BIGEY
Raphaël

Ex 1.4.9 :

1) $\nabla(fg) = (fg)_{,i}\, e_i = (f_{,i}\, g + f g_{,i}) e_i$
$= f_{,i}\, g\, e_i + f g_{,i}\, e_i$
$= \underline{\nabla f g + f \nabla g}$.

2) $\mathrm{div}\,(fu) = \sum_{i=1}^{3} \frac{\partial (fu)_i}{\partial x_i} = (fu)_{i,i}$
$= f u_{i,i} + f_{,i}\, u_i$
$= \underline{f\, \mathrm{div}(u) + \nabla f \cdot u}$

3) $\mathrm{rot}(fu) = \varepsilon_{ijk}\, (fu)_{k,j}\, e_i$
$= \varepsilon_{ijk}\, (f_{,j}\, u_k + f u_{k,j}) e_i$
$= \varepsilon_{ijk} f_{,j}\, u_k\, e_i + \varepsilon_{ijk}\, f u_{k,j}\, e_i$
$= \nabla f \wedge u + f\, \mathrm{rot}(u)$

4) $\mathrm{div}\,(u \wedge v) = (u \wedge v)_{i,i}$
$= (\varepsilon_{ijk}\, u_j v_k\, e_i)_{i,i}$
$= (\varepsilon_{ijk}\, u_j v_k)_{,i}$
$= \varepsilon_{ijk}\, u_{j,i}\, v_k + \varepsilon_{ijk}\, u_j\, v_{k,i}$
$= v \cdot (\varepsilon_{ijk}\, u_{j,i}\, e_k) + (\varepsilon_{ijk}\, v_{k,i}\, e_j) \cdot u$
$= v \cdot (\varepsilon_{jki}\, u_{k,j}\, e_i) + (-\varepsilon_{jik}\, v_{k,j}\, e_i) \cdot u$
$= \underline{v \cdot \mathrm{rot}(u) - (\mathrm{rot}(v)) \cdot u}$.

5) $[\mathrm{rot}(u \wedge v)]_i = \varepsilon_{ijk}\, (u \wedge v)_{k,j}$
$= \varepsilon_{ijk} (\varepsilon_{kqr}\, u_q v_r)_{,j}$
$= \varepsilon_{ijk}\, \varepsilon_{kqr}\, (u_{q,j}\, v_r + u_q\, v_{r,j})$
$= \varepsilon_{ijk}\, \varepsilon_{rqk}\, (u_{q,j}\, v_r + u_q\, v_{r,j})$
$= (\delta_{iq}\, \delta_{jr} - \delta_{ir}\, \delta_{jq}) (u_{q,j}\, v_r + u_q\, v_{r,j})$
$= \delta_{iq}\, \delta_{jr} (u_{q,j}\, v_r + u_q\, v_{r,j}) - \delta_{ir}\, \delta_{jq} (u_{q,j}\, v_r + u_q\, v_{r,j})$
$= u_{i,j}\, v_j + u_i\, v_{j,j} - u_{j,j}\, v_i - u_j\, v_{i,j}$
$= \underline{(u_i v_j)_{,j} - (u_j v_i)_{,j}}$

$$\mathrm{rot}(u\wedge v) = (u_i v_j)_{,j} e_i - (u_j v_i)_{,j} e_i$$
$$= u_{i,j} v_j e_i + u_i v_{j,j} e_i - u_{j,j} v_i e_i - u_j v_{i,j} e_i$$
$$= u_{i,j} (e_i \otimes e_j) v + \mathrm{div}(v) u - \mathrm{div}(u) v - v_{i,j} (e_i \otimes e_j) u$$
$$= \underline{(\nabla u) v + \mathrm{div}(v) u - \mathrm{div}(u) v - (\nabla v) u}$$

6) $\nabla(u\cdot v) = (u_j v_j)_{,i} e_i$
$$= u_{j,i} v_j e_i + u_j v_{j,i} e_i$$
$$= (\nabla u) v + (\nabla v) u$$
$$= (\nabla u) v + (\nabla v) u + \overbrace{u\wedge \mathrm{rot}(v) + v \wedge \mathrm{rot}(u)}^{=0}$$
car $(v_1 \wedge v_2 = - v_2 \wedge v_1)$

7) u vecteur ?
$1^{ère}$ partie admise.
On déduit $\Delta v = \Delta(u(Ox)) = (\Delta u)(Ox)$
$$\underline{= 0}.$$