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State Constrained Stochastic Optimal Control Using LSTMs

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Problem Setup

A system with dynamics that involves stochastic processes can be described using a stochastic differential equation (SDE) as follows

$$dx(t) = f(x(t), t)dt + G(x(t), t)u(t)dt + \Sigma(x(t), t)dw(t)$$

we want to find the control that minimizes the control objective

$$J^u(x,t) = \mathbb{E}\Big[g(x(T)) + \int_t^T \Big(q(x(s)) + \frac{1}{2}u(s)^T R u(s)\Big) ds \Big| x(t) = x\Big]$$

with state constraints

$$c_{\min} \le c_s(x) \le c_{\max}$$

and control saturation

$$u \in \mathcal{U} = \{u \mid |u_i| \le U_{i,\max}\}$$

State & Control Constraint

The control is saturated as

$$u^*(x,t) = U_{\max} * \mathrm{sig}(-R^{-1}G^T(t,x)V_x)$$

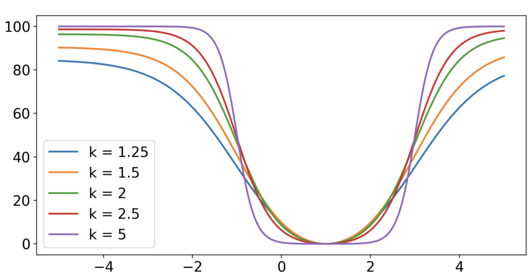
the control cost would then become

$$S_i(u_i) = c_i \int_0^{u_i} \mathrm{sig}^{-1}\Big(\frac{v}{U_{i,\max}}\Big) dv$$

The state constrained is applied via a penatly function

$$p(x) = \frac{L}{1 + e^{-k(c_s(x) - c_{\max})}} - \frac{L}{1 + e^{-k(c_s(x) - c_{\min})}} + L - \frac{2L}{1 + e^{-k(\mu - c_{\max})}}$$

For a state constraint of [-1, 3], the penalty function under different k values looks like:



Taking both state constraints and control saturation into consideration the overall cost function has the form

$$\mathbb{E}\Bigg[g(x(T)) + \int_t^T \Big(q(x(s)) + p(x(s)) + \sum_{i=1}^m S_i(u_i(s))\Big) ds \Big| x(t) = x\Bigg]$$

Adaptive Update Scheme

To ensure numerical stability we use the square root of state cost variance over a fixed number of iterations as the update threshold, and gradually harden the penalty function p(x). Since the state cost variance would never decrease to zero we also set a minimum value for the threshold.

$$k \leftarrow k + \delta$$
$$\delta \leftarrow \delta - \Delta_\delta$$
$$\beta \leftarrow \gamma\beta$$
$$\gamma \leftarrow \gamma + \Delta$$

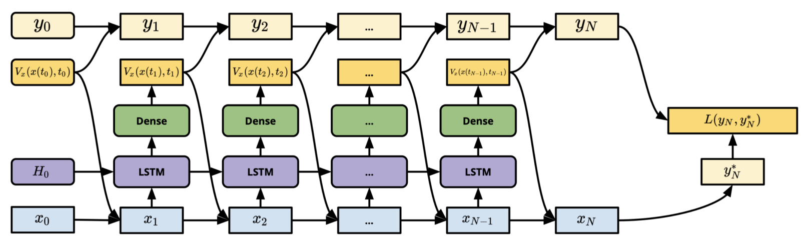
Deep FBSDE

We can write the problem mentioned in the "Problem Setup" under the updated cost function in "State Constraint and Control Saturation" as a forward-backward stochastic differential equation (FBSDE) as shown on the right, where Vx is the partial derivative of the value function w.r.t. the state, and the Hamiltonian is defined as

$$h(x, V_x, t, u^*) = q(x) + V_x^T G(x,t) u^*(x,t) + \sum_{i=1}^m S_i(u_i^*).$$

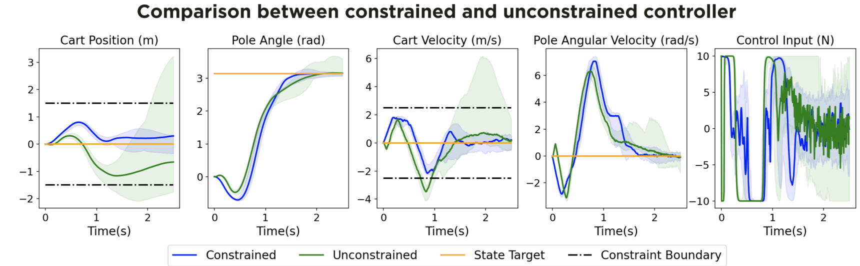
$$\begin{aligned}
dy(t) &= \Big(-h(x(t), V_x(x(t), t; \theta), t, u(t)) \\
&\quad + V_x^T(x(t), t; \theta) G(x(t), t) u(x(t), t)\Big) dt \\
&\quad + V_x^T(x(t), t; \theta)\Sigma(x(t), t) dw(t) \\
dx(t) &= \Big(f(x(t), t) + G(x(t), t) u(x(t), t)\Big) dt \\
&\quad + \Sigma(x(t), t) dw(t) \\
u(t) &= U_{\max} * \mathrm{sig}(-R^{-1}G^T(x(t), t) V_x(x(t), t; \theta)) \\
y(0) &= V(\phi) \\
dy(0) &= V_x(\phi) \\
x(0) &= x_0.
\end{aligned}$$

The corresponding neural network architecture is

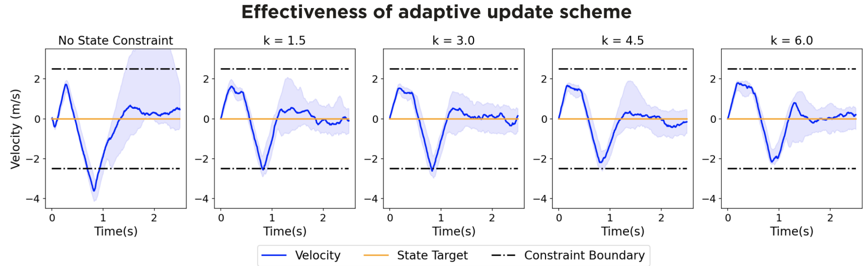


Experiments

Comparison between constrained and unconstrained controller



Effectiveness of adaptive update scheme



All experiments were conducted on the cart-pole swing-up task. Two state constraint settings were tested: (i) constraining cart position and cart velocity; (ii) constraining the sum of kinetic and potential energy. We see that in both settings the learned controller is able to respect the constraint boundaries.

Energy constraint comparison