Germs for scalar conservation laws: the Hamilton-Jacobi equation point of view

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Abstract

We prove that the entropy solution to a scalar conservation law posed on the real line with a flux that is discontinuous at one point (in the space variable) coincides with the derivative of the solution to a Hamilton-Jacobi (HJ) equation whose Hamiltonian is discontinuous. Flux functions (Hamiltonians) are not assumed to be convex in the state (gradient) variable. The proof consists in proving the convergence of two numerical schemes. We rely on the theory developed by B. Andreianov, K. H. Karlsen and N. H. Risebro (*Arch. Ration. Mech. Anal.*, 2011) for such scalar conservation laws and on the viscosity solution theory developed by the authors (*arxiv*, 2023) for the corresponding HJ equation. This study allows us to characterise certain germs introduced in the AKR theory (namely maximal and complete ones) and relaxation operators introduced in the viscosity solution framework.

Contents

1 Introduction

We are interested in scalar conservation laws (SCL) with discontinuous flux posed on the real line. The discontinuity arises at the origin in space variable, with a flux on the left and a possible different flux on the right. As far as entropy solutions for such equations are concerned, we adopt here the (AKR) point of view of B. Andreianov, K. H. Karlsen and N. H. Risebro [3]. The condition imposed to the entropy solution at the discontinuity concerns its strong traces from the right and the left. They are imposed to belong to a set that is called a *germ*. Following the AKR theory, uniqueness of the solution is known for maximal germs (in the sense of inclusion), and existence of the solution is known for complete germs (for which the Riemann problem can be solved).

We can assert at least formally that the solution of a SCL is the derivative of the solution of a Hamilton-Jacobi (HJ) equation whose Hamiltonian coincides with the flux function of the conservation law. In our framework, this Hamiltonian is thus discontinuous (in the spatial variable). In the viscosity solution theory developed for the corresponding HJ equations, conditions imposed at the discontinuity are in correspondance with a family of monotone nonlinearities. The relaxation of such nonlinearities creates naturally some $\mathcal{G}$-Godunov fluxes for a certain germ $\mathcal{G}$.

When flux functions / Hamiltonians are convex in the state / gradient variable, it was recently proved by P. Cardaliaguet, T. Girard and the first and third authors [10] that the spatial derivative of the viscosity solution is the entropy solution for a germ associated with the boundary nonlinearity. We prove that this result still holds true for general Hamiltonians, not necessarily convex, but coercive. The problem is significantly more difficult because conditions at the discontinuity (for both equations) are much richer. Indeed, the proof of the result consists in approximating solutions of the two equations by numerical schemes and in proving their convergence. The difficulty lies in identifying the germ selected by the numerical scheme associated with the SCL.

It is one of the main contributions of this work to show that the desired condition at the discontinuity of the limit of the SCL numerical scheme is necessarily relaxed. Such a phenomena was exhibited by the authors at the level of HJ equations [22, 16]. Its understanding is used to address the relaxation of the condition at the discontinuity for the SCL.

Another contribution of this work is to show that the germs selected by this approximation procedure coincide with maximal (in the sense of inclusion) and complete (for which the Riemann problem can be solved) ones. The derivation of this formula is based on a Hamilton-Jacobi point of view on the problem.

1.1 Scalar conservation laws and Hamilton-Jacobi equations

In this article, we consider a scalar conservation law of the form

$$\begin{cases} v_t + H_L(v)_x = 0, & t > 0, x < 0, \\ v_t + H_R(v)_x = 0, & t > 0, x > 0, \\ (v(t,0-), v(t,0+)) \in \mathcal{G}, & t > 0 \end{cases} \tag{1.1}$$

where $H_L, H_R \colon \mathbb{R} \to \mathbb{R}$ satisfy,

$$\begin{cases} H_\alpha \text{ are } \mathfrak{L}_\alpha\text{-Lipschitz continuous,} \\ H_\alpha \text{ is not constant on any open interval,} \\ H_\alpha(p_\alpha) \to +\infty \quad \text{as } |p_\alpha| \to +\infty \end{cases} \tag{1.2}$$

with $\alpha \in \{L, R\}$.

The condition at $x = 0$ for entropy solutions that we will work with in this article was introduced by B. Andreianov, K. H. Karlsen and N. H. Risebro in [3]. It is necessary to supplement the equation with a condition at $x = 0$ (even if $H_L = H_R$). This condition amounts to impose that the couple of traces $(v(t,0+), v(t,0-))$ lie in a given set $\mathcal{G}$, called the *germ*.

In this work, we make precise the link between such entropy solutions of (1.1) associated to a germ $\mathcal{G}$ and viscosity solutions of the following Hamilton-Jacobi equation,

$$\begin{cases} u_t + H_L(u_x) = 0, & t > 0, x < 0, \\ u_t + H_R(u_x) = 0, & t > 0, x > 0, \\ u_t + F_0(u_x(t,0-), u_x(t,0+)) = 0, & t > 0, x = 0. \end{cases} \tag{1.3}$$

We assume that the function F_0 satisfies

$$\begin{cases} F_0 \text{ is } \mathfrak{L}_0\text{-Lipschitz continuous,} \\ p_L \mapsto F_0(p_L, p_R) \text{ is non-decreasing,} \\ p_R \mapsto F_0(p_L, p_R) \text{ is non-increasing,} \\ F_0(p) \to +\infty \text{ as } (p_L)_+ + (p_R)_- \to +\infty \end{cases} \tag{1.4}$$

with $(p_\alpha)_- = \max(0, -p_\alpha)$.

Both equations are supplemented with an initial condition,

$$u \quad = u_0 \quad \text{in} \quad \{0\} \times \mathbb{R}, \tag{1.5}$$

$$v \quad = v_0 \quad \text{in} \quad \{0\} \times \mathbb{R} \tag{1.6}$$

with $v_0 = (u_0)_x \in L^\infty \cap \mathrm{BV}(\mathbb{R})$.
In short, we say that v is a $\mathcal{G}$-entropy solution to (1.1), while u is a F_0-viscosity solution to (1.3).

1.2 Numerical schemes

We now describe the numerical scheme used to solve the Hamilton-Jacobi equation (1.3). Given a time step $\Delta t > 0$ and a space step $\Delta x > 0$, we consider the discrete time $t_n = n\Delta t$ for $n \in \mathbb{N}$ and the discrete point $x_j = j\Delta x$ for $j \in \mathbb{Z}$. We denote by u_j^n the numerical approximation of $u(t_n, x_j)$. In order to discretize (1.3), we will use a Godunov approximation. More precisely, we introduce the following Godunov numerical Hamiltonians, for $\alpha = L, R$

$$g^{H_\alpha}(p^-, p^+) = \begin{cases} \min\limits_{p \in [p^-, p^+]} H_\alpha(p) & \text{if } p^- \le p^+, \\ \max\limits_{p \in [p^+, p^-]} H_\alpha(p) & \text{if } p^+ \le p^-. \end{cases} \tag{1.7}$$

We remark that the functions g^{H_α} are non-decreasing in the first variable and non-increasing in the second one. Moreover, $g^{H_\alpha}(p,p) = H_\alpha(p)$ for $\alpha = R, L$. Given $n \ge 1$, we define for all $j \in \mathbb{Z}$,

$$v_{j+\frac{1}{2}}^n = \frac{u_{j+1}^n - u_j^n}{\Delta x}. \tag{1.8}$$

The numerical scheme is then given by

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + g^{H_L}\left(v_{j-\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n\right) = 0 & \text{for } j \le -1, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + g^{H_R}\left(v_{j-\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n\right) = 0 & \text{for } j \ge 1, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + F_0\left(v_{j-\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n\right) = 0 & \text{for } j = 0 \end{cases} \tag{1.9}$$

completed with the initial condition

$$u_j^0 = u_0(j\Delta x) \quad \text{for } j \in \mathbb{Z}.$$

Given u_0, we consider $v_0 := (u_0)_x$ and its discretized version

$$v_{j+\frac{1}{2}}^0 = \frac{u_{j+1}^0 - u_j^0}{\Delta x} = \frac{u_0(x_{j+1}) - u_0(x_j)}{\Delta x} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} v_0(y)\,\mathrm{d}y. \tag{1.10}$$

The scheme for (1.1) is directly derived from the scheme (1.9). Indeed, recalling the definition of $v_{j+1/2}^n$ in (1.8), we can write

$$v_{j+\frac{1}{2}}^{n+1} = v_{j+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x}\left(f_{j+1}(v_{j+\frac{1}{2}}^n, v_{j+\frac{3}{2}}^n) - f_j(v_{j-\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n)\right) \tag{1.11}$$

where

$$f_j(a,b) = \begin{cases} g^{H_L}(a,b) & \text{if } j \le -1 \\ g^{H_R}(a,b) & \text{if } j \ge 1 \\ F_0(a,b) & \text{if } j = 0. \end{cases} \tag{1.12}$$

It is convenient to introduce the functions $\mathcal{F}_j$ from the right-hand side of the above scheme: for any n, j, we have

$$v^{n+1}_{j+\frac{1}{2}} = \mathcal{F}_j(v^n_{j-\frac{1}{2}}, v^n_{j+\frac{1}{2}}, v^n_{j+\frac{3}{2}}). \tag{1.13}$$

We shed light on the fact that $\mathcal{F}_j(v^n_{j-\frac{1}{2}}, v^n_{j+\frac{1}{2}}, v^n_{j+\frac{3}{2}})$ is monotone with respect to $v^n_{j-\frac{1}{2}}$ and $v^n_{j+\frac{3}{2}}$ only if $F_0(a,b)$ satisfies the monotonicity properties given in (1.4). A condition of CFL type ensures that it is monotone with respect to $v^n_{j+\frac{1}{2}}$,

$$\frac{\Delta t}{\Delta x} \le (2\mathfrak{L})^{-1} =: C_{\text{CFL}} \tag{1.14}$$

with

$$\mathfrak{L} = \max_{\alpha \in \{0,L,R\}} \mathfrak{L}_\alpha.$$

We recall that for $\alpha \in \{L, R\}$, $\mathfrak{L}_\alpha$ denotes the Lipschitz constant of H_α (and g^{H_α} in both variables) and $\mathfrak{L}_0$ denotes the Lipschitz constant of F_0 (in both variables).

