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CONSERVATION LAWS AND HAMILTON-JACOBI EQUATIONS ON A JUNCTION: THE CONVEX CASE

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Abstract. The goal of this paper is to study the link between the solution to an Hamilton-Jacobi (HJ) equation and the solution to a Scalar Conservation Law (SCL) on a special network. When the equations are posed on the real axis, it is well known that the space derivative of the solution to the Hamilton-Jacobi equation is the solution to the corresponding scalar conservation law. On networks, the situation is more complicated and we show that this result still holds true in the convex case on a 1:1 junction. The correspondence between solutions to HJ equations and SCL on a 1:1 junction is done showing the convergence of associated numerical schemes. A second direct proof using semi-algebraic approximations is also given.

Here a 1:1 junction is a simple network composed of two edges and one vertex. In the case of three edges or more, we show that the associated HJ germ is not a L^1-dissipative germ, while it is the case for only two edges.

As an important byproduct of our numerical approach, we get a new result on the convergence of numerical schemes for scalar conservation laws on a junction. For a general desired flux condition which is discretized, we show that the numerical solution with the general flux condition converges to the solution of a SCL problem with an effective flux condition at the junction. Up to our knowledge, in previous works the effective condition was directly implemented in the numerical scheme. In general the effective flux condition differs from the desired one, and is its relaxation, which is very natural from the point of view of Hamilton-Jacobi equations. Here for SCL, this effective flux condition is encoded in a germ that we characterize at the junction.

1. **Introduction.** In one space dimension, it's well known that the space derivative of the viscosity solution to a Hamilton-Jacobi (HJ) equation is the solution to a scalar conservation law (SCL). We refer for example to [16, 19, 29, 23] for this kind of results. In this paper, we want to investigate this relation in the case of simple junctions composed of two edges and one vertex (referred later as 1:1 junctions),

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for which, up to our knowledge, this result is completely open. Scalar Conservation Laws and Hamilton-Jacobi equations on networks have been largely studied in the last decade. Concerning SCL, the 1:1 case has been studied following many different approaches during the last 20 years (see the two surveys [32] and [3] and references therein for an overview on the subject). In this paper, we choose to focus mostly on the germ approach (see [7, 24]) as it is suitable for the correspondence result. Concerning Hamilton-Jacobi equations on networks, the theory has been largely developed since the pioneer works of Achdou, Camilli, Cutrì, Tchou [1] and Imbert, Monneau, Zidani [28]. We refer in particular to [27], where a general comparison principle has been developed using PDE tools and a classification of the junction condition has been proposed, to [30, 31] for an extension to the non-convex case and to the monograph [12] for a general review on the topic.

Even if the theories are now well understood both for scalar conservation laws and Hamilton-Jacobi equations, the relation between these two theories has never been addressed on junctions until now. In this paper, we will give an answer for 1:1 junctions with convex Hamiltonians and we will also show that the situation is much more complicated when the junction is composed of more than three branches. The main difficulty comes from the junction condition and we will explain how the junction condition of the HJ equation, namely a flux limiter condition as in [27], can be interpreted as a condition on a germ, as in [7].

1.1. **The main result.** The aim of this paper is to make the link between viscosity solutions to Hamilton-Jacobi equations posed on the real line with a discontinuity at the origin and entropy solutions to a suitable conservation law. We consider here the case where the fluxes are convex but the result remains valid in the concave case (just changing the solution u by $-u$). Namely, we start with the flux-limited viscosity solution u, as in [27], of

$$\begin{cases} u_t + H_L(u_x) = 0 & \text{if } x < 0 \\ u_t + H_R(u_x) = 0 & \text{if } x > 0 \\ u_t + \bar{F}_A(u_x(t,0^-), u_x(t,0^+)) = 0 & \text{if } x = 0 \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{1}$$

where u_0 is a Lipschitz continuous initial condition. For $\alpha = L, R$, let $a_\alpha < b_\alpha < c_\alpha$. We make the following assumptions on the (convex) Hamiltonians for some $\delta > 0$

$$\begin{cases} \text{For } \alpha = L, R, \text{ the Hamiltonian } H_\alpha : \mathbb{R} \to \mathbb{R} \text{ is of class } C^2, \text{ with } H''_\alpha \geqslant \delta > 0, \\ \text{decreasing on } (-\infty, b_\alpha] \text{ and increasing on } [b_\alpha, +\infty), \text{ with } H_\alpha(a_\alpha) = H_\alpha(c_\alpha) = 0. \end{cases} \tag{2}$$

Notice here that the zero value of the Hamiltonians on the boundary of the intervals of definition is a convenient normalization, but is not essential.

We define the two associated monotone envelopes

$$H_\alpha^+(p) = \begin{cases} H_\alpha(b_\alpha) & \text{for} \quad p \in (-\infty, b_\alpha] \\ H_\alpha(p) & \text{for} \quad p \in [b_\alpha, +\infty) \end{cases}, \quad H_\alpha^-(p) = \begin{cases} H_\alpha(p) & \text{for} \quad p \in (-\infty, b_\alpha] \\ H_\alpha(b_\alpha) & \text{for} \quad p \in [b_\alpha, +\infty). \end{cases}$$

Concerning the initial data, we make the following assumption

$$\begin{aligned} &u_0 \text{ is Lipschitz continuous on } \mathbb{R} \text{ and a.e.} \\ &(u_0)_x \in [a_L, c_L] \text{ if } x < 0 \text{ and } (u_0)_x \in [a_R, c_R] \text{ if } x > 0. \end{aligned} \tag{3}$$

We set

$$H_0 := \max_{\alpha = L,R} \min_p H_\alpha(p)$$

and for $A \in [H_0, 0]$, we define the effective junction condition $\bar{F}_A : \mathbb{R}^2 \to \mathbb{R}$ by

$$\bar{F}_A(p_L, p_R) := \max\{A, H_L^+(p_L), H_R^-(p_R)\} \tag{4}$$

The goal is then to understand the equation satisfied by

$$\rho := u_x.$$

Heuristics. By [16, 19], we first note that it is known (even if it is not obvious) that ρ is an entropy solution to

$$\begin{cases} \rho_t + H_L(\rho)_x = 0 & \text{if } x < 0 \\ \rho_t + H_R(\rho)_x = 0 & \text{if } x > 0 \\ \rho(0,x) = \rho_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{5}$$

where $\rho_0 = (u_0)_x$ a.e.. We then focus on the main difficulty which is to understand what is the appropriate junction condition satisfied by ρ at $x = 0$. For solutions to conservation laws with strongly convex fluxes, we recall the existence of strong traces of ρ at $x = 0$ (see (12) and also [36]). We denote by $\rho(t, 0^-)$ and $\rho(t, 0^+)$ these traces respectively on the left and on the right. In order to fix a condition at $x = 0$ for the scalar conservation law, following the works of [7] and [24, 35], we look for stationary solutions to (5), that is solutions of the form

$$\rho(t,x) = \begin{cases} k_L & \text{if } x < 0 \\ k_R & \text{if } x > 0 \end{cases} \qquad \text{where } (k_L, k_R) \in Q := [a_L, c_L] \times [a_R, c_R]. \tag{6}$$

Let us note that, if we set

$$u(t,x) = (k_L x - tH_L(k_L))\mathbb{1}_{\{x<0\}} + (k_R x - tH_R(k_R))\mathbb{1}_{\{x>0\}},$$

then $\rho = u_x$ and u is solution to the Hamilton-Jacobi equation (1) on $(0, +\infty) \times \mathbb{R}\backslash\{0\}$. Since we want u to be continuous at 0, this implies that the k_α have to satisfy the Rankine-Hugoniot condition

$$H_L(k_L) = H_R(k_R).$$

Moreover, u satisfies the junction condition in (1) iff

$$H_R(k_R) = H_L(k_L) = \max(A, H_L^+(k_L), H_R^-(k_R)).$$

Following [7, 24, 35], we then define the the set $\mathcal{G}_A$ (that we call a germ) as

$$\mathcal{G}_A := \left\{(k_L, k_R) \in Q,\ H_R(k_R) = H_L(k_L) = \max(A, H_L^+(k_L), H_R^-(k_R))\right\}, \tag{7}$$

where Q is defined in (6). We will explain in Proposition 2.10 that this germ is maximal, L^1-dissipative and complete. Hence the following scalar conservation law

$$\begin{cases} \rho_t + H_L(\rho)_x = 0 & \text{if } x < 0 \\ \rho_t + H_R(\rho)_x = 0 & \text{if } x > 0 \\ (\rho(t,0^-), \rho(t,0^+)) \in \mathcal{G}_A & \text{a.e} \\ \rho(0,x) = \rho_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{8}$$

is well-posed (see [7]).

Remark 1.1. Even if this formal calculation does give some kind of heuristics, we recall that u is a viscosity solution to (1) whereas ρ is an entropy solution to (8). The theories for these notions of solutions both emerge from the vanishing viscosity method. As solutions of non-linear first order equations, both solutions lack regularity in order to use proper derivatives (one can typically expect $u \in W^{1,\infty}(\mathbb{R}^+ \times \mathbb{R})$ and $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$). But, in order to recover well-posedness of these equations, the theories differ greatly. Entropy solutions must verify a family of integral inequalities (the so-called entropy inequalities) whereas viscosity solutions must verify pointwise conditions when "touched" by above or below by regular test functions (the viscosity sub. or supersolution conditions). As such, even with the formal calculations, the correspondence result can seem quite mysterious.

The first main result of this paper is the following theorem, which makes rigorous the previous computations.

Theorem 1.2 (Viscosity versus entropy solutions: flux limited conditions)**.** *Let u_0 satisfy* (3) *and let us set $\rho_0 = (u_0)_x$. Let $H_{L,R}$ satisfying* (2)*. Let u be the unique viscosity solution of* (1) *in the sense of Definition 2.1 and ρ be the unique $\mathcal{G}_A$-entropy solution of* (8) *in the sense of Definition 2.7. Then, in the distributional sense, we have*

$$u_x = \rho.$$

We propose two different proofs for this result. The first proof consists to prove Theorem 1.3 below which implies Theorem 1.2. This proof uses numerical schemes for (1) and (8). More precisely, we propose a finite difference scheme for (1) and we consider the numerical derivative of the numerical solution, which gives a finite volume scheme for (8). Since we have the convergence for the two schemes, we recover the result by passing to the limit. The first interest of this proof is that it will be convenient to generalize it in a future work to the non-convex case. The second interest is that it can be extended to the important situation of a more general junction condition, as presented below.

The second proof is a direct approach which uses approximations by semi-algebraic functions (see Section 5), and the nice properties of the derivatives of semi-algrebraic functions.

General junction conditions. Up to this point, we only considered a flux-limiter type of junction condition (with flux-limiter A) at the junction point $x = 0$. However it is known that, in the specific setting considered here, a large class of coupling conditions can be equivalently treated as a flux-limiter. Then we present our result in this larger class. More precisely, we want to consider the general problem

$$\begin{cases} u_t + H_L(u_x) = 0 & \text{if } x < 0 \\ u_t + H_R(u_x) = 0 & \text{if } x > 0 \\ u_t + F_0(u_x(t,0^-), u_x(t,0^+)) = 0 & \text{if } x = 0 \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{9}$$

where the function $F_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called a *desired coupling condition* and satisfies the following conditions

$$\begin{cases} \textbf{(Regularity)} & F_0 \text{ is Lipschitz continuous and piecewise } C^1(\mathbb{R}^2) \\ \textbf{(Monotonicity)} & \begin{array}{c} F_0 \text{ is non decreasing in the first variable} \\ \text{and non increasing in the second one} \end{array} \\ \textbf{(Semi-coercivity)} & \displaystyle\lim_{\max\{p_L, -p_R\} \to +\infty} F_0(p_L, p_R) = +\infty \\ \textbf{(Boundedness of the solution)} & F_0(a_L, a_R) = F_0(c_L, c_R) = 0 \end{cases} \tag{10}$$

In (10), the required zero value of F_0 in the last line, is consistent with the zero values of the Hamiltonians in (2). Note that assumption (10) is naturally satisfied if the junction condition is of the form (4). More generally, assumption (10) will imply that the solutions stay in the box $Q = [a_L, c_L] \times [a_R, c_R]$, if the initial data belongs to this box. In particular, only the value of F_0 restricted to the box Q will then play a role. Working with functions defined on the whole $\mathbb{R}^2$ and satisfying in particular the third condition of (10) is only technical to get existence of a solution, and can afterwards be removed. Still it is convenient to assume (10) in order to use safely the results that we invoke for instance from [27].

It is well-known that, in general, one cannot expect to have a strong viscosity solution for (9), in the sense that the junction condition is satisfied in the viscosity sense (see Definition 2.1 below). Nevertheless, it is always possible to define a weak viscosity solution, meaning that either the equation or the junction condition is satisfied at $x = 0$ (see Definition 2.2 below). We are now interested in the corresponding SCL. Formally, we can make the following calculation with $\rho := u_x$ (say with $H(\rho) = 0$ at $x = \pm\infty$)

$$u_t = \partial_t \int_{-\infty}^{x} \rho \,\mathrm{d}x = \int_{-\infty}^{x} \partial_t \rho \,\mathrm{d}x = -\int_{-\infty}^{x} (H(\rho))_x \,\mathrm{d}x = -H(\rho).$$

Then for a solution u of problem (9), we expect $\rho := u_x$ to solve the scalar conservation law problem

$$\begin{cases} \rho_t + H_L(\rho)_x = 0 & \text{if } x < 0 \\ \rho_t + H_R(\rho)_x = 0 & \text{if } x > 0 \\ H_L(\rho(t,0^-)) = H_R(\rho(t,0^+)) = F_0(\rho(t,0^-), \rho(t,0^+)) & \text{if } x = 0 \\ \rho(0,x) = \rho_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{11}$$

However, this problem does not admit a solution whose traces satisfy the third equation of (11) in general for any given F_0 satisfying (10) and one has to relax the junction condition. We recall in Subsection 2.2 how this problem has to be solved.

We then have the following result which generalizes Theorem 1.2.

Theorem 1.3 (Viscosity versus entropy solutions: desired conditions)**.** *Let u_0 satisfy (3) and denote by $\rho_0 = (u_0)_x$. Let $H_{L,R}$ satisfying (2) and F_0 satisfying (10). Let u be the unique weak viscosity solution to (9) in the sense of Definition 2.2 and ρ be the unique F_0-admissible solution to (11) in the sense of Definition 2.14. Then, in the distributional sense, we have*

$$u_x = \rho.$$

As an important byproduct of our proof of Theorem 1.3 by numerical schemes, we also have the following meta-theorem, which statement is made precise in Theorem 3.3.

Theorem 1.4 (Numerical approximation for SCL: desired condition)**.** *Let ρ^Δ (with $\Delta = (\Delta t, \Delta x)$) be the numerical solution of (11) (with the junction condition given by F_0). Then, there exists a flux limiter A_{F_0} depending on F_0 such that, as Δ goes to zero, ρ^Δ converges to the unique solution to (8) with A replaced by A_{F_0}.*

Note that this result was already known at the Hamilton-Jacobi level (see [26]). By contrast, this result is completely new here at the level of conservation laws, for finite volume schemes with a junction with general transmission condition F_0. The main point is the following: in the numerical scheme, we put an approximation of the junction condition F_0, but at the limit when the space and time steps go to 0, we recover the relaxed flux-limited junction condition $\bar{F}_{A_{F_0}}$ instead of F_0, where the flux limiter A_{F_0} is given in Lemma 2.3.

Remark 1.5. It is also possible to consider an even simpler junction constituted of only one edge and one vertex. In that case, our result remains valid with analogous proofs. For instance, the analogue of Theorem 1.2 is precisely the following.

Theorem 1.6 (Viscosity versus entropy solutions: the half line)**.** *Let u_0 satisfy* (3) *on* $(0,+\infty)$ *and let us set* $\rho_0 = (u_0)_x$*. Let* H_R *satisfying* (2) *and* $A \in [\min H_R, 0]$*. Let* u *be the unique viscosity solution to*

$$\begin{cases} u_t + H_R(u_x) &= 0 & \textit{if} & x > 0 \\ u_t + \max\left\{A, H_R^-(u_x(t,0^+))\right\} &= 0 & \textit{if} & x = 0 \\ u(0,x) &= u_0(x) & \textit{for} & x \in (0,+\infty) \end{cases}$$

and ρ *be the unique* $\mathcal{G}_A^1$*-entropy solution to*

$$\begin{cases} \rho_t + H_R(\rho)_x &= 0 & \textit{if} & x > 0 \\ \rho(t,0^+) &\in \mathcal{G}_A^1 & \textit{if} & x = 0 \quad \textit{and for a.e. } t \in (0,+\infty) \\ \rho(0,x) &= \rho_0(x) & \textit{for} & x \in (0,+\infty). \end{cases}$$

with

$$\mathcal{G}_A^1 := \left\{k_R \in \mathbb{R}, \quad H_R(k_R) = \max\left\{A, H_R^-(k_R)\right\}\right\}$$

Then, in the distributional sense, we have

$$u_x = \rho.$$

The above notion of solution for a scalar conservation law with boundary condition for $A := H_R^+(u_D)$ is equivalent to the Dirichlet condition $u(t,0^+) = u_D$ in the sense of standard Bardos-Leroux-Nedelec approach (see [11, 20]).

1.2. **Outline.** Our work is mostly based on the known results of [7, 26, 27]. Then our paper is presented as follows.

In Section 2, we recall the different definitions of solutions used throughout the paper and we detail the links between them.

Subsection 2.1 is dedicated to the definitions of the notions of viscosity solutions for (9) or (1). The well-posedness results for these notions are not detailled but can be found in [27]. We also detail the link between viscosity solutions to (9) and viscosity solutions to (1) and we explain how the flux limiter A_{F_0} is constructed from a general function F_0.

Subsection 2.2 deals with the notions of entropy solutions to (8) or (11). We prove that the germ $\mathcal{G}_A$ enters in the framework of [7]. This guarantees existence and uniqueness for our entropy solutions. We also prove Lemma 2.13 which shows that the germ $\mathcal{G}_A$ can be "generated" by a strictly smaller subset $\mathcal{E}_A$ (which only contains three points). In the integral formulation of a finite volume scheme approximation of the entropy solution, the fact that some error term R_{F_0} (defined in (33)) vanishes on this set $\mathcal{E}_A$ will be used later on, as a key property for the proof of Theorem 3.3.

Section 3 is devoted to the study of solutions u_Δ to a finite difference numerical scheme for (9) (Subsection 3.1) and solution ρ_Δ to a finite volume scheme for (11) (Subsection 3.1), where $\Delta = (\Delta t, \Delta x)$. We mainly prove Theorems 1.2 and 1.3 using these numerical schemes in Section 4, and show the commutative diagram

$$\begin{array}{lcl} u_\Delta & \underset{\Delta\to 0}{\to} & u \\ \downarrow_{D_{\Delta x}} & & \downarrow_{\partial_x} \\ D_{\Delta x} u_\Delta =: \rho_\Delta & \underset{\Delta\to 0}{\to} & \rho = \partial_x u \end{array}$$

where $D_{\Delta x} u_\Delta(t,x) := \frac{u_\Delta(t,x+\Delta x) - u_\Delta(t,x)}{\Delta x}$. Hence the correspondence holds true at the level of numerical solutions, and stays true at the limit $\Delta \to 0$, because of the uniqueness theories for both Hamilton-Jacobi and Conservation Laws equations with junctions.

In Section 4 we use the numerical schemes studied previously to prove Theorem 1.2 and Theorem 1.3.

