

Notes on Homotopy Theory

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Contents

1	Categorical preliminaries	2
1.1	Some important categories	2
1.2	Categorical constructions	3
1.2.1	Product	3
1.2.2	Coproduct	3
1.2.3	Pullback	4
1.2.4	Pushout	4
2	Homotopical Constructions	5
2.1	Homotopy groupoid	5
2.2	Mapping cylinder	5
2.3	Suspension	6

1 Categorical preliminaries

In this section, we'll define the categories we'll be dealing with in the rest of the notes. We'll also define some categorical constructions: in particular the *pushout* and the *pullback*.

1.1 Some important categories

SET: This is the category of sets, where the objects are sets, and the morphisms between objects are set maps.

TOP: This is the category of topological spaces, where the objects are topological spaces, and the maps are continuous maps between topological spaces.

hTOP: This is the category with the objects being topological spaces, but the maps are homotopy classes of continuous maps, rather than being continuous maps themselves.

TOP⁰: This is the category of pointed spaces, i.e. the objects are tuples of spaces and a basepoint in them, and morphisms are continuous maps that take basepoints to basepoints.

hTOP⁰: This is the homotopy category of pointed spaces, i.e. the objects are the same as in TOP⁰, but the maps are homotopy classes of maps between pointed spaces.

TOP(2): This is the category of pairs of spaces. The objects here are (X, A) , where $A \subset X$, and a morphism from (X, A) to (Y, B) is a continuous map $f : X \rightarrow Y$ such that $f(A) \subset B$.

$W(X, Y)$: Here, X and Y are two topological spaces. The objects of $W(X, Y)$ are the continuous maps between X and Y , and the morphisms are homotopies between maps.

TOP_B: Given a fixed topological space B , an object in the category TOP_B is a topological space X along with a map $f : X \rightarrow B$. Given two objects $(X, f : X \rightarrow B)$ and $(Y, g : Y \rightarrow B)$, a morphism from the former to the latter is a continuous map h from X to Y such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ & \searrow h & \uparrow g \\ & & Y \end{array}$$

This is the *category of spaces over B*.

hTOP_B: This is the homotopy category of TOP_B, where the objects are the same, but the maps are quotiented out by homotopies.

TOP^A: Given a fixed topological space A , an object in the category TOP^A is a space X along with a map $f : A \rightarrow X$. Given two objects $(X, f : A \rightarrow X)$ and $(Y, g : A \rightarrow Y)$,

a morphism between these objects is a map $h : X \rightarrow Y$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ g \downarrow & \swarrow h & \\ Y & & \end{array}$$

This is the *category of spaces under A*.

\mathbf{hTOP}^A : This is the homotopy category of \mathbf{TOP}^A , described in a manner similar to \mathbf{hTOP}_B .

1.2 Categorical constructions

1.2.1 Product

Definition 1.1. Given two objects A and B in a category \mathcal{C} , their product is an object $A \times B$ along with maps $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that for any object F with maps $f_1 : F \rightarrow A$ and $f_2 : F \rightarrow B$, there exists a unique map from F to $A \times B$ making the following diagram commute.

$$\begin{array}{ccccc} & & F & & \\ & f_1 \swarrow & \downarrow \exists! & \searrow f_2 & \\ A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \end{array}$$

Products may not exist in all categories, but when they do, they are unique. They exist in SET and TOP, are the usual cartesian product.

1.2.2 Coproduct

Definition 1.2. In a category \mathcal{C} , the coproduct of objects A and B is the object $A \coprod B$ along with maps $i_1 : A \rightarrow A \coprod B$ and $i_2 : B \rightarrow A \coprod B$ such that for any pair of maps $g_1 : A \rightarrow G$ and $g_2 : B \rightarrow G$, there exists a unique factorization via $A \coprod B$.

$$\begin{array}{ccccc} A & \xrightarrow{i_1} & A \coprod B & \xleftarrow{i_2} & B \\ & \searrow g_1 & \downarrow \exists! & \swarrow g_2 & \\ & & G & & \end{array}$$

Coproducts exist in SET and TOP and are the disjoint union in these two categories. In \mathbf{TOP}^0 , the coproduct is the wedge sum along the basepoint.

