Proofs of some technical results

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Contents

1 Blakers-Massey Theorem

Fill in later

2 Comparison theorem for cohomology theories

Fill in later

3 Brown's representability theorem

In this section, we shall see that all reduced cohomology theories that satisfy the wedge sum (DV) axiom are representable functors, i.e. they are naturally isomorphic to the hom functor in the homotopy category hCW_*. In particular, for a given reduced cohomology theory h^*, we'll construct a sequence of spaces $\mathcal{Z}(n)$, which we'll call a spectrum, such that $\widetilde{h}^n(X)$ is naturally isomorphic to $[X, \mathcal{Z}(n)]$.

3.1 Spectra and cohomology theories

Definition 3.1 (Ω-Prespectrum)**.** A prespectrum is a $\mathbb{Z}$ indexed sequence of pointed spaces $\mathcal{Z}(n)$ together with structure maps $\sigma_n : \Sigma\mathcal{Z}(n) \to \mathcal{Z}(n+1)$. If the adjoints of the structure maps, i.e. the maps $\widetilde{\sigma}_n : \mathcal{Z}(n) \to \Omega\mathcal{Z}(n+1)$ are homotopy equivalences, then the prespectrum is called an Ω-prespectrum.

Proposition 3.1. *Given a Ω-prespectrum $\mathcal{Z}$, one can define the following functor.*

$$\widetilde{h}^n(X;\mathcal{Z}) = [X, \mathcal{Z}(n)]$$

This is a contravariant functor which satisfies the homotopy invariance (H), suspension (S), exactness (E), and the wedge sum (DV) axiom. It is therefore a reduced cohomology theory.

Proof. We'll deal with the axioms one at a time.

Homotopy invariance (H): This is obvious, because we are looking at homotopy classes of maps.

Suspension (S): We need to show there is a natural isomorphism from $\widetilde{h}^n(X)$ to $\widetilde{h}^{n+1}(\Sigma X)$. Note that the adjoint of the structure maps are homotopy equivalences. We therefore have a natural isomorphism.

$$[X, \mathcal{Z}(n)] \cong [X, \Omega\mathcal{Z}(n+1)]$$

On the other hand, since Σ are Ω are adjoints, we have the following natural isomorphism.

$$[X, \Omega\mathcal{Z}(n+1)] \cong [\Sigma X, \mathcal{Z}(n+1)]$$

Composing the two natural isomorphisms, we get our required isomorphism.

Exactness (E): We need to show for any cofibration $i : A \hookrightarrow X$, the following sequence is exact.

$$\widetilde{h}^n(A) \leftarrow \widetilde{h}^n(X) \leftarrow \widetilde{h}^n\left(X/A\right)$$

Using the cofiber sequence, we get that following sequence is exact.

$$[A, \mathcal{Z}(n)] \leftarrow [X, \mathcal{Z}(n)] \leftarrow [(X/A), \mathcal{Z}(n)]$$

Wedge sum (DV): The functor $[\cdot, \mathcal{Z}(n)]$ satisfies (DV) axiom. This is fairly easy to check. That means $\widetilde{h}^*$ satisfies (DV) axiom.

□

To construct easy examples of spectra, one needs to check that filtered colimits commute with the loop space functor, at least for nice enough spaces.

3.2 Proof of Brown's representability theorem

In the previous section, we saw that if we are given an Ω-prespectrum, we can construct a reduced cohomology theory using the prespectrum. Brown's representability theorem is the converse of the previous theorem, i.e. given a reduced cohomology theory which satisfies the (DV) axiom, it can be represented by an Ω-prespectrum, which is unique up to homotopy. This theorem is fairly technical, and will require the use of the theorem on Milnor exact sequence (theorem B.7).

Theorem 3.2 (Brown's representability theorem)**.** *Let $\widetilde{h}^*$ be a reduced cohomology theory satisfying the (DV) axiom. Then there is an Ω-prespectrum $\mathcal{Z}$ such that $\widetilde{h}^n$ is naturally isomorphic to $[\cdot, \mathcal{Z}(n)]$. Furthermore, this Ω-prespectrum is unique up to homotopy.*

Proof. The proof will have two main parts. The first part will involve constructing the spaces $\mathcal{Z}(n)$ for each n such that there is a natural isomorphism from $\widetilde{h}^n(X)$ to $[X, \mathcal{Z}(n)]$ for all CW complexes X. The second part will involve constructing the structure maps from $\Sigma\mathcal{Z}(n) \to \mathcal{Z}(n+1)$.

