Supplementary Material of Stagewise Implementations of Sequential Quadratic Programming for Model-Predictive Control

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Let's consider the following Quadratic Program:

$$\min_{x_{1:T}, u_{0:T-1}} x_T^T Q_T x_T + x_T^T q_T + \sum_{k=0}^{T-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \begin{bmatrix} q_k \\ r_k \end{bmatrix}^T \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{1}$$

$$\begin{aligned} \text{subject to} \quad & x_0 = 0, \\ & x_{k+1} = A_k x_k + B_k u_k + \gamma_{k+1}, \qquad 0 \leq k < T. \end{aligned}$$

The associated Lagrangian has the following form:

$$\begin{aligned} \mathcal{L}(x_{1:T}, u_{0:T-1}, \lambda_{1:T}) = x_T^T Q_T x_T + x_T^T q_T + \sum_{k=0}^{T-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \begin{bmatrix} q_k \\ r_k \end{bmatrix}^T \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ - \lambda_{k+1}^T (x_{k+1} - A_k x_k - B_k u_k - \gamma_{k+1}) \end{aligned} \tag{2}$$

The KKT condition can be written as a set of linear equations:

$$Q_T x_T + q_T = \lambda_T \tag{3}$$

$$Q_k x_k + S_k u_k + q_k + A_k^T \lambda_{k+1} = \lambda_k \qquad \forall k \leq 1 \tag{4}$$

$$R_k u_k + S_k^T x_k + r_k + B_k^T \lambda_{k+1} = 0 \qquad \forall k < T \tag{5}$$

$$x_{k+1} = A_k x_k + B_k u_k + \gamma_{k+1}. \tag{6}$$

Proposition 1. *The KKT conditions can be written as block tri-diagonal symmetric matrix equation. More precisely:*

$$\begin{bmatrix} \Gamma_1 & M_1^T & 0 & 0 & \cdots & 0 \\ M_1 & \Gamma_2 & M_2^T & 0 & \cdots & 0 \\ 0 & M_2 & \Gamma_3 & M_3^T & \cdots & 0 \\ 0 & 0 & M_3 & \Gamma_4 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \Gamma_T \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ \vdots \\ s_T \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ \vdots \\ g_T \end{bmatrix} \tag{7}$$

where:

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$$\Gamma_k = \begin{bmatrix} R_{k-1} & 0 & -B_{k-1}^T \\ 0 & Q_k & I \\ -B_{k-1} & I & 0 \end{bmatrix}, \quad M_k = \begin{bmatrix} 0 & S_k^T & 0 \\ 0 & 0 & 0 \\ 0 & -A_k & 0 \end{bmatrix}$$
$$\textit{and} \quad s_k = \begin{bmatrix} u_{k-1} \\ x_k \\ -\lambda_k \end{bmatrix}, \quad g_k = \begin{bmatrix} -r_{k-1} \\ -q_k \\ \gamma_k \end{bmatrix} \tag{8}$$

.

Proof. We find that:

- For $k = 1$, as $x_0 = 0$:

$$\begin{aligned} \Gamma_1 s_1 + M_1^T s_2 &= \begin{bmatrix} R_0 u_0 + B_0^T \lambda_1 \\ Q_1 x_1 - \lambda_1 \\ -B_0 u_0 + x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ S_1 u_1 + A_1^T \lambda_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} R_0 u_0 + S_0^T x_0 + B_0^T \lambda_1 \\ Q_1 x_1 - \lambda_1 + S_1 u_1 + A_1^T \lambda_2 \\ -A_0 x_0 - B_0 u_0 + x_1 \end{bmatrix} = \begin{bmatrix} -r_0 \\ -q_1 \\ \gamma_1 \end{bmatrix} \end{aligned} \tag{9}$$

- For $1 < k < T$:

$$\begin{aligned} M_{k-1} s_{k-1} + \Gamma_k s_k + M_k^T s_{k+1} &= \begin{bmatrix} S_{k-1}^T x_{k-1} \\ 0 \\ -A_{k-1} x_{k-1} \end{bmatrix} + \begin{bmatrix} R_{k-1} u_{k-1} + B_{k-1}^T \lambda_k \\ Q_k x_k - \lambda_k \\ -B_{k-1} u_{k-1} + x_k \end{bmatrix} + \begin{bmatrix} 0 \\ S_k u_k + A_k^T \lambda_{k+1} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -r_{k-1} \\ -q_k \\ \gamma_k \end{bmatrix} = g_k \end{aligned} \tag{10}$$

