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Shape-from-Shading, viscosity solutions and edges

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Summary. This article deals with the so-called Shape-from-Shading problem which arises when recovering a shape from a single image. The general case of a distribution of light sources illuminating a Lambertian surface is considered. This involves original definitions of three types of edges, mainly the apparent contours, the grazing light edges and the shadow edges. The elevation of the shape is expressed in terms of viscosity solution of a first-order Hamilton-Jacobi equation with various boundary conditions on these edges. Various existence and uniqueness results are presented.

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1. Introduction

We discuss here various aspects of the so-called Shape-from-Shading problem. This problem, which is classical in Vision Theory, corresponds, roughly speaking, to the reconstruction of a shape (a surface) from a two-dimensional image and more precisely from the brightness of the two-dimensional image. We shall look at a somewhat idealized case namely the case of a Lambertian surface. Let us briefly explain the mathematical formulation of this model case. First of all, the shape of the surface is related to the image brightness by the Horn image irradiance equation (see Horn [6], chap. 10) which relates the brightness of the image $I(x_1, x_2)$ to the reflectance

$$R(n) = I(x_1, x_2) \tag{1}$$

where R is the reflectance map which specifies the reflectance of a surface as a function of its orientation (or unit normal) n. The reflectance depends in general on the reflectance properties of the surface and on the distribution of light sources.

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Of course, the surface is (essentially) given locally by $x_3 = u(x_1, x_2)$ (and u is thus the elevation or height of the surface) so that $n = n(x_1, x_2)$ may be taken to be

(2) $$n = (-\nabla u(x), 1)(1 + |\nabla u|^2)^{-1/2}$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)$, $x = (x_1, x_2)$ and $|p|$ denotes the euclidian norm of p.

Next, we shall assume that the light sources are distant (mathematically, infinitely away from the surface) at least compared to the size of the piece of the surface we wish to recover. Thus, light sources can be considered as elements of the northern hemisphere of the unit sphere S^2 of $\mathbb{R}^3$ i.e. $S^2_+ = \{\omega \in \mathbb{R}^3 / |\omega| = 1, \omega_3 > 0\}$ and the light rays are just $X + \lambda\omega$ $(\lambda \in \mathbb{R})$ for any $X \in \mathbb{R}^3$. Therefore, a point of the surface $(x, u(x))$ will be illuminated by the "light source" ω if

(3) $$u(x + t\omega') < u(x) + t\omega_3 \quad \text{for all } t > 0$$

(where $\omega' = (\omega_1, \omega_2) \in \mathbb{R}^2$ and $\omega = (\omega', \omega_3)$), in other ways, $(x, u(x))$ is illuminated if no other part of the surface blocks the light ray that goes through this point.

And we may finally write Eq. (1) "explicitly" in terms of the unknown function u

(4) $$\int_{S^2_+} \cos(n, \omega) 1_K \, d\mu(\omega) = I(x)$$

or

(5) $$\int_{S^2_+} (\omega_3 - \omega' \cdot \nabla u(x))(1 + |\nabla u|^2)^{-1/2} 1_K \, d\mu(\omega) = I(x) .$$

Here and everywhere below, μ is (up to some irrelevant normalization constant) a distribution of light sources i.e. a probability measure on S^2_+, 1_K is the indicator function of the set K(=1 on K, = 0 outside), $(\xi, \eta) = \xi \cdot \eta$ denotes the usual scalar product in $\mathbb{R}^2$ or $\mathbb{R}^3$ and the set $K = K(u, x) \subset S^2_+$ corresponds to the possibility of having light rays blocked by some parts of the surface. And thus

(6) $$K = \{\omega \in S^2_+ / u(x + t\omega') < u(x) + t\omega_3 \text{ for all } t > 0\} .$$

Of course, Eq. (5) is still not really precise since we do need to specify the region where u is defined and the equation holds, possible boundary conditions. . . Notice also that the choice of a strict inequality instead of a large inequality in the definition of K is somewhat arbitrary and that the very formulation of the equation seems to require some smoothness of the surface namely that it is of class C^1, an assumption we shall always make even if it does not mean that we shall work only with C^1 functions u as we shall see later on.

But if we are willing to forget these additional difficulties, Eq. (5) is a rather complicated first-order, nonlinear, nonlocal (integro-differential) partial differential equation. And it is certainly useful to present some "simple" examples of this general class of equations.