It is convenient to define a function u_Δ in continuous variables $(t,x) \in (0,+\infty)\times\mathbb{R}$ by linear interpolation: for $t > 0$ and $x \in \mathbb{R}$,

$$u_\Delta(t,x) := \sum_{n\in\mathbb{N}} \mathbb{1}_{[t_n,t_{n+1})}(t)\mathbb{1}_{[x_j,x_{j+1})}(x) \left[u^n_j + \frac{u^n_{j+1} - u^n_j}{\Delta x}(x - x_j)\right]. \tag{1.15}$$

where Δ stands for $(\Delta t, \Delta x)$. Similarly, we define v_Δ on $(0,+\infty)\times\mathbb{R}$ by,

$$v_\Delta := \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{Z}} v^n_{j+\frac{1}{2}} \mathbb{1}_{[t_n,t_{n+1})\times[x_j,x_{j+1})}. \tag{1.16}$$

1.3 Main results

The first main result of this article asserts that the spatial derivative of the viscosity solution of the Hamilton-Jacobi (HJ) equation coincides with the entropy solution of the corresponding scalar conservation law (SCL). More precisely, if at the HJ level, the junction condition is encoded by the nonlinearity F_0, then at the SCL level, the associated germ is $\mathcal{G}_{\mathcal{R}F_0}$ that is represented by the "relaxed" nonlinearity $\mathcal{R}F_0$. The definition of the relaxation operator is recalled in the next section.

Theorem 1.1 (Link between HJ and SCL)**.** *Let u_0 be Lipschitz continuous and $v_0 = (u_0)_x$ be of bounded variation in $\mathbb{R}$. Let $u\colon (0,+\infty)\times\mathbb{R} \to \mathbb{R}$ be the unique F_0-viscosity solution of (1.3)-(1.5) and v be the unique $\mathcal{G}$-entropy solution of (1.1),(1.6) with the germ $\mathcal{G} := \mathcal{G}_F$ defined by*

$$\mathcal{G}_F = \{(p_L, p_R) \in \mathbb{R}^2 : H_L(p_L) = H_R(p_R) = F(p_L, p_R)\} \tag{1.17}$$

where $F := \mathcal{R}F_0$ (defined in (2.5)) is the relaxed junction condition at $x = 0$. Then $v = u_x$ in L^∞.

Remark 1.2. Since we want to relate solutions for scalar conservation laws with the ones for Hamilton-Jacobi equations, the functions H_L and H_R are both flux functions and Hamiltonians. We make assumptions on H_L and H_R that are convenient to work with for both equations. More precisely, the continuity and coercivity of the H_α's is used in the HJ framework while the Lipschitz continuity and the non-degeneracy (no open interval on which H_α is constant) is used at the level of the conservation law.

Remark 1.3. The assumptions we make on u_0 are also necessary either for solving the Hamilton-Jacobi equation of for solving the scalar conservation law (with the initial data $v_0 = (u_0)_x$). It is convenient to assume that u_0 is globally Lipschitz so that viscosity solutions are also globally Lipschitz and entropy solutions are essentially bounded. The BV assumption on v_0 is convenient to prove the convergence of the numerical scheme associated with the scalar conservation law.

This theorem derives from the convergence of the two numerical schemes associated with each equation.

Theorem 1.4 (Convergence of the numerical scheme for SCL)**.** *Let Δt and Δx satisfy the CFL condition* (1.14)*. The function v_Δ weakly converges in $L^1_{\text{loc}}((0,+\infty)\times\mathbb{R})$ as $\Delta x \to 0$ towards the unique $\mathcal{G}_F$-entropy solution v of* (1.1),(1.6)*, with $F := \mathcal{R}F_0$ defined in* (2.5)*.*

Remark 1.5. Let us point out that even if we put the desired junction condition F_0 in the numerical scheme, we recover at the limit $\Delta x \to 0$ the relaxed junction condition $\mathcal{G}_{\mathcal{R}F_0}$.

Theorem 1.6 (Convergence of the numerical scheme for HJ)**.** *Let Δt and Δx satisfy the CFL condition* (1.14)*. The function u_Δ converges locally uniformly as $\Delta x \to 0$ towards the unique F_0-viscosity solution u of* (1.3),(1.5)*.*

Remark 1.7. Let us note that even if the results in Theorem 1.4 and Theorem 1.6 seems different since, in the last one, we recover a F_0-viscosity solution and not a $\mathcal{R}F_0$-viscosity solution, this is not the case. Indeed, we know from [18] that u is a F_0-viscosity solution if and only if u is $\mathcal{R}F_0$-viscosity solution.

Open question. Theorem 1.4 is proved in the case where the junction condition is not relaxed (*i.e.* $F_0 \neq \mathcal{R}F_0$), using the result of Theorem 1.6. Is it possible to show the result of Theorem 1.4 directly at the level of scalar conservation laws, without using the HJ framework? Moreover, is it possible to get an error estimate on the difference $|v_\Delta - v|_{L^1}$?

The next theorem concerns germs. Roughly speaking, a germ is a collection of admissible strong traces for entropy solutions of the conservation law (1.1). It is maximal if it is not contained in a bigger germ and complete if the Riemman problem can be solved for all admissible strong traces. Precise definitions are given in the next section. We prove that a germ is maximal and complete if and only if it is represented by a "self-relaxed nonlinearity". By doing so, we enrich the AKR theory from [3].

Theorem 1.8 (Classification of maximal and complete germs)**.** *Let $\mathcal{G}$ be a germ for* (1.1)*. The following properties are equivalent.*

(i) there exists a function F_0 satisfying (1.4) *such that $\mathcal{G} = \mathcal{G}_{\mathcal{R}F_0}$.*

(ii) The germ $\mathcal{G}$ is maximal and complete.

1.4 Brief review of literature

Scalar conservation laws with discontinuous flux. Contributions to the study of scalar conservation laws with discontinuous flux are numerous. We can cite for instance the work by F. Bachmann and J. Vovelle [4] where the flux function is only assumed to be C^1 and their uniqueness proof do not require the existence of strong traces. The reader is referred to the introduction of this work for the reference containing the model or for previous mathematical contributions under stronger assumptions. The book by M. Garavello and B. Piccoli [19] was also influential: the network geometrical setting involves to consider flux functions with discontinuities at edges. B. Andreianov, K. H. Karlsen and N. H. Risebro in [3] developed a general theory of semi-groups of entropy solutions associated with a scalar conservation law on the real line with a discontinuity. In particular, they shed light on the fact that several conditions can be imposed at the discontinuity and they can be characterized in terms of a set that they refer to as a *germ*. The interested reader is refer to recent survey articles such as [26, 2] for more references about this line of research. We also refer to [28, 14] for the extension to junctions.

Hamilton-Jacobi equations with discontinuous Hamiltonians. The study of Hamilton-Jacobi equations with discontinuous (in x) and convex (in $p = u_x$) Hamiltonians developed with the study of these equations on networks [1, 23]. These first contributions are closely related to optimal control of trajectories in a two-domains framework [6]. Many contributions followed these three articles and the reader is referred to the book by G. Barles and E. Chasseigne [7] for an up-to-date state of the art, including original contributions to the topic.

Boundary conditions. The fact that the boundary condition can be lost by solutions of first order equations is classical. Beyond transport effect and the fact that characteristics can exit the domain, an important contribution to this subject is the work by C. Bardos, A.-Y. Le Roux and J. C. Nédélec about the Dirichlet problem for a scalar conservation law [5]. They gave a weak formulation of the problem by passing to the limit in the viscous approximation. As far as Hamilton-Jacobi equations are concerned, the boundary condition that is effectively obtained when imposing one that is compatible with the maximum principle were first described for convex Hamiltonians [22]. In this case, any relaxed boundary condition is characterized by a real number, amounting for the limitation of the "flux" at the discontinuity (see also [24]). The non-convex case is much richer, the class of relaxed boundary conditions is much larger. It was first studied by J. Guerand [20] and recently revisited by the authors [16](see also [18] for the case of junctions). As observed in [16], it is remarkable that the relaxation operator can be described in terms of Godunov fluxes appearing in the BLN condition.

Comparison principles for Hamilton-Jacobi equations. In order to prove that the numerical scheme converges, it is important to prove a comparison principle. The first results in this direction are contained in [25]. More recently, the authors developed a new strategy to prove such a strong uniqueness result [15, 17]. See also [18] for the case of several branches.

Numerical schemes. A general theorem for numerical schemes for Hamilton-Jacobi equation (and more generally second order nonlinear parabolic equations) [8] asserts that they converge as soon as they are monotone, stable and consistent. A numerical scheme for convex Hamilton-Jacobi equations on a junction was first studied in [11]. It was motivated by applications to traffic flow. An error estimate was obtained in [21].

As far as scalar conservation laws are concerned, the convex case of the problem adressed in the present article was recently treated in [10]. More generally, for examples of numerical schemes on junctions with $N \geq 1$ branches, we refer the reader to [28, 14].

1.5 Organisation of the article

The article is organised as follows. In Section 2, definitions of germs, solutions and relaxation operators are recalled. We also show that germs $\mathcal{G}_{\mathcal{R}F_0}$ are maximal. In Section 3, we recall a criterion for checking that a function is a viscosity solution of the Hamilton-Jacobi equation (1.3) and we establish a similar criterion for entropy solutions of the scalar conservation law (1.1). The numerical approximation of Hamilton-Jacobi equations is addressed in Section 4, where the convergence of the numerical solution is done (proof of Theorem 1.6). The numerical approximation of the scalar conservation law is studied in Section 5, in particular, Theorem 1.4 is proved. This section also contains the (short) proof of Theorem 1.1. The final (short) Section 6 is devoted to the proof of Theorem 1.8.

Notation. For two real numbers a, b, the maximum of a and b is denoted by $a \vee b$ while the minimum is denoted by $a \wedge b$. The positive part of a is defined by $a \vee 0$ and is denoted by a_+. The negative part a_- of a is defined by $\max(0, -a)$.