- For $k = T$:

$$M_{T-1} s_{T-1} + \Gamma_T s_T = \begin{bmatrix} S_{T-1}^T x_{T-1} \\ 0 \\ -A_{T-1} x_{T-1} \end{bmatrix} + \begin{bmatrix} R_{T-1} u_{T-1} + B_{T-1}^T \lambda_T \\ Q_T x_T - \lambda_T \\ -B_{T-1} u_{T-1} + x_T \end{bmatrix} = \begin{bmatrix} -r_{T-1} \\ -q_T \\ \gamma_T \end{bmatrix} = g_T \tag{11}$$

□

Proposition 2. *By applying Thomas algorithm, we recover the well-known Ricatti recursions. Specifically, the* ***backward pass*** *can be done by initializing $V_T = Q_T$ and $v_T = q_T$, and then by applying the following equations:*

$$h_k = r_k + B_k^T (v_{k+1} + V_{k+1} \gamma_{k+1}) \tag{12}$$
$$G_k = S_k^T + B_n^T V_{k+1} A_k \qquad K_k = -H_k^{-1} G_k$$
$$H_k = R_k + B_k^T V_{k+1} B_k \qquad k_k = -H_k^{-1} h_k$$
$$\begin{aligned} V_k &= Q_k + A_k^T V_{k+1} A_k - K_k^T H_k K_k \\ v_k &= q_k + K_k^T r_k + (A_k + K_k B_k)^T (v_{k+1} + V_{k+1} \gamma_{k+1}) \end{aligned} \tag{13}$$

Then, the ***forward pass*** *initializes $\Delta x_0 = 0$ and unrolls the linearized dynamics:*

$$\Delta x_{k+1} = (A_k + B_k K_k) \Delta x_k + B_k k_k + \gamma_{k+1} \tag{14}$$
$$\Delta u_k = K_k \Delta x_k + k_k \tag{15}$$
$$\lambda_k = V_k \Delta x_k + v_k \tag{16}$$

Proof. For this proof, we will extensively use the following lemma:

Lemma 0.1.

$$\begin{bmatrix} R_k & 0 & -B_k^T \\ 0 & V_{k+1} & I \\ -B_k & I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} H_{k+1}^{-1} & H_{k+1}^{-1}B_k^T & -H_{k+1}^{-1}B_k^T V_{k+1} \\ B_k H_{k+1}^{-1} & B_k H_{k+1}^{-1} B_k^T & I - B_k H_{k+1}^{-1} B_k^T V_{k+1} \\ -V_{k+1}B_k H_{k+1}^{-1} & I - V_{k+1}B_k H_{k+1}^{-1} B_k^T & -V_{k+1}(I - B_k H_{k+1}^{-1} B_k^T V_{k+1}) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & -V_{k+1} \end{bmatrix} + \begin{bmatrix} H_{k+1}^{-1} & 0 & 0 \\ 0 & B_k H_{k+1}^{-1} & 0 \\ 0 & 0 & -V_{k+1}B_k H_{k+1}^{-1} \end{bmatrix} \begin{bmatrix} I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \end{bmatrix}$$

where $H_k = R_k + B_k^T V_{k+1} B_k$

Thomas algorithm uses a forward recursion in order to find a equivalent linear system of the form:

$$\begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ \bar{\Gamma}_2^{-1}M_1 & I & 0 & 0 & \cdots & 0 \\ 0 & \bar{\Gamma}_3^{-1}M_2 & I & 0 & \cdots & 0 \\ 0 & 0 & \bar{\Gamma}_4^{-1}M_3 & I & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & I \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ \vdots \\ s_T \end{bmatrix} = \begin{bmatrix} \bar{g}_1 \\ \bar{g}_2 \\ \bar{g}_3 \\ \bar{g}_4 \\ \vdots \\ \bar{g}_T \end{bmatrix} \tag{17}$$

Then, a forward recursion recovers the sequence: $s_1, \cdots s_T$. This can be formalized in the following way:

