Rigid-body motion

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1 Rotation

A rotation is a rigid-body tranformation that keeps the origin unchanged. A rotation can be represented by a 3×3 matrix R satisfying:

$$RR^T = I_3 \text{ and } \det(R) = 1 \tag{1}$$

where I_3 is the identity matrix. The set of rotation is called *Special orthogonal group* and is denoted by $SO(3)$.

1.1 rotation motion

Let us consider a mapping R from $\mathbb{R}$ to $SO(3)$ representing a time-varying rotation. A point $\mathbf{u} \in \mathbb{R}^3$ is mapped at time t to $R(t)\mathbf{u}$. The velocity at time t of this point is given by $\dot{R}(t)\mathbf{u}$.

Let us differentiate (1):

$$\dot{R}R^T + R\dot{R^T} = 0 \tag{2}$$

$$\dot{R}R^T + (\dot{R}R^T)^T = 0 \tag{3}$$

The latter equality states that $\dot{R}R^T$ is a skew-symmetric matrix.

Let $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ be a vector. We denote by

$$[\omega]_\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{4}$$

Let us notice that for any ω and any $\mathbf{u}$ in $\mathbb{R}^3$,

$$[\omega]_\times \vec{u} = \omega \times \vec{u}.$$

where $\times$ denotes the cross product. From (3), we can state that there exists $\omega \in \mathbb{R}^3$ such that

$$\begin{aligned} \dot{R}R^T &= [\omega]_\times \\ \dot{R} &= [\omega]_\times R \end{aligned}$$

The velocity of point $\mathbf{u}$ defined above along the rotation motion is therefore given by

$$\dot{R}(t)\mathbf{u} = [\omega]_\times R\,\mathbf{u} = \omega \times (R\,\mathbf{u}).$$

which is the well known formula giving the velocity of a point moving on an object with angular velocity ω.

1.2 Exponential

Let us consider a rotation motion $R(t)$ starting at I_3 and with constant angular velocity $\omega \in \mathbb{R}^3$:

$$\begin{aligned} R(0) &= I_3 \\ \dot{R}(t) &= [\omega]_\times R(t) \end{aligned}$$

It is easy to state that the solution to this differential equation is given by

$$R(t) = \exp(t\,[\omega]_\times) = \sum_{i=0}^{\infty} \frac{1}{i!} t^i\,[\omega]_\times^i$$

1.3 Logarithm

The exp mapping from the space of skew-symmetric matrices to $SO(3)$ is surjective. This means that for any $R \in SO(3)$, there exists $\omega \in \mathbb{R}^3$ such that

$$R = \exp([\omega]_\times)$$

we define $\log(R)$ as the skew-symmetric matrix of smaller norm among all solutions of the above equation.

By extension, we will sometimes confuse the skew-symmetric matrix with the underlying vector and write

$$\omega = \log(R) \text{ instead of } [\omega]_\times = \log(R)$$

2 Rigid-body tranformation

A rigid-body tranformation M is a mapping from $\mathbb{R}^3$ to itself that preserves distances and angles. Any rigid-body tranformation can be expressed as:

$$\forall \mathbf{p} \in \mathbb{R}^3,\ M(\mathbf{p}) = R\,\mathbf{p} + \mathbf{t}$$

where $R \in SO(3)$ is a rotation matrix and $\mathbf{t}$ is a vector.

The space of rigid-body tranformations is called the *special euclidean group* and is denoted by $SE(3)$.

2.1 Homogeneous matrix

A rigid-body tranformation can be represented by a 4×4 matrix as follows:

$$H = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}.$$

By adding a one as the fourth componentof $\mathbf{p}$, the rigid-body tranformation can be represented by a matrix-vector product as follows:

$$\begin{pmatrix} M(\mathbf{p}) \\ 1 \end{pmatrix} = H \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} R\,\mathbf{p} + \mathbf{t} \\ 1 \end{pmatrix}$$

H is called a homogeneous matrix.

3 Rigid-body motion

Let us consider a rigid-body motion represented by a time varying homogeneous matrix $H(t)$. At time t, point $\mathbf{p}$ is moved to $R(t)\,\mathbf{p} + \mathbf{t}(t)$.

The velocity of this point at time t is thus given by

$$\begin{pmatrix} \dot{\mathbf{p}} \\ 0 \end{pmatrix} = \dot{H}(t) \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} \quad \text{or} \quad \dot{\mathbf{p}} = [\omega]_\times\, R\,\mathbf{p} + \dot{\mathbf{t}}$$

$\dot{\mathbf{t}}$ is the velocity of the image of the origin at time t. Let us denote by $\mathbf{v}$ this velocity expressed in the moving frame:

$$\mathbf{v} = R^T \dot{\mathbf{t}}.$$

Similarly, let us denote by Ω the angular velocity expressed in the moving frame:

$$\Omega = R^T \omega$$

If we admit that

$$\left[R^T \omega\right] = R^T\, [\omega]\, R.$$

Then, we can write

$$\dot{H} = \begin{pmatrix} R\,[\Omega]_\times & R\mathbf{v} \\ 0 & 0 \end{pmatrix} = H \begin{pmatrix} [\Omega]_\times & \mathbf{v} \\ 0 & 0 \end{pmatrix} \tag{5}$$

If we consider a motion with constant linear velocity of the origin and angular velocity expressed in the moving frame, and starting at identity, Equation (5) can be seen as a differential equation in H. The solution to this equation is

$$H(t) = \exp t \begin{pmatrix} [\Omega]_\times & \mathbf{v} \\ 0 & 0 \end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i \begin{pmatrix} [\Omega]_\times & \mathbf{v} \\ 0 & 0 \end{pmatrix}^i$$

The motion defined by $H(t)$ is called a screw motion. It has the following properties:

- if $\Omega = 0$, this is a translation of constant velocity $\mathbf{v}$,
- otherwise there exists a straight line called the *axis* of the screw motion such that the motion of the points of the axis is a pure translation of constant velocity along the axis.

Similarly, we define the logarithm of a rigid-body transformation as the screw motion of minimal norm whose exponential is the rigid-body transformation. For some singular rigid-body transformations, the logarithm may not be uniquely defined.

4 What you need to remember

1. The derivative of a rigid-body motion is a screw represented by the linear and angular velocities of the image of the origin by the rigid-body motion and expressed in the moving frame.
2. the exponential maps screw velocities to rigid-body transformations.
3. The logarithm maps the other way back and is uniquely defined in a neighborhood of $0 \in \mathbb{R}^6$.
4. this mapping defines a distance in $SE(3)$ as follows:

$$d(M_1, M_2) = \| \log M_1^{-1} M_2 \|.$$

   The distance is indeed equal to zero if and only if $M_1 = M_2$.