The Lipschitz constant of the Hamiltonian/flux function H_α for $\alpha \in \{R, L\}$ is denoted by $\mathfrak{L}_\alpha$ while $\mathfrak{L}_0$ denotes the Lipschitz constant of F_0 (in all variables).

2 Germs, entropy solutions, viscosity solutions

2.1 Germs

In their study of scalar conservation laws with discontinous flux, B. Andreianov, K. H. Karlsen and N. H. Riserbro introduced the notion of germs. In order to recall their definition, we first recall the definition of the entropy flux functions q_L and q_R associated with the fluxes (or nonlinearities) H_L and H_R.

$$\forall a, b \in \mathbb{R}, \quad q_\alpha(a, b) = \operatorname{sgn}(a - b)(H_\alpha(a) - H_\alpha(b)), \qquad \alpha \in \{L, R\}. \tag{2.1}$$

The definition of germs relies on the notion of dissipation. We recall that it is defined as follows,

$$\forall P = (p_L, p_R), P' = (p'_L, p'_R) \in \mathbb{R}^2, \quad D(P, P') = q_L(p_L, p'_L) - q_R(p_R, p'_R).$$

Definition 2.1 (Germs)**.** *A set $\mathcal{G} \subset \mathbb{R}^2$ is a* germ *for* (1.1) *if it satisfies*

- *the* Rankine-Hugoniot condition*: for all $(p_L, p_R) \in \mathcal{G}$, we have $H_L(p_L) = H_R(p_R)$.*
- *the* dissipation condition*: for all $P, P' \in \mathcal{G}$, we have $D(P, P') \geq 0$.*

Remark 2.2. In [3], a set $\mathcal{G}$ is called an admissible germ if it only satisfies the Rankine-Hugoniot condition and is called L^1-dissipative if it also satisfies the dissipation condition. We will simply call them germs.

Important examples of germs are the ones coming from a junction function F_0,

$$\mathcal{G}_{F_0} = \{(p_L, p_R) \in \mathbb{R}^2 : H_L(p_L) = H_R(p_R) = F_0(p_L, p_R)\}. \tag{2.2}$$

Definition 2.3 (Maximal germs)**.** *A germ $\mathcal{G}$ is* maximal *if any germ containing $\mathcal{G}$ coincide with $\mathcal{G}$.*

In order to define complete germs, we recall what the Riemann problem associated with $(v_-, v_+) \in \mathbb{R}^2$ is. It consists in solving (1.1) with the initial condition:

$$v(0, x) = \begin{cases} v_- & \text{if } x < 0, \\ v_+ & \text{if } x > 0. \end{cases} \tag{2.3}$$

Given $(v_-, v_+) \in \mathbb{R}^2$, a solution of the $\mathcal{G}$-Riemann problem with initial data (2.3) consists, for some suitable data $(p_L, p_R) \in \mathcal{G}$, in a standard Kruzhkov self-similar solution in $x < 0$ joining v_- at $t = 0, x < 0$ and p_L at $t > 0, x = 0-$, with a jump at $x = 0, t > 0$ from p_L to p_R and a standard Kruzhkov self-similar solution in $x > 0$ joining p_R at $x = 0+, t > 0$ and v_+ at $x > 0, t = 0$.

Definition 2.4 (Complete germs)**.** *A germ $\mathcal{G}$ is* complete *if for all $(v_-, v_+) \in \mathbb{R}^2$, there exists a $\mathcal{G}$-entropy solution v of* (1.1) *with initial data* (2.3)*.*

Remark 2.5. In particular the traces (p_L, p_R) at $x = 0-, 0+$ of the solution v of the $\mathcal{G}$-Riemann problem lies in the germ $\mathcal{G}$.

2.2 Entropy solutions

Definition 2.6 (Strong traces)**.** *Let $T > 0$ and $v\colon (0, T) \times \mathbb{R} \to \mathbb{R}$ be essentially bounded. We say that v admits a* strong trace *at $x = 0-$ (resp. $x = 0+$) if the function $x \mapsto v(\cdot, x) \in L^1((0, T))$ has an essential limit in $L^1((0, T))$ as $x \to 0-$ (resp. $x \to 0+$).*

E. Yu. Panov [29] proved that classical entropy solutions in $(0, T) \times (0, +\infty)$ (resp. $(0, T) \times (-\infty, 0)$) admit strong traces at $x = 0+$ (resp. $x = 0-$) as soon as there is no open interval on which flux functions H_α are constant.

Definition 2.7 ($\mathcal{G}$-entropy solutions – [3])**.** *Let $\mathcal{G}$ be a germ and $T > 0$, let $v\colon (0,T) \times \mathbb{R} \to \mathbb{R}$ be essentially bounded and such that, for almost every $t \in (0,T)$, the function $v(t,\cdot)$ has strong traces at $x = 0$, from the left $v(t,0-)$ and from the right $v(t,0+)$. It is a* $\mathcal{G}$-entropy solution *of* (1.1)-(1.6) *if*

- *it is a classical entropy solution in $(0,T) \times (-\infty,0)$ and $(0,T) \times (0,+\infty)$;*
- *for almost all $t \in (0,T)$, we have $(v(t,0-), v(t,0+)) \in \mathcal{G}$;*
- *$v(t,\cdot) \to v_0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $t \to 0+$.*

Remark 2.8. $\mathcal{G}$-entropy solutions are unique ([3, Theorem 3.11]) as soon as the germ is *definite*. In our case, this reduces to impose that the germ is *maximal* since we only consider germs that are, following the terminology introduced in [3], L^1-dissipative.

2.3 Viscosity solutions

In order to define viscosity solutions for the Hamilton-Jacobi equation, we have to specify the class of test functions we will work with. Following for instance [22], we use the following class.

Definition 2.9 (Test functions)**.** *A test function $\varphi\colon (0,T) \times \mathbb{R} \to \mathbb{R}$ is continuous and its restriction to $(0,T) \times [0,+\infty)$ and $(0,T) \times (-\infty,0]$ are continuously differentiable. For such a function φ and $X_0 = (t_0,0)$, $\partial_x^L \varphi(X_0)$ and $\partial_x^R \varphi(X_0)$ denote derivatives in x at X_0 of the restrictions of φ to $(0,T) \times (-\infty,0]$ and $(0,T) \times [0,+\infty)$ respectively.*

Let Q_T denote $(0,T) \times \mathbb{R}$ and $C^1_\wedge(Q_T)$ denote the set of test functions.

We also say that a test function φ touches a function $u\colon Q_T \to \mathbb{R}$ from above (resp. from below) at X_0 if $u(X_0) = \varphi(X_0)$ and it there exists a neighbourhood $\mathcal{V}$ such that $u \le \varphi$ (resp. $u \ge \varphi$) in $\mathcal{V}$.

Definition 2.10 (F_0-viscosity solutions)**.** *Let $T > 0$ and let $u\colon (0,T) \times \mathbb{R}$ be locally bounded.*

- *The function u is a* sub-solution *of* (1.3) *if it is upper semi-continuous on $(0,T) \times \mathbb{R}$ and if, for all test function $\varphi \in C^1_\wedge(Q_T)$ touching u from above at $X_0 = (t_0,x_0) \in (0,T) \times \mathbb{R}$, we have*

$$\begin{aligned} \partial_t \varphi + H_L(\partial_x \varphi) &\le 0 \ \text{at } X_0 \text{ if } x_0 < 0,\\ \partial_t \varphi + H_R(\partial_x \varphi) &\le 0 \ \text{at } X_0 \text{ if } x_0 > 0,\\ \partial_t \varphi + \min(H_L(\partial_x^L \varphi), H_R(\partial_x^R \varphi), F_0(\partial_x^L \varphi, \partial_x^R \varphi)) &\le 0 \ \text{at } X_0 \text{ if } x_0 = 0, \end{aligned}$$

- *The function u is a* super-solution *of* (1.3) *if it is lower semi-continuous on $(0,T) \times \mathbb{R}$ and if, for all test function $\varphi \in C^1_\wedge(Q_T)$ touching u from below at $X_0 = (t_0,x_0) \in (0,T) \times \mathbb{R}$, we have*

$$\begin{aligned} \partial_t \varphi + H_L(\partial_x \varphi) &\ge 0 \ \text{at } X_0 \text{ if } x_0 < 0,\\ \partial_t \varphi + H_R(\partial_x \varphi) &\ge 0 \ \text{at } X_0 \text{ if } x_0 > 0,\\ \partial_t \varphi + \max(H_L(\partial_x^L \varphi), H_R(\partial_x^R \varphi), F_0(\partial_x^L \varphi, \partial_x^R \varphi)) &\ge 0 \ \text{at } X_0 \text{ if } x_0 = 0, \end{aligned}$$

- *The function u is a* solution *of* (1.3) *if its upper semi-continuous envelope u^* is a sub-solution and its lower semi-continuous envelope u_* is a super-solution.*

Remark 2.11. Since we will work with various nonlinearities F_0, it is convenient to simply say that u is an F_0-sub-solution of the Hamilton-Jacobi equation if u is a sub-solution of (1.3). The same remark applies to super-solutions and solutions.

Remark 2.12. (Mapping the line onto the two half lines on the right)
Define

$$\begin{cases} \bar{u}^1(t,x) := u(t,x) & \text{for} \quad x > 0, \qquad H_1(p) = H_R(p)\\ \bar{u}^2(t,x) := u(t,-x) & \text{for} \quad x > 0, \qquad H_2(p) = H_L(-p)\\ \bar{F}(p_1,p_2) := F(p_1,-p_2) & \end{cases}$$

Then HJ equation (1.3) is equivalent to

$$\begin{cases} \bar{u}^i_t + H_i(\bar{u}^i_x) & = 0, \qquad x > 0, \quad i = 1,2 \\ \bar{u}^1(t,0) = \bar{u}^2(t,0) & =: \bar{u}(t) \qquad x = 0 \\ \bar{u}_t + \bar{F}(\bar{u}^1_x, \bar{u}^2_x) & = 0 \qquad x = 0 \end{cases}$$

which is the natural framework for HJ equations. This explains the signs that the reader may find strange in the definitions below.

2.4 Characteristic points

Characteristic points are first defined in the framework of HJ equations. They are associated with the nonlinearity F_0 at the origin. They are first introduced in [22] in the convex case and in [20] for non-convex Hamiltonians. See also in [16].