Example 1 (vertical light). We consider the case of a single vertical light source i.e. $\mu = \delta_{\omega^0}$ and $\omega^0 = (0, 0, 1)$ in which case Eq. (5) becomes

(7) $$(1 + |\nabla u|^2)^{-1/2} = I(x) .$$

Example 2 (oblique light, no shadows). We consider now the case of a single oblique light source i.e. $\mu = \delta_{\omega^0}$ and $\omega^0 = (-\alpha, -\beta, \gamma)$ with $\gamma > 0$. Now, if there are no shadows (at least on the portion of the surface we want to reconstruct), Eq. (5) becomes

$$\left(\alpha \frac{\partial u}{\partial x_1} + \beta \frac{\partial u}{\partial x_2} + \gamma \right)(1 + |\nabla u|^2)^{-1/2} = I(x)\,. \tag{8}$$

Example 3 (multiple light sources, concave surface). Suppose now that the surface is concave i.e. u is a concave function then $\omega \in K$ as long as $\cos(\omega, n) > 0$. Therefore, (5) may be written

$$\int_{S^2_+} (\omega_3 - \omega' \cdot \nabla u(x))^+ (1 + |\nabla u|^2)^{-1/2} \, d\mu(\omega) = I(x) \tag{9}$$

and the right-hand side is a given function $F(\nabla u)$.

In all these examples, (5) becomes a local, nonlinear, scalar first-order equation of the Hamilton-Jacobi type. But this is clearly not always the case.

Let us now explain the various goals we want to achieve here. The main one is to present enough evidence that this general problem is not an ill-posed mathematical problem, to the contrary of many inverse problems. More precisely that in the various situations we can analyse, the shape that is u can be determined from Eq. (5) appropriately complemented with boundary conditions. As we shall see however, some indeterminations subsist and have to subsist: the simplest being that u can only be determined up to the addition of a constant. More serious ones exist like for instance in Example 1, $-u$ is automatically a solution if u is one. But, see also Rouy and Tourin [19], this lack of uniqueness can be completely analysed and thus somewhat circumvented.

A related issue that this work will clarify (at least, we hope so) is the question of boundary conditions. Indeed, in order to reduce the number of indeterminations, it would be useful to localize the reconstruction, therefore creating edges on which boundary conditions have to be imposed. And we want to show in this paper that there are three different categories of edges which lead to three different boundary conditions (whose formulations will be mentioned later on). The first one concerns *apparent contours*: beyond those edges, the surface cannot be seen even if it might be illuminated (recall that the camera or eye or observer is "facing the surface" or in other words can be thought to be located at "infinity" in the $x_3 = +\infty$ direction) and we thus expect the exterior normal derivative of u to become $-\infty$ on the edge which corresponds to the two-dimensional (x_1, x_2) projection of the apparent contour.

The second type of boundaries corresponds to *grazing light edges* on the surface: more precisely, the boundary is the two-dimensional (x_1, x_2) projection of such edges. These edges can be defined in the case of a single light source (to simplify) as the set of points $(x_1, x_2, u(x_1, x_2))$ where — with the notations of the Example 2 — $\alpha \frac{\partial u}{\partial x_1} + \beta \frac{\partial u}{\partial x_2} + \gamma = 0$.

The third type of boundaries consists of the two-dimensional (x_1, x_2) *projection of the edge of the* (projected) *shadow*. Of course, inside the shadow region, no information is available but, as we shall see, on its edge or more precisely on the projection of this edge, we can formulate a boundary condition that will allow some analysis of the reconstruction problem in the case when shadows occur. It is a nonlocal boundary condition.

Let us mention how this paper is organized: Sections 2, 3 and 4 are respectively devoted to the three types of boundaries and thus boundary conditions we mentioned above in the case of a single light source. Section 5 concerns some examples of results with multiple light sources. Finally, we present in Sect. 6 some numerical experiments illustrating the rigorous results proved in this paper: these experiments were made using the numerical method used in Rouy and Tourin [19].

We next want to explain the main mathematical tool we shall be using essentially everywhere below. We use here the same approach as in Rouy and Tourin [19] — this work is in fact, in some sense, the sequel of [19]—namely the theory of viscosity solutions. The notion of viscosity solutions has been introduced by Crandall and Lions [4] and has allowed a rather general and complete understanding of fully nonlinear, scalar, first or second-order, degenerate elliptic equations. Many applications of the theory to a large variety of fields and subjects exist by now and we just want to mention here the intrinsic links with optimal control theory (see [5, 10, 11, . . .]). The reader interested in the PDE aspects of the theory should consult the survey by Crandall et al. [3]. The relationships between Shape-from-Shading and viscosity solutions are, at a formal level, obvious since, at least in the single light source case (Examples 1 and 2, Eqs. (7) and (8)) and even in Example 3 [Eq. (9)], the equations are first-order equations of Hamilton-Jacobi type. And this class of equations falls into the scope of viscosity solutions theory.

Next, let us explain what we achieve with the help of the viscosity solutions. We show that viscosity solutions of the (various) Shape-from-Shading equations can be classified and characterized. In particular, the solutions of primary interest namely C^1 solutions (corresponding to C^1 surfaces) can be characterized in the set of all viscosity solutions (they are extremal in a sense explained in Sect. 2), in the case when uniqueness does not hold. And the possible losses of uniqueness are identified and classified. This leads to numerical algorithms that are stable, robust and efficient for the effective reconstruction of the surface. The idea is thus to avoid working with the seemingly natural C^1 solutions which are in fact "isolated accidents" for such Hamilton-Jacobi equations (and thus unstable, sensitive to small perturbations like a noisy measurement of I . . .). We embed C^1 solutions into viscosity solutions, find all of them and recover the C^1 solutions as extremal points.