Algorithm 1: Thomas algorithm

$\bar{\Gamma}_T \leftarrow \Gamma_T$

$\bar{g}_T \leftarrow \Gamma_T^{-1} g_T$

/* backward pass */

for $k \leftarrow 1$ *to* $T-1$ **do**

  $\bar{\Gamma}_k \leftarrow \Gamma_k - M_k^T \bar{\Gamma}_{k+1}^{-1} M_k$

  $\bar{g}_k \leftarrow \bar{\Gamma}_k^{-1}(g_k - M_k^T \bar{g}_{k+1})$

/* forward pass */

$s_1 \leftarrow \bar{g}_1$

for $k \leftarrow 1$ *to* $T-1$ **do**

  $s_{k+1} \leftarrow \bar{g}_{k+1} - \bar{\Gamma}_{k+1}^{-1} M_k s_k$

Let's now show by recursion that:

$$\bar{\Gamma}_k = \begin{bmatrix} R_{k-1} & 0 & -B_{k-1}^T \\ 0 & V_k & I \\ -B_{k-1} & I & 0 \end{bmatrix} \tag{18}$$

and that:

$$\bar{g}_k = \bar{\Gamma}_k^{-1} \begin{bmatrix} -r_{k-1} \\ -v_k \\ \gamma_k \end{bmatrix} \tag{19}$$

Backward pass:

- $k = T$

As $V_T = Q_T$ and $v_T = q_T$ by definition, the property is true for $k = T$.

- If the property is true for $k+1$,

$$
\begin{aligned}
\bar{\Gamma}_k &= \Gamma_k - M_k^T \bar{\Gamma}_{k+1}^{-1} M_k \\
&= \begin{bmatrix} R_{k-1} & 0 & -B_{k-1}^T \\ 0 & Q_k & I \\ -B_{k-1} & I & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ S_k & 0 & -A_k^T \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_k & 0 & -B_k^T \\ 0 & V_{k+1} & I \\ -B_k & I & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & S_k^T & 0 \\ 0 & 0 & 0 \\ 0 & -A_k & 0 \end{bmatrix} \\
&= \begin{bmatrix} R_{k-1} & 0 & -B_{k-1}^T \\ 0 & Q_k & I \\ -B_{k-1} & I & 0 \end{bmatrix} \\
&- \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(S_k H_{k+1}^{-1} + A_k^T V_{k+1} B_k H_{k+1}^{-1}\right) S_k^T - \left(-S_k H_{k+1}^{-1} B_k^T V_{k+1} + A_k^T V_{k+1}(I - B_k H_{k+1}^{-1} B_k^T V_{k+1})\right) A_k & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R_{k-1} & 0 & -B_{k-1}^T \\ 0 & V_k & I \\ -B_{k-1} & I & 0 \end{bmatrix}
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
V_k &= Q_k + A_k^T V_{k+1} A_k - S_k H_{k+1}^{-1}(S_k^T + B_k^T V_{k+1} A_k) - A_k^T V_{k+1} B_k H_{k+1}^{-1}(S_k^T + B_k^T V_{k+1} A_k) \\
&= Q_k + A_k^T V_{k+1} A_k - (S_k + A_k^T V_{k+1} B_k)(R_k + B_k^T V_{k+1} B_k)^{-1}(S_k^T + B_k^T V_{k+1} A_k)
\end{aligned} \tag{21}
$$

then, we have:

$$
\begin{aligned}
&- \bar{\Gamma}_k \bar{g}_k = -g_k + M_k^T \bar{g}_{k+1} \\
&= -g_k - \begin{bmatrix} 0 & 0 & 0 \\ S_k & 0 & -A_k^T \\ 0 & 0 & 0 \end{bmatrix} \bar{\Gamma}_{k+1}^{-1} \begin{bmatrix} r_k \\ v_{k+1} \\ -\gamma_{k+1} \end{bmatrix} \\
&= -g_k - \begin{bmatrix} 0 & 0 & 0 \\ S_k & 0 & -A_k^T \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & -V_{k-1} \end{bmatrix} + \begin{bmatrix} H_{k+1}^{-1} & 0 & 0 \\ 0 & B_k H_{k+1}^{-1} & 0 \\ 0 & 0 & -V_{k+1} B_k H_{k+1}^{-1} \end{bmatrix} \begin{bmatrix} I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \end{bmatrix} \right) \begin{bmatrix} r_k \\ v_{k+1} \\ -\gamma_{k+1} \end{bmatrix} \\
&= -g_k - \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & -A_k^T & A_k^T V_{k+1} \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ S_k H_{k+1}^{-1} & 0 & A_k^T V_{k+1} B_k H_{k+1}^{-1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \end{bmatrix} \right) \begin{bmatrix} r_k \\ v_{k+1} \\ -\gamma_{k+1} \end{bmatrix} \\
&= -g_k - \left(\begin{bmatrix} 0 \\ -A_k^T v_{k+1} - A_k^T V_{k+1} \gamma_{k+1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ S_k H_{k+1}^{-1} & 0 & A_k^T V_{k+1} B_k H_{k+1}^{-1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_k + B_k^T v_{k+1} + B_k^T V_{k+1} \gamma_{k+1} \\ r_k + B_k^T v_{k+1} + B_k^T V_{k+1} \gamma_{k+1} \\ r_k + B_k^T v_{k+1} + B_k^T V_{k+1} \gamma_{k+1} \end{bmatrix} \right)
\end{aligned}
$$

