# ME-GY 7943/ECE-GY9273
# Network Robotic Systems, Cooperative Control and Swarming

## Exercise series 1

Please typeset your answers (e.g. using LaTeX). For all questions, justify clearly your answers.

## Exercise 1

For matrix $A_1$,

a) the three columns are linearly independent therefore its rank is 3

b) since the rank is 3, the nullspace dimension is 0: the nullspace contains only the 0 vector $\{0\}$.

c) The range space is $\mathbb{R}^3$ (it is possible to find a $x$ such that $A_1 x = y$ for any $y$, since the columns of $A_1$ form a basis of $\mathbb{R}^3$. A possible orthonormal basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and its dimension is 3.

For matrix $A_2$,

a) the three columns are linearly independent therefore its rank is 3

b) since the rank is 3, the nullspace dimension is 0: the nullspace contains only the 0 vector $\{0\}$.

c) The range space is made of any linear combination of the columns of $A_2$, an orthonormal basis for this can be $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\}$. We verify that it is a basis for the range space as these vectors can reconstruct each columns of $A_2$. The dimension of the range space is 3.

For matrix $A_3$,

a) there are 3 independent columns (for example the first three ones) so its rank is 3.

b) The dimension of the nullspace is therefore 1. We can find a basis for the nullspace by finding a solution for the equation $A_3 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$. An orthonormal basis for the nullspace is $\left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3}) \end{pmatrix} \right\}$.

c) the range space is

For matrix $A_4$,

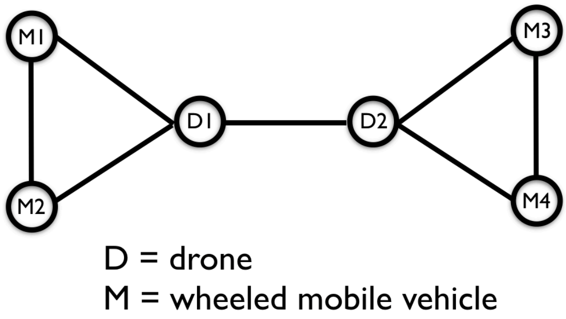



Figure 1: Communication graph

a) the first and last column are linearly dependent and therefore the rank of the matrix is 2.

b) This implies that the dimension of the nullspace is 1. We can find a basis for the nullspace by finding a solution for the equation $A_3 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$. An orthonormal basis for the nullspace is $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2}) \end{pmatrix}$.

c) Since there are 2 independent vectors, the range space dimension is 2. A possible orthonormal basis for the range space is $\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \right\}$.

# Exercise 2

For $A_1$, there are 4 complex and 1 real eigenvalues and no eigenvalue is 0. Therefore the rank of the matrix is 5 and its nullspace is empty.

For $A_2$, there are 5 real eigenvalues and one is 0. Therefore the rank of the matrix is 4 and its nullspace has dimension 1.

# Exercise 3

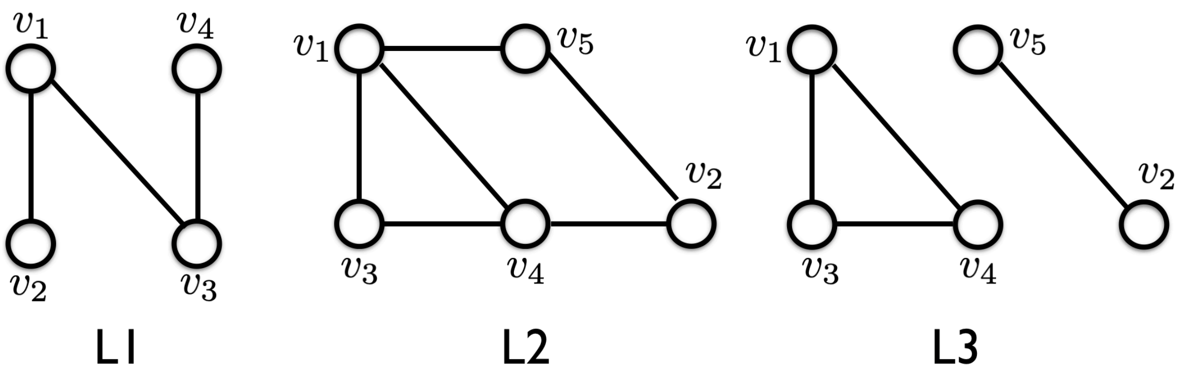
- The communication graph is shown in Figure 1
- The graph is the tuple $(V, E)$ where

$$V = \{M1, M2, M3, M4, D1, D2\}$$
$$E = \{(M1, M2), (M1, D1), (M2, D1), (D1, D2), (M3, M4), (M3, D2), (M4, D2)\}$$

  The graph is undirected.

- The graph is connected because there is a path from any vertex to any other vertex.
- The adjacency and degree matrices (Assuming the ordering shown in $V$) are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$
$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$




Choosing edge orientation according to the tuple ordering shown in $E$, we have the following incidence matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

- The graph Laplacian is

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

One necessary and sufficient condition for having a connected graph is that its Laplacian has a nullspace of dimension 1, i.e. that it has only one 0 eigenvalue. The eigenvalues of the Laplacian are approximately 0, 0.438, 4.56, 3, 3, 3 and therefore the graph is connected.

## Exercise 4

Consider the following matrices

$$L_1 = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 2 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}, \quad L_5 = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

In order for a matrix to be a Laplacian, it needs to have 0 of >0 values on its diagonal and 0 or $-1$ for all off-diagonal terms. This eliminates already $L_2$. Moreover the sum of its columns need to be 0 which eliminates $L_4$. The other matrices could be graph Laplacian.

a) Results are shown in Figure **??**.

b) The graphs generated by $L_1$ and $L_3$ are connected. The graph generated by $L_5$ is not.

c) There should be only positive or 0 eigenvalues. For $L_1$ there should be one 0 eigenvalues and 3 positive. For $L_3$ there should be one 0 eigenvalue and 4 positive and for $L_5$ there should be two 0 eigenvalues and 3 positive.