In Section 5, we also propose an independent and direct proof of Theorem 1.2, using regularization with semi-algebraic functions, instead of numerical approximations. This provides second interesting tool that can be used in other situations, when necessary.

Finally, Section A is an appendix where we collect complementary results, which are either new, or not accessible with full details in the literature. In Subsection A.1 we give discrete entropy inequalities on a junction, in Subsection A.2 we give a local compactness result for numerical solutions of conservation laws with strictly convex flux, and in Subsection A.3 we show that Hamilton-Jacobi germs are not L^1-dissipative for $N \geqslant 3$ branches.

2. Notions of solution.

In this section we recall the definition of viscosity solutions to equation (9) in Subsection 2.1. We also explain how we construct the flux limiter A_{F_0} from a general condition F_0. In Subsection 2.2, we recall the definition of entropy solutions to equation (11).

2.1. Definition of weak and strong solutions for Hamilton-Jacobi equations.

We begin to recall the notion of weak viscosity solutions to (9). We consider the set of test functions on the junction $J_T := (0,T) \times \mathbb{R}$:

$$\begin{aligned} C^1_\wedge(J_T) := \{&\varphi \in C^0(J_T), \\ &\text{the restrictions of } \varphi \text{ to } (0,T)\times(-\infty,0] \text{ and to } (0,T)\times[0,\infty) \text{ are } C^1\}. \end{aligned}$$

We also recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined on $[0,T) \times \mathbb{R}$,

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y).$$

We begin with the notion of strong viscosity solution for which the junction condition is satisfied in a strong sense.

Definition 2.1 (Strong viscosity solution)**.** Let us consider a function $u : J_T \to \mathbb{R}$.
i) (Strong viscosity subsolution) We say that u is a strong viscosity subsolution to (9) if for any point $(t_0,x_0) \in J_T$ and any function $\varphi \in C^1_\wedge(J_T)$ such that $u^* - \varphi$ reaches a local maximum at (t_0,x_0) we have

$$\begin{cases} \varphi_t(t_0,x_0) + H_L(\varphi_x(t_0,x_0)) \leqslant 0 & \text{if } x_0 < 0 \\ \varphi_t(t_0,x_0) + H_R(\varphi_x(t_0,x_0)) \leqslant 0 & \text{if } x_0 > 0 \end{cases}$$

when $x_0 \neq 0$ and

$$\varphi_t(t_0,x_0) + F_0(\varphi_x(t_0,x_0^-), \varphi_x(t_0,x_0^+)) \leqslant 0$$

when $x_0 = 0$. We call u a strong F_0-subsolution.
ii) (Strong viscosity supersolution) We say that u is a strong viscosity supersolution to (9) if for any point $(t_0,x_0) \in J_T$ and any function $\varphi \in C^1_\wedge(J_T)$ such that

$u_* - \varphi$ reaches a local minimum at (t_0, x_0) we have

$$\begin{cases} \varphi_t(t_0,x_0) + H_L(\varphi_x(t_0,x_0)) \geqslant 0 & \text{if } x_0 < 0 \\ \varphi_t(t_0,x_0) + H_R(\varphi_x(t_0,x_0)) \geqslant 0 & \text{if } x_0 > 0 \end{cases}$$

when $x_0 \neq 0$ and

$$\varphi_t(t_0,x_0) + F_0(\varphi_x(t_0,x_0^-), \varphi_x(t_0,x_0^+)) \geqslant 0$$

when $x_0 = 0$. We call u a strong F_0-supersolution.
iii) (Strong viscosity solution) We say that u is a strong viscosity solution to (9), if u is a strong viscosity subsolution to (9), and u is a strong viscosity supersolution to (9). We call u a strong F_0-solution.

A first result of Imbert, Monneau [27] is that when the junction condition is of the form $\bar{F}_A$ in (4), then the junction condition is satisfied strongly as in the previous definition. Nevertheless, this is not true for general junction condition and one has to consider weak viscosity solutions for which either the junction condition or the equation is satisfied at $x = 0$.

Definition 2.2 (Weak viscosity solution)**.** Let us consider a function $u : J_T \to \mathbb{R}$
i) (Weak viscosity subsolution) We say that u is a weak viscosity subsolution to (9) if for any point $(t_0, x_0) \in J_T$ and any function $\varphi \in C^1_\wedge(J_T)$ such that $u^* - \varphi$ reaches a local maximum at (t_0, x_0) we have

$$\begin{cases} \varphi_t(t_0,x_0) + H_L(\varphi_x(t_0,x_0)) \leqslant 0 & \text{if } x_0 < 0 \\ \varphi_t(t_0,x_0) + H_R(\varphi_x(t_0,x_0)) \leqslant 0 & \text{if } x_0 > 0 \end{cases}$$

when $x_0 \neq 0$ and

$$\varphi_t(t_0,x_0) + H_L(\varphi_x(t_0,x_0^-)) \leqslant 0 \quad \text{or} \quad \varphi_t(t_0,x_0) + H_R(\varphi_x(t_0,x_0^+)) \leqslant 0$$
$$\text{or} \quad \varphi_t(t_0,x_0) + F_0(\varphi_x(t_0,x_0^-), \varphi_x(t_0,x_0^+)) \leqslant 0$$

when $x_0 = 0$. We call u a weak F_0-subsolution.
ii) (Weak viscosity supersolution) We say that u is a weak viscosity supersolution to (9) if for any point $(t_0, x_0) \in J_T$ and any function $\varphi \in C^1_\wedge(J_T)$ such that $u_* - \varphi$ reaches a local minimum at (t_0, x_0) we have

$$\begin{cases} \varphi_t(t_0,x_0) + H_L(\varphi_x(t_0,x_0)) \geqslant 0 & \text{if } x_0 < 0 \\ \varphi_t(t_0,x_0) + H_R(\varphi_x(t_0,x_0)) \geqslant 0 & \text{if } x_0 > 0 \end{cases}$$

when $x_0 \neq 0$ and

$$\varphi_t(t_0,x_0) + H_L(\varphi_x(t_0,x_0^-)) \geqslant 0 \quad \text{or} \quad \varphi_t(t_0,x_0) + H_R(\varphi_x(t_0,x_0^+)) \geqslant 0$$
$$\text{or} \quad \varphi_t(t_0,x_0) + F_0(\varphi_x(t_0,x_0^-), \varphi_x(t_0,x_0^+)) \geqslant 0$$

when $x_0 = 0$. We call u a weak F_0-supersolution.
iii) (Weak viscosity solution) We say that a locally bounded function u is a weak viscosity solution to (9), if u is a weak viscosity subsolution to (9), and u is a weak viscosity supersolution to (9). We call u a weak F_0-solution.

For completeness' sake, let us mention that the existence and uniqueness (using a comparison principle) of the solutions of (1) and (9) is proven in [27]. Another important result of Imbert, Monneau [27] is that it is possible to relax the junction condition in order to make the solution satisfy the junction condition strongly. We give the details of the construction of the relaxation below.

Given a desired junction condition F_0 satisfying (10), we want to define the relaxed junction condition such that the weak viscosity solution to (9) satisfies the relaxed junction condition strongly. This junction condition is of the form $\bar{F}_{A_{F_0}}$ (see (4)), where the constant A_{F_0} depends on F_0 and is defined as the unique constant such that there exists $\bar{p} = (\bar{p}_l, \bar{p}_R)$ such that

$$A_{F_0} = F_0(\bar{p}) = H_R^+(\bar{p}_R) = H_L^-(\bar{p}_L).$$

More precisely, we have the following lemma (see also [27, Lemma 2.13]):

Lemma 2.3 (Definition of the flux limiter A_{F_0}). *Let F_0 and H_α, $\alpha = L, R$ satisfy respectively (10) and (2). We denote by*

$$H_0 := \max_{\alpha=L,R} \min H_\alpha(p) = \max(H_L(b_L), H_R(b_R)),$$

where we recall that b_α is the point of minimum of H_α.

Let $\bar{b}_R$ be the maximal p such that $H_R(p) = H_0$, and $\bar{b}_L$ be the minimal p such that $H_L(p) = H_0$. If $F_0(\bar{b}_L, \bar{b}_R) < H_0$, we set $A_{F_0} := H_0$. If $F_0(\bar{b}_L, \bar{b}_R) \geqslant H_0$, then we define the set

$$\Lambda := \{\lambda \in \mathbb{R}, \exists \bar{p} = (\bar{p}_L, \bar{p}_R) \text{ s.t. } \lambda = F_0(\bar{p}) = H_R^+(\bar{p}_R) = H_L^-(\bar{p}_L)\}.$$

Then Λ is non-empty and is reduced to a singleton. We denote by A_{F_0} the unique constant such that

$$\Lambda = \{A_{F_0}\}.$$

We also have $A_{F_0} \in [H_0, 0]$.

Moreover, if $F_0 = \bar{F}_A$ with $A \in [H_0, 0]$, then $A_{F_0} = A$.

Proof. If $F_0(\bar{b}_L, \bar{b}_R) < H_0$, then there is nothing to prove. Hence we now assume that $F_0(\bar{b}_L, \bar{b}_R) \geqslant H_0$ and consider the set Λ of the statement.

Step 1. Λ is non empty.

Given $\lambda > H_0$, we define p_α^λ such that

$$H_L^-(p_L^\lambda) = H_R^+(p_R^\lambda) = \lambda.$$

For $\lambda = H_0$, we set

$$p_\alpha^{H_0} := \lim_{\lambda \to (H_0)^+} p_\alpha^\lambda$$

which satisfies $p_R^{H_0} = \bar{b}_R$ and $p_L^{H_0} = \bar{b}_L$. In particular, the map $\lambda \mapsto p_R^\lambda$ is continuous and increasing, while the map $\lambda \mapsto p_L^\lambda$ is continuous and decreasing. Since F_0 is non-decreasing in the first variable and non-increasing in the second one, the map $\lambda \mapsto F_0(p_L^\lambda, p_R^\lambda)$ is non-increasing.

We then define the application $K : \lambda \mapsto F_0(p_L^\lambda, p_R^\lambda) - \lambda$. When $F_0(p_L^{H_0}, p_R^{H_0}) = F_0(\bar{b}_L, \bar{b}_R) \geqslant H_0$, we get that $K(H_0) \geqslant 0$. Using the fact of $\lambda \mapsto F_0(p_L^\lambda, p_R^\lambda)$ is non-increasing, we also have

$$K(\lambda) \leqslant F_0(\bar{b}_L, \bar{b}_R) - \lambda$$

and so for λ large enough, we have $K(\lambda) < 0$. By continuity, this implies that there exists $\bar{\lambda} \geqslant H_0$ such that $K(\bar{\lambda}) = 0$. We set $\bar{p} = (p_L^{\bar{\lambda}}, p_R^{\bar{\lambda}})$ and we get

$$F_0(\bar{p}) = \bar{\lambda} = H_L^-(\bar{p}_L) = H_R^+(\bar{p}_R),$$

i.e. $\bar{\lambda} \in \Lambda$.

Notice also that $(p_L^\lambda, p_R^\lambda)_{|\lambda=0} = (a_L, c_R)$ and then

$$K(0) = F_0(a_L, c_R) \leqslant F_0(c_L, c_R) = 0$$

where we have used the monotonicities of F_0 and condition (10). Therefore we deduce that $\bar{\lambda} \in [H_0, 0]$.

Step 2. Λ is reduced to a singleton and conclusion.

Assume that there exists $\bar{\lambda}_1, \bar{\lambda}_2 \in \Lambda$ such that $\bar{\lambda}_1 > \bar{\lambda}_2$. Hence, there exists p_R^i and p_L^i such that

$$\bar{\lambda}_1 = F_0(p_L^1, p_R^1) = H_L^-(p_L^1) = H_R^+(p_R^1) > \bar{\lambda}_2 = F_0(p_L^2, p_R^2) = H_L^-(p_L^2) = H_R^+(p_R^2)$$

In particular, we have $p_L^1 < p_L^2$ and $p_R^1 > p_R^2$. By monotonicity of F_0, this implies that

$$F_0(p_L^1, p_R^1) \leqslant F_0(p_L^2, p_R^2),$$

which is a contradiction. Therefore $\Lambda = \{\bar{\lambda}\}$ with $A_{F_0} := \bar{\lambda} \in [H_0, 0]$. □

Remark 2.4. Notice that the fourth line of assumption (10) on F_0 has been used to insure that $A_{F_0} \leqslant 0$ (and indeed only $F_0(a_L, c_R) \leqslant 0$ is really used at this stage).

As explained before, the solution of (9) is satisfied in a weak sense for general F_0. Nevertheless, it is possible to relax the junction condition in order to make the solution satisfy the junction condition strongly. More precisely, we have the following theorem, given in [27, Proposition 2.12].

Theorem 2.5 (General junction conditions reduce to flux limited ones)**.** *Assume that H_L and H_R satisfy (2) and that F_0 satisfies (10). Then u is a continuous weak viscosity solution to (9), if and only if u is a strong viscosity solution to (1) with $\bar{F}_A$ for $A := A_{F_0}$ defined above in Lemma 2.3.*

We also have:

Theorem 2.6 (A priori bounds on the gradient of u)**.** *Assume that H_L and H_R satisfy (2), that F_0 satisfies (10) and that the initial data u_0 is uniformly continuous. Then there exists a unique weak viscosity solution u to (9) in the sense of Definition 2.2.*

i) *Moreover the solution u is continuous on $[0, +\infty) \times \mathbb{R}$.*

ii) *If u_0 satisfies (3), then u is Lipschitz continuous on $[0, +\infty) \times \mathbb{R}$ and such that for all $t \geqslant 0$, the function $u(t, \cdot)$ satisfies (3).*

Proof of Theorem 2.6. The existence and uniqueness of a weak viscosity solution u of (9) follows from Theorem 1.5 in [27]. Point i) follows from the comparison principle Theorem 1.4 in [27].

In order to show point ii), we now work with strong viscosity solutions. From Theorem 2.5, the function u is also a strong viscosity solution of (1) with $\bar{F}_A$ for $A := A_{F_0} \leqslant 0$ defined in Lemma 2.3. Using $H_R(a_R) = H_L(c_L) = 0 \geqslant A_{F_0}$, we simply remark that $v(t, x) := u_0(x)$ is a strong subsolution to (1) in the sense of Definition 2.1. Because every strong viscosity subsolution is in particular a weak viscosity subsolution, and because the comparison principle Theorem 1.4 in [27] holds true for weak viscosity sub/supersolutions, we deduce that $u(h, \cdot) \geqslant u_0$ for all times $h \geqslant 0$. Applying again the comparison principle for the solutions of the Cauchy problems respectively with initial data $u(h, \cdot)$ and u_0, we deduce that $u(t+h, \cdot) \geqslant u(t, \cdot)$ for all $t, h \geqslant 0$. This implies that $u_t \geqslant 0$. From the PDE, we have $u_t = -H_\alpha(u_x)$ on each branch α. This implies $H_\alpha(u_x) \leqslant 0$ and the coercivity of H_α implies that u is Lipschitz continuous in x. More precisely, we get $u_x \in [a_\alpha, c_\alpha]$ a.e. on the branch α, which means precisely that $u(t, \cdot)$ satisfies (3). Now from the PDE, we also deduce that $0 \leqslant u_t \leqslant \max_\alpha \{-\min H_\alpha\}$, which itself implies that u is Lipschitz continuous in time. □

2.2. **Definition of solution for conservation law.** We first recall that any solution to a Scalar Conservation Law for $x \in (0,+\infty)$ with strongly convex flux has a strong trace at $x=0$ (see Panov [36, Theorem 2.4]). For any real function $f \in L^\infty((0,T)\times\mathbb{R})$, we denote by $\gamma_{L,R} f$ the strong left and right traces of f at $x=0$ when they exist. For instance for the left trace, this means that

$$\operatorname{ess}\lim_{x\to 0^-} \int_0^T |f(t,x) - \gamma_L f(t)|\,\mathrm{d}t = 0. \tag{12}$$

Here we present the notion of solution we will consider for (8). We consider an effective junction condition F_A as defined in (4) and we recall that the corresponding germ $\mathcal{G}_A$ is given by (7).

Definition 2.7 (Strong entropy solution)**.** Let u_0 satisfying (3) and denote by $\rho_0 = (u_0)_x$. We say that $\rho \in L^\infty((0,T)\times\mathbb{R})$ is a "strong" $\mathcal{G}_A$-entropy solution to (8) if

1. ρ is a weak solution to
$$\begin{cases} \rho_t + H_L(\rho)_x = 0 & \text{if } x<0 \\ \rho_t + H_R(\rho)_x = 0 & \text{if } x>0. \end{cases}$$
2. For any $\phi_L \in C_c^\infty([0,T)\times(-\infty,0))$ (resp. $\phi_R \in C_c^\infty([0,T)\times(0,+\infty))$) that is non-negative, for any $k_L \in [a_L,c_L]$ (resp. $k_R \in [a_R,c_R]$) the following entropy inequalities hold
$$\iint_{(0,T)\times\mathbb{R}^-} |\rho - k_L|(\phi_L)_t + \operatorname{sign}(\rho-k_L)\left[H_L(\rho)-H_L(k_L)\right](\phi_L)_x\,\mathrm{d}t\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^-} |\rho_0(x) - k_L|\phi_L(0,x)\,\mathrm{d}x \geqslant 0$$
$$\Bigg(\text{resp.}$$
$$\iint_{(0,T)\times\mathbb{R}^+} |\rho - k_R|(\phi_R)_t + \operatorname{sign}(\rho-k_R)\left[H_R(\rho)-H_R(k_R)\right](\phi_R)_x\,\mathrm{d}t\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^+} |\rho_0(x) - k_R|\phi_R(0,x)\,\mathrm{d}x \geqslant 0\Bigg).$$
3. The strong traces satisfy the germ condition
$$(\gamma_L\rho(t), \gamma_R\rho(t)) \in \mathcal{G}_A \quad \text{for a.e. } t \in (0,T).$$

As proved in [7], this notion of solution grants existence and uniqueness as soon as the germ $\mathcal{G}_A$ is L^1 dissipative, maximal and complete. We begin by recalling the notion of L^1-dissipativity, maximality and completeness of a germ.

Definition 2.8 (Germ and properties)**.**
i) (germ) We say that a set $\mathcal{G} \subset \mathbb{R}^2$ is a germ if any element of $\mathcal{G}$ satisfies the Rankine-Hugoniot condition, i.e.

$$H_L(k_L) = H_R(k_R) \quad \forall k = (k_L,k_R) \in \mathcal{G}.$$

ii) (L^1-dissipative germ) We say that a germ $\mathcal{G}$ is L^1-dissipative if for any pair of elements in the germ, i.e. $k=(k_L,k_R), \hat{k} = (\hat{k}_L,\hat{k}_R) \in \mathcal{G}$, we have

$$\operatorname{sgn}(k_L-\hat{k}_L)(H_L(k_L)-H_L(\hat{k}_L)) \geqslant \operatorname{sgn}(k_R-\hat{k}_R)(H_R(k_R)-H_R(\hat{k}_R)).$$

iii) (maximal L^1-dissipative germ) A L^1-dissipative germ $\mathcal{G}$ is called maximal if there is no L^1-dissipative germ $\bar{\mathcal{G}}$ having $\mathcal{G}$ as a strict subset.

iv) (complete L^1-dissipative germ) A L^1-dissipative germ $\mathcal{G}_A$ is called complete (on the box Q), if for every $\hat{k} = (\hat{k}_L, \hat{k}_R) \in Q$, there exists a strong $\mathcal{G}_A$-entropy solution of (8), with initial data $\rho_0 = \hat{k}_L 1_{(-\infty,0)} + \hat{k}_R 1_{(0,+\infty)}$.