Definition 2.13 (Characteristic points for germs)**.** *Let $\mathcal{G}$ be a germ for (1.1).*

- *A point $P = (p_L, p_R)$ lies in $\overline{\chi}(\mathcal{G})$ if $P \in \mathcal{G}$ and if there exists $\varepsilon > 0$ such that $H_\alpha(q_\alpha) > H_L(p_L) = H_R(p_R)$ for $\alpha \in \{L, R\}$ and $q_L \in (p_L - \varepsilon, p_L)$ and $q_R \in (p_R, p_R + \varepsilon)$.*
- *A point $P = (p_L, p_R)$ lies in $\underline{\chi}(\mathcal{G})$ if $P \in \mathcal{G}$ and if there exists $\varepsilon > 0$ such that $H_\alpha(q_\alpha) < H_L(p_L) = H_R(p_R)$ for $\alpha \in \{L, R\}$ and $q_L \in (p_L, p_L + \varepsilon)$ and $q_R \in (p_R - \varepsilon, p_R)$.*
- *We set $\chi(\mathcal{G}) := \underline{\chi}(\mathcal{G}) \cup \overline{\chi}(\mathcal{G})$.*

Definition 2.14 (Characteristic points for nonlinearities)**.** *Let $F_0 \colon \mathbb{R} \times \mathbb{R}$ be such that (1.4) holds true. A point $P = (p_L, p_R)$ lies in $\underline{\chi}(F_0)$ (resp. $\overline{\chi}(F_0)$) if $P \in \underline{\chi}(\mathcal{G}_{F_0})$ (resp. $\overline{\chi}(\mathcal{G}_{F_0})$) where we recall that $\mathcal{G}_{F_0}$ is defined in (2.2).*

2.5 The relaxation operator

We now define the relaxation operator $\mathcal{R}$. It associates with any function F_0 satisfying (1.4) a new function $\mathcal{R}F_0$. In order to define it, it is convenient to define

$$\underline{H}(p_L, p_R) := \max\{H_{L,+}(p_L), H_{R,-}(p_R)\}$$

with

$$H_{L,+}(p_L) := \inf_{q_L \geq p_L} H_L(q_L) \quad \text{and} \quad H_{R,-}(p_R) := \inf_{q_R \leq p_R} H_R(q_R).$$

If $F_0 \geq \underline{H}$, then it is possible to define the Godunov relaxation of F_0 as the map $F_0 G : \mathbb{R}^2 \to \mathbb{R}$ where

$$(F_0 G)(p_L, p_R) = \lambda \quad \text{s.t. there exists } (q_L, q_R) \in \mathbb{R}^2 \text{ s.t.} \quad g^{H_L}(p_L, q_L) = F_0(q_L, q_R) = g^{H_R}(q_R, p_R) =: \lambda \tag{2.4}$$

where g^{H_α} are Godunov fluxes associated to each flux H_α (see (1.7)). It is important to notice that there may be several admissible values of $Q = (q_L, q_R)$, but it is possible to show that the value of λ is uniquely defined. Following [18], we define for any F_0 satisfying (1.4)

$$\mathcal{R}F_0 := (\max\{H_0, \underline{H}\})G. \tag{2.5}$$

We recall some properties of the relaxation operator.

Proposition 2.15 (Properties of the relaxation operator, [18])**.** *Let F_0 be continuous, non-decreasing in the first variable et non-increasing in the second one. Then $\mathcal{R}F_0$ satisfies (1.4) and we have*

(i) $\mathcal{R}F_0 = F_0$ *on* $\{F_0 = H_L = H_R\}$,

(ii) $\mathcal{R}^2 = \mathcal{R}$,

(iii) $\mathcal{R}F_0 \geq \underline{H}$

Remark 2.16. Notice that for $F_\varepsilon(p_L,p_R) := \varepsilon^{-1}(p_L - p_R)$, we have $\mathcal{R}F_\varepsilon \to F_0$ as $\varepsilon \to 0$, where

$$F_0(p_L,p_R) := g^{H_L}(p_L,z) = g^{H_R}(z,p_R) \quad \text{for some } z \in \mathbb{R}^2$$

which is exactly Diehl's condition (see [12, p. 28]), obtained by vanishing viscosity. This shows that the natural relaxation operator that we identified in (2.4), can be seen as a sort of generalization of Diehl's condition.

2.6 Maximality of germs associated to $\mathcal{R}F_0$

Lemma 2.17 (Germs associated to $\mathcal{R}F_0$)**.** *Assume that H^L, H^R satisfy (1.2) and that F_0 satisfies (1.4). Then the set $\mathcal{G} := \mathcal{G}_{\mathcal{R}F_0}$ defined in (1.17) is a maximal germ in the sense of Definitions 2.1 and 2.3.*

Proof. The proof proceeds in two steps.
Step 1: $\mathcal{G}$ is a germ. From Proposition 2.15, recall that $F := \mathcal{R}F_0$ satisfies (1.4). We then notice that by definition, the set $\mathcal{G}$ satisfies Rankine-Hugoniot relation, and that the monotonicity of F implies the dissipation $D \geq 0$ on $\mathcal{G} \times \mathcal{G}$. Therefore $\mathcal{G}$ is a germ.

Step 2: $\mathcal{G}$ is maximal. Assume by contradiction that $\mathcal{G}$ is not maximal. Then there exists $\bar{P} = (\bar{p}_L, \bar{p}_R) \in \mathbb{R}^2 \backslash \mathcal{G}$ such that

$$\hat{\mathcal{G}} := \mathcal{G} \cup \left\{\bar{P}\right\} \quad \text{is a germ.} \tag{2.6}$$

Hence

$$\bar{\lambda} := H_L(\bar{p}_L) = H_R(\bar{p}_R) \neq F(\bar{P}) =: \lambda^*$$

From Proposition 2.15, recall that $F = \mathcal{R}F \geq \underline{H}$. Hence by construction there exists $\bar{Q} = (\bar{q}_L, \bar{q}_R) \in \mathbb{R}^2$ such that

$$F(\bar{p}_L,\bar{p}_R) = \lambda^* = g^{H_L}(\bar{p}_L,\bar{q}_L) = F(\bar{q}_L,\bar{q}_R) = g^{H_R}(\bar{q}_R,\bar{p}_R).$$

We first assume that $\bar{\lambda} > \lambda^*$. Then $g^{H_R}(\bar{p}_R,\bar{p}_R) = H_R(\bar{p}_R) = \bar{\lambda} > \lambda^* = F(\bar{P}) = g^{H_R}(\bar{q}_R,\bar{p}_R)$, which implies that $\bar{q}_R < \bar{p}_R$. Similarly, we get $\bar{q}_L > \bar{p}_L$. Since $F(\bar{p}_L,\bar{p}_R) = F(\bar{q}_L,\bar{q}_R)$, the monotonicities of F imply that

$$F = const = \lambda^* \quad \text{on} \quad [\bar{p}_L,\bar{q}_L] \times [\bar{q}_R,\bar{p}_R]$$

In particular, the expressions of the Godunov fluxes g^{H_α} implies that there exists a unique

$$Q^* := (q_L^*, q_R^*) \in (\bar{p}_L,\bar{q}_L] \times [\bar{q}_R,\bar{p}_R) \quad \text{such that} \quad \begin{cases} H_R > \lambda_* = H_R(q_R^*) & \text{on} \quad (q_R^*,\bar{p}_R] \\ H_L > \lambda_* = H_L(q_L^*) & \text{on} \quad [\bar{p}_L,q_L^*) \end{cases}$$

Hence $Q^* \in \mathcal{G} \subset \hat{\mathcal{G}}$, and then

$$\begin{aligned} 0 &\leq D(\bar{P},Q^*) \\ &= \operatorname{sign}(\bar{p}_L - q_L^*) \cdot \{H_L(\bar{p}_L) - H_L(q_L^*)\} - \operatorname{sign}(\bar{p}_R - q_R^*) \cdot \{H_R(\bar{p}_R) - H_R(q_R^*)\} \\ &= -2(\bar{\lambda} - \lambda^*) \\ &< 0. \end{aligned}$$

Assume now that $\bar{\lambda} < \lambda^*$. We can argue as in the previous case and get a contradiction too.
Hence we conclude that (2.6) is false, and then $\mathcal{G}$ is maximal. ∎

3 Criteria for viscosity and entropy solutions

We begin by defining the reduced set of test functions.

Definition 3.1 (Reduced set of test functions)**.** *Let F_0 be such that (1.4) holds true. The reduced set of test functions associated to F_0 for subsolutions (resp. supersolutions) is made of test functions $\varphi \in C^1_\wedge(Q_T)$ such that $(\partial_x^L \varphi(t_0,0), \partial_x^R \varphi(t_0,0)) \in \underline{\chi}(F_0)$ (resp. $(\partial_x^L \varphi(t_0,0), \partial_x^R \varphi(t_0,0)) \in \overline{\chi}(F_0)$) for all $t_0 \in (0,T)$. It is denoted by $C^1_\wedge(Q_T, F_0, SUB)$ (resp. $C^1_\wedge(Q_T, F_0, SUP)$).*

We now recall that for viscosity solutions, we have the following criterion.

Proposition 3.2 (A criterion for F_0-sub- and F_0-super-solutions)**.** *Let $u\colon Q_T \to \mathbb{R}$.*

- *If u is upper-semi continuous, and u is a (classical) viscosity sub-solution of (1.3) away from $x = 0$, and for all $t_0 \in (0,T)$,*
$$u(t_0,0) = \limsup_{(t,x)\to(t_0,0)} u(t,x), \tag{3.1}$$
and for all $\varphi \in C^1_\wedge(Q_T, F_0, SUB)$ touching u from above, we have
$$\partial_t \varphi + F_0(\partial_x^L \varphi, \partial_x^R \varphi) \le 0 \quad \text{at } (t_0,0).$$
Then u is an F_0-sub-solution.

- *If u is lower-semi continuous and u is a (classical) viscosity super-solution of (1.3) away from $x = 0$, and for all $\varphi \in C^1_\wedge(Q_T, F_0, SUP)$ touching u from below, we have*
$$\partial_t \varphi + F_0(\partial_x^L \varphi, \partial_x^R \varphi) \ge 0 \quad \text{at } (t_0,0).$$
Then u is an F_0-super-solution.

