B$  and  $g : Y \rightarrow B$ , the pullback of  $f$  and  $g$  is the following diagram

$$\begin{array}{ccc} W & \xrightarrow{F} & Y \\ G \downarrow & & \downarrow g \\ X & \xrightarrow{f} & B \end{array}$$

along with the universal property that for any  $V$  with maps  $F_V$  and  $G_V$  to  $X$  and  $Y$ ,  $F_V$  and  $G_V$  factor uniquely through  $W$ .

$$\begin{array}{ccccc} V & & \xrightarrow{F_V} & & Y \\ & \searrow \exists! & & \searrow & \\ & & W & \xrightarrow{F} & Y \\ & & G \downarrow & & \downarrow g \\ & & X & \xrightarrow{f} & B \\ & \swarrow G_V & & \swarrow & \\ & & & & \end{array}$$

In TOP, the pullback exists, and is given by the following subspace.

$$W = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

Alternatively, a pullback can be shown to be the product in the category  $\text{TOP}_B$ .

### 1.2.4 Pushout

**Definition 1.4.** A pushout is the dual notion to a pullback. Given a category  $\mathcal{C}$ , and maps  $f : A \rightarrow X$  and  $g : A \rightarrow Y$ , the pushout of  $f$  and  $g$  is the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ g \downarrow & & \downarrow G \\ Y & \xrightarrow{F} & W \end{array}$$

$W$  must also satisfy the following universal property.

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & & \\ g \downarrow & & \downarrow G & & \\ Y & \xrightarrow{F} & W & \xrightarrow{G_V} & V \\ & \searrow F_V & & \searrow \exists! & \\ & & & & \end{array}$$

In TOP, the pushout  $W$  is the following space.

$$W = \frac{(X \amalg Y)}{f(a) \sim g(a)}$$

Alternatively, a pushout can be seen as a coproduct in the category  $\text{TOP}^A$ .

## 2 Homotopical Constructions

In this section, we'll cover the construction of the essential groups and spaces in homotopy theory: the homotopy groupoid, mapping cylinder, cones, suspensions, and loop spaces.

### 2.1 Homotopy groupoid

**Definition 2.1.** Let  $X$  and  $Y$  be topological spaces. The category  $\Pi(X, Y)$  has its objects as maps from  $X$  to  $Y$ , and its morphisms are homotopies between maps quotiented by the following relation. Two homotopies between maps  $f$  and  $g$ ,  $\mathcal{P}$  and  $\mathcal{Q}$  are the same morphism if there is a homotopy  $\mathcal{M}$  from  $\mathcal{P}$  to  $\mathcal{Q}$  relative to  $^1 X \times \partial I$ .

The quotienting gives the collection of morphisms a groupoid structure. In particular, associativity only works out because of the quotienting. The fundamental groupoid is a special case of the homotopy groupoid  $\Pi(X, Y)$ , when  $X$  is just a point. Similarly, we can describe the pointed version of the homotopy groupoid, which we denote by  $\Pi^0(X, Y)$  for pointed spaces  $X$  and  $Y$ .

### 2.2 Mapping cylinder

**Definition 2.2.** Given a map  $f : X \rightarrow Y$ , the mapping cylinder  $Z(f)$  is constructed via the following pushout.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_1^X \downarrow & & \downarrow j \\ X \times I & \xrightarrow{a} & Z(f) \end{array}$$

Topologically, the mapping cylinder is the disjoint union of  $X \times I$  and  $Y$  quotiented with the relation  $(x, 1) \sim f(x)$ .

We construct some more maps.

$$\begin{aligned} q : Z(f) &\rightarrow Y \\ q(x, t) &:= f(x) \\ q(y) &:= y \end{aligned}$$

$$\begin{aligned} j : X &\rightarrow Z(f) \\ j(x) &:= (x, 0) \end{aligned}$$

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<sup>1</sup>A homotopy relative to a subspace is a homotopy that is constant on that subspace.

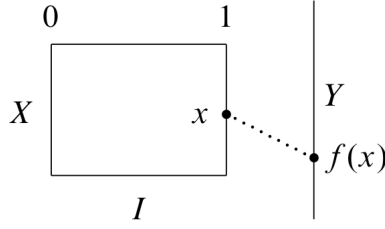


Figure 1: The mapping cylinder (Temporary. Put citation.)

We now have the following relations.

$$\begin{aligned} qj &= f \\ qJ &= \text{id}_Y \end{aligned}$$

We can also see the map  $Jq$  is homotopic to  $\text{id}_{Z(f)}$  relative to the  $Y$  subspace. This means  $Z(f)$  is homotopy equivalent to  $Y$  and  $q$  and  $J$  are the homotopy equivalence. Note that  $j$  is a closed embedding. We have thus decomposed  $f$  into a closed embedding  $j$ , and a homotopy equivalence  $q$ .

## 2.3 Suspension

**Definition 2.3.** content...

# 3 Generalized (Co)homology Theories

## 3.1 Correspondence between unreduced and reduced homology theory

**Definition 3.1** (Unreduced homology). An unreduced homology theory is a functor  $h_*$  from the category  $\text{hCW}^2$  to the category of  $\mathbb{Z}$  graded abelian groups  $\mathbb{Z} - \text{Ab}$  such that  $h_*$  along with the following natural transformations.

$$\partial_* : h_* \rightarrow h_{*-1} \circ I$$

Here,  $I$  sends  $(X, A)$  to  $(A, \emptyset)$ . The functor must satisfy the following axioms.

**Homotopy equivalence (H):** If two spaces in  $\text{hCW}^2$  are homotopy equivalent via a map of pairs  $f$ , then  $h_*(f)$  is a natural isomorphism.

**Exactness (E):** We have the following long exact sequence.

$$\dots \longrightarrow h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X, A) \xrightarrow{\partial_*} h_{n-1}(A) \longrightarrow \dots$$

**Excision (A):** For subcomplexes  $A$  and  $B$ , the map  $i : (A, A \cap B) \hookrightarrow (A \cup B, B)$  induces an isomorphism from  $h_*(A, A \cap B) \rightarrow h_*(A \cup B, B)$ .

**Direct union (DV):** For any indexing set  $\Omega$ , and a space  $X = \coprod_{\alpha \in \Omega} X_\alpha$ , the induced maps  $h_*(X_\alpha) \rightarrow h_*(X)$  induce the following isomorphism.

$$\bigoplus_{\alpha \in \Omega} h_*(X_\alpha) \rightarrow h_*(X)$$

**Definition 3.2** (Reduced homology). A reduced homology theory is a functor  $\tilde{h}_*$  from  $\text{hCW}^0$  to  $\mathbb{Z}$  graded abelian groups along with the following natural transformation  $s_*$ .

$$\tilde{h}_*(X) \rightarrow \tilde{h}_{*+1}(SX)$$

This functor satisfies the following axioms.

**Homotopy invariance (H):** A pointed homotopy between spaces induces isomorphisms in the homology groups.

**Suspension (S):** The map  $s_*$  is a natural isomorphism.

**Exactness (E):** If  $i : A \hookrightarrow X$  is a cofibration, then the following sequence is exact.

$$\tilde{h}_*(A) \rightarrow \tilde{h}_*(X) \rightarrow \tilde{h}_*(X/A)$$

**Direct union (DV):** If  $X = \bigvee_{\alpha} X_\alpha$ , then the inclusions induce the following isomorphism.

$$\bigoplus_{\alpha} \tilde{h}_*(X_\alpha) \rightarrow \tilde{h}_*(X)$$

Unreduced and reduced cohomology are defined in a similar manner, by using contravariant functors and reversing all the arrows.