We then have the following theorem.

Theorem 2.9 (Existence and uniqueness for (8), [7])**.** *Let ρ_0 be an initial data satisfying $\rho_0((-\infty, 0)) \times \rho_0((0, +\infty)) \subset Q$.*

- (i) *If the germ $\mathcal{G}_A$ is L^1-dissipative and maximal, there exists at most one solution to (8) in the sense of Definition 2.7.*
- (ii) *Furthermore, if the germ $\mathcal{G}_A$ is also complete (on the box Q), then there exists a unique solution to (8) in the sense of Definition 2.7.*

In order to apply this result to (8), it remains to show that the germ $\mathcal{G}_A$ defined in (7) is L^1-dissipative maximal and complete.

Proposition 2.10 ($\mathcal{G}_A$ is L^1-dissipative, maximal and complete)**.** *Let $A \in [H_0, 0]$. We recall that $F_A(k_L, k_R) = \max\{A, H_L^-(k_L), H_R^+(k_R)\}$. Then, the set $\mathcal{G}_A$ defined by*

$$\begin{aligned}\mathcal{G}_A &= \left\{(k_L, k_R) \in \mathbb{R}^2,\ H_R(k_R) = H_L(k_L) = \bar{F}_A(k_L, k_R)\right\}\\ &= \left\{\begin{array}{c}(k_L, k_R) \in \mathbb{R}^2,\ H_R(k_R) = H_L(k_L) \geqslant A \text{ and}\\ [\text{either } H_R(k_R) = A, \text{or } H_R(k_R) = H_R^-(k_R), \text{or } H_L(k_L) = H_L^+(k_L)]\end{array}\right\}\end{aligned} \tag{13}$$

is a maximal and complete L^1-dissipative germ.

Remark 2.11. This Definition of the germ $\mathcal{G}_A$ is close to the definition of viscosity solution for Hamilton-Jacobi equations. One can also relate this germ to the classical flux limited notion of solution for scalar conservation law with applications to traffic (see [17] and [6]).

This germ is also the unique maximal L^1-dissipative germ containing $(\bar{p}_L, \bar{p}_R)$ where $(\bar{p}_L, \bar{p}_R)$ is the unique couple such that $A = H_R^+(\bar{p}_R) = H_L^-(\bar{p}_L)$. This corresponds to the so called $(\mathcal{A}, \mathcal{B})$-connection if one takes $(\mathcal{A}, \mathcal{B}) := (\bar{p}_L, \bar{p}_R)$ (see [2]). Notice also that contrarily to [2] and [7], we do not need any crossing condition to be satisfied.

Finally, we can also link this definition with the monotone graph approach introduced in [3]. If one takes $\Gamma_0 := \{(p_L, p_R, F_0(p_L, p_R), F_0(p_L, p_R)),\quad (p_L, p_R) \in \mathbb{R}^2\}$ then the projected maximal monotone graph is $\Gamma = \{(p_L, p_R, \bar{F}_A(p_L, p_R), \bar{F}_A(p_L, p_R)),$ $(p_L, p_R) \in \mathcal{G}_{A_{F_0}}\}$.

Proof of Proposition 2.10. We begin to prove that the germ is L^1-dissipative. Let $k = (k_L, k_R), \hat{k} = (\hat{k}_L, \hat{k}_R) \in \mathcal{G}_A$. We have to show that

$$\operatorname{sgn}(k_L - \hat{k}_L)(H_L(k_L) - H_L(\hat{k}_L)) \geqslant \operatorname{sgn}(k_R - \hat{k}_R)(H_R(k_R) - H_R(\hat{k}_R)). \tag{14}$$

The result is obvious if $H_L(k_L) - H_L(\hat{k}_L) = H_R(k_R) - H_R(\hat{k}_R) = 0$ or if $H_L(k_L) - H_L(\hat{k}_L) = H_R(k_R) - H_R(\hat{k}_R) > 0$ and $k_L > \hat{k}_L$. Let us now assume to fix the ideas that $H_L(k_L) - H_L(\hat{k}_L) = H_R(k_R) - H_R(\hat{k}_R) > 0$ and $k_L < \hat{k}_L$ (the case

$H_L(k_L) - H_L(\hat{k}_L) = H_R(k_R) - H_R(\hat{k}_R) < 0$ and $k_L > \hat{k}_L$ is obtained exchanging k and $\hat{k}$). We need to check that $k_R < \hat{k}_R$. Note that

$$H_L(k_L) > H_L(\hat{k}_L) \geqslant H_L^+(\hat{k}_L) \geqslant H_L^+(k_L).$$

Since $H_R(k_R) = H_L(k_L) > H_L(\hat{k}_L) \geqslant A$, and since $k \in \mathcal{G}_A$, we necessarily have $H_R^-(k_R) = H_R(k_R)$. Therefore

$$H_R^-(k_R) = H_R(k_R) > H_R(\hat{k}_R) \geqslant H_R^-(\hat{k}_R),$$

which implies that $\hat{k}_R > k_R$. This proves (14) and the L^1 dissipativity of $\mathcal{G}_A$.

To prove the maximality of $\mathcal{G}_A$, let us now fix some $k \in \mathbb{R}^2$ such that $H_L(k_L) = H_R(k_R)$ and assume that (14) holds for any $\hat{k} \in \mathcal{G}_A$. We have to check that $k \in \mathcal{G}_A$. We first check that $H_L(k_L) = H_R(k_R) \geqslant A$. By contradiction, assume that $H_L(k_L) = H_R(k_R) < A$. We take $\hat{k}$ such that $\hat{k}_L$ is the smallest element in $(H_L)^{-1}(\{A\})$ and $\hat{k}_R$ the largest in $(H_R)^{-1}(A)$. Then $\hat{k} \in \mathcal{G}_A$, $\hat{k}_L < k_L$, $\hat{k}_R > k_R$ and $H_L(k_L) < A = H_L(\hat{k}_L)$ and similarly $H_R(k_R) < A = H_R(\hat{k}_R)$, which contradicts (14). So $H_L(k_L) = H_R(k_R) \geqslant A$.

We now prove that

$$H_L(k_L) = A \text{ or } H_R(k_R) = H_R^-(k_R) \text{ or } H_L(k_L) = H_L^+(k_L). \tag{15}$$

By contradiction, assume that

$$H_L(k_L) > A,\ H_L^+(k_L) < H_L(k_L) \text{ and } H_R^-(k_R) < H_R(k_R).$$

Let us choose $\hat{k} \in \mathcal{G}_A$ such that $H_L(\hat{k}_L) = H_L^+(\hat{k}_L) = A$ and $H_R(\hat{k}_R) = H_R^-(\hat{k}_R) = A$. Then, as H_L and H_R are convex and as $H_L^+(k_L) < H_L(k_L)$ and $H_R^-(k_R) < H_R(k_R)$, we have

$$H_L^+(k_L) = \min H_L \leqslant A = H_L^+(\hat{k}_L),$$

which implies that $\hat{k}_L > k_L$ (equality cannot hold because $H_L(k_L) > A = H_L(\hat{k}_L)$) while

$$H_R^-(k_R) = \min H_R \leqslant A = H_R^-(\hat{k}_R),$$

which implies that $\hat{k}_R < k_R$ (because $H_R(k_R) = H_L(k_L) > A = H_R(\hat{k}_R)$). This yields a contradiction with (14). Therefore k satisfies (15) and belongs to $\mathcal{G}_A$. This shows the maximality of $\mathcal{G}_A$.

The proof of the completeness of the germ $\mathcal{G}_A$ is postponed to Lemma 3.9, where we show the existence of a solution using the convergence of the numerical scheme introduced in Subsection 3.2. □

Remark 2.12. In the case of a junction with $N \geqslant 3$ branches, it is possible to show that the Hamilton-Jacobi germ is never L^1-dissipative (except in the special case where the limiter $A = 0$ which corresponds to no flux at the junction point). See Lemma A.8.

We now present an important result (that will be used in order to prove Theorem 3.3) telling that the gem $\mathcal{G}_A$ is generated by a set of three points :

$$\mathcal{E}_A := \{(a_L, a_R), (c_L, c_R), (\bar{p}_L^A, \bar{p}_R^A)\}, \tag{16}$$

where $(\bar{p}_L^A, \bar{p}_R^A)$ is such that

$$H_L(\bar{p}_L^A) = H_L^-(\bar{p}_L^A) = A = H_R^+(\bar{p}_R^A) = H_R(\bar{p}_R^A).$$

This fact was already mentioned in [2].

Lemma 2.13 ($\mathcal{E}_A$ generates $\mathcal{G}_A$ on Q)**.** *Assume that $A \in [H_0, 0]$. Then the set $\mathcal{E}_A$ generates $\mathcal{G}_A$ on the box Q: namely, for any $(k_L, k_R) \in Q$,*

$$\Big(q_L(k_L, \bar{k}_L) - q_R(k_R, \bar{k}_R) \geqslant 0 \qquad \forall (\bar{k}_L, \bar{k}_R) \in \mathcal{E}_A \Big) \qquad \Longrightarrow \qquad (k_L, k_R) \in \mathcal{G}_A,$$

where, for $\alpha = L, R$, q_α are the entropy fluxes defined by

$$q_\alpha(q, p) = sign(q - p)(H_\alpha(q) - H_\alpha(p)).$$

Proof. We choose $(k_L, k_R) \in Q$ and we will test it with the elements $(\bar{k}_L, \bar{k}_R) \in \mathcal{E}_A$ using the dissipation condition in order to show that $(k_L, k_R) \in \mathcal{G}_A$.

Step 1. recovering Rankine-Hugoniot condition.

We choose $(\bar{k}_L, \bar{k}_R) = (a_L, a_R)$. We then have

$$0 \leqslant q_L(k_L, \bar{k}_L) - q_R(k_R, \bar{k}_R) = \mathrm{sign}(k_L - a_L) H_L(k_L) - \mathrm{sign}(k_R - a_R) H_R(k_R).$$

Since $k_L \geqslant a_L$ and $k_R \geqslant a_R$, we recover that $H_L(k_L) \geqslant H_R(k_R)$. In the same way, taking $(\bar{k}_L, \bar{k}_R) = (c_L, c_R)$, we get $H_L(k_L) \leqslant H_R(k_R)$, which implies that

$$H_L(k_L) = H_R(k_R).$$

Step 2. $H_L(k_L) = H_R(k_R) \geqslant A$**.**

We choose $(\bar{k}_L, \bar{k}_R) = (\bar{p}_L^A, \bar{p}_R^A)$ and by contradiction, we assume that

$$H_L(k_L) = H_R(k_R) < A = H_L(\bar{k}_L) = H_L(\bar{k}_R).$$

Since

$$H_L^-(\bar{k}_L) = H_L(\bar{k}_L) > H_L(k_L) \geqslant H_L^-(k_L),$$

we deduce that $\bar{k}_L < k_L$. In the same way, we get $\bar{k}_R > k_R$. Using that

$$\begin{aligned} 0 &\leqslant q_L(k_L, \bar{k}_L) - q_R(k_R, \bar{k}_R) \\ &= \mathrm{sign}(k_L - \bar{k}_L)(H_L(k_L) - A) - \mathrm{sign}(k_R - \bar{k}_R)(H_R(k_R) - A) < 0 \end{aligned}$$

we get a contradiction.

Step 3. $H_L(k_L) = H_R(k_R) = \bar{F}_A(k_L, k_R)$**.**

We choose $(\bar{k}_L, \bar{k}_R) = (\bar{p}_L^A, \bar{p}_R^A)$ and by contradiction, we assume that

$$H_R(k_R) = H_L(k_L) > A \quad \text{and} \quad H_L(k_L) > H_L^+(k_L) \quad \text{and} \quad H_R(k_R) > H_R^-(k_R).$$

Using that $H_L(k_L) = H_L^-(k_L) > A = H_L^-(\bar{k}_L)$, we deduce that $k_L < \bar{k}_L$. In the same way, we have $k_R > \bar{k}_R$. This implies that

$$\begin{aligned} 0 &\leqslant q_L(k_L, \bar{k}_L) - q_R(k_R, \bar{k}_R) \\ &= \mathrm{sign}(k_L - \bar{k}_L)(H_L(k_L) - A) - \mathrm{sign}(k_R - \bar{k}_R)(H_R(k_R) - A) < 0 \end{aligned}$$

which is a contradiction. □

General junction condition for SCL. We now explain how the Scalar Conservation Law (11) should be treated. Following the approach of [4], the idea to understand this problem is to study two half-space problems for two given Dirichlet boundary condition (k_L, k_R),

$$\begin{cases} \rho_t + H_L(\rho)_x = 0 & \text{if } x < 0 \\ \rho(t, 0^-) = k_L(t) & \\ \rho(0, x) = \rho_0(x) & \text{for } x < 0 \end{cases} \qquad \begin{cases} \rho_t + H_R(\rho)_x = 0 & \text{if } x > 0 \\ \rho(t, 0^+) = k_R(t) & \\ \rho(0, x) = \rho_0(x) & \text{for } x > 0 \end{cases} \tag{17}$$

where the couple of boundary conditions $(k_L(t), k_R(t))$ satisfies the following transmission condition

$$H_L(\rho(t, 0-)) = H_R(\rho(t, 0+)) = F_0(k_L(t), k_R(t)). \tag{18}$$

Moreover, the Dirichlet boundary conditions in (17) have to be understood in the sense of Bardos-Leroux-Nedelec (see [11]), i.e.

$$\begin{aligned} H_L(\rho(t,0-)) &= g^{H_L}(\rho(t,0-),k_L(t)), \\ H_R(\rho(t,0+)) &= g^{H_R}(k_R(t),\rho(t,0+)) \quad \text{for a.e. } t\in(0,T) \end{aligned}$$

where for a general Hamiltonian H, g^H is the Godunov flux defined by

$$g^H(p_1,p_2) = \begin{cases} \min_{p\in[p_1,p_2]} H(p) & \text{if} \quad p_1 \leqslant p_2 \\ \max_{p\in[p_2,p_1]} H(p) & \text{if} \quad p_2 \leqslant p_1. \end{cases} \tag{19}$$

We say that a solution to (17)-(18) is a F_0-admissible solution to (11).
In our specific setting, due to the monotonicity of F_0, for any couple $(\rho^L,\rho^R)\in\mathbb{R}^2$ verifying $H_L(\rho^L) = H_R(\rho^R)$, there exists a unique value $F(\rho^L,\rho^R)\in\mathbb{R}$ such that there exists $(k_L,k_R)\in\mathbb{R}^2$ satisfying

$$\begin{cases} F_0(k_L,k_R) = F(\rho^L,\rho^R) \\ H_L(\rho_L) = H_R(\rho_R) = F_0(k_L,k_R) \end{cases}$$

This was proven in [9] and [4]. Moreover, one can show (the reader can try to check it directly, but this result will be addressed in a much more generality in a future work) that

$$F(\rho(t,0^-),\rho(t,0^+)) = \max(A_{F_0}, H_L^+(\rho(t,0^-)), H_R^-(\rho(t,0^+))),$$

where A_{F_0} is constructed in Lemma 2.3 below. Then, solving (18) rewrites as

$$H_R(\rho(t,0^-)) = H_L(\rho(t,0^+)) = \bar{F}_{A_{F_0}}(\rho(t,0^-),\rho(t,0^+)).$$

which is exactly the junction condition that ρ must satisfy in (8).
We then define the solution of (11) as follow.

Definition 2.14 (Definition of solutions to (11))**.** We say that $\rho\in L^\infty((0,T)\times\mathbb{R})$ is a F_0-admissible solution to (11) if ρ is a $\mathcal{G}_{A_{F_0}}$-entropy solution to (8) with A_{F_0} uniquely defined in Lemma 2.3.

3. Numerical schemes.

3.1. Numerical scheme for the Hamilton-Jacobi equation (9).

In this subsection, we describe the numerical scheme used to solve the Hamilton-Jacobi equation (9). Given a time step $\Delta t > 0$ and a space step $\Delta x > 0$, we consider the discrete time $t_n = n\Delta t$ for $n\in\mathbb{N}$ and the discrete point $x_j = j\Delta x$ for $j\in\mathbb{Z}$. We denote by u_j^n the numerical approximation of $u(t_n,x_j)$. In order to discretize (9), we will use a Godunov approximation. More precisely, we use the Godunov flux introduced earlier in (19): for $\alpha = L,R$

$$g^{H_\alpha}(p^-,p^+) = \begin{cases} \min_{p\in[p^-,p^+]} H_\alpha(p) & \text{if} \quad p^- \leqslant p^+ \\ \max_{p\in[p^+,p^-]} H_\alpha(p) & \text{if} \quad p^+ \leqslant p^- \end{cases}$$

We remark that g^{H_α} are non-decreasing in the first variable and non-increasing in the second one. Moreover, $g^{H_\alpha}(p,p) = H_\alpha(p)$ for $\alpha = R,L$. For $j\in\mathbb{Z}$, we define

$$p^n_{j+\frac{1}{2}} = \frac{u^n_{j+1} - u^n_j}{\Delta x}.$$

The numerical scheme is then given by

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} + g^{H_L}\left(p^n_{j-\frac12}, p^n_{j+\frac12}\right) = 0 & \text{for } j \leqslant -1, \\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t} + g^{H_R}\left(p^n_{j-\frac12}, p^n_{j+\frac12}\right) = 0 & \text{for } j \geqslant 1, \\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t} + F_0\left(p^n_{j-\frac12}, p^n_{j+\frac12}\right) = 0 & \text{for } j = 0 \end{cases} \tag{20}$$

completed with the initial condition

$$u_j^0 = u_0(j\Delta x) \quad \text{for } j \in \mathbb{Z}.$$

For $\Delta = (\Delta t, \Delta x)$, let

$$u_\Delta(t,x) := \sum_{n\in\mathbb{N}} \mathbb{1}_{[t_n,t_{n+1})}(t)\mathbb{1}_{[x_j,x_{j+1})}(x)\left[u_j^n + \frac{u_{j+1}^n - u_j^n}{\Delta x}(x-x_j)\right].$$

We then have the following convergence result (where we recall that all functions F_0 of the form (4) indeed satisfy (10)).

Theorem 3.1 (Numerical approximation for Hamilton-Jacobi equations)**.** *Let $T > 0$ and u_0 be globally Lipschitz continuous. We assume that the H_α satisfy (2) and F_0 satisfies (10). Let u_j^n be the solution of the scheme (20) and u be the viscosity solution of the Hamilton-Jacobi equation (1) with the relaxed junction condition $F_{A_{F_0}}$ for A_{F_0} given by Lemma 2.3. Let*

$$L_{\mathcal{H}} := \max(L_{H_L}, L_{H_R}, ||\partial_{p_1}F_0||_\infty, ||\partial_{p_2}F_0||_\infty) \tag{21}$$

where L_{H_α} is the Lipschitz constant of H_α. We also assume that the CFL condition

$$\frac{\Delta x}{\Delta t} \geqslant 2L_{\mathcal{H}} \tag{22}$$

holds. Then u_Δ converges locally uniformly to u.