Fix an $n \in \mathbb{Z}$. We will construct the space $\mathcal{Z}(n)$ as a CW complex, using finite dimensional skeletons $\mathcal{Z}(n)_k$. For each k, we will also pick a cohomology class $c_n(k)$ in $\widetilde{h}^n(\mathcal{Z}(n)_k)$ such that the map $d_n^m(k) : [S^m, \mathcal{Z}(n)_k] \to \widetilde{h}^n(S^m)$ is an isomorphism for $m < k$ and surjection for $m = k$.

$$\begin{aligned} d_n^m(k) &: [S^m, \mathcal{Z}(n)_k] \to \widetilde{h}^n(S^m) \\ d_n^m(k) &: [f] \mapsto f^*(c_n(k)) \end{aligned}$$

For $k = 0$, we define $\mathcal{Z}(n)_0$ as follows.

$$\mathcal{Z}(n)_0 := \bigvee_{\alpha \in \widetilde{h}^n(S^0)} S^0_\alpha$$

The cohomology group of $\mathcal{Z}(n)_0$ is given by a direct product, since $\widetilde{h}^n$ satisfies the (DV) axiom.

$$\widetilde{h}^n(Z(n)_0) \cong \prod_{\alpha \in \widetilde{h}^n(S^0)} \widetilde{h}^n(S^0_\alpha)$$

Pick the following element as $c_n(0)$.

$$c_n(0) := \prod_{\alpha \in \widetilde{h}^n(S^0)} \alpha$$

Since $k = 0$, we only need to show that $d_n^0(0)$ is a surjection. Pick any $\alpha \in \widetilde{h}^n(S^0)$. Corresponding to this α, there's a copy of S^0 sitting inside $\mathcal{Z}(n)_0$. Let f be the inclusion map of this copy of S^0 into $\mathcal{Z}(n)_0$. Then the induced map on cohomology is the projection map on the α^{th} coordinate, since the cohomology theory satisfies the (DV) axiom. Applying this induced map on $c_n(0)$, we see that in the α^{th} coordinate, it has α, because of the way we defined it. This shows the map is surjective.

To prove the induction step, suppose we have defined the space $\mathcal{Z}(n)_k$ and $c_n(k)$ that satisfy the required properties. Let $K_k \trianglelefteq \left[S^k, \mathcal{Z}(n)_k\right]$ be the kernel of the map $d_n^k(k)$. We construct the following map.

$$\phi_n(k) : \bigvee_{x \in K_k} S^k \to \mathcal{Z}(n)_k \vee \bigvee_{y \in \widetilde{h}^n(S^{k+1})} S^{k+1}$$

This map is obtained by taking the wedge of maps from S^k to $\mathcal{Z}(n)_k$ which are contained in K_k. This is a cofibration (Not sure how to show this, or whether this is entirely correct. Need to check later). By the (DV) axiom, we have the following cohomology groups.

$$\widetilde{h}^n\left(\mathcal{Z}(n)_k \vee \bigvee_{y \in \widetilde{h}^n(S^{k+1})} S^{k+1}\right) = \widetilde{h}^n(\mathcal{Z}(n)_k) \times \prod_{y \in \widetilde{h}^n(S^{k+1})} \widetilde{h}^n(S^{k+1})$$

From this, we can immediately see that the elements of $\widetilde{h}^n\left(\mathcal{Z}(n)_k \vee \bigvee_{y \in \widetilde{h}^n(S^{k+1})} S^{k+1}\right)$ of the form $(c_n(k), \bullet)$ (where $\bullet$ is any arbitrary element) is in the kernel of $\phi_n^*(k)$. Define $\mathcal{Z}(n)_{k+1}$ to be the cofiber of the map $\phi_n(k)$, and let the map from to the cofiber be $b_n(k)$. By the exactness axiom, we have that the following sequence is exact.

$$\widetilde{h}^n(\mathcal{Z}(n)_{k+1}) \xrightarrow{b_n^*(k)} \widetilde{h}^n(\mathcal{Z}(n)_k) \times \prod_{y \in \widetilde{h}^n(S^{k+1})} \widetilde{h}^n(S^{k+1}) \xrightarrow{\phi_n^*(k)} \prod_{x \in K_k} \widetilde{h}^n(S^k)$$

Pick the following element $A \in \prod_{y \in \widetilde{h}^n(S^{k+1})} \widetilde{h}^n(S^{k+1})$.

$$A := \prod_{\alpha \in \widetilde{h}^n(S^{k+1})} \alpha$$

The element $(c_n(k), A)$ lies in the kernel of $\phi_n^*(k)$, which means it lies in the image of $b_n^*(k)$. We define $c_n(k+1)$ to be a pre-image of $(c_n(k), A)$. Seeing that the associated map $d_n^m(k)$ is surjective for $m = k+1$ is easy enough. The proof is the same as that for $k = 0$. $\square$

A Definitions and notation

Definition A.1 (Suspension of a pointed space)**.** The suspension ΣX of a pointed space X is the smash product $S^1 \wedge X$.