Remark 3.3. Notice the dissymmetry between subsolutions and supersolutions in the result of Proposition 3.2, with assumption (3.1) only for subsolutions. This comes from the fact that the Hamiltonians H^L, H^R are assumed to be coercive (last line of condition (1.2)).

Proposition 3.4 (Germs from characteristic points)**.** *Let $\mathcal{G}_F$ be a germ associated with a junction function $F = \mathcal{R}F_0$ and consider $P = (p_L, p_R) \in \mathbb{R}^2$ satisfying $H_L(p_L) = H_R(p_R)$. If for all $Q \in \chi(\mathcal{G}_F)$, we have $D(P,Q) \ge 0$, then $P \in \mathcal{G}_F$.*

Remark 3.5. Later we will use this result, by assuming more, namely assuming that $D(P,Q) \ge 0$ for all $Q \in \mathcal{G}_F$.

Proof. We consider the function $u\colon (0,+\infty) \times \mathbb{R}$ defined by
$$u(t,x) := \begin{cases} -\lambda t + p_L x & \text{if } x > 0, \\ -\lambda t + p_R x & \text{if } x < 0 \end{cases}$$
with $\lambda = H_L(p_L) = H_R(p_R)$. The function u is a viscosity solution of (1.3) in $(0,+\infty) \times \mathbb{R} \setminus \{0\}$. We claim that the assumption ensures that it is also a viscosity solution at $x = 0$, i.e.
$$\lambda = F(p_L, p_R).$$
This precisely means that $P \in \mathcal{G}_F$.

We are thus left with checking that u is a viscosity solution at $x = 0$. We only prove that it is a sub-solution, the other case being similar. It is enough to consider a test-function φ of the form
$$\varphi(t,x) = \Psi(t) + \begin{cases} q_L x & \text{if } x < 0, \\ q_R x & \text{if } x > 0 \end{cases}$$

with $(q_L, q_R) \in \underline{\chi}(F)$. We assume that $u \leq \varphi$ in $(0, +\infty) \times \mathbb{R}$ and $u(t_0, 0) = \varphi(t_0, 0)$ for some $t_0 > 0$. In particular, we have

$$\Psi'(t_0) = -\lambda, \quad q_L \leq p_L, \quad q_R \geq p_R.$$

The assumption of the proposition ensures that

$$\operatorname{sgn}(p_L - q_L)(H_L(p_L) - H_L(q_L)) \geq \operatorname{sgn}(p_R - q_R)(H_R(p_R) - H_R(q_R)).$$

Keeping in mind that $H_L(p_L) = H_R(p_R) = \lambda$ and $H_L(q_L) = H_R(q_R) = F(q_L, q_R)$, if $p_L > q_L$ or $p_R < q_R$, then we get

$$(\lambda - F(q_L, q_R)) \geq 0.$$

The result is also true if $p_L = q_L$ and $p_R = q_R$ since $\lambda = F(p_L, p_R)$. Since $\Psi'(t_0) = -\lambda$, we finally get the desired viscosity inequality: $\Psi'(t_0) + F(q_L, q_R) \leq 0$. □

4 The numerical scheme for the Hamilton-Jacobi equation

Before proving the convergence of the numerical scheme for the scalar conservation law, we study the one associated with the Hamilton-Jacobi equation. It is necessary to study it first, since we will use it to prove the convergence of the numerical scheme for the scalar conservation law in the case where the nonlinearity F_0 is not necessarily relaxed (*i.e.* $F_0 \neq \mathcal{R}F_0$).

4.1 Stability

Lemma 4.1 (Stability of the numerical scheme)**.** *For all $t \geq 0$ and $x \in \mathbb{R}$, we have $|u_\Delta(t,x) - u_\Delta(0,x)| \leq C_0 t$ for $C_0 = \max(C_L, C_R, C_{F_0})$ with $C_\alpha = \max_{|a| \leq \|u_0\|_{\mathrm{Lip}}} |H_\alpha(a)|$ and $C_{F_0} = \max_{|a|,|b| \leq \|u_0\|_{\mathrm{Lip}}} |F_0(a,b)|$.*

In particular, the function u_Δ is locally bounded in L^∞, uniformly in Δ.

Proof. Since u_Δ is constant in time on intervals $[t_n, t_{n+1})$, it is enough to prove that $|u_\Delta(t_n, x) - u_\Delta(0,x)| \leq C_0 t_n$. We prove it by induction on n.

We only prove $u_\Delta(t_n, x) \leq u_\Delta(0,x) + C_0 t_n$ since the other inequality can be proved in the same way. It is true for $n = 0$. We assume it is true for $n \geq 0$ and we prove it for $n+1$. In order to do so, we combine the induction assumption with the monotonicity of the scheme. Recalling the definition of f_j, see (1.12), we have

$$\begin{aligned}
u_j^{n+1} &= u_j^n + (\Delta t) f_j(v_{j-\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n) \\
&=: \mathcal{H}_j(u_{j-1}^n, u_j^n, u_{j+1}^n) \\
&\leq \mathcal{H}_j(u_{j-1}^0 + C_0 t_n, u_j^0 + C_0 t_n, u_{j+1}^0 + C_0 t_n) \\
&= (u_j^0 + C_0 t_n) + (\Delta t) f_j(v_{j-\frac{1}{2}}^0, v_{j+\frac{1}{2}}^0) \\
&\leq (u_j^0 + C_0 n \Delta t) + C_0 (\Delta t).
\end{aligned}$$

where in the third line, we have used the fact that the numerical scheme $\mathcal{H}_j$ for HJ equation is monotone under our CFL condition. We conclude that $u_\Delta(t_{n+1}, x) \leq u_\Delta(0,x) + C_0 t_{n+1}$. □

4.2 Consistency

The following lemma is very classical and straightforward. We skip the proof.

Lemma 4.2 (Consistency of the numerical scheme)**.** *Let (1.14) hold true. Let $(t,x) \in (0,T) \times \mathbb{R}$ and $\phi \in C^1_\wedge(Q_T)$ for some $T > 0$. Assume that there exists $(t_\Delta, x_\Delta) = (n\Delta t, j\Delta x) \to (t,x)$ as $\Delta x \to 0$ such that*

$$\frac{\phi(t_\Delta + \Delta t, x_\Delta) - \phi(t_\Delta, x_\Delta)}{\Delta t} + f_j \left(\frac{\phi(t_\Delta, x_\Delta) - \phi(t_\Delta, x_\Delta - \Delta x)}{\Delta x}, \frac{\phi(t_\Delta, x_\Delta + \Delta x) - \phi(t_\Delta, x_\Delta)}{\Delta x} \right) \leq 0 \quad (4.1)$$

where f_j is defined in (1.12).

- *If* $x > 0$, *then* $\phi_t + H_R(\phi_x) \leq 0$ *at* (t, x).
- *If* $x < 0$, *then* $\phi_t + H_L(\phi_x) \leq 0$ *at* (t, x).
- *If* $x = 0$, *then* $\phi_t + \min(H_R(\phi_x^R), H_L(\phi_x^L), F_0(\phi_x^L, \phi_x^R) \leq 0$ *at* (t, x).

4.3 Convergence of the scheme

We checked that the numerical scheme is monotone (thanks to the CFL condition (1.14)), stable (by Lemma 4.1) and consistent (Lemma 4.2). It is then known [8] that it converges towards the unique solution of (1.3)-(1.5). Let us give some details for the reader's convenience.

Proof of Theorem 1.6. Let u^+ denote the upper relaxed limit of u_Δ as $\Delta x \to 0$ (recall (1.14)). Thanks to the stability of the scheme (Lemma 4.1), we know that u^+ is finite and $u^+(0, x) \leq u_0(x)$. Let us prove that it is a F_0-sub-solution of (1.3) in $(0, T) \times \mathbb{R}$ for all $T > 0$. In order to do so, let $T > 0$ and $\phi \in C^1_\wedge(Q_T)$ touching u^+ from above at (t, x). We can assume without loss of generality that the contact is strict. We thus know that there exists (t_Δ, x_Δ) such that

$$u_\Delta - u_\Delta(t_\Delta, x_\Delta) \leq \phi - \phi(t_\Delta, x_\Delta).$$

Let $\psi = \phi + C_\Delta$ with $C_\Delta = u_\Delta(t_\Delta, x_\Delta) - \phi(t_\Delta, x_\Delta)$. The monotonicity of the scheme implies that (4.1) holds true for ψ, and thus for ϕ. Then Lemma 4.2 allows us to get the viscosity inequality.

Analogously, we can prove that the lower relaxed limit u_- of u_Δ as $\Delta x \to 0$ is a F_0-super-solution of (1.3) and $u_-(0, x) \geq u_0(x)$. The comparison principle (see [18]) then implies that $u_+ \leq u_-$ and this implies that u_Δ converges locally uniformly towards u. □

5 The numerical scheme for the scalar conservation law

5.1 Maximum principle

It is classical that a monotone scheme enjoys a maximum principle.

Lemma 5.1 (Maximum principle). *We have:* $\|v_\Delta\|_{L^\infty((0,+\infty)\times\mathbb{R})} \leq \|v_0\|_{L^\infty(\mathbb{R})}$.

Proof. Let $M_0 = \|v_0\|_{L^\infty(\mathbb{R})}$. The monotonicity of the scheme implies that $|v_j^n| \leq M_0$ by arguing by induction on n. □

5.2 Discrete entropy inequalities and L^1-contraction

An immediate consequence of the monotonicity of the scheme is the fact that the maximum of two discrete sub-solutions is still a discrete sub-solution.