Proof. Recalling that by Theorem 2.5, the solution to (1) with $A = A_{F_0}$ is also the solution to (9), the proof is a consequence of [26, Theorem 1.1 or Theorem 1.2] remarking that the two schemes are identical. The main difference with the result in [26] is that in that paper, the network is composed of two outgoing edges, but it's rather easy to come back to this setting. Indeed, if we set, for $x \geqslant 0$,

$$v^\alpha(t,x) = \begin{cases} u(t,-x) & \text{if} \quad \alpha = L \\ u(t,x) & \text{if} \quad \alpha = R \end{cases}$$

then v^α is solution of

$$\begin{cases} v_t^\alpha + \tilde{H}_\alpha(v_x) = 0 & \text{in} \quad (0,T)\times(0,+\infty),\ \alpha = R, L \\ v_t^\alpha + \tilde{F}_0(v_x^L, v_x^R) = 0 & \text{in} \quad (0,T)\times\{0\} \end{cases} \tag{23}$$

where $\tilde{H}_L(p) = H_L(-p)$, $\tilde{H}_R = H_R$, $\tilde{F}_0(p_1,p_2) = F_0(-p_1,p_2)$. Setting $v_j^{L,n} = u_{-j}^n$ and $v_j^{R,n} = u_j^n$ for $j \geqslant 0$, an easy computation, using that $g^{\tilde{H}_L}(p_1,p_2) =$

$g^{H_L}(-p_2,-p_1)$, shows that $v_j^{\alpha,n}$ is solution of the following scheme

$$\begin{cases} \dfrac{v_j^{\alpha,n+1}-v_j^{\alpha,n}}{\Delta t} + g^{\tilde{H}_\alpha}\left(\tilde{p}^{\alpha,n}_{j-\frac12},\tilde{p}^{\alpha,n}_{j+\frac12}\right) = 0 \quad \text{for } j \geqslant 1, \quad \alpha = L,R \\ \dfrac{v_j^{\alpha,n+1}-v_j^{\alpha,n}}{\Delta t} + \tilde{F}_0\left(\tilde{p}^{L,n}_{j+\frac12},\tilde{p}^{R,n}_{j+\frac12}\right) = 0 \quad \text{for } j=0,\ \alpha = L,R,\ \text{with } v_0^{L,n} = v_0^{R,n} \end{cases}$$

where

$$\tilde{p}^{\alpha,n}_{j+\frac12} = \frac{v^{\alpha,n}_{j+1}-v^{\alpha,n}_j}{\Delta x}. \tag{24}$$

On the other hand, the scheme proposed in [26] to solve (23) writes

$$\begin{cases} \dfrac{v_j^{\alpha,n+1}-v_j^{\alpha,n}}{\Delta t} + \max\left(\tilde{H}^+_\alpha\left(\tilde{p}^{\alpha,n}_{j-\frac12}\right),\tilde{H}^-_\alpha\left(\tilde{p}^{\alpha,n}_{j+\frac12}\right)\right) = 0 \quad \text{for } j \in \mathbb{N}, \quad \alpha = 1,2, \\ \dfrac{v_j^{\alpha,n+1}-v_j^{\alpha,n}}{\Delta t} + \tilde{F}_0\left(\tilde{p}^{L,n}_{j+\frac12},\tilde{p}^{R,n}_{j+\frac12}\right) = 0 \quad \text{for } j=0,\ \alpha = 1,2 \ \text{with } v_0^{1,b} = v_0^{2,n}. \end{cases} \tag{25}$$

The rest of the proof is then a direct consequence of the following lemma.

Lemma 3.2 (Equivalent formulation of the Godunov flux)**.** *For a general convex hamiltonian H, we have*

$$g^H(p_1,p_2) = \max(H^+(p_1),H^-(p_2)).$$

This shows that the two schemes for (23) are equivalent and so, using [26, Theorem 1.1 or Theorem 1.2], this shows that "$v_i^{\alpha,n}$ converges to v^α" locally uniformly and so, by a change of variable, "u_i^n converges to u" in the sense of Theorem 3.1. This ends the proof of the theorem. □

It remains to show the lemma.

Proof of Lemma 3.2. We denote by p_0 the minimum point of H so that H is non-increasing on $(-\infty,p_0]$ and non-decreasing on $[p_0,+\infty)$ and we distinguish several cases:

Case 1: $p_1 \leqslant p_0 \leqslant p_2$. In that case $H^+(p_1) = H(p_0) = H^-(p_2)$ and

$$g^H(p_1,p_2) = \min_{[p_1,p_2]} H = H(p_0) = \max(H^+(p_1),H^-(p_2)).$$

Case 2: $p_0 \leqslant p_1 \leqslant p_2$. In that case $H^+(p_1) = H(p_1)$, $H^-(p_2) = H(p_0)$ and

$$g^H(p_1,p_2) = \min_{[p_1,p_2]} H = H(p_1) = \max(H^+(p_1),H^-(p_2)).$$

Case 3: $p_1 \leqslant p_2 \leqslant p_0$. This case is similar to the previous one.

Case 4: $p_2 \leqslant p_0 \leqslant p_1$. In that case $H^+(p_1) = H(p_1)$, $H^-(p_2) = H(p_2)$ and

$$\max(H(p_1),H(p_2)) = \max_{[p_2,p_1]} H = g^H(p_1,p_2).$$

Case 5: $p_0 \leqslant p_2 \leqslant p_1$. In that case $H^+(p_1) = H(p_1)$, $H^-(p_2) = H(p_0)$ and

$$\max(H(p_1),H(p_0)) = H(p_1) = \max_{[p_2,p_1]} H = g^H(p_1,p_2).$$

Case 6: $p_2 \leqslant p_1 \leqslant p_0$. This case is similar to the previous one. □

3.2. **Numerical scheme for the scalar conservation law equation** (11)**.** Given u_0 satisfying (3), we consider $\rho_0 := (u_0)_x$ and its discretized version

$$p^0_{j+1/2} = \frac{u^0_{j+1} - u^0_j}{\Delta x} = \frac{u_0(x_{j+1}) - u_0(x_j)}{\Delta x} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \rho_0(y)\,\mathrm{d}y.$$

We now want to describe the numerical scheme for (11). This scheme is directly derived from the scheme (20). Indeed, recalling the definition of $p^n_{j+1/2}$ in (24), we get from (25) the following scheme

$$\begin{cases} p^{n+1}_{j+\frac12} = p^n_{j+\frac12} - \dfrac{\Delta t}{\Delta x}\left(g^{H_L}\left(p^n_{j+\frac12}, p^n_{j+\frac32}\right) - g^{H_L}\left(p^n_{j-\frac12}, p^n_{j+\frac12}\right)\right) & \text{for } j < -1 \\ p^{n+1}_{j+\frac12} = p^n_{j+\frac12} - \dfrac{\Delta t}{\Delta x}\left(g^{H_R}\left(p^n_{j+\frac12}, p^n_{j+\frac32}\right) - g^{H_R}\left(p^n_{j-\frac12}, p^n_{j+\frac12}\right)\right) & \text{for } j \geqslant 1 \\ p^{n+1}_{j+\frac12} = p^n_{j+\frac12} - \dfrac{\Delta t}{\Delta x}\left(g^{H_R}\left(p^n_{j+\frac12}, p^n_{j+\frac32}\right) - F_0\left(p^n_{j-\frac12}, p^n_{j+\frac12}\right)\right) & \text{for } j = 0 \\ p^{n+1}_{j+\frac12} = p^n_{j+\frac12} - \dfrac{\Delta t}{\Delta x}\left(F_0\left(p^n_{j+\frac12}, p^n_{j+\frac32}\right) - g^{H_L}\left(p^n_{j-\frac12}, p^n_{j+\frac12}\right)\right) & \text{for } j = -1. \end{cases} \tag{26}$$

For notations' sake, we also denote by $\mathcal{F}_j$ the right-hand side of the above scheme such that for any n, j, we have

$$p^{n+1}_{j+1/2} = \mathcal{F}_j(p^n_{j-1/2}, p^n_{j+1/2}, p^n_{j+3/2})$$

and this scheme can be identified as a standard finite volume scheme (at least outside the junction point). We denote $\Delta = (\Delta t, \Delta x)$ and

$$p_\Delta := \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}} p^n_{j+1/2}\mathbb{1}_{[t_n,t_{n+1})\times[x_j,x_{j+1})}. \tag{27}$$

For this scheme we have the following convergence result

Theorem 3.3 (Numerical approximation for SCL)**.** *Let u_0 satisfy (3), $H_{L,R}$ satisfy (2) and F_0 satisfy (10). Suppose also that the CFL condition (22) is satisfied and that*

$$\frac{\Delta t}{\Delta x}\frac{\delta}{2}M \leqslant 1, \tag{28}$$

where $M = \max(|a_L|, |c_L|, |a_R|, |c_R|)$ and δ is introduced in (2). Then $(p_\Delta)_\Delta$ converges almost everywhere as $\Delta \longrightarrow (0,0)$ to $\rho \in L^\infty$, the unique solution to (8), in the sense of Definition 2.7, with $A = A_{F_0}$ and A_{F_0} given in Lemma 2.3.

Remark 3.4. This result is rather classical if we take $F_0 := \bar{F}_A$ for $A \in [H_0, 0]$ in the numerical scheme (26) and the proof of convergence has been written in a similar setting in various sources including [6], [8] and [37]. The result we present here is stronger. Indeed, we put the desired condition F_0 in the scheme and we show that the numerical solution converges to the solution with the relaxed junction condition $\bar{F}_{A_{F_0}}$. The strategy of the proof is similar to the classical case, but for completeness' sake we rewrite it here, putting most of the heavy computations in Appendix.

We first present the different lemmas that we piece together in order to get Theorem 3.3.

Lemma 3.5. (Monotonicity and stability)
Let u_0 satisfy (3), $H_{L,R}$ satisfy (2) and F_0 satisfy (10). Suppose also that the CFL condition (22) is satisfied. Then the numerical scheme (26) is monotone. That is to say $\mathcal{F}_j$ is non-decreasing with respect to each of its three variables. Furthermore, the scheme is stable, namely we have

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z},\ p^n_{j+1/2} \in \begin{cases} [a_L, c_L] & \text{if } j \leqslant -1 \\ [a_R, c_R] & \text{if } j \geqslant 0. \end{cases} \tag{29}$$

Proof. We begin to prove the monotonicity. Fix n, j. Using the definition of the Godunov flux g^H and the junction condition F_0 it is clear that both are non-decreasing with respect to their first argument and non-increasing with respect to their second one. Then,

$$\forall v, w \in \mathbb{R},\ u \mapsto \mathcal{F}_j(u, v, w) \text{ is non-decreasing,}$$
$$\forall u, v \in \mathbb{R},\ w \mapsto \mathcal{F}_j(u, v, w) \text{ is non-decreasing.}$$

Notice that, for a given H, the derivative of the Godunov flux g^H is bounded by the Lipschitz constant L_H of H,

$$\partial_{p_1} g^H(p_1, p_2) \in [0; L_H], \quad \partial_{p_2} g^H(p_1, p_2) \in [-L_H, 0].$$

Recalling from (21) that $L_{\mathcal{H}} := \max(L_{H_L}, L_{H_R}, ||\partial_{p_1} F_0(p_1, p_2)||_\infty, ||\partial_{p_2} F_0(p_1, p_2)||_\infty)$, we also have

$$\partial_{p_1} F_0(p_1, p_2) \in [0, L_{\mathcal{H}}], \quad \partial_{p_1} F_0(p_1, p_2) \in [-L_{\mathcal{H}}, 0].$$

Then

$$\begin{aligned} \partial_v \mathcal{F}_j(u, v, w) &\geqslant 1 - \frac{\Delta_t}{\Delta_x}(L_{\mathcal{H}} - (-L_{\mathcal{H}})) \\ &\geqslant 1 - 2\frac{\Delta_t}{\Delta_x} L_{\mathcal{H}}. \end{aligned}$$

Since the CFL condition (22) is satisfied, we recover that $v \mapsto \mathcal{F}_j(u, v, w)$ is non-decreasing.

We now prove the stability result by induction on n. First, by assumption (3), the property (29) is true for $n = 0$. Fix $n \in \mathbb{N}$ such that (29) holds true for n. We recall that

$$p^{n+1}_{j+1/2} = \mathcal{F}_j(p^n_{j-1/2}, p^n_{j+1/2}, p^n_{j+3/2}).$$

If $j \geqslant 1$, by monotonicity of the scheme, we then have

$$p^{n+1}_{j+1/2} \geqslant \mathcal{F}_j(a_R, a_R, a_R) = a_R - \frac{\Delta t}{\Delta x}(H_R(a_R) - H_R(a_R)) = a_R$$

and

$$p^{n+1}_{j+1/2} \leqslant \mathcal{F}_j(c_R, c_R, c_R) = c_R - \frac{\Delta t}{\Delta x}(H_R(c_R) - H_R(c_R)) = c_R.$$

In the same way, if $j = 0$, we get

$$p^{n+1}_{j+1/2} \geqslant \mathcal{F}_j(a_L, a_R, a_R) = a_R - \frac{\Delta t}{\Delta x}(H_R(a_R) - F_0(a_L, a_R)) = a_R$$

and

$$p^{n+1}_{j+1/2} \leqslant \mathcal{F}_j(c_L, c_R, c_R) = c_R - \frac{\Delta t}{\Delta x}(H_R(c_R) - F_0(c_L, c_R)) = c_R,$$

where we used Assumption (10) to get that $F_0(a_L, a_R) = F_0(c_L, c_R) = 0$. Using the same arguments, we get also the result for $j = 0$, $j = -1$ and $j \leqslant -2$. This ends the proof of the lemma. □

Recall that, associated to the entropy $p \mapsto |p-k|$, is the entropy flux

$$p \mapsto \operatorname{sign}(p-k) \cdot \{H(p) - H(k)\} = H(p \vee k) - H(p \wedge k)$$

where we used the notation, for any $a, b \in \mathbb{R}$, $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. This naturally suggests the following result.

Lemma 3.6 (Discrete entropy inequalities)**.** *Let u_0 satisfy (3), $H_{L,R}$ satisfy (2) and F_0 satisfy (10). Suppose also that the CFL condition (22) is satisfied. Let $T > 0$ and $(p_\Delta)_\Delta$ be defined by (27). For any $(k_L, k_R) \in Q$, writing $k_\Delta = k_L \mathbb{1}_{j \leqslant -1} + k_R \mathbb{1}_{j \geqslant 0}$, we set*

$$\Phi_j^n(k_\Delta) = \begin{cases} g^{H_L}(p_{j-1/2}^n \vee k_L, p_{j+1/2}^n \vee k_L) - g^{H_L}(p_{j-1/2}^n \wedge k_L, p_{j+1/2}^n \wedge k_L) & \text{if} \quad j \leqslant -1 \\ g^{H_R}(p_{j-1/2}^n \vee k_R, p_{j+1/2}^n \vee k_R) - g^{H_R}(p_{j-1/2}^n \wedge k_R, p_{j+1/2}^n \wedge k_R) & \text{if} \quad j \geqslant 1 \\ F_0(p_{j-1/2}^n \vee k_L, p_{j+1/2}^n \vee k_R) - F_0(p_{j-1/2}^n \wedge k_L, p_{j+1/2}^n \wedge k_R) & \text{if} \quad j = 0. \end{cases} \tag{30}$$

We also set

$$\Phi_\Delta(k_\Delta) := \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \Phi_j^n(k_\Delta) \mathbb{1}_{[t_n, t_{n+1}) \times [x_j, x_{j+1})}. \tag{31}$$

Then, for any $\phi \in C_c^\infty((0,T) \times \mathbb{R})$ non-negative, we have, with p_Δ defined in (27),

$$\int_0^T \int_{\mathbb{R}} \left(|p_\Delta - k_\Delta| \, \phi_t + \Phi_\Delta(k_\Delta) \phi_x \right) \, dt \, dx + \int_{\mathbb{R}} |p_\Delta(0,x) - k_\Delta| \, \phi(0,x) \, dx \\ + \int_0^T R_{F_0}(k_L, k_R) \phi(t, 0) \, dt \geqslant O(\Delta x) + O(\Delta t), \tag{32}$$

where

$$R_{F_0}(k_L, k_R) := |H_L(k_L) - F_0(k_L, k_R)| + |H_R(k_R) - F_0(k_L, k_R)|. \tag{33}$$

Remark 3.7. The proof of this lemma is pretty straightforward and derives directly from the monotonicity proven in Lemma 3.5. Since it contains long computations, we postponed it to the Apppendix.

Finally, in order to get the desired convergence, we also need the compactness of $(p_\Delta)_\Delta$. We use the following lemma, which proof is also postponed to the Appendix.

Lemma 3.8 (Compactness of ρ_Δ)**.** *Let u_0 satisfy (3), $H_{L,R}$ satisfy (2) and F_0 satisfy (10). For any l, let $(\Delta_l)_l$ verify the CFL condition (22) and (28). Then, there exists $\rho \in L^\infty$ and a subsequence also denoted $(p_{\Delta_l})_l$ such that*

$$p_{\Delta_l} \longrightarrow \rho \qquad a.e. \;\; as \; \Delta_l \to 0.$$

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3.

Step 1. Preliminaries

First, using Lemma 3.8, we take a subsequence of p_Δ that converges to $\rho \in L^\infty$ a.e.. We now want to prove that ρ is a solution to (8) in the sense of Definition 2.7. The first point of Definition 2.7 is classical and we skip it (the reader can for instance find the complete details in [8]).