Definition A.2 (Loop space of a pointed space)**.** The loop spaces ΩX of a pointed space X is the set of all pointed maps from S^1 to X with the compact-open topology.

Definition A.3 ($\lim^1$)**.** Let T be the category of towers of abelian groups, i.e. $\mathbb{N}$ indexed set of abelian groups G_i with maps $f_i : G_i \to G_{i-1}$, and maps are set of arrows that make the whole thing commute (Check that this category has enough injectives). Then $\lim$ is a left exact functor from T to AbGrp (This is easy to check), and we define $\lim^1$ to be the first right derived functor of $\lim$.

B Some useful lemmas and theorems

Note: Although we state many of the lemmas here for TOP, they are also true for TOP_*, and the proof is similar.

Lemma B.1. *If $i : A \hookrightarrow X$ is a cofibration (in the category TOP), then the mapping cone $C(i)$ is homotopy equivalent to X/A.*

Proof. We will first construct the maps to and from $C(i)$ to X/A. The maps from $C(i)$ to X/A is the map that collapses the cone of A to a point corresponding to A in X/A. Now consider a map from $H : A \times I$ to $C(i)$, such that H contracts A to a point in $C(i)$, starting from the inclusion of A in X. Let the map j from X to $C(i)$ be the inclusion map. Since i is a cofibration, we can extend H with the initial condition j to a map $J : X \times I \to C(i)$. But $J(\cdot, 1)$ collapses A to a point. That means it factors through a X/A. This gives us a map k from X/A to $C(i)$.

The fact that these maps are homotopy inverses can be verified using the homotopy J. (Not sure of this. Verify later.) □

Lemma B.2. *In the category TOP, the following sequence is h-coexact.*

$$A \xrightarrow{f} B \xrightarrow{i} C(f)$$

That means for any space Z, the following sequence of abelian groups is exact.

$$[A, Z] \leftarrow [B, Z] \leftarrow [C(f), Z]$$

Proof. If an element $[c] \in [B, Z]$ goes to 0 in $[A, Z]$, that means $c \circ f : A \to Z$ is nullhomotopic, where c is a representative of $[c]$. But that means there is some function $d \in C(f)$ such that $c = d \circ i$. This shows the exactness of the sequence. □

Lemma B.3. *If K is a compact space, let A_i be a sequence of spaces where points are closed, and A is the colimit of the following diagram:*

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots$$

where all the embeddings are closed, then a map from K to A factors finitely through some A_i.

Proof. Let $J = f(K)$ be the compact image of K in A. For each set $A_i \setminus A_{i-1}$, pick an element c_i of J in the set, if J intersects $A_i \setminus A_{i-1}$. Since A_i's are closed, that means the subset c_i has the discrete topology. Furthermore, since points are closed, the set $\{\cup c_i\}$ is a closed subset of J, hence compact. And compact spaces with discrete topology are finite. That means only finitely many $A_i \setminus A_{i-1}$ intersect J. This means the map factors through at some finite stage. □

Theorem B.4 (Cofiber sequence). *Put in result*

Lemma B.5 (Characterizing cofibrations). *If $i : A \hookrightarrow X$ is a closed inclusion, and there is a retract $r : X \to A$, then i is a cofibration.*

Proof. fill in later □

Theorem B.6 (Alternative characterization of $\lim^1$). *If F is an object in the tower category, then $\lim^1(F)$ is the cokernel of the following map.*

$$\alpha_F : \prod_{i \in \mathbb{N}} F_i \to \prod_{i \in \mathbb{N}} F_i$$
$$\alpha_F : (g_0, g_1, g_2, \dots) \mapsto (g_0 - f_1(g_1), g_1 - f_2(g_2), \dots)$$

Proof. fill in later □

Theorem B.7 (Milnor exact sequence). *If $\{i_n : X_n \hookrightarrow X_{n+1}\}$ for $n \in \mathbb{N}$ are a sequence of nested CW subcomplexes such that $X = \bigcup_n X_n$, and $\widetilde{h}^*$ is a reduced cohomology theory, then we have the following exact sequence for all $i \geq 1$.*

$$0 \to \lim_n{}^1 \widetilde{h}^{i-1}(X_n) \to \widetilde{h}^i(X) \to \lim_n \widetilde{h}^i(X_n) \to 0$$

Proof. fill in later □