Lemma 5.2 (Maximum of discrete sub-solutions). *Let* $v^n_{j+\frac{1}{2}}$ *and* $w^n_{j+\frac{1}{2}}$ *be such that*

$$v^{n+1}_{j+\frac{1}{2}} \leq \mathcal{F}_j(v^n_{j-\frac{1}{2}}, v^n_{j+\frac{1}{2}}, v^n_{j+\frac{3}{2}})$$
$$w^{n+1}_{j+\frac{1}{2}} \leq \mathcal{F}_j(w^n_{j-\frac{1}{2}}, w^n_{j+\frac{1}{2}}, w^n_{j+\frac{3}{2}})$$

where $\mathcal{F}_j$ *denotes the monotone scheme, see* (1.13). *Then* $V^n_{j+\frac{1}{2}} = \max(v^n_{j+\frac{1}{2}}, w^n_{j+\frac{1}{2}})$ *satisfies the same inequality.*

Similarly, the minimum of discrete super-solutions is a discrete super-solutions. Combining these two facts, we get the following discrete version of entropy inequalities (using $|a - b| = a \vee b - a \wedge b$).

Lemma 5.3 (Discrete entropy inequalities). *Let $v^n_{j+\frac12}$ and $w^n_{j+\frac12}$ be two solutions of the numerical scheme,*

$$\begin{cases} v^{n+1}_{j+\frac12} &= \mathcal{F}_j(v^n_{j-\frac12}, v^n_{j+\frac12}, v^n_{j+\frac32}) \\ w^{n+1}_{j+\frac12} &= \mathcal{F}_j(w^n_{j-\frac12}, w^n_{j+\frac12}, w^n_{j+\frac32}) \end{cases} \tag{5.1}$$

where $\mathcal{F}_j$ denotes the monotone scheme, see (1.13). Then $V^n_{j+\frac12} = |v^n_{j+\frac12} - w^n_{j+\frac12}|$ satisfies

$$\frac{V^{n+1}_{j+\frac12} - V^n_{j+\frac12}}{\Delta t} + \frac{\mathcal{Q}^n_{j+1} - \mathcal{Q}^n_j}{\Delta x} \leq 0$$

where

$$\mathcal{Q}^n_j = Q_j(v^n_{j-\frac12}, v^n_{j+\frac12}; w^n_{j-\frac12}, w^n_{j+\frac12})$$

with $Q_j(a,b;c,d) = f_j(a \vee c, b \vee d) - f_j(a \wedge c, b \wedge d)$ and f_j is given by formula (1.12).

We now state the L^1-contraction property of the scheme.

Lemma 5.4 (Discrete L^1-contraction). *Let $v^n_{j+\frac12}$ and $w^n_{j+\frac12}$ be two bounded solutions of the numerical scheme (5.1), and assume that $|v_\Delta(0,\cdot) - w_\Delta(0,\cdot)|$ is integrable.*

Then for all $t = n\Delta t$ with $n \geq 1$, we have

$$\int_{\mathbb{R}} |v_\Delta(t,x) - w_\Delta(t,x)| \,\mathrm{d}x \leq \int_{\mathbb{R}} |v_\Delta(0,x) - w_\Delta(0,x)| \,\mathrm{d}x.$$

Proof. We first assume that $|v_\Delta(0,x) - w_\Delta(0,x)|$ has compact support. In this case, there exists $J \geq 1$ such that for all $s \in [0,t]$, $|v_\Delta(s,x) - w_\Delta(s,x)| = 0$ for $|x| \geq x_J$. In particular,

$$Q_j(v^m_{j-\frac12}, v^m_{j+\frac12}; w^m_{j-\frac12}, w^m_{j+\frac12}) = 0$$

for $|j| \geq J+1$.

We write the discrete entropy inequalities from Lemma 5.3 at time $m \in \{0,\ldots,n-1\}$ and sum over $j \in \{-J-1,\ldots,J+1\}$,

$$\sum_{j\in -J-1}^{J+1} \frac{V^{m+1}_{j+\frac12} - V^m_{j+\frac12}}{\Delta t} \leq 0. \tag{5.2}$$

This yields the result if $|v_\Delta(0,x) - w_\Delta(0,x)|$ has compact support.

We can now consider the sequence of numerical solutions associated with $v_\Delta(0,x)$ supported in the interval $[-N\Delta x, N\Delta x]$ for $N \geq 1$. We can then easily pass to the limit in the inequality that we obtained in the first case. □

5.3 Continuous BV estimates

In order to prove that the numerical solution associated with the scalar conservation law converges towards the entropy solution, we derive discrete BV estimates in the time and the space variables. The computations at the discrete level follow their continuous counterpart closely. This is the reason why we first explain how to derive BV estimates at the continuous level without justification.

Let v be an entropy solution of (1.1).

Time BV estimate. Given a time increment h, the function $w(t,x) = v(t+h,x)$ is also an entropy solution of (1.1) and the L^1-contraction property (with finite speed of propagation) implies that

$$\int_{\mathbb{R}} |v(t+h,x) - v(t,x)| \,\mathrm{d}x \leq \int_{\mathbb{R}} |v(h,x) - v(0,x)| \,\mathrm{d}x.$$

Dividing by h and letting $h \to 0$, we obtain

$$\int_{\mathbb{R}} |\partial_t v(t,x)|\,\mathrm{d}x \le \int_{\mathbb{R}} |\partial_t v(0,x)|\,\mathrm{d}x.$$

We now use that $\partial_t v = -\partial_x(H_\alpha(v)) = -H'_\alpha(v)v_x$ and in particular,

$$|\partial_t v(0,x)| \le L|(v_0)_x|.$$

We conclude that

$$\int_{\mathbb{R}} |\partial_t v(t,x)|\,\mathrm{d}x \le L\|v_0\|_{\mathrm{BV}}.$$

Space BV estimate. Given a spatial increment h, since v is an entropy solution of (1.1), we know that it satisfies for $t > 0$ and $x > 0$,

$$\partial_t |v(t,x+h) - v(t,x)| + \partial_x q_R(v(t,x), v(t,x+h) \le 0.$$

Integrating this inequality on $[t_1,t_2] \times [a_R,b_R]$ with $0 < t_1 < t_2$ and $0 < a_R < b_R$, we get

$$\begin{aligned}\int_{[a_R,b_R]} |v(t_2,x+h) - v(t_2,x)|\,\mathrm{d}x \le \int_{[a_R,b_R]} &|v(t_1,x+h) - v(t_1,x)|\,\mathrm{d}x \\ &+ \sum_{c=a_R,b_R} \int_{t_1}^{t_2} |q_R(v(t,c), v(t,c+h))|\,\mathrm{d}t.\end{aligned}$$

We now estimate the right hand side of the previous inequality. Recalling the definition of q_R, see (2.1), we have

$$\begin{aligned}\int_{t_1}^{t_2} |q_R(v(t,c), v(t,c+h))|\,\mathrm{d}t &\le \int_{t_1}^{t_2} |H_R(v(t,c)) - H_R(v(t,c+h))|\,\mathrm{d}t \\ &\le \int_{t_1}^{t_2} \left| \int_{[c,c+h]} \partial_t v(t,x)\,\mathrm{d}x \right| \mathrm{d}t \\ &\le \int_{[c,c+h]} \|v(\cdot,x)\|_{\mathrm{BV}([t_1,t_2])}\,\mathrm{d}x.\end{aligned}$$

We thus get,

$$\int_{[a_R,b_R]} |v(t_2,x+h) - v(t_2,x)|\,\mathrm{d}x \le \int_{[a_R,b_R]} |v(t_1,x+h) - v(t_1,x)|\,\mathrm{d}x + \sum_{c=a_R,b_R} \int_{[c,c+h]} \|v(\cdot,x)\|_{\mathrm{BV}([t_1,t_2])}\,\mathrm{d}x.$$

Dividing by $h \to 0$, we formally get

$$\int_{[a_R,b_R]} |\partial_x v(t_2,x)|\; dx \le \int_{[a_R,b_R]} |\partial_x v(t_1,x)|\; dx + \sum_{c=a_R,b_R} ||v(\cdot,c)||_{BV([t_1,t_2])}$$

5.4 Discrete BV estimates

We first show that discrete solutions of the scalar conservation law have bounded variation in the time variable. This is a classical consequence of the L^1-contraction property.

Lemma 5.5 (Discrete time BV estimate)**.** *Let $v^n_{j+\frac{1}{2}}$ be a solution of the discrete numerical scheme such that (1.10) holds true for some $v_0 \in \mathrm{BV}(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \frac{|v_\Delta(t+\Delta t,x) - v_\Delta(t,x)|}{\Delta t}\,\mathrm{d}x \le 2\mathfrak{L}\|v_0\|_{\mathrm{BV}(\mathbb{R})}$$

where $\mathfrak{L} = \max_{\alpha\in\{0,L,R\}} \mathfrak{L}_\alpha$.

Proof. We first use the formula for $v^1_{j+\frac12}$ and the fact that f_j is $\mathfrak{L}$-Lipschitz continuous in order to get,

$$\frac{\left|v^1_{j+\frac12}-v^0_{j+\frac12}\right|}{\Delta t}\le\frac{\mathfrak{L}}{\Delta x}\left(|v^0_{j+\frac12}-v^0_{j-\frac12}|+|v^0_{j+\frac32}-v^0_{j+\frac12}|\right).$$

We now use the formula for v^0_j, see (1.10), and we get,

$$|v^0_{j+\frac32}-v^0_{j+\frac12}|\le\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}}|v_0(y+\Delta x)-v_0(y)|\,\mathrm{d}y.$$

This shows that $v_\Delta(\Delta t,x)-v_\Delta(0,x)$ is integrable.

Then the discrete L^1-contraction (Lemma 5.4) applied to $v^n_{j+\frac12}$ and $w^n_{j+\frac12}=v^n_{j+1+\frac12}$ yields the result for all $n\ge0$, using the fact that $\displaystyle\int_{\mathbb{R}}\left|\frac{v_\Delta(x+\Delta x)-v_0(x)}{\Delta x}\right|\,dx\le||v_0||_{BV(\mathbb{R})}$. ☐

We now turn to BV estimates in the spatial variable. Such estimates are less classical but known, see for instance [9, Lemma 4.2]. We provide an alternative proof following the reasoning at the continuous level presented above.