Step 2. The limit PDE outside the junction point $x = 0$

Let $\phi \in C_c^\infty([0,T) \times (-\infty, 0))$ be non-negative and $(k_L, k_R) \in Q$. We first want to prove that

$$\int_0^T \int_{\mathbb{R}^-} \Phi_\Delta(k_\Delta)\phi_x \, \mathrm{d}t \, \mathrm{d}x \longrightarrow \int_0^T \int_{\mathbb{R}^-} \operatorname{sign}(\rho - k_L) \left[H_L(\rho) - H_L(k_L)\right] \phi_x \, \mathrm{d}t \, \mathrm{d}x. \quad (34)$$

Let $x < 0$ and $t \in [0,T)$ such that $p_\Delta(t,x) \longrightarrow \rho$. Then, for any Δx, there exists $j \leqslant -1$ such that $p_\Delta(t,x) = p^n_{j+1/2}$ and

$$\begin{aligned}
&\Phi_\Delta(k_\Delta)(t,x)\\
=&g^{H_L}(p_\Delta(t,x-\Delta x) \vee k_L, p_\Delta(t,x) \vee k_L) - g^{H_L}(p_\Delta(t,x-\Delta x) \wedge k_L, p_\Delta(t,x) \wedge k_L)\\
=&g^{H_L}(p_\Delta(t,x-\Delta x) \vee k_L, p_\Delta(t,x) \vee k_L) - g^{H_L}(p_\Delta(t,x) \vee k_L, p_\Delta(t,x) \vee k_L)\\
&+ \operatorname{sign}(p_\Delta(t,x) - k_L) \left[H_L(p_\Delta(t,x)) - H_L(k_L)\right]\\
&+ g^{H_L}(p_\Delta(t,x) \wedge k_L, p_\Delta(t,x) \wedge k_L) - g^{H_L}(p_\Delta(t,x-\Delta x) \wedge k_L, p_\Delta(t,x) \wedge k_L).
\end{aligned}$$

Using the Lipschitz bound on the Godunov flux, we recover:

$$\int_0^T \int_{\mathbb{R}^-} \Phi_\Delta(k_\Delta)\phi_x \, \mathrm{d}t \, \mathrm{d}x = \int_0^T \int_{\mathbb{R}^-} \operatorname{sign}(p_\Delta - k_L) \left[H_L(p_\Delta) - H_L(k_L)\right] \phi_x \, \mathrm{d}t \, \mathrm{d}x + \mathcal{I}_1$$

where, using the same arguments as in the proof of Lemma 3.5 with $L_{\mathcal{H}}$ given in (21), we have:

$$\begin{aligned}
|\mathcal{I}_1| &\leqslant 2L_{\mathcal{H}} \int_0^T \int_{\mathbb{R}^-} |p_\Delta(t,x) - p_\Delta(t,x-\Delta x)| \, |\phi_x(t,x)| \, \mathrm{d}t \, \mathrm{d}x\\
&\leqslant 2L_{\mathcal{H}} \int_0^T \int_{\mathbb{R}^-} |p_\Delta(t,x)\phi_x(t,x) - p_\Delta(t,x-\Delta x)\phi_x(t,x)| \, \mathrm{d}t \, \mathrm{d}x\\
&\leqslant 2L_{\mathcal{H}} \Big[\int_0^T \int_{\mathbb{R}^-} |p_\Delta(t,x)\phi_x(t,x) - p_\Delta(t,x-\Delta x)\phi_x(t,x-\Delta x)| \, \mathrm{d}t \, \mathrm{d}x\\
&\quad + \int_0^T \int_{\mathbb{R}^-} |p_\Delta(t,x-\Delta x)\phi_x(t,x) - p_\Delta(t,x-\Delta x)\phi_x(t,x-\Delta x)| \, \mathrm{d}t \, \mathrm{d}x \Big]\\
&=: \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

First, notice that

$$\begin{aligned}
\mathcal{I}_3 \leqslant& 2L_{\mathcal{H}} \int_0^T \int_{\mathbb{R}^-} \left| p_\Delta(t,x-\Delta x) \int_{x-\Delta x}^x \phi_{xx}(t,y) \, \mathrm{d}y \right| \mathrm{d}t \, \mathrm{d}x\\
\leqslant& 2L_{\mathcal{H}} ||p_\Delta||_{L^\infty} ||\phi_{xx}||_{L^1} \Delta x.
\end{aligned}$$

Now, since $p_\Delta \longrightarrow \rho$ a.e. and $|p_\Delta \phi_x| \leqslant C|\phi_x| \in L^1((0,T) \times \mathbb{R})$, the sequence $(p_\Delta \phi_x)_\Delta$ is convergent to $\rho\phi_x$ in $L^1((0,T) \times \mathbb{R})$. From continuity of the translations in L^1, we deduce that

$$\lim_{\Delta x \to 0} ||\tau_{\Delta x}(\rho\phi_x) - \rho\phi_x||_{L^1((0,T)\times\mathbb{R})} = 0,$$

where $\tau_{\Delta x} f(x) = f(x - \Delta x)$. It is then easy to conclude that $\mathcal{I}_2 = o(1)$ when $\Delta \longrightarrow (0,0)$. This implies (34). Then, for $\phi \in C_c^\infty([0,T) \times (-\infty, 0))$, passing to the limit in (32), we get

$$\begin{aligned}
&\iint_{(0,T)\times\mathbb{R}^-} |\rho - k_L|\phi_t + \operatorname{sign}(\rho - k_L) \left[H_L(\rho) - H_L(k_L)\right] \phi_x \, \mathrm{d}t \, \mathrm{d}x\\
&+ \int_{\mathbb{R}^-} |\rho_0(x) - k_L| \phi(0,x) \, \mathrm{d}x \geqslant 0.
\end{aligned}$$

The analogous result holds if ϕ is compactly supported in $[0,T)\times(0,+\infty)$. Note that, when treating this case, we need to consider a Δ_0 such that for any $\Delta x \leqslant \Delta_0$, $p_\Delta(t,x) = p^n_{j+1/2}$ with $j \geqslant 1$. We can however choose Δx to be small enough such that $\phi = 0$ on $(0,\Delta x)$ and recover the analogous inequalities. So the second condition in Definition 2.7 is satisfied.

Step 3: Recovering the junction condition

We now want to prove the third point of Definition 2.7, i.e. that the trace of the solution is in the germ. To this end, we first consider a non-negative test function $\phi \in C_c^\infty((0,T)\times\mathbb{R})$ which satisfies furthermore

$$\phi_x = 0 \quad \text{on} \quad (0,T)\times(-\delta,\delta), \quad \text{for some small } \delta > 0 \tag{35}$$

Let us consider the set $\mathcal{E}_{A_{F_0}}$ defined in (16), which generates the germ $\mathcal{G}_{A_{F_0}}$. Now notice that the numerical error term defined in (33) satisfies $R_{F_0}(k_L,k_R) = 0$ for $(k_L,k_R) \in \mathcal{E}_{A_{F_0}}$, which is a key fact for us. Hence for the choice of $(k_L,k_R) \in \mathcal{E}_{A_{F_0}}$, inequality (32) reads

$$\int_0^T \int_{\mathbb{R}} \left(|p_\Delta - k_\Delta|\, \phi_t + \Phi_\Delta(k_\Delta)\phi_x \right) \mathrm{d}t\, \mathrm{d}x \geqslant O(\Delta x) + O(\Delta t)$$

Using the notation $\mathcal{H}(x,p) := H_L(p)\cdot \mathbb{1}_{\mathbb{R}^-}(x) + H_R(p)\cdot \mathbb{1}_{\mathbb{R}^+}(x)$ and $k(x) := k_L \cdot \mathbb{1}_{\mathbb{R}^-}(x) + k_R \cdot \mathbb{1}_{\mathbb{R}^+}(x)$, we get as in Step 2 that

$$\int_0^T \int_{\mathbb{R}} \Phi_\Delta(k_\Delta)\phi_x \, \mathrm{d}t\, \mathrm{d}x \longrightarrow \int_0^T \int_{\mathbb{R}\setminus\{0\}} \operatorname{sign}(\rho - k)\left[\mathcal{H}(x,\rho) - \mathcal{H}(x,k)\right]\phi_x \, \mathrm{d}t\, \mathrm{d}x$$

and then

$$\iint_{(0,T)\times\mathbb{R}} |\rho - k|\phi_t + \operatorname{sign}(\rho-k)\left[\mathcal{H}(x,\rho) - \mathcal{H}(x,k)\right]\phi_x \, \mathrm{d}t\, \mathrm{d}x \geqslant 0 \tag{36}$$

Up to replace ϕ, by a sequence of approximations ϕ_δ satisfying (35), we can assume that (36) holds for $\phi^\varepsilon(t,x) = \phi(t,0)\max\left\{0, 1 - \varepsilon^{-1}|x|\right\}$ for $\varepsilon > 0$ with non-negative $\phi(\cdot,0) \in C_c((0,T))$. In the limit $\varepsilon \to 0$, the sequence of test functions ϕ^ε focuses on $x = 0$ (like in the proof of Proposition 2.12 in [35]), and we get

$$\int_0^T \left[\operatorname{sign}(\rho - k)\left\{\mathcal{H}(x,\rho) - \mathcal{H}(x,k)\right\}\right]_{x=0^-}^{x=0^+} \, dt \geqslant 0$$

which writes rigorously as

$$\int_0^T \left[q_L(\gamma_L\rho, k_L) - q_R(\gamma_R\rho, k_R)\right]\phi(t,0)\, \mathrm{d}t \geqslant 0,$$

where the q_α are defined in Lemma 2.13, and $\gamma_L\rho, \gamma_R\rho$ are the left and right traces of ρ at $x = 0$. Because the non-negative function $\phi(\cdot,0) \in C_c((0,T))$ is arbitrary, we deduce that for almost every t,

$$q_L(\gamma_L\rho, k_L) - q_R(\gamma_R\rho, k_R) \geqslant 0 \quad \text{for all} \quad (k_L,k_R) \in \mathcal{E}_{A_{F_0}}.$$

Using the fact that $(\gamma_L\rho, \gamma_R\rho) \in Q$ a.e. on $(0,T)$ and the fact that $\mathcal{E}_{A_{F_0}}$ generates the whole germ $\mathcal{G}_{A_{F_0}}$ (see Lemma 2.13) , we deduce that $(\gamma_L\rho, \gamma_R\rho) \in \mathcal{G}_{A_{F_0}}$ a.e. on $(0,T)$ which is nothing else than the third condition of Definition 2.7, now satisfied by ρ. Finally the uniqueness of ρ follows from the first point of Theorem 2.9. $\square$

We now state and prove the following result.

Lemma 3.9 (Completeness of $\mathcal{G}_A$)**.** *Under the assumptions of Proposition 2.10, the L^1-dissipative germ $\mathcal{G}_A$ is complete.*

Proof of Lemma 3.9. Consider any $k = (k_L, k_R) \in Q$. In order to show the completeness of $\mathcal{G}_A$, we simply have to show the existence of a $\mathcal{G}_A$-entropy solution to (8) with initial data $\rho_0 = k_L 1_{(-\infty,0)} + k_R 1_{[0,+\infty)}$. The existence of such a solution follows from the construction of the function ρ in the proof of Theorem 3.3. This insures that $\mathcal{G}_A$ is complete and ends the proof. □

4. **Proof of Theorem 1.2 and Theorem 1.3 using numerical schemes.** We are now able to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Fix $\Delta := (\Delta t, \Delta x)$ satisfying the CFL condition (22) and (28). Denote by $(u_j^n)_{n\in\mathbb{N}, j\in\mathbb{Z}}$ the solution of the scheme (20). Recall that

$$u_\Delta(t,x) := \sum_{n\in\mathbb{N}} \mathbb{1}_{[t_n,t_{n+1})}(t)\mathbb{1}_{[x_j,x_{j+1})}(x)\left[u_j^n + \frac{u_{j+1}^n - u_j^n}{\Delta x}(x - x_j)\right].$$

Then, by construction (see (24)), for any Δ,

$$(u_\Delta)_x = p_\Delta$$

where p_Δ is the solution of the scheme (26) with ρ_0 as initial datum. Let $\phi \in C_c^1([0,+\infty) \times \mathbb{R})$. Then we have

$$\iint u_\Delta \phi_x \,\mathrm{d}t\,\mathrm{d}x = -\iint p_\Delta \phi \,\mathrm{d}t\,\mathrm{d}x.$$

Using Theorem 3.1, we know that the scheme (20) with u_0 as initial datum converges locally uniformly to u the unique weak viscosity solution to (9). Furthermore, $\phi_x \in C_c^0([0,+\infty)\times\mathbb{R})$ so we can pass to the limit in the left-hand side as $\Delta \longrightarrow (0,0)$ satisfying the CFL condition to get

$$\iint u_\Delta \phi_x \,\mathrm{d}t\,\mathrm{d}x \longrightarrow \iint u \phi_x \,\mathrm{d}t\,\mathrm{d}x.$$

On the other hand, using Theorem 3.3, we get that p_Δ converges a.e. to ρ the unique solution of (11) in the sense of Definition 2.7. Also, thanks to Lemma 3.5, we know that $(p_\Delta)_\Delta$ is uniformly bounded. By dominated convergence, we also pass to the limit in the right-hand side and get that, for any test function $\phi \in C_c^1([0,+\infty)\times\mathbb{R})$,

$$\iint u \phi_x \,\mathrm{d}t\,\mathrm{d}x = -\iint \rho \phi \,\mathrm{d}t\,\mathrm{d}x.$$

This gives the desired result. □

The proof of Theorem 1.2 can be obtained exactly in the same way.

5. **An alternative proof of Theorem 1.2 using semi-algebraic functions.** Let u be the viscosity solution of (1) for $A \in [H_0, 0]$ and ρ be defined by

$$\rho(t,x) := u_x(t,x). \tag{37}$$

We would like to give a more direct proof of Theorem 1.2 and show that ρ is an entropy solution of (8). It is easy to check that ρ is already an entropy solution outside $\{x = 0\}$ (see for instance [18, 29]). We then focus on the junction condition at $x = 0$. In all this section, we assume that $H_{L,R}$ satisfy (2). We denote by ρ_L and ρ_R the strong traces of ρ at 0 (see [36] and (12)): $\rho_L := \gamma_L\rho$ and $\rho_R := \gamma_R\rho$.

We first note that formally

$$H_L(u_x(t,0^-)) = H_R(u_x(t,0^+)) \qquad \forall t>0. \tag{38}$$

Equality (38) can be rewritten rigorously as

Lemma 5.1 (The Rankine-Hugoniot condition)**.** *We have*

$$H_L(\rho_L(t)) = H_R(\rho_R(t)) \qquad a.e.\ t>0. \tag{39}$$

This common value is equal to $-u_t(t,0)$.

Equality (39) makes sense since $\rho_\alpha(t,\cdot)$ (and then also $H_\alpha(\rho(t,\cdot))$ have strong traces at $x=0$. Note also that equality (39) is nothing else the Rankine-Hugoniot condition at $x=0$.

Proof of Lemma 5.1. For any $\xi \in C_c^\infty((0,+\infty))$ and $h>0$ small, we have, after integrating the equation of u which is satisfied a.e. (since u is Lipschitz continuous from Theorem 2.6)

$$h^{-1}\int\int_{(t,x)\in(0,+\infty)\times(0,h)} \xi(t)H_R(u_x(t,x))\ dxdt = h^{-1}\int_0^\infty \int_0^h \xi'(t)u(t,x)\ dxdt.$$

By continuity of u, the right-hand side converges, as $h\to 0^+$, to $\int_0^\infty \xi'(t)u(t,0)dt$. The left-hand side can be rewritten as

$$h^{-1}\int_0^\infty \int_0^h \xi(t)H_R(\rho(t,x))\ dxdt$$

and converges to $\int_0^\infty \xi(t)H_R(\rho_R(t))\ dt$ as $h\to 0^+$ (where $\rho_R(t)$ is the strong trace of ρ at 0^+). This implies that

$$\int_0^\infty \xi(t)H_R(\rho_R(t))\ dt = \int_0^\infty \xi'(t)u(t,0)dt.$$

In the same way, we have

$$\int_0^\infty \xi(t)H_L(\rho_L(t))\ dt = \int_0^\infty \xi'(t)u(t,0)dt.$$

This shows (39). Note in addition that, as u is Lipschitz continuous,

$$\int_0^\infty \xi(t)u_t(t,0)dt = -\int_0^\infty \xi'(t)u(t,0)dt = -\int_0^\infty \xi(t)H_\alpha(\rho_\alpha(t))\ dt$$

for $\alpha = L,R$, which proves that the common value in (39) is equal to $-u_t(t,0)$. □

We continue by showing that the traces of ρ satisfy the first line in the second equivalent definition of $\mathcal{G}_A$ in (13).

Lemma 5.2 (ρ satisfies the first property of the germ $\mathcal{G}_A$)**.** *Assume that u is a solution to* (1)*. Then ρ defined by* (37) *satisfies*

$$H_L(\rho_L(t)) \geqslant A \qquad a.e.\ t>0.$$

Proof. We know by [27, Theorem 2.11] that $w(t) := u(t,0)$ is a viscosity subsolution of $w_t + A \leqslant 0$. Thus it satisfies $w(t+\tau)-w(t) \leqslant -A\tau$ for any $t,\tau>0$. Let us integrate the equation satisfied by u against the test function $(s,y)\to (\tau h)^{-1}\mathbf{1}_{[t,t+\tau]\times[-h,0]}(s,y)$ for $\tau,h>0$. We have, by Lipschitz continuity of u,

$$0 = (\tau h)^{-1}\int_t^{t+\tau}\int_{-h}^0 (u_t(s,y)+H_L(u_x(s,y)))dyds$$

$$
\begin{aligned}
&= (\tau h)^{-1}\int_{-h}^{0}(u(t+\tau,y)-u(t,y))dy + (\tau h)^{-1}\int_t^{t+\tau}\int_{-h}^{0} H_L(\rho(s,y))dyds\\
&\leqslant \tau^{-1}(u(t+\tau,0)-u(t,0)) + C\frac{h}{\tau} + (\tau h)^{-1}\int_t^{t+\tau}\int_{-h}^{0} H_L(\rho(s,y))dyds\\
&\leqslant -A + C\frac{h}{\tau} + (\tau h)^{-1}\int_t^{t+\tau}\int_{-h}^{0} H_L(\rho(s,y))dyds.
\end{aligned}
$$

We let $h \to 0^+$ and obtain

$$\int_t^{t+\tau} H_L(\rho_L(s))ds \geqslant A\tau,$$

which gives the claim. □

Lemma 5.3 (The traces are in the germ)**.** *Assume that the Lipschitz continuous viscosity solution u to (1) satisfies*

$$
\begin{aligned}
&\textit{for a.e. } t\in(0,T),\\
&u(t,\cdot) \textit{ has a left derivative } u_x(t,0^-) \textit{ and a right derivative } u_x(t,0^+) \textit{ at } 0.
\end{aligned}
\tag{40}
$$

Then

$$(\rho_L(t),\rho_R(t)) = (u_x(t,0^-),u_x(t,0^+)) \in \mathcal{G}_A \quad \textit{for a.e.} \quad t\geqslant 0. \tag{41}$$

Remark 5.4. The forthcoming paper [33] shows that (40) actually holds in a very general set-up (and in particular under our standing conditions). Below we prove it for semi-algebraic data only by using a representation formula.

Proof. **Step 1. proof of equality in (41)**

Using the definition of strong traces, we have

$$\text{ess-}\lim_{x\to 0^+}\int_0^T |\rho_L(t)-\rho(t,-x)| + |\rho_R(t)-\rho(t,x)|\; dt = 0. \tag{42}$$

This implies that

$$\text{ess-}\lim_{x\to 0^+}\int_0^T |\rho_L(t)-u_x(t,-x)| + |\rho_R(t)-u_x(t,x)|\; dt = 0.$$

Therefore, for any $\varepsilon>0$ there exists $x_\varepsilon>0$ such that

$$\int_0^T |\rho_L(t)-u_x(t,-x)| + |\rho_R(t)-u_x(t,x)|\; dt \leqslant \varepsilon \quad \text{for a.e.} \quad x\in(0,x_\varepsilon).$$

Thus, after integration in space, we get

$$\int_0^T |\rho_L(t)x+u(t,-x)-u(t,0)|+|\rho_R(t)x-u(t,x)+u(t,0)|dt \leqslant \varepsilon x \quad \text{for all} \quad x\in(0,x_\varepsilon).$$

Using that u is Lipschitz continuous, assumption (40) and Lebesgue Theorem, we get therefore

$$\int_0^T \left|\rho_L(t) - \lim_{x\to 0^+}\frac{u(t,-x)-u(t,0)}{-x}\right| + \int_0^T \left|\rho_R(t) - \lim_{x\to 0^+}\frac{u(t,x)-u(t,0)}{x}\right|\; dt = 0.$$

This means that, for a.e. t,

$$
\begin{aligned}
&u_x(t,0^-) = \lim_{x\to 0^+}\frac{u(t,-x)-u(t,0)}{-x} = \rho_L(t) \qquad \text{and}\\
&u_x(t,0^+) = \lim_{x\to 0^+}\frac{u(t,x)-u(t,0)}{x} = \rho_R(t).
\end{aligned}
\tag{43}
$$

Step 2. proof of the inclusion in (41)

We already know that $-u_t(t,0) = H_L(\rho_L(t)) = H_R(\rho_R(t)) \geqslant A$ for a.e. time t (see Lemma 5.1 and Lemma 5.2). Let us fix such a time $t > 0$. Our aim is to check that $(\rho_L(t), \rho_R(t)) \in \mathcal{G}_A$. We argue by contradiction, assuming that

$$H_L(\rho_L(t)) > A, \quad H_L^+(\rho_L(t)) < H_L(\rho_L(t)) \quad \text{and} \quad H_R^-(\rho_R(t)) < H_R(\rho_R(t)).$$

Let us fix $\varepsilon > 0$ so small that $\lambda := H_L(\rho_L(t)) - \varepsilon > A$. We then choose k_L^ε as the smallest solution to $H_L(k_L^\varepsilon) := \lambda$ and k_R^ε as the largest solution to $H_R(k_R^\varepsilon) := \lambda$. As H_L and H_R are convex and $H_L^+(\rho_L(t)) < H_L(\rho_L(t))$ and $H_R^-(\rho_R(t)) < H_R(\rho_R(t))$, we have $k_L^\varepsilon > \rho_L(t)$ and $k_R^\varepsilon < \rho_R(t)$. Moreover, $H_L^+(k_L^\varepsilon) = \min H_L$, while $H_R^-(k_R^\varepsilon) = \min H_R$. Let us define the map $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$w(s,x) = u(t,0) + \begin{cases} k_L^\varepsilon x - \lambda s & \text{if } x \leqslant 0 \\ k_R^\varepsilon x - \lambda s & \text{if } x \geqslant 0 \end{cases}$$

Then w is a test function which is a subsolution of the Hamilton-Jacobi equation (1) because,

$$H_L(k_L^\varepsilon) = H_R(k_R^\varepsilon) = \lambda = -w_s$$

and using $A \in [H_0, 0]$, we get

$$\max\{A, H_L^+(k_L^\varepsilon), H_R^-(k_R^\varepsilon)\} = \max\{A, \min H_L, \min H_R\} = A \leqslant \lambda = -w_s.$$

Moreover, by (43) and the fact that $k_L^\varepsilon > \rho_L(t)$ and $k_R^\varepsilon < \rho_R(t)$, we get that $u(t,x) \geqslant w(0,x)$ if $|x|$ is small enough. Thus, by finite speed of propagation and comparison, we have $u(t+h,0) \geqslant w(h,0)$ for $h > 0$ small enough. Therefore

$$-H_L(\rho_L(t)) = u_t(t,0) \geqslant w_s(0,0) = -\lambda = -H_L(\rho_L(t)) + \varepsilon,$$

which contradicts our assumption. This proves that $(\rho_L(t), \rho_R(t)) \in \mathcal{G}_A$. $\square$

We are now ready to give an alternative proof of Theorem 1.2. This proof relies on semi-algebraic functions. For the reader's convenience, we recall below some useful facts about semi-algebraic sets and functions and we refer to [21] for a complete reference (see also [22]).