1.2.3 Pullback

Definition 1.3. In a category \mathcal{C} , given two maps $f : X \rightarrow B$ and $g : Y \rightarrow B$, the pullback of f and g is the following diagram

$$\begin{array}{ccc} W & \xrightarrow{F} & Y \\ G \downarrow & & \downarrow g \\ X & \xrightarrow{f} & B \end{array}$$

along with the universal property that for any V with maps F_V and G_V to X and Y , F_V and G_V factor uniquely through W .

$$\begin{array}{ccccc} V & & \xrightarrow{F_V} & & Y \\ & \searrow \exists! & & \downarrow g & \\ & & W & \xrightarrow{F} & Y \\ & \swarrow G_V & \downarrow G & & \\ & & X & \xrightarrow{f} & B \end{array}$$

In TOP, the pullback exists, and is given by the following subspace.

$$W = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

Alternatively, a pullback can be shown to be the product in the category TOP_B .

1.2.4 Pushout

Definition 1.4. A pushout is the dual notion to a pullback. Given a category \mathcal{C} , and maps $f : A \rightarrow X$ and $g : A \rightarrow Y$, the pushout of f and g is the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ g \downarrow & & \downarrow G \\ Y & \xrightarrow{F} & W \end{array}$$

W must also satisfy the following universal property.

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & & \\ g \downarrow & & \downarrow G & & \\ Y & \xrightarrow{F} & W & \xrightarrow{G_V} & V \\ & \searrow F_V & & \swarrow \exists! & \\ & & & & \end{array}$$

In TOP, the pushout W is the following space.

$$W = \frac{(X \amalg Y)}{f(a) \sim g(a)}$$

Alternatively, a pushout can be seen as a coproduct in the category TOP^A .

2 Homotopical Constructions

In this section, we'll cover the construction of the essential groups and spaces in homotopy theory: the homotopy groupoid, mapping cylinder, cones, suspensions, and loop spaces.

2.1 Homotopy groupoid

Definition 2.1. Let X and Y be topological spaces. The category $\Pi(X, Y)$ has its objects as maps from X to Y , and its morphisms are homotopies between maps quotiented by the following relation. Two homotopies between maps f and g , \mathcal{P} and \mathcal{Q} are the same morphism if there is a homotopy \mathcal{M} from \mathcal{P} to \mathcal{Q} relative to $^1 X \times \partial I$.

The quotienting gives the collection of morphisms a groupoid structure. In particular, associativity only works out because of the quotienting. The fundamental groupoid is a special case of the homotopy groupoid $\Pi(X, Y)$, when X is just a point. Similarly, we can describe the pointed version of the homotopy groupoid, which we denote by $\Pi^0(X, Y)$ for pointed spaces X and Y .

2.2 Mapping cylinder

Definition 2.2. Given a map $f : X \rightarrow Y$, the mapping cylinder $Z(f)$ is constructed via the following pushout.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_1^X \downarrow & & \downarrow j \\ X \times I & \xrightarrow{a} & Z(f) \end{array}$$

Topologically, the mapping cylinder is the disjoint union of $X \times I$ and Y quotiented with the relation $(x, 1) \sim f(x)$.

We construct some more maps.

$$\begin{aligned} q : Z(f) &\rightarrow Y \\ q(x, t) &:= f(x) \\ q(y) &:= y \end{aligned}$$

$$\begin{aligned} j : X &\rightarrow Z(f) \\ j(x) &:= (x, 0) \end{aligned}$$

¹A homotopy relative to a subspace is a homotopy that is constant on that subspace.

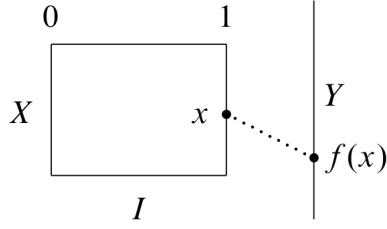


Figure 1: The mapping cylinder (Temporary. Put citation.)

We now have the following relations.

$$\begin{aligned} qj &= f \\ qJ &= \text{id}_Y \end{aligned}$$

We can also see the map Jq is homotopic to $\text{id}_{Z(f)}$ relative to the Y subspace. This means $Z(f)$ is homotopy equivalent to Y and q and J are the homotopy equivalence. Note that j is a closed embedding. We have thus decomposed f into a closed embedding j , and a homotopy equivalence q .

2.3 Suspension

Definition 2.3. content...