Lemma 5.6 (Discrete space BV estimate)**.** *Let $v^n_{j+\frac12}$ be a solution of the discrete numerical scheme. Let $j_R,J_R\ge1$ and $j_L,J_L\le-2$ with $J_L\le j_L$ and $j_R\le J_R$. Then for any integers n_1,n_2 such that $1\le n_1<n_2$, we have,*

$$\sum_{j=j_R}^{J_R}\frac{|v^{n_2}_{j+\frac12+1}-v^{n_2}_{j+\frac12}|}{\Delta x}\le\sum_{j=j_R}^{J_R}\frac{|v^{n_1}_{j+\frac12+1}-v^{n_1}_{j+\frac12}|}{\Delta x}+C_{\mathrm{CFL}}\sum_{n=n_1}^{n_2-1}\frac{\left|v^{n+1}_{j_R+\frac12}-v^n_{j_R+\frac12}\right|}{\Delta t}+C_{\mathrm{CFL}}\sum_{n=n_1}^{n_2-1}\frac{\left|v^{n+1}_{J_R+1+\frac12}-v^n_{J_R+1+\frac12}\right|}{\Delta t},$$

$$\sum_{j=J_L}^{j_L}\frac{|v^{n_2}_{j+\frac12+1}-v^{n_2}_{j+\frac12}|}{\Delta x}\le\sum_{j=J_L}^{j_l}\frac{|v^{n_1}_{j+\frac12+1}-v^{n_1}_{j+\frac12}|}{\Delta x}+C_{\mathrm{CFL}}\sum_{n=n_1}^{n_2-1}\frac{\left|v^{n+1}_{J_L+\frac12}-v^n_{J_L+\frac12}\right|}{\Delta t}+C_{\mathrm{CFL}}\sum_{n=n_1}^{n_2-1}\frac{\left|v^{n+1}_{j_L+1+\frac12}-v^n_{j_L+1+\frac12}\right|}{\Delta t}.$$

Remark 5.7. In order to use the time BV estimate from Lemma 5.5, it is necessary to consider a mean (i.e. to integrate) in the x variable in the right hand side, that is to say in j_R and J_R.

Proof. We only do the proof at the right hand side of the origin since the estimate on the other side is identical. Considering $w^n_{j+\frac12}=v^n_{j+1+\frac12}$ and integrating in the discrete variables n and j the estimate from Lemma 5.3 yields the following discrete BV estimate away from $x=0$.

$$\begin{aligned}\sum_{j=j_R}^{J_R}|v^{n_2}_{j+\frac12+1}-v^{n_2}_{j+\frac12}|&\le\sum_{j=j_R}^{J_R}|v^{n_1}_{j+\frac12+1}-v^{n_1}_{j+\frac12}|\\&\quad+\sum_{n=n_1}^{n_2-1}Q^R(v^n_{j_R-\frac12},v^n_{j_R+\frac12};v^n_{j_R+\frac12},v^n_{j_R+\frac32})\frac{\Delta t}{\Delta x}-\sum_{n=n_1}^{n_2-1}Q^R(v^n_{J_R+1-\frac12},v^n_{J_R+1+\frac12};v^n_{J_R+1+\frac12},v^n_{J_R+1+\frac32})\frac{\Delta t}{\Delta x}.\end{aligned}$$

where $Q^R(a,b;c,d)=g^{H_R}(a\vee c,b\vee d)-g^{H_R}(a\wedge c,b\wedge d)$. Thanks to the technical lemma 5.8, we get,

$$\begin{aligned}\sum_{j=j_R}^{J_R}|v^{n_2}_{j+\frac12+1}-v^{n_2}_{j+\frac12}|\le\sum_{j=j_R}^{J_R}&|v^{n_1}_{j+\frac12+1}-v^{n_1}_{j+\frac12}|\\&+\sum_{n=n_1}^{n_2-1}\left|g^{H_R}(v^n_{j_R-\frac12},v^n_{j_R+\frac12})-g^{H_R}(v^n_{j_R+\frac12},v^n_{j_R+\frac32})\right|\frac{\Delta t}{\Delta x}\\&+\sum_{n=n_1}^{n_2-1}\left|g^{H_R}(v^n_{J_R+1-\frac12},v^n_{J_R+1+\frac12})-g^{H_R}(v^n_{J_R+1+\frac12},v^n_{J_R+1+\frac32})\right|\frac{\Delta t}{\Delta x}.\end{aligned}$$

Recalling the definition of the scheme, see (1.11), and the CFL condition, see (1.14), we obtain the desired estimate. ☐

We used the following technical lemma in the proof of the spatial BV estimates. It can be viewed as the discrete counterpart of the elementary inequality $|q_\alpha(a,c)| \le |H_\alpha(a) - H_\alpha(c)|$ at the continuous level. Recall that $q_\alpha(\cdot, b)$ is the flux function associated with the entropy function $|\cdot - b|$, see (2.1).

Lemma 5.8. *For all $a,b,c \in \mathbb{R}$, we have: $|Q^\alpha(a,b;b,c)| \le \left|g^{H_\alpha}(a,b) - g^{H_\alpha}(b,c)\right|$.*

Proof. We want to study

$$D_Q := Q^\alpha(a,b;b,c) = g^{H_\alpha}(a \vee b, b \vee c) - g^{H_\alpha}(a \wedge b, b \wedge c).$$

We distinguish cases by examining the values taken by $g^{H_\alpha}(a \vee b, b \vee c)$.

If $g^{H_\alpha}(a \vee b, b \vee c) = g^{H_\alpha}(a,b)$, then $a \ge b$ and $c \le b$. In particular, $g^{H_\alpha}(a \wedge b, b \wedge c) = g^{H_\alpha}(b,c)$ and we get the desired estimate.

If $g^{H_\alpha}(a \vee b, b \vee c) = g^{H_\alpha}(a,c)$, then $b \le a$ and $b \le c$. In particular, $g^{H_\alpha}(a \wedge b, b \wedge c) = g^{H_\alpha}(b,b)$. In this case, we have

$$\begin{aligned} D_Q &= g^{H_\alpha}(a,c) - g^{H_\alpha}(b,b) \le g^{H_\alpha}(a,b) - g^{H_\alpha}(b,c), \\ D_Q &= g^{H_\alpha}(a,c) - g^{H_\alpha}(b,b) \ge g^{H_\alpha}(b,c) - g^{H_\alpha}(a,b). \end{aligned}$$

These inequalities also imply the desired result.

If $g^{H_\alpha}(a \vee b, b \vee c) = g^{H_\alpha}(b,c)$, then $a \le b$ and $b \le c$. In particular, $g^{H_\alpha}(a \wedge b, b \wedge c) = g^{H_\alpha}(a,b)$ and $D_Q = g^{H_\alpha}(b,c) - g^{H_\alpha}(a,b)$ and we conclude in this case too.

Finally if $g^{H_\alpha}(a \vee b, b \vee c) = g^{H_\alpha}(b,b)$, then $a \le b$ and $c \le b$. In particular, $g^{H_\alpha}(a \wedge b, b \wedge c) = g^{H_\alpha}(a,c)$ and we have

$$\begin{aligned} D_Q &= g^{H_\alpha}(b,b) - g^{H_\alpha}(a,c) \le g^{H_\alpha}(b,c) - g^{H_\alpha}(a,b), \\ D_Q &= g^{H_\alpha}(b,b) - g^{H_\alpha}(a,c) \ge g^{H_\alpha}(a,b) - g^{H_\alpha}(b,c). \end{aligned}$$

The proof of the lemma is now complete. ☐

5.5 Proof of convergence

We prove simultaneously Theorems 1.4 and 1.1.

Proof of Theorems 1.4 and 1.1. We first consider any F_0.

In order to prove that v_Δ converges towards the entropy solution v of (1.1) in L^1 locally in time and space, we use the maximum principle (Lemma 5.1) and the discrete BV estimates in time and space from Lemmas 5.5 and 5.6 (see Remark 5.7). These estimates give

$$\forall \delta \in (0,1), \quad \forall T > 0, \quad |v_\Delta|_{BV(\Omega_{\delta,T})} \le C_{\delta,T} \quad \text{with} \quad \Omega_{\delta,T} := ((-\delta^{-1}, -\delta) \cup (\delta, \delta^{-1})) \times (0,T)$$

where the constant $C_{\delta,T}$ is independent on Δ small. Because v_Δ is also bounded in L^∞, this implies that v_Δ is compact in $L^1(K)$ for any compact set $K \subset [0,+\infty) \times \mathbb{R}$ (see for instance [13, Theorem 5.5]). Consequently, we can extract a subsequence (still denoted by Δ) such that v_Δ converges in L^1_{loc} and almost everywhere as $\Delta \to 0$. We are going to prove that the limit v is the unique entropy solution of (1.1) submitted to the initial condition (1.6).

Deriving the entropy inequalities away from the origin. Let $\kappa \in \mathbb{R}$. Using Lemma 5.3 with $w^n_{j+\frac{1}{2}} = \kappa$, we know that we have for all $x > \Delta x$,

$$\frac{|v_\Delta(t+\Delta t, x) - \kappa| - |v_\Delta(t,x) - \kappa|}{\Delta t} + \frac{1}{\Delta x}\left(Q^R(v_\Delta(t,x), v_\Delta(t, x+\Delta x); \kappa, \kappa) - Q^R(v_\Delta(t, x-\Delta x), v_\Delta(t,x); \kappa, \kappa)\right) \leq 0$$

where we recall that

$$Q^R(a, b; \kappa, \kappa) = g^{H_R}(a \vee \kappa, b \vee \kappa) - g^{H_R}(a \wedge \kappa, b \wedge \kappa).$$

In particular,

$$Q^R(v_\Delta(t, x-\Delta x), v_\Delta(t,x); \kappa, \kappa) \to H_R(v(t,x) \vee \kappa) - H_R(v(t,x) \wedge \kappa) = q_R(v(t,x), \kappa)$$

almost everywhere. Integrating against a non-negative test function $\phi \in C^\infty_c([0,+\infty) \times (0,+\infty))$ and using the dominated convergence theorem then implies that for all $\kappa \in \mathbb{R}$,

$$\int_{\mathbb{R}^2} (|v - \kappa|\partial_t \phi + q_R(v,\kappa)\partial_x \phi)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \phi(0,x)|v_0(x) - \kappa|\,\mathrm{d}x \geq 0.$$

Similarly, we have for all $\kappa \in \mathbb{R}$ and all non-negative test function $\phi \in C^\infty_c([0,+\infty) \times (-\infty, 0))$,

$$\int_{\mathbb{R}^2} (|v - \kappa|\partial_t \phi + q_R(v,\kappa)\partial_x \phi)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \phi(0,x)|v_0(x) - \kappa|\,\mathrm{d}x \geq 0.$$

Then it is classical (using the finite speed of propagation and localized L^1 contraction) that this implies that the essential limit of $v(t,\cdot)$ is v_0 in $L^1_{\text{loc}}(0,+\infty)$ as $t \to 0+$. The same argument for the left side allows us to conclude that v has initial data v_0 in the sense of Definition 2.7.