Remark 5.5. We recall that a basic semi-algebraic set is a set defined by a finite number of polynomial equalities and polynomial inequalities, and a semi-algebraic set is a finite union of basic semi-algebraic sets. The class $\mathcal{SA}_n$ of semi-algebraic subsets of $\mathbb{R}^n$ has the following properties:

- All algebraic subsets of $\mathbb{R}^n$ (i.e., zeros of a finite number of polynomial equalities) are in $\mathcal{SA}_n$.
- $\mathcal{SA}_n$ is stable by finite intersection, finite union and taking complement.
- The cartesian products of semi-algebraic sets are semi-algebraic.
- The Tarski-Seidenberg Theorem says that the image by the canonical projection $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ of a semi-algebraic set of $\mathbb{R}^{n+1}$ is a semi-algebraic set of $\mathbb{R}^n$.
- By [21, Proposition 1.12], the closure and the interior of a semi-algebraic subset of $\mathbb{R}^n$ are semi-algebraic.
- By definition, a semi-algebraic map is a map defined on a semi-algebraic set and whose graph is a semi-algebraic set.
- An important property of semi-algebraic functions is given in [21, Theorem 2.1] (Monotonicity Theorem): If $f : (a,b) \to \mathbb{R}$ is semi-algebraic, then there exists a finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ such that, on each interval (a_i, a_{i+1}), f is continuous and either constant or strictly monotone.

- An important consequence of the monotonicity Theorem is given in [21, Lemma 6.1]: left and right derivatives of a continuous semi-algebraic map on an open interval exist (with values in $\mathbb{R} \cup \{\pm\infty\}$).

Sketch of proof of Theorem 1.2. As Theorem 1.2 has already been established by using numerical schemes, we only sketch the proof. Recall that u is a viscosity solution of (1). We have to prove that $\rho := u_x$ is an entropy solution to (8). Following for instance [18, 29], we know that ρ solves the equation in $\{x \neq 0\}$. It remains to check the junction condition at $x = 0$. From Lemma 5.3, we just need to show that the left and right derivatives $u_x(t, 0^-)$ and $u_x(t, 0^+)$ are well defined for a.e. time t. To do so, in Step 1, we will use a representation formula, as it is classical for convex Hamiltonian and optimal control interpretation (see for instance the classical reference [10], or Section 3.2 and Chapter 14 in [12], and also Section 6 of [27]). In Step 2, using this representation formula, we show the existence of these derivatives when the initial datum and hamiltonians are semi-algebraic. Finally, in Step 3, we approximate general data by semi-algebraic ones and conclude.

Step 1. representation formula of the solution

In order to use a representation formula, we reverse the time direction of trajectories, and for this reason, we set $\hat{u}(t, x) = u(T - t, x)$ and

$$L_\alpha(q) = \sup_{p\in\mathbb{R}} (-qp - H_\alpha(p))$$

where more generally we denote by $H_\alpha : \mathbb{R} \to \mathbb{R}$ a C^1 function, strictly convex and superlinear, for $\alpha = L, R$. This implies that $L_\alpha : \mathbb{R} \to \mathbb{R}$ is also C^1, strictly convex and superlinear. Let us now define

$$L(x, q) := \begin{cases} L_L(q) & \text{if} \quad x < 0 \\ -A & \text{if} \quad x = 0 \\ L_R(q) & \text{if} \quad x > 0 \end{cases}$$

Following [27, Proposition 6.3], for $t_0 \leqslant T$, we have

$$\hat{u}(t_0, x_0) = \inf_{\gamma(t_0)=x_0} \int_{t_0}^{T} L(\gamma(t), \dot{\gamma}(t))\, dt + u_0(\gamma(T)),$$

where the infimum is taken over $\gamma \in W^{1,\infty}([t_0, T], \mathbb{R})$.

If $\hat{\gamma}$ is optimal for x_0, then $\hat{\gamma}$ is a straight-line on each interval where it does not vanish (by optimality conditions using L_α strictly convex). As a consequence, the minimization problem boils down to minimize for $t_0 < T$ and if, for instance $x_0 < 0$:

$$\begin{aligned}
&\hat{u}(t_0, x_0)\\
&= \min\Big\{ \min_{y\leqslant 0} (T - t_0) L_L\left(\frac{y - x_0}{T - t_0}\right) + u_0(y),\\
&\quad \min_{t_0<\tau_1\leqslant\tau_2<T, y\geqslant 0} (\tau_1 - t_0) L_L\left(\frac{0 - x_0}{\tau_1 - t_0}\right)\\
&\qquad - A(\tau_2 - \tau_1) + (T - \tau_2) L_R\left(\frac{y - 0}{T - \tau_2}\right) + u_0(y),\\
&\quad \min_{t_0<\tau_1\leqslant\tau_2<T, y\leqslant 0} (\tau_1 - t_0) L_L\left(\frac{0 - x_0}{\tau_1 - t_0}\right)\\
&\qquad - A(\tau_2 - \tau_1) + (T - \tau_2) L_L\left(\frac{y - 0}{T - \tau_2}\right) + u_0(y)\Big\}
\end{aligned}$$

$$= \min\{f_1(x_0), f_2(x_0), f_3(x_0)\}, \tag{44}$$

where f_1 corresponds to trajectories ending at $y \leqslant 0$ while f_2 (resp. f_3) corresponds to trajectories ending at $y \geqslant 0$ (resp. $y \leqslant 0$) and remaining in $x = 0$ during the time interval $[\tau_1, \tau_2]$. Notice that (44) is still true for $x_0 = 0$, with each minimum replaced by an infimum.

Step 2. Argument for semi-algebraic data

Here we assume that the data (L_R, L_L and u_0) are semi-algebraic. We claim that the map $\hat{u}(t_0, \cdot)$ given by (44) is also semi-algebraic. Let us mention that in the case of analytic data, Trlat proved in [39, 40] that the solution to the Hamilton-Jacobi equation is subanalytic. To prove our claim, let us show for instance that f_2 is semi-algebraic. Let us define the semi-algebraic set A_2 by

$$\begin{aligned} A_2 := \Big\{ &(t_0, x_0, \tau_1, \tau_2, y, z, u, v) \in \mathbb{R}^8, \\ &0 < t_0 < \tau_1 \leqslant \tau_2 < T,\ (\tau_1 - t_0)u = x_0 < 0,\ (T - \tau_2)v = y, \\ &y \geqslant 0,\ z \geqslant (\tau_1 - t_0)L_L(-u) - A(\tau_2 - \tau_1) + (T - \tau_2)L_R(v) + u_0(y) \Big\}. \end{aligned}$$

Let C_2 denotes the projection of A_2 onto the components (t_0, x_0, z). Then, by the Tarski-Seidenberg Theorem, C_2 is a semi-algebraic set. Note that C_2 is also, by definition, the epigraph of f_2. Therefore the subgraph of f_2 (which is the closure of the complement of C_2) and its graph (intersection of the epigraph and subgraph) are also semi-algebraic. Thus f_2 is a semi-algebraic map. By stability of semi-algebraic sets by finite union, we deduce that $\hat{u}(t_0, \cdot)$ is semi-algebraic on $(-\infty, 0)$. Moreover the function $\hat{u}(t_0, \cdot)$ is continuous at $x_0 = 0$. Hence $\hat{u}(t_0, \cdot)$ is also semi-algebraic on $(-\infty, 0]$. A similar argument shows that it is also semi-algebraic on $[0, \infty)$. Because the union of semi-algebraic sets is semi-algebraic, we deduce that $\hat{u}(t_0, \cdot)$ is semi-algebraic on $\mathbb{R}$. This implies that $u(t, \cdot)$ is semi-algebraic on $\mathbb{R}$ for any $t \in (0, T)$.

Using [21, Lemma 6.1], we then deduce that the limits

$$u_x(t, 0^-) := \lim_{h \to 0^-} \frac{u(t,h) - u(t,0)}{h} \quad \text{and} \quad u_x(t, 0^+) := \lim_{h \to 0+} \frac{u(t,h) - u(t,0)}{h}$$

exist at any time $t \in (0, T)$. Therefore (40) holds. We can then conclude by Lemma 5.3 that $(\rho_L(t), \rho_R(t)) \in \mathcal{G}_A$ for a.e. $t \in [0, T]$.

Step 3. argument in the general case

One can check[1] that it is possible to approximate our data $(H_\alpha)_{|[a_\alpha, c_\alpha]}$ for $\alpha = L, R$, u_0 by semi-algebraic data H_α^ε and u_0^ε satisfying our standing assumptions (with locally uniform convexity for H_L^ε and H_L^ε, which can be for instance locally piecewise linear). By the previous step, we know that, if u^ε is the solution to the HJ equation associated with these perturbed data, then $\rho^\varepsilon = u_x^\varepsilon$ solves the associated SCL. To conclude, we only need to pass to the limit: indeed, u^ε converges locally uniformly to the solution u of the HJ equation (1), while ρ^ε converges in L^1_{loc} to the entropy solution ρ of (8). We infer therefore that u_x, which is the weak limit of u_x^ε, is equal to the solution ρ of (8). □

[1] This is the point where the proof is sketchy: the actual construction of H_L^ε, H_R^ε, and u_0^ε—basically based on Weierstrass theorem—requires some work (using finite velocity to approximate u_0 only in a compact set containing $x = 0$) and has to be done with care. Notice that functions which are piecewise linear with a finite number of pieces can also be used instead of Weierstrass theorem.

Remark 5.6. Notice that that the existence of pointwise derivatives $u_x(t,0^-)$ and $u_x(t,0^+)$ for a.e. t is not a straightforward consequence of the existence of strong traces for the entropy solution u_x outside $x=0$. For more details see [33, 34].

Appendix A.

A.1. **Proof of the discrete entropy inequalities for the SCL numerical scheme.**

Before proving that the scheme satisfies the discrete entropy inequalities stated in Lemma 3.6, we prove the following discrete entropy inequalities, independent of the test function.

Lemma A.1 (First discrete entropy inequalities)**.** *The numerical scheme* (26) *satisfies the following discrete entropy inequalities: for all $n \in \mathbb{N}$, $j \in \mathbb{N}$ and $(k_L,k_R) \in Q$, set $k_\Delta = k_L \mathbb{1}_{j\leqslant -1} + k_R \mathbb{1}_{j\geqslant 0}$. Then*

$$\frac{|p^{n+1}_{j+1/2}-k_\Delta| - |p^{n}_{j+1/2}-k_\Delta|}{\Delta t} + \frac{\Phi^n_{j+1}(k_\Delta) - \Phi^n_j(k_\Delta)}{\Delta x} \leqslant \begin{cases} \dfrac{R_L}{\Delta x} & \text{if } j=-1 \\[2ex] \dfrac{R_R}{\Delta x} & \text{if } j=0 \\[2ex] 0 & \text{otherwise}\,, \end{cases}$$

where

$$R_\alpha = |H_\alpha(k_\alpha) - F_0(k_L,k_R)|\,, \quad \alpha = L,R,$$

and $\Phi^n_j(k_\Delta)$ is defined in (30)*.*

Proof. Let $k \in \mathbb{R}$. Fix $n \in \mathbb{N}$, $j \in \mathbb{Z}$ such that $j \neq 0,-1$. We have, using the monotonicity of the scheme,

$$\begin{aligned}
&|p^{n+1}_{j+1/2}-k| \\
&= p^{n+1}_{j+1/2} \vee k - p^{n+1}_{j+1/2} \wedge k \\
&= \mathcal{F}_j(p^n_{j-1/2},p^n_{j+1/2},p^n_{j+3/2}) \vee \mathcal{F}_j(k,k,k) - \mathcal{F}_j(p^n_{j-1/2},p^n_{j+1/2},p^n_{j+3/2}) \wedge \mathcal{F}_j(k,k,k) \\
&\leqslant \mathcal{F}_j(p^n_{j-1/2}\vee k,p^n_{j+1/2}\vee k,p^n_{j+3/2}\vee k) - \mathcal{F}_j(p^n_{j-1/2}\wedge k,p^n_{j+1/2}\wedge k,p^n_{j+3/2}\wedge k) \\
&= |p^n_{j+1/2}-k| + \frac{\Delta t}{\Delta x}(\Phi^n_j(k) - \Phi^n_{j+1}(k)).
\end{aligned}$$

This is exactly the third inequality. Now we treat the case $j=0$. We have

$$\mathcal{F}_0(k_L,k_R,k_R) = k_R - \frac{\Delta t}{\Delta x}\left(H_R(k_R) - F_0(k_L,k_R)\right).$$

Then,

$$\begin{aligned}
k_R &\geqslant \mathcal{F}_0(p^n_{j-1/2}\wedge k_L, p^n_{j+1/2}\wedge k_R, p^n_{j+3/2}\wedge k_R) - \frac{\Delta t}{\Delta x}\left(H_R(k_R)-F_0(k_L,k_R)\right)^-, \\
k_R &\leqslant \mathcal{F}_0(p^n_{j-1/2}\vee k_L, p^n_{j+1/2}\vee k_R, p^n_{j+3/2}\vee k_R) + \frac{\Delta t}{\Delta x}\left(H_R(k_R)-F_0(k_L,k_R)\right)^+,
\end{aligned}$$

where $a^\pm = \max(\pm a, 0)$, and we can adapt the previous argument in the following way

$$\begin{aligned}
&|p^{n+1}_{1/2}-k_R| \\
&= p^{n+1}_{1/2}\vee k_R - p^{n+1}_{1/2}\wedge k_R \\
&= \mathcal{F}_0(p^n_{-1/2},p^n_{1/2},p^n_{3/2})\vee k_R - \mathcal{F}_0(p^n_{-1/2},p^n_{1/2},p^n_{3/2})\wedge k_R
\end{aligned}$$

$$\begin{aligned}
&\leqslant \mathcal{F}_0(p^n_{-1/2}\vee k_L, p^n_{1/2}\vee k_R, p^n_{3/2}\vee k_R) + \frac{\Delta t}{\Delta x}\left(H_R(k_R) - F_0(k_L,k_R)\right)^+\\
&\quad - \mathcal{F}_0(p^n_{-1/2}\wedge k_L, p^n_{1/2}\wedge k_R, p^n_{3/2}\wedge k_R) + \frac{\Delta t}{\Delta x}\left(H_R(k_R) - F_0(k_L,k_R)\right)^-\\
&= |p^n_{1/2} - k_R| + \frac{\Delta t}{\Delta x}(\Phi^n_0(k_\Delta) - \Phi^n_1(k_\Delta) + R_R).
\end{aligned}$$

We conclude for the case $j = -1$ with the same procedure. This ends the proof of the lemma. □

We are now ready to prove Lemma 3.6.

Proof of Lemma 3.6. Let $\phi \in C^\infty_c([0,T)\times\mathbb{R})$ be non-negative. For all $j\in\mathbb{Z}$ and $n\in\mathbb{N}$, we define

$$\phi^n_{j+1/2} := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \phi(t_n,x)\,\mathrm{d}x.$$

We also denote by $N := \inf\{n\in\mathbb{N}, t_n > T\}$. By Lemma A.1 and since $\phi^n_{j+\frac12}\geqslant 0$ $\forall n,j$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left[|p^{n+1}_{j+1/2} - k_\Delta| - |p^n_{j+1/2} - k_\Delta|\right]\Delta x\ \phi^{n+1}_{j+1/2}\\
&\leqslant -\sum_{j\neq 0,-1}\left[\Phi^n_{j+1}(k_\Delta) - \Phi^n_j(k_\Delta)\right]\Delta t\ \phi^{n+1}_{j+1/2} - \left[\Phi^n_1(k_\Delta) - \Phi^n_0(k_\Delta) - R_R\right]\Delta t\ \phi^{n+1}_{1/2}\\
&\quad - \left[\Phi^n_0(k_\Delta) - \Phi^n_{-1}(k_\Delta) - R_L\right]\Delta t\ \phi^{n+1}_{-1/2}.
\end{aligned}$$

Using the Abel's transformation and rearranging the terms, we get

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left[|p^{n+1}_{j+1/2} - k_\Delta| - |p^n_{j+1/2} - k_\Delta|\right]\Delta x\ \phi^{n+1}_{j+1/2}\\
&\leqslant \left[R_L\phi^{n+1}_{-1/2} + R_R\phi^{n+1}_{1/2}\right]\Delta t + \sum_{j\in\mathbb{Z}}\Phi^n_j(k_\Delta)\Delta t\ \left[\phi^{n+1}_{j+1/2} - \phi^{n+1}_{j-1/2}\right] =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}\tag{45}$$

First, we estimate $\mathcal{I}_2$.

$$\begin{aligned}
\mathcal{I}_2 &= \sum_{j\in\mathbb{Z}}\Phi^n_j(k_\Delta)\Delta t\ \left[\phi^{n+1}_{j+1/2} - \phi^{n+1}_{j-1/2}\right]\\
&= \sum_{j\in\mathbb{Z}}\Phi^n_j(k_\Delta)\frac{\Delta t}{\Delta x}\ \left[\int_{x_j}^{x_{j+1}}\phi(t_{n+1},x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j}\phi(t_{n+1},x)\,\mathrm{d}x\right]\\
&= \sum_{j\in\mathbb{Z}}\Phi^n_j(k_\Delta)\frac{\Delta t}{\Delta x}\ \int_{x_j}^{x_{j+1}}\left[\phi(t_{n+1},x) - \phi(t_{n+1},x-\Delta x)\right]\mathrm{d}x\\
&= \sum_{j\in\mathbb{Z}}\Phi^n_j(k_\Delta)\frac{\Delta t}{\Delta x}\ \int_{x_j}^{x_{j+1}}\Bigg(\phi_x(t_{n+1},x)\Delta x\\
&\qquad\qquad + \int_{x-\Delta x}^{x}(x-\Delta x - y)\phi_{xx}(t_{n+1},y)\,\mathrm{d}y\Bigg)\,\mathrm{d}x\\
&= \int_{\mathbb{R}}\Phi_\Delta(k_\Delta)(t_n,x)\ \Delta t\phi_x(t_{n+1},x)\,\mathrm{d}x\\
&\quad + \sum_{j\in\mathbb{Z}}\Phi^n_j(k_\Delta)\frac{\Delta t}{\Delta x}\int_{x_j}^{x_{j+1}}\int_{x-\Delta x}^{x}(x-\Delta x-y)\phi_{xx}(t_{n+1},y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \Phi_\Delta(k_\Delta)(t_n,x) \int_{t_n}^{t_{n+1}} \left[\phi_x(t,x) + \int_t^{t_{n+1}} \phi_{tx}(s,x)\,\mathrm{d}s\right] \mathrm{d}t\,\mathrm{d}x \\
&\quad + \sum_{j\in\mathbb{Z}} \Phi_j^n(k_\Delta)\frac{\Delta t}{\Delta x}\int_{x_j}^{x_{j+1}}\int_{x-\Delta x}^{x}(x-\Delta x-y)\phi_{xx}(t_{n+1},y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \Phi_\Delta(k_\Delta)(t_n,x) \int_{t_n}^{t_{n+1}} \phi_x(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \Phi_\Delta(k_\Delta)(t_n,x) \int_{t_n}^{t_{n+1}}\int_t^{t_{n+1}} \phi_{tx}(s,x)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \sum_{j\in\mathbb{Z}} \Phi_j^n(k_\Delta)\frac{\Delta t}{\Delta x}\int_{x_j}^{x_{j+1}}\int_{x-\Delta x}^{x}(x-\Delta x-y)\phi_{xx}(t_{n+1},y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

From Lemma 3.5, we know that the discrete gradient lives in Q which is compact. Then the continuity of H_α and F_0 and the definition of Φ_Δ in (30)-(31) imply the existence of a constant C (independent on Δ) such that, if we take $(k_L,k_R)\in Q$, then $|\Phi_\Delta| \leqslant C$.