and we get:

$$
\begin{aligned}
v_k &= q_k + A_k^T(v_{k+1} + V_{k+1}\gamma_{k+1}) - (S_k + A_k^T V_{k+1} B_k) H_{k+1}^{-1}(r_k + B_k^T v_{k+1} + B_k^T V_{k+1} \gamma_{k+1}) \\
&= q_k + A_k^T(v_{k+1} + V_{k+1}\gamma_{k+1}) + K_k^T(r_k + B_k^T v_{k+1} + B_k^T V_{k+1} \gamma_{k+1}) \\
&= q_k + K_k^T r_k + (A_k + B_k K_k)^T (v_{k+1} + V_{k+1}\gamma_{k+1})
\end{aligned} \tag{22}
$$

Hence, the property is also true for k.

Forward pass:

- $k = 1$:

$$
s_1 = \bar{g}_1 \tag{23}
$$

implies that:

$$
\begin{aligned}
\begin{pmatrix} u_0 \\ x_1 \\ \lambda_1 \end{pmatrix} &= \bar{\Gamma}_1^{-1} \begin{bmatrix} -r_0 \\ -v_1 \\ \gamma_1 \end{bmatrix} \\
&= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & -V_1 \end{bmatrix} + \begin{bmatrix} H_0^{-1} & 0 & 0 \\ 0 & B_0 H_0^{-1} & 0 \\ 0 & 0 & -V_1 B_0 H_1^{-1} \end{bmatrix} \begin{bmatrix} I & B_0^T & -B_0^T V_1 \\ I & B_0^T & -B_0^T V_1 \\ I & B_0^T & -B_0^T V_1 \end{bmatrix} \right) \begin{bmatrix} -r_0 \\ -v_1 \\ \gamma_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \gamma_1 \\ -v_1 - V_1\gamma_1 \end{bmatrix} - \begin{bmatrix} H_0^{-1} & 0 & 0 \\ 0 & B_0 H_0^{-1} & 0 \\ 0 & 0 & -V_1 B_0 H_1^{-1} \end{bmatrix} \begin{bmatrix} r_0 + B_0^T(v_1 + V_1\gamma_1) \\ r_0 + B_0^T(v_1 + V_1\gamma_1) \\ r_0 + B_0^T(v_1 + V_1\gamma_1) \end{bmatrix} \\
&= \begin{bmatrix} k_0 \\ B_0 k_0 + \gamma_1 \\ -V_1 B_0 k_0 - V_1 \gamma_1 - v_1 \end{bmatrix} \qquad (24)
\end{aligned}
$$

Hence, $\lambda_1 = -V_1 x_1 - v_1$

- $k \geq 1$:

Then:

$$
\bar{s}_k = \bar{g}_k - \bar{\Gamma}_k^{-1} M_{k-1} s_{k-1} \qquad (25)
$$

implies that:

$$
\begin{aligned}
\begin{pmatrix} u_k \\ x_{k+1} \\ \lambda_{k+1} \end{pmatrix} &= \bar{\Gamma}_{k+1}^{-1} (\begin{bmatrix} -r_k \\ -v_{k+1} \\ \gamma_{k+1} \end{bmatrix} - M_k s_k) \\
&= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & -V_{k+1} \end{bmatrix} + \begin{bmatrix} H_{k+1}^{-1} & 0 & 0 \\ 0 & B_k H_{k+1}^{-1} & 0 \\ 0 & 0 & -V_{k+1} B_k H_{k+1}^{-1} \end{bmatrix} \begin{bmatrix} I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \\ I & B_k^T & -B_k^T V_{k+1} \end{bmatrix} \right) \begin{bmatrix} -r_k - S_k^T x_k \\ -v_{k+1} \\ \gamma_{k+1} + A_k x_k \end{bmatrix} \\
&= \begin{bmatrix} k_k + K_k x_k \\ (A_k + B_k K_k) x_k + B_k k_k + \gamma_{k+1} \\ V_{k+1}((A_k + B_k K_k) x_k + B_k k_k + \gamma_{k+1}) + v_{k+1} \end{bmatrix} \qquad (26)
\end{aligned}
$$

and we get:

$$
\lambda_{k+1} = V_{k+1} x_{k+1} + v_{k+1} \qquad (27)
$$

□