Weak formulation in the special case $F_0 = \mathcal{R}F_0$. Since v is an entropy solution of a scalar conservation law away from $x = 0$, we can apply Panov's theorem [29] and deduce that v admits strong traces on both sides (see Definition 2.6).

If $v(t, 0\pm)$ denotes the strong traces of $v(t,\cdot)$ at the discontinuity $x = 0$, we are left with checking that $P = (v(t,0-), v(t,0+))$ is in the germ $\mathcal{G} = \mathcal{G}_{\mathcal{R}F_0}$. It is convenient to write $P = (p_L, p_R)$.

In order to do so, we prove that

$$\forall K \in \mathcal{G}_{F_0}, \qquad D(P,K) \geq 0. \tag{5.3}$$

Let $K = (\kappa_L, \kappa_R)$, let $\phi \in C^\infty_c((0,+\infty) \times \mathbb{R})$ be non-negative and let

$$w_\Delta(t,x) = \kappa_\Delta(x) = \begin{cases} \kappa_L & \text{if } x < 0, \\ \kappa_R & \text{if } x > 0. \end{cases}$$

It is a solution of the numerical scheme for the scalar conservation law if and only if $K \in \mathcal{G}_{F_0}$.

We now integrate the discrete entropy inequality from Lemma 5.3 with $w_\Delta = \kappa_\Delta$ against ϕ,

$$\int_{(0,+\infty)\times\mathbb{R}} \frac{|v_\Delta - \kappa_\Delta|(t+\Delta t, x) - |v_\Delta - \kappa_\Delta|(t,x)}{\Delta t}\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_{(0,+\infty)\times\mathbb{R}} \frac{\mathcal{Q}_\Delta(t, x+\Delta x) - \mathcal{Q}_\Delta(t,x)}{\Delta x}\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x \leq 0$$

where

$$\mathcal{Q}_\Delta(t,x) = \mathcal{Q}^m_j \text{ for } (t,x) \in [t_m, t_{m+1}) \times [x_j, x_{j+1})$$

with $\mathcal{Q}_j^m$ defined in the statement of Lemma 5.3.

For Δx (and then Δt) small enough so that ϕ is supported in $[t_1, +\infty)$, we get after integrating by parts,

$$\int_{(0,+\infty)\times\mathbb{R}} |v_\Delta - \kappa_\Delta|(t,x) \frac{\phi(t,x) - \phi(t-\Delta t, x)}{\Delta t} \,\mathrm{d}t\,\mathrm{d}x + \int_{(0,+\infty)\times\mathbb{R}} \mathcal{Q}_\Delta(t,x) \frac{\phi(t,x) - \phi(t, x-\Delta x)}{\Delta x} \,\mathrm{d}t\,\mathrm{d}x \geq 0.$$

We examine the function $\mathcal{Q}_\Delta(t,x)$. We have,

$$\mathcal{Q}_\Delta(t,x) = \begin{cases} Q^R(v_\Delta(t,x-\Delta x), v_\Delta(t,x); \kappa_R, \kappa_R) & \text{if } x > \Delta x, \\ Q^L(v_\Delta(t,x-\Delta x), v_\Delta(t,x); \kappa_L, \kappa_L) & \text{if } x < 0, \\ Q_0(v_\Delta(t,x-\Delta x), v_\Delta(t,x); \kappa_L, \kappa_R) & \text{if } 0 < x < \Delta x. \end{cases}$$

We thus can write

$$\int_{(0,+\infty)\times\mathbb{R}} \mathcal{Q}_\Delta(t,x) \left(\frac{\phi(t,x) - \phi(t,x-\Delta x)}{\Delta x} \right) \mathrm{d}t\,\mathrm{d}x = \mathcal{D}_R + \mathcal{D}_L + \mathcal{D}_0$$

with

$$\mathcal{D}_R = \int_{(0,+\infty)\times[\Delta x,+\infty)} Q^R(v_\Delta(t,x-\Delta x), v_\Delta(t,x); \kappa_R, \kappa_R) \left(\frac{\phi(t,x) - \phi(t,x-\Delta x)}{\Delta x} \right) \mathrm{d}t\,\mathrm{d}x$$
$$\mathcal{D}_L = \int_{(0,+\infty)\times(-\infty,0]} Q^L(v_\Delta(t,x-\Delta x), v_\Delta(t,x); \kappa_L, \kappa_L) \left(\frac{\phi(t,x) - \phi(t,x-\Delta x)}{\Delta x} \right) \mathrm{d}t\,\mathrm{d}x$$
$$\mathcal{D}_0 = \int_{(0,+\infty)\times[0,\Delta x]} Q^0(v_\Delta(t,x-\Delta x), v_\Delta(t,x); \kappa_L, \kappa_L) \left(\frac{\phi(t,x) - \phi(t,x-\Delta x)}{\Delta x} \right) \mathrm{d}t\,\mathrm{d}x.$$

We now pass to the limit in the resulting inequality,

$$\int_{(0,+\infty)\times\mathbb{R}} |v_\Delta - \kappa_\Delta|(t,x) \left(\frac{\phi(t,x) - \phi(t-\Delta t,x)}{\Delta t} \right) \mathrm{d}t\,\mathrm{d}x + \mathcal{D}_R + \mathcal{D}_L + \mathcal{D}_0 \geq 0.$$

It is easy to pass to the limit in the first three terms. As far as $\mathcal{D}_0$ is concerned, it goes to 0 as $\Delta x \to 0$. We finally get,

$$\int_{(0,+\infty)\times(0,+\infty)} (|v - \kappa_R|\phi_t + q_R(v,\kappa_R)\phi_x) \,\mathrm{d}t\,\mathrm{d}x + \int_{(0,+\infty)\times(-\infty,0)} (|v - \kappa_L|\phi_t + q_L(v,\kappa_L)\phi_x) \,\mathrm{d}t\,\mathrm{d}x \geq 0.$$

Now choosing a test function of the form $\phi(t,x) = \phi_\varepsilon(t,x) = \psi(t) \cdot \max\left\{0, 1 - \varepsilon^{-1}|x|\right\}$, which focuses on the interface $x = 0$ as $\varepsilon \to 0$, we get a boundary term, which is well defined from the existence of strong traces. This gives an inequality for all $0 \leq \psi \in C_c^1(0,T)$, which implies that

$$D(P,K) \geq 0.$$

We thus proved (5.3) and we can apply Proposition 3.4 (recall that $F_0 = \mathcal{R}F_0$ and see Remark 3.5) and obtain that $P = (v(t,0-), v(t,0+)) \in \mathcal{G}_{F_0}$. Therefore v is a $\mathcal{G}_{F_0}$-entropy solution (1.1) with initial data v_0. The uniqueness of v follows from the maximality of the germ $\mathcal{G}_{F_0}$ (see Lemma 2.17).

General case when $F_0 \not\equiv \mathcal{R}F_0$. We now treat the general case, that is to say we do not assume anymore that $F_0 = \mathcal{R}F_0$. In this case, we consider the numerical solution $\bar{u}_\Delta$ for the HJ equation associated with $\mathcal{R}F_0$ and the numerical solution $\bar{v}_\Delta$ of the conservation law associated with $\mathcal{R}F_0$. We have in particular $\bar{v}_\Delta = \partial_x \bar{u}_\Delta$. We know that $\bar{u}_\Delta$ converges towards the unique $\mathcal{R}F_0$-viscosity solution $\bar{u}$ of (1.3)-(1.5) and $\bar{v}_\Delta$ converges towards the unique $\mathcal{G}_{\mathcal{R}F_0}$-entropy solution $\bar{v}$ of (1.1),(1.6) and $(\bar{u})_x = \bar{v}$. We now also consider u_Δ and v_Δ the numerical schemes associated with F_0. We know that u_Δ converges towards the unique F_0-viscosity solution u of (1.3),(1.5), which is also the unique $\mathcal{R}F_0$-viscosity solution (by [18]). Hence $u = \bar{u}$. Moreover, v_Δ converges in $L^1_{\mathrm{loc}}([0,T) \times \mathbb{R})$ towards $u_x = (\bar{u})_x = \bar{v}$. We thus proved that v_Δ converges towards the unique $\mathcal{G}_{\mathcal{R}F_0}$-entropy solution of (1.1),(1.6). □

6 Classification of maximal complete germs: proof of Theorem 1.8

We start with the following independent result (whose proof is quite long and then can not be reproduced here)

Theorem 6.1 (Complete germs are classified, [27])**.** *Assume* (1.2)*. Let $\mathcal{G} \subset \mathbb{R}^2$ be a complete germ in the sense of Definitions 2.1 and 2.4. Then $\mathcal{G}$ is maximal in the sense of Definition 2.3. Moreover, there exists a function F satisfying* (1.4) *such that $F = \mathcal{R}F$ and $\mathcal{G} = \mathcal{G}_F$ with $\mathcal{G}_F$ defined in* (2.2)*.*

Proof of Theorem 1.8. We set $F := \mathcal{R}F_0$.
Step 1: $\mathcal{G}_F$ is a maximal and complete germ. From Lemma 2.17, we already know hat $\mathcal{G}_F$ is a maximal germ. Now Theorem 1.4 shows the existence of a $\mathcal{G}_F$-entropy solution for any suitable initial data, including the ones for the Riemann problem. This shows the completeness of the germ $\mathcal{G}_F$.
Step 2: identification of any (maximal) complete germ $\mathcal{G}$. If $\mathcal{G}$ is a complete germ, then $\mathcal{G} = \mathcal{G}_F$ with $F = \mathcal{R}F$ follows from Theorem 6.1. ☐

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