Consequently,

$$
\mathcal{I}_2 = \int_{\mathbb{R}} \Phi_\Delta(k_\Delta)(t_n,x) \int_{t_n}^{t_{n+1}} \phi_x(t,x)\,\mathrm{d}t\,\mathrm{d}x + \mathcal{I}_2' + \mathcal{I}_2''
$$

where

$$
|\mathcal{I}_2'| \leqslant C \sup_t ||\phi_{tx}(t,\cdot)||_{L^1}(\Delta t)^2, \quad |\mathcal{I}_2''| \leqslant C \sup_t ||\phi_{xx}(t,\cdot)||_{L^1}\Delta t\Delta x.
$$

We then have

$$
\mathcal{I}_2 = \int_{\mathbb{R}} \Phi_\Delta(k_\Delta)(t_n,x) \int_{t_n}^{t_{n+1}} \phi_x(t,x)\,\mathrm{d}t\,\mathrm{d}x + O(\Delta t^2) + O(\Delta t\Delta x). \tag{46}
$$

We now estimate $\mathcal{I}_1$. Recalling that $R_{F_0}(k_L,k_R) := |H_L(k_L) - F_0(k_L,k_R)| + |H_R(k_R) - F_0(k_L,k_R)| = R_L + R_R$, we have

$$
\begin{aligned}
\mathcal{I}_1 =& \Delta t\left[R_L\phi_{-1/2}^{n+1} + R_R\phi_{1/2}^{n+1}\right] \\
=& \frac{\Delta t}{\Delta x}\left[R_L\int_{x_{-1}}^{x_0}\phi(t_{n+1},x)\,\mathrm{d}x + R_R\int_{x_0}^{x_1}\phi(t_{n+1},x)\,\mathrm{d}x\right] \\
=& \frac{\Delta t}{\Delta x}\Big[(R_L+R_R)\phi(t_{n+1},0)\Delta x \\
& +R_L\int_{x_{-1}}^{x_0}\int_0^x \phi_x(t_{n+1},y)\,\mathrm{d}y\,\mathrm{d}x + R_R\int_{x_0}^{x_1}\int_0^x\phi_x(t_{n+1},y)\,\mathrm{d}y\,\mathrm{d}x\Big] \\
=& R_{F_0}(k_L,k_R)\int_{t_n}^{t_{n+1}}\phi(t,0)\,\mathrm{d}t + R_{F_0}(k_L,k_R)\int_{t_n}^{t_{n+1}}\int_t^{t_{n+1}}\phi_t(s,0)\,\mathrm{d}s\,\mathrm{d}t \\
& + \frac{\Delta t}{\Delta x}R_L\int_{x_{-1}}^{x_0}\int_0^x\phi_x(t_{n+1},y)\,\mathrm{d}y\,\mathrm{d}x + \frac{\Delta t}{\Delta x}R_R\int_{x_0}^{x_1}\int_0^x\phi_x(t_{n+1},y)\,\mathrm{d}y\,\mathrm{d}x \\
=:& R_{F_0}(k_L,k_R)\int_{t_n}^{t_{n+1}}\phi(t,0)\,\mathrm{d}t + \mathcal{I}_1' + \mathcal{I}_1'' + \mathcal{I}_1'''
\end{aligned}
$$

and there exist a constant C such that

$$
|\mathcal{I}_1'| \leqslant C||\phi_t||_\infty(\Delta t)^2, \;\; |\mathcal{I}_1'' + \mathcal{I}_1'''| \leqslant C||\phi_x||_\infty\Delta t\Delta x.
$$

This implies that

$$\mathcal{I}_1 = R_{F_0}(k_L, k_R) \int_{t_n}^{t_{n+1}} \phi(t,0)\,\mathrm{d}t + O(\Delta t^2) + O(\Delta t \Delta x). \tag{47}$$

Combining (45), (46) and (47), we finally get

$$\begin{aligned}
&\int_{\mathbb{R}} \Phi_\Delta(k_\Delta)(t_n, x) \int_{t_n}^{t_{n+1}} \phi_x(t,x)\,\mathrm{d}t\,\mathrm{d}x + R_{F_0}(k_L, k_R) \\
&\int_{t_n}^{t_{n+1}} \phi(t,0)\,\mathrm{d}t + O(\Delta t^2) + O(\Delta x \Delta t) \\
&\geqslant \sum_{j\in\mathbb{Z}} \Big[|p_{j+1/2}^{n+1} - k_\Delta| - |p_{j+1/2}^{n} - k_\Delta| \Big] \Delta x\, \phi_{j+1/2}^{n+1}.
\end{aligned}$$

We sum up with respect to $0 \leqslant n \leqslant N$ and use once again Abel's transformation to get

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^T \Phi_\Delta(k_\Delta)(t,x)\, \phi_x(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_0^T R_{F_0}(k_L, k_R) \phi(t,0)\,\mathrm{d}t + O(\Delta t) + O(\Delta x) \\
&\geqslant \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \Big[|p_{j+1/2}^{n+1} - k_\Delta| - |p_{j+1/2}^{n} - k_\Delta| \Big] \Delta x\, \phi_{j+1/2}^{n+1} \\
&\geqslant \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} |p_{j+1/2}^{n} - k_\Delta| \Big[\phi_{j+1/2}^{n} - \phi_{j+1/2}^{n+1} \Big] \Delta x \\
&\quad - \sum_{j\in\mathbb{Z}} |p_{j+1/2}^{0} - k_\Delta| \phi_{j+1/2}^{0} \Delta x + \sum_{j\in\mathbb{Z}} |p_{j+1/2}^{N+1} - k_\Delta| \phi_{j+1/2}^{N+1} \Delta x.
\end{aligned}$$

Recalling that $\phi \in C_c^\infty([0,T) \times \mathbb{R})$, we get that $\phi_{j+1/2}^{N+1} = 0$ for all j. Hence

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^T \Phi_\Delta(k_\Delta)(t,x)\, \phi_x(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_0^T R_{F_0}(k_L, k_R) \phi(t,0)\,\mathrm{d}t + O(\Delta t) + O(\Delta x) \\
&\geqslant \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} |p_{j+1/2}^{n} - k_\Delta| \int_{x_j}^{x_{j+1}} \int_{t_n}^{t_{n+1}} -\phi_t(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad - \sum_{j\in\mathbb{Z}} |p_{j+1/2}^{0} - k_\Delta| \int_{x_j}^{x_{j+1}} \phi(0,x)\,\mathrm{d}x \\
&\geqslant - \int_{\mathbb{R}} \int_0^T |p_\Delta - k_\Delta| \phi_t(t,x)\,\mathrm{d}t\,\mathrm{d}x \; - \int_{\mathbb{R}} |p_\Delta(0,x) - k_\Delta| \phi(0,x)\,\mathrm{d}x
\end{aligned}$$

and we recover the desired discrete entropy inequality. □

A.2. **Local compactness for a numerical scheme of a conservation law.** The proof of Lemma 3.8 is a direct consequence of the following lemma, stated on one branch:

Proposition A.2 (Local compactness on one branch)**.** *Let $f \in C^2(\mathbb{R})$ be Lipschitz continuous and such that*

$$f'' \geqslant \delta > 0. \tag{48}$$

For $n \geqslant 0$, we assume that $q^n_{j+\frac{1}{2}}$ is given for $j=0$, and for $j \geqslant 1$ we assume that $q^{n+1}_{j+\frac{1}{2}}$ is solution of the following scheme

$$q^{n+1}_{j+\frac{1}{2}} = q^n_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\left(g^f(q^n_{j+\frac{1}{2}}, q^n_{j+\frac{3}{2}}) - g^f(q^n_{j-\frac{1}{2}}, q^n_{j+\frac{1}{2}})\right) \tag{49}$$

where we recall that the Godunov flux associated to f is given by

$$g^f(p,q) = \begin{cases} \min_{x\in[p,q]}(f(x)) & \text{if } p \leqslant q \\ \max_{x\in[q,p]}(f(x)) & \text{if } p \geqslant q. \end{cases}$$

We assume that $\left|q^n_{j+\frac{1}{2}}\right| \leqslant M$ for some $M > 0$ and for all $j, n \geqslant 0$ and that $\Delta = (\Delta t, \Delta x)$ satisfies

$$\frac{\Delta x}{\Delta t} \geqslant 2L_f \qquad \text{and} \qquad \gamma := \frac{\Delta t}{\Delta x}\frac{\delta}{2}M \leqslant 1 \tag{50}$$

where L_f is the Lipschitz constant of f. We set

$$q_\Delta := \sum_{n\in\mathbb{N}}\sum_{j\geqslant 1} q^n_{j+1/2}\mathbb{1}_{[t_n,t_{n+1})\times[x_j,x_{j+1})}.$$

Then, there exists $\rho \in L^\infty$ and a subsequence also denoted $(q_{\Delta_k})_k$ such that

$$q_{\Delta_k} \longrightarrow \rho \ a.e..$$

Proof of Lemma 3.8. The proof is a direct consequence of the previous proposition applied on $(0,+\infty)$ to $q^{n,+}_{j+\frac{1}{2}} = p^n_{j+\frac{1}{2}}$ and on $(-\infty,0)$ to $q^{n,-}_{j+\frac{1}{2}} = p^n_{-j-\frac{1}{2}}$ for $j \geqslant 1$. $\square$

The rest of this section is devoted to the proof of Proposition A.2. The idea consists to use a localized discrete Oleinik estimate, see Lemma A.5. To prove this estimate, we first need to prove the following discrete ODE on the discrete gradient.

Lemma A.3 (A discrete ODE on the discrete Gradient)**.** *For $j \geqslant 1$, let*

$$w^n_j := \frac{q^n_{j+1/2} - q^n_{j-1/2}}{\Delta x}$$

and for $j \geqslant 2$

$$\hat{w}^n_j := \max\{0, w^n_{j-1}, w^n_j, w^n_{j+1}\}.$$

Then, for all $j \geqslant 2$ and for all $n \geqslant 0$

$$\frac{\max(0, w^{n+1}_j) - \hat{w}^n_j}{\Delta t} \leqslant -\frac{\delta}{8}|\hat{w}^n_j|^2. \tag{51}$$

Proof. First, fix $n \in \mathbb{N}$ and $j \geqslant 2$. we have

$$\begin{aligned} w^{n+1}_j &= w^n_j - \frac{\Delta t}{(\Delta x)^2}\Big[g^f(q^n_{j+1/2}, q^n_{j+3/2}) - g^f(q^n_{j-1/2}, q^n_{j+1/2}) \\ &\quad - g^f(q^n_{j-1/2}, q^n_{j+1/2}) + g^f(q^n_{j-3/2}, q^n_{j-1/2})\Big] \\ &= w^n_j - \frac{\Delta t}{(\Delta x)^2}\Big[g^f(q^n_{j-1/2} + w^n_j\Delta x, q^n_{j+1/2} + w^n_{j+1}\Delta x) - 2g^f(q^n_{j-1/2}, q^n_{j+1/2}) \\ &\quad + g^f(q^n_{j-1/2} - w^n_{j-1}\Delta x, q^n_{j+1/2} - w^n_j\Delta x)\Big] \\ &=: G(w^n_{j-1}, w^n_j, w^n_{j+1}, q^n_{j-1/2}, q^n_{j+1/2}). \end{aligned}$$

Due to the monotonicity of g^f, we know that G is non-decreasing with respect to its first and third variables. We now prove that G is also non-decreasing with respect to its second variable. Indeed, we have

$$\begin{aligned}&\partial_w G(a,w,b,q_{-1},q_1)\\&= 1-\frac{\Delta t}{\Delta x}\left[\partial_1 g^f(q_{-1}+w\Delta x, q_1+b\Delta x)-\partial_2 g^f(q_{-1}-a\Delta x, q_1-w\Delta x)\right]\\&\geqslant 1-2\frac{\Delta t}{\Delta x}L_f\geqslant 0,\end{aligned}$$

by (50). This implies that

$$w_j^{n+1}=G(w_{j-1}^n,w_j^n,w_{j+1}^n,q_{j+1/2}^n,q_{j-1/2}^n)\leqslant G(\hat{w}_j^n,\hat{w}_j^n,\hat{w}_j^n,q_{j+1/2}^n,q_{j-1/2}^n).$$

Moreover,

$$0=G(0,0,0,q_{j+1/2}^n,q_{j-1/2}^n)\leqslant G(\hat{w}_j^n,\hat{w}_j^n,\hat{w}_j^n,q_{j+1/2}^n,q_{j-1/2}^n).$$

This implies that

$$\max(0,w_j^{n+1})\leqslant G(\hat{w}_j^n,\hat{w}_j^n,\hat{w}_j^n,q_{j+1/2}^n,q_{j-1/2}^n). \tag{52}$$

For clarity's sake, we omit the n dependency when not necessary. Set

$$Q_j:=\begin{pmatrix}q_{j-1/2}^n\\ q_{j+1/2}^n\end{pmatrix},\quad W_j:=\begin{pmatrix}\hat{w}_j^n\\ \hat{w}_j^n\end{pmatrix}.$$

We then get

$$\frac{\max(0,w_j^{n+1})-\hat{w}_j^n}{\Delta t}\leqslant -\frac{1}{(\Delta x)^2}\left[g^f(Q_j+W_j\Delta x)-2g^f(Q_j)+g^f(Q_j-W_j\Delta x)\right]. \tag{53}$$

We now want to estimate the right hand term. Using (56) in Lemma A.4 below (with $P=Q_j$, $W=W_j$ and $\alpha=\pm\Delta x$), we have

$$\frac{\max(0,w_j^{n+1})-\hat{w}_j^n}{\Delta t}\leqslant -I_j=-(I_j^+ + I_j^-) \tag{54}$$

where for $\beta=\pm$,

$$I_j^\beta=\int_0^1(1-t)\,\mathbf{Hess}(g^f)(Q_j+t\beta\Delta x W_j)W_j\cdot W_j dt.$$

To estimate $I_j^\pm$, we use the explicit form of $\mathbf{Hess}(g^f)(Q_j+t\alpha\Delta x W_j)$ given in Lemma A.4 below. We assume for the moment that $\hat{w}_j^n>0$. We then have

$$I_j^+\geqslant \delta|\hat{w}_j^n|^2\int_0^1(1-t)\mathbb{1}_{\{f^-(q)<f(p),f'(p)>0\}}dt$$

where $p=p(t)=q_{j-\frac12}^n+t\Delta x\hat{w}_j^n$ and $q=q(t)=q_{j+\frac12}^n+t\Delta x\hat{w}_j^n$, and

$$I_j^-\geqslant \delta|\hat{w}_j^n|^2\int_0^1(1-t)\mathbb{1}_{\{f(q')>f^+(p'),f'(q')<0\}}dt$$

where $p'=p'(t)=q_{j-\frac12}^n-t\Delta x\hat{w}_j^n$ and $q'=q'(t)=q_{j+\frac12}^n-t\Delta x\hat{w}_j^n$. We now want to prove that

$$\mathbb{1}_{\{f^-(q)<f(p),f'(p)>0\}}+\mathbb{1}_{\{f(q')>f^+(p'),f'(q')<0\}}\geqslant 1\quad \forall t\in]\frac12,1]. \tag{55}$$

Since $\hat{w}_j^n > 0$, we have $q' - p' = w_j^n \Delta x - 2t\Delta x\ \hat{w}_j^n \leqslant (1-2t)\Delta x\ \hat{w}_j^n < 0$ if $t > \frac{1}{2}$. Moreover, by definition of p, q, p', q', we have $p' < p$ and $q' < q$.

By contradiction assume that (55) is not satisfied, i.e.

$$\begin{cases} & f'(p) \leqslant 0 \qquad \text{or} \quad f^-(q) \geqslant f(p) \\ \text{and} & \\ & f(q') \leqslant f^+(p') \qquad \text{or} \quad f'(q') \geqslant 0. \end{cases}$$

On the one hand, if $f'(p) \leqslant 0$, since $q' < p' < p$, we deduce that $f'(q') < 0$. Hence $f(q') \leqslant f^+(p')$. Since $p' \leqslant p$, we also have $f^+(p') = \inf f$ and so $f(q') = \inf f$ which contradicts the fact that $f'(q') < 0$. On the other hand, if $f'(p) > 0$ and $f^-(q) \geqslant f(p)$, then $f'(q) < 0$. Since $q' < q$ and $p' < p$, we then get

$$f(q') = f^-(q') > f^-(q) \geqslant f(p) = f^+(p) \geqslant f^+(p')$$

which is a contradiction. We then deduce that (55) holds true. This implies that

$$I_j \geqslant \delta |\hat{w}_j^n|^2 \int_{1/2}^1 (1-t)dt = \frac{1}{8}\delta |\hat{w}_j^n|^2.$$

Notice that this inequality is also true if $\hat{w}_j^n = 0$. Injecting this in (54), we get the result. ☐

It remains to show the following lemma concerning some properties of the Godunov flux.

Lemma A.4 (Regularity of the Godunov flux)**.** *Define*

$$\Gamma := \{(p,q)\ s.t.\ f^+(p) = f^-(q) > \inf_{\mathbb{R}} f\}.$$

Then g^f is $C^1(\mathbb{R}^2 \backslash \Gamma)$ and

$$\nabla g^f(p,q) = \begin{pmatrix} f'(p)\mathbb{1}_{\{f^-(q)<f(p), f'(p)>0\}} \\ f'(q)\mathbb{1}_{\{f(q)>f^+(p), f'(q)<0\}}. \end{pmatrix}$$

Moreover g^f is $W^{2,\infty}(\mathbb{R}^2\backslash\Gamma)$ and for all $(p,q) \notin \Gamma$

$$\mathbf{Hess}(g^f)(p,q) = \begin{pmatrix} \mathbb{1}_{\{f^-(q)<f(p), f'(p)>0\}} f''(p) & 0 \\ 0 & \mathbb{1}_{\{f(q)>f^+(p), f'(q)<0\}} f''(q) \end{pmatrix}.$$

Finally, if $P = (p,q)$ and $W = (w,w)$, then for all $\alpha \in \mathbb{R}$ and for any subgradient $\nabla g^f(P) \in \partial g^f(P)$ (which is a true gradient if $P \notin \Gamma$)

$$g^f(P+\alpha W) - g^f(P) \geqslant \alpha W \cdot \nabla g^f(P) + \alpha^2 \int_0^1 (1-t)\,\mathbf{Hess}(g^f)(P+t\alpha W)W\cdot W dt \quad (56)$$

Proof. We just prove (56), the proof of the other properties being direct consequences of the reformulation of the Godunov flux, in the convex case, $g^f(p,q) = \max(f^+(p), f^-(q))$, given in Lemma 3.2.

If $w = 0$, the result is obvious. Assume that $w \neq 0$. We set $U = [-M,M]^2\backslash\Gamma$. Since f is convex, g^f is also convex and we have $D^2 g^f \geqslant \{D^2 g^f\}_{|U} \cdot \mathbb{1}_U$, where $\{D^2 g^f\}_{|U}$ is the classical derivative part of $D^2 g^f$ given by $\mathbf{Hess}(g^f)$. So to prove (56), it's sufficient to show that $\mathbb{1}_U(Q+\alpha t W) = 1$ for a.e. t. To show this, we claim that for all t

$$\Gamma \cap (\Gamma + tW) = \varnothing.$$

Indeed, if there exists $Q = (q_1, q_2) \in \Gamma \cap (\Gamma + tW)$ for some $t \neq 0$ (assume that $w > 0$ and $t > 0$ to fix the idea, the other cases being similar), then

$$f^-(q_2 + tw) = f^+(q_1 + tw) > f^+(q_1) = f^-(q_2) > f^-(q_2 + tw)$$

which is a contradiction. This implies that the curve $t \mapsto Q + \alpha t W$ can cross Γ at most one time and so $\mathbb{1}_U(Q + \alpha t W) = 1$ for a.e. t. $\square$

Lemma A.5 (Discrete Oleinik estimate)**.** *Under the same assumptions as Proposition A.2, let $R_2 > R_1 > 0$ and $J_2 > J_1 \geqslant 2$ be such that $(J_1\Delta x, J_2\Delta x) \subset (R_1, R_2)$. Then for w_j^n defined in Lemma A.3 and for $0 \leqslant n \leqslant \frac{1}{2}(J_2 - J_1)$, we have*

$$\frac{\delta}{8} \sup_{J_1+n \leqslant j \leqslant J_2-n} w_j^n \leqslant \frac{1}{(n+1)\Delta t}. \tag{57}$$

Remark A.6. We provide here a proof of the localized estimate (57). A similar estimate (with possible different constants) can also be deduce from the proofs of the known global results. For Godunov flux, it can be deduced either from [25], or from [13] for an optimal constant with a nice proof (which simply uses the fact that Godunov scheme is equivalent to solve exactly the Riemann problem (i.e. solve the exact PDE), and then average the solution). See also [38] for the case of Lax-Friedrichs schemes.

Proof of Lemma A.5. Step 1. Initial condition

We first check that (57) holds true for $n = 0$. We have

$$w_j^n = \frac{q_{j+1/2}^n - q_{j-1/2}^n}{\Delta x} \quad \text{with} \quad |q_{j\pm 1/2}^n| \leqslant M$$

Hence

$$\left(\Delta t \frac{\delta}{8}\right) \sup_{j \in [J_1, J_2]} w_j^0 \leqslant \left(\Delta t \frac{\delta}{8}\right) \frac{2M}{\Delta x} = \frac{\gamma}{2} \leqslant \frac{1}{2} \leqslant 1$$

and (57) is satisfied for $n = 0$.

Step 2. The supersolution

Recall that, by Lemma A.3, we have, with $\hat{w}_j^n := \max(0, w_{j-1}^n, w_j^n, w_{j+1}^n)$, for $j \geqslant 2$

$$\frac{\max(0, w_j^{n+1}) - \hat{w}_j^n}{\Delta t} \leqslant -\frac{\delta}{8} |\hat{w}_j^n|^2 \tag{58}$$

Notice that

$$\frac{1}{m+1} - \frac{1}{m} \geqslant -\left|\frac{1}{m}\right|^2 \quad \text{for} \quad m \geqslant 1$$

and then we see immediately that

$$h^n := \frac{1}{\left(\Delta t \frac{\delta}{8}\right)} \frac{1}{(n+1)}$$

is a supersolution of the equation with equality in (58), whose w^n is itself a subsolution. Moreover h^n satisfies the equality in the inequality (57) for $n = 0$.

Step 3. Time evolution and comparison

Now assume that (57) is true at step $n \geqslant 0$ and let us show it is also true at step $n + 1$.

We then assume that

$$\sup_{j \in [J_1+n, J_2-n]} w_j^n \leqslant h^n$$

i.e.

$$\sup_{j \in [J_1+n+1, J_2-(n+1)]} \hat{w}_j^n \leqslant h^n.$$

Then (58) implies that

$$\begin{aligned}
&\sup_{j\in[J_1+(n+1),J_2-(n+1)]} \max(0,w_j^{n+1})\\
&\leqslant \sup_{j\in[J_1+n+1,J_2-(n+1)]} \Psi(\hat{w}_j^n) \quad \text{with} \quad \Psi(w):=w-\Delta t\frac{\delta}{8}|w|^2.
\end{aligned}$$

Because Ψ is nondecreasing on $\left[0,\left(\Delta t\frac{\delta}{4}\right)^{-1}\right]$, and

$$0\leqslant \hat{w}_j^n \leqslant \frac{2M}{\Delta x}\leqslant \left(\Delta t\frac{\delta}{4}\right)^{-1}=\frac{1}{2}h^0 \quad \text{because} \quad \gamma\leqslant 1,$$

we deduce, using that h^n is a supersolution, that

$$\Psi(\hat{w}_j^n)\leqslant \begin{cases} \Psi(h^n)\leqslant h^{n+1} & \text{if} \quad n\geqslant 1, \qquad \text{because} \quad h^n\leqslant \left(\Delta t\frac{\delta}{4}\right)^{-1}\\ \Psi(\frac{1}{2}h^0)=\frac{1}{4}h^0\leqslant \frac{1}{2}h^0=h^1 & \text{if} \quad n=0 \qquad \text{because} \quad \hat{w}_j^0\leqslant \frac{1}{2}h^0\end{cases}$$

for all $j\in[J_1+(n+1),J_2-(n+1)]$. This implies that

$$\sup_{j\in[J_1+(n+1),J_2-(n+1)]} \max(0,w_j^{n+1})\leqslant h^{n+1}.$$

This ends the proof fo the lemma. $\square$

Lemma A.7. (Total variation estimates). *Assume that for $J_2\geqslant J_1\geqslant 2$ and for $B\geqslant 0$*

$$\begin{cases} \dfrac{q_{j+1/2}^n-q_{j-1/2}^n}{\Delta x}\leqslant B & \textit{for all} \quad j\in[J_1,J_2-1]\\ |q_{j-\frac{1}{2}}^n|\leqslant M & \textit{for all} \quad j\in[J_1,J_2].\end{cases}$$

Then we have

$$\sum_{j\in[J_1,J_2-1]}|q_{j+1/2}^n-q_{j-1/2}^n|\leqslant 2M+2B(J_2-J_1)\Delta x$$

and

$$\begin{aligned}
\sum_{j\in[J_1+1,J_2-1]}|q_{j-1/2}^{n+1}-q_{j-1/2}^n| &\leqslant 2L_f\frac{\Delta t}{\Delta x}\sum_{j\in[J_1,J_2-1]}|q_{j+1/2}^n-q_{j-1/2}^n|\\
&\leqslant 2L_f\frac{\Delta t}{\Delta x}\cdot(2M+2B(J_2-J_1)\Delta x).
\end{aligned}$$

where L_f is the Lipschitz constant of f.

Proof. The result easily follows from a picture with worse cases (and from the scheme for the last bound). We skip the details. This ends the proof of the lemma. $\square$

We are now in a position to prove Proposition A.2.

Proof of Proposition A.2. We simply apply the bounds of Lemma A.7, which shows that for all $\theta>0$ and $0<R_1<R_2$

$$|q_\Delta|_{BV(\Omega_{\theta,R_1,R_2})}\leqslant C_\theta, \quad |q_\Delta|_{L^\infty(0,+\infty)\times(0,+\infty)}\leqslant M$$

for the triangle

$$\Omega_{\theta,R_1,R_2} := \left\{(t,x)\in(0,+\infty)^2 \text{ s.t. } t\in\left(\theta,\frac{R_2-R_1}{2}+\theta\right),\ x\in(R_1+t-\theta,R_2-(t-\theta))\right\}.$$

Recall that we have compactness of the injection $(BV\cap L^\infty)(\Omega)\to L^1(\Omega)$ for any triangle Ω. Recovering $(0,+\infty)\times(0,+\infty)$ by triangles possibly arbitrary small, we deduce the result from a standard diagonal extraction argument. This ends the proof of the proposition. ☐

A.3. **Hamilton-Jacobi germs are not L^1-dissipative for $N\geqslant 3$ branches.** In this subsection, for convenience of an (undeveloped) traffic interpretation/motivation, we prefer to work with concave fluxes instead of convex fluxes (which is indeed equivalent by a simple change of sign and by replacing max with min).

Notation.

Let I and J be two non-empty finite sets (of indices) with $I\cap J=\varnothing$. For $\alpha\in I\cup J$, we consider real numbers $a_\alpha<c_\alpha$, and non constant concave functions $f^\alpha:[a_\alpha,c_\alpha]\to[0,+\infty)$ with $f^\alpha(a_\alpha)=0=f^\alpha(c_\alpha)$. We consider $\lambda^\alpha_{max}:=\max_{Q_\alpha} f^\alpha>0$ and $Q_\alpha:=[a_\alpha,c_\alpha]$. We set

$$f^{\alpha,+}(q)=\sup_{[a_\alpha,q]} f^\alpha,\quad f^{\alpha,-}(q)=\sup_{[q,c_\alpha]} f^\alpha,\quad \text{for}\quad q\in Q_\alpha$$

and, for all $\lambda\in[0,\lambda^\alpha_{max}]$,

$$q^\alpha_\pm(\lambda):=q\quad \text{where } q\in Q_\alpha \text{ is defined by}\quad f^\alpha(q)=\lambda=f^{\alpha,\pm}(q) \tag{59}$$

We consider weights

$$\theta_\alpha\in(0,1]\quad \text{for all}\quad \alpha\in I\cup J\quad \text{such that}\quad 1=\sum_{i\in I}\theta_i=\sum_{j\in J}\theta_j. \tag{60}$$

The weights $(\theta_\alpha)_{\alpha\in I}$ represent the relative ratio of car entering the junction from each branch $i\in I$. Whereas the weights $(\theta_\alpha)_{\alpha\in J}$ represent the relative ratio of car leaving the junction into each branch $j\in J$. Notice that for $\alpha\in I\cup J$, the equality $\theta_\alpha=1$ implies that $\mathrm{Card}(I)=1$ (if $\alpha\in I$) or $\mathrm{Card}(J)=1$ (if $\alpha\in J$).

We also define

$$A_0:=\min_{\alpha\in I\cup J}\theta_\alpha^{-1}\lambda^\alpha_{max}.$$

HJ problem

We consider the following Hamilton-Jacobi problem on a junction with incoming branches indexed by I and outgoing branches indexed by J

$$\begin{cases} u^i_t+\theta_i^{-1}f^i(\theta_i u^i_x) &= 0 & x<0 & i\in I\\ u^j_t+\theta_j^{-1}f^j(\theta_j u^j_x) &= 0 & x>0 & j\in J\\ u^i=u^j &=: u & x=0 & i\in I,\ j\in J\\ u_t+\min\left\{A,\min_{i\in I}\theta_i^{-1}f^{i,+}(\theta_i u^i_x),\min_{j\in J}\theta_j^{-1}f^{j,-}(\theta_j u^j_x)\right\} &= 0 & x=0 & \end{cases} \tag{61}$$

where $A \in [0, A_0]$ is the flux limiter. We define $\rho^\alpha := \theta_\alpha u_x^\alpha$ for $\alpha \in I \cup J$, which satisfies (at least formally)

$$\left\{\begin{array}{rlll} \rho_t^i + f^i(\rho^i)_x & = 0 & x < 0 & i \in I \\ \rho_t^j + f^j(\rho^j)_x & = 0 & x > 0 & j \in J \\ \rho = ((\rho^i)_{i\in I}, (\rho^j)_{j\in J}) & \in \mathcal{G}_A^{HJ} & x = 0 & \text{for a.e. time } t \end{array}\right. \tag{62}$$

with the HJ germ defined by the set

$$\mathcal{G}_A^{HJ} := \left\{ \begin{array}{l} p = (p_\alpha)_{\alpha\in I\cup J} \in \prod_{\alpha\in I\cup J} Q_\alpha, \quad \text{such that there exists } \lambda \in \mathbb{R} \text{ with} \\ \theta_\alpha^{-1} f^\alpha(p_\alpha) = \lambda = \min\left\{A, \quad \min_{i\in I} \ \theta_i^{-1} f^{i,+}(p_i), \quad \min_{j\in J} \ \theta_j^{-1} f^{j,-}(p_j)\right\} \\ \qquad\qquad\qquad \text{for all} \quad \alpha \in I \cup J \end{array} \right\} \tag{63}$$

By (60) we recover the Rankine-Hugoniot relation

$$\sum_{i\in I} f^i(p_i) = \sum_{j\in J} f^j(p_j) \quad \text{for all} \quad p \in \mathcal{G}_A^{HJ}.$$

Lemma A.8. (Lack of dissipation for Hamilton-Jacobi germs with 3 branches or more) *Set $n := Card(I)$ and $m := Card(J)$ with $n, m \geqslant 1$. Under the previous assumptions, we have:*
i) *The set $\mathcal{G}_A^{HJ}$ is L^1-dissipative if $A \in [0, A_0]$ and $n = m = 1$, or if $A = 0$ and $n, m \geqslant 1$.*
ii) *For $A \in (0, A_0]$, the set $\mathcal{G}_A^{HJ}$ is not L^1-dissipative if $n + m \geqslant 3$.*

Proof of Lemma A.8. Recall that the germ $\mathcal{G}_A^{HJ}$ is L^1-dissipative (on the box $Q := \prod_{\alpha\in I\cup J} Q_\alpha$) if and only if the entropy flux satisfies IN $\geqslant$ OUT, i.e. for all $p', p \in \mathcal{G}_A^{HJ}$, we have

$$\sum_{i\in I} \text{sign}(p_i' - p_i) \cdot \left\{f^i(p_i') - f^i(p_i)\right\} \geqslant \sum_{j\in J} \text{sign}(p_j' - p_j) \cdot \left\{f^j(p_j') - f^j(p_j)\right\} \tag{64}$$

The case $A = 0$ is trivial, and we now assume that $A \in (0, A_0]$. We choose

$$p_i' := q_+^i(\theta_i A), \quad p_j' := q_-^j(\theta_j A), \quad i \in I, \quad j \in J,$$

where the map $q_\pm^\alpha(\cdot)$ is defined in (A.3). Now we choose $\alpha_0 \in I \cup J$ and for some $\lambda \in (0, A)$, we set

$$p_i := \left\{ \begin{array}{ll} q_+^i(\theta_i\lambda) & \text{if } i = \alpha_0 \in I, \\ q_-^i(\theta_i\lambda) & \text{if } i \in I\backslash\{\alpha_0\}, \end{array}\right. \qquad \text{and} \qquad p_j := \left\{ \begin{array}{ll} q_+^j(\theta_j\lambda) & \text{if } j = \alpha_0 \in J \\ q_-^i(\theta_i\lambda) & \text{if } j \in J\backslash\{\alpha_0\} \end{array}\right.$$

Then we have

$$\text{sign}(p_i' - p_i) = \left\{ \begin{array}{ll} +1 & \text{if } i = \alpha_0 \in I, \\ -1 & \text{if } i \in I\backslash\{\alpha_0\}, \end{array}\right. \qquad \text{and} \quad \text{sign}(p_j' - p_j) = \left\{ \begin{array}{ll} -1 & \text{if } j = \alpha_0 \in J, \\ +1 & \text{if } j \in J\backslash\{\alpha_0\} \end{array}\right.$$

and

$$\left\{f^\alpha(p_\alpha') - f^\alpha(p_\alpha)\right\} = \theta_\alpha(A - \lambda) > 0 \quad \text{for all} \quad \alpha \in I \cup J.$$

Notice that $p, p' \in \mathcal{G}_A^{HJ}$. Indeed for p' this follows from $\theta_i^{-1} f^{i,+}(q_+^i(\theta_i A)) = A$ and $\theta_j^{-1} f^{j,-}(q_-^j(\theta_j A)) = A$. While for p, this follows for $i = \alpha_0$ or $j = \alpha_0$ from $f^{i,+}(q_-^i(\theta_i\lambda)) = \lambda_{max}^i$ and $f^{j,-}(q_+^j(\theta_j\lambda)) = \lambda_{max}^j$, and then $\theta_i^{-1} f^{i,+}(q_-^i(\theta_i\lambda))$, $\theta_j^{-1} f^{j,-}(q_+^j(\theta_j\lambda)) \geqslant A_0 \geqslant A > \lambda$.

Now dividing (64) by $(A-\lambda)>0$, and using (60), this leads to:

$$\begin{cases} \{-1+2\theta_{\alpha_0}\} & \geqslant \{+1\} \qquad \text{if} \quad \alpha_0 \in I \\ \{-1\} & \geqslant \{+1-2\theta_{\alpha_0}\} \qquad \text{if} \quad \alpha_0 \in J \end{cases}$$

which forces $\theta_{\alpha_0} \geqslant 1$. This contradicts (60) if $\mathrm{Card}(I) \geqslant 2$ or $\mathrm{Card}(J) \geqslant 2$. The fact that $\mathcal{G}_A^{HJ}$ is L^1-dissipative for $\mathrm{Card}(I) = 1 = \mathrm{Card}(J)$ is proved in Proposition 2.10. This ends the proof of the lemma. □

To conclude, the correspondence result is false when we consider 3 or more branches. However, we still have wellposedness for Hamilton-Jacobi problem (61). Also it is possible to show that the derivative ρ of the solution u to (61) along each branch, is still an entropy solution of (62) outside the node. Moreover, from [33] for strictly convex or concave Hamiltonians/fluxes, we know that the trace of ρ at the node $x=0$ belongs to the germ for almost every time, i.e. $\gamma\rho(x=0) \in \mathcal{G}_A^{HJ}$ with $\mathcal{G}_A^{HJ}$ defined by (63). For such a notion of solution, we do not know how to prove uniqueness, since the germ condition is not L^1-dissipative. Still, because we use [33], this shows the correspondence, namely that the derivative of the unique solution u of (61), is *a* solution ρ of (62).

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