But viscosity solutions allow more. It turns out — and we shall explain below this is no miracle — that the three types of boundary conditions we mentioned above can be imposed in a natural way in the context of viscosity solutions. The boundary condition corresponding to apparent contours is related to state-constraints ("subsolution on the boundary") as introduced by Soner [20] (see also [2]). The boundary condition corresponding to the grazing light edges is the Neumann boundary condition as introduced by Lions [12] (see also the survey [3] . . .). Finally, the third boundary condition corresponding to the edge of the shadow is not a standard one but falls into viscosity solutions theory since it would correspond in optimal control problems to processes jumping when they reach a certain boundary. This will be made more precise in Sect. 4.

Finally, viscosity solutions theory allows to consider the general situation of multiple light sources by straightforward adaptations of the classical notion (at least to formulate the problem).

The reader might wonder at this stage why viscosity solutions appear in the Shape-from-Shading problem and we wish to convince him (or her) by a simple "physical" argument that they had to appear. Indeed, the notion of viscosity solutions is natural and useful (and makes sense!) for general equations of the form

$$A[u] = I \tag{10}$$

where I is a given function and A is a nonlinear operator acting on continuous (or smooth) scalar functions u, whenever the operator A has an "order-preserving" property (or "satisfies the maximum principle", or is formally "accretive for the sup norm") that can be written as follows

$$A[u](x^0) \geqq A[v](x^0) \quad \text{if } (u - v)(x^0) = \sup(u - v)^+ \tag{11}$$

for all test functions u, v.

Next, we claim that the Shape-from-Shading operators do satisfy this property and that in fact this property is natural and "physical". Indeed, we notice first that the Shape-from-Shading operator is invariant by the addition of a constant (this is due to the fact that the source(s) of light is (are) assumed to be infinitely away from the surface). In other words

$$A[u + C] = A[u], \quad \text{for all } C \in \mathbb{R} . \tag{12}$$

This invariance corresponds to the freedom of choosing the altitude 0! Therefore, if u, v define two surfaces and if $(u - v)(x^0) = \sup(u - v)^+$ at some point x^0, we may assume without loss of generality that $u(x^0) = v(x^0)$. In other words, the surface given by $(x_3 = u(x_1, x_2))$ is everywhere below the surface given by $(x_3 = v(x_1, x_2))$ and they touch exactly at the point x^0. Then, $A[u](x^0)$, respectively $A[v](x^0)$, in the Shape-from-Shading problem, measures the quantity of light received by (the surface) u (resp. v) at x^0. Thus, u being lower than v everywhere else will receive more light at x^0 and we find the desired inequality (11). We could of course check this directly on Eq. (5) and the verification is straightforward observing that

i) u and v being C^1, $\nabla u(x^0) = \nabla v(x^0)$ since x^0 is a maximum point of $u - v$
ii) $K(u, x^0) \supset K(v, x^0)$ since $u(x^0 + t\omega') - u(x^0) \leqq v(x^0 + t\omega') - v(x^0)$
iii) $\omega_3 - \omega' \cdot \nabla u(x^0) \geqq 0$ if $\omega \in K(u, x^0)$.

Let us conclude this long presentation by recalling a few previous works on the Shape-from-Shading problem. Horn [6] first solved the equation (in the case of a single light source) by a global method under some assumptions on the surface providing the necessary additional information in order to characterize the shape. He then developed another formulation of the problem by a minimization of a certain functional involving either the orientation or the height of the surface and a regularization term (a viscosity term in the terminology of viscosity solutions). Still in the case of a single distant light source, Pentland proposed a method that estimates the shape from local variations in the image intensity by computing the solution in the Fourier domain (the reflectance map is then approximated by a linear function of the partial derivatives of u). The algorithm seems to give a good estimate of the surface for high frequencies and may be united with a stereo treatment for low ones (see [15–17]). Our approach (as in [19]) is thus radically different, leading to efficient numerical methods and allowing to investigate cases with edges (apparent contours, shadows . . .) or with multiple sources of light.

2. Apparent contours

To simplify the presentation, we will consider essentially always a model case involving a single source of vertical light (Example 1) and to fix ideas a volumic object resting on an arbitrary background. We denote by Ω the two-dimensional projection on the plane (x_1, x_2) of the part of the object (a smooth surface) that is lit so that $\partial\Omega$ is the projection of the apparent contour (see Fig. 1). Again, to simplify the presentation, we assume that Ω is smooth (of class $C^{1,1}$ for instance) and that u is continuous on $\bar{\Omega}$ (this is an assumption at $\partial\Omega$ only if we consider smooth surfaces). More general situations can be analysed but we shall not do so here.

Equation (7) then becomes

$$(1 + |\nabla u|^2)^{-1/2} = I \quad \text{in } \Omega \tag{13}$$

where I is given on $\bar{\Omega}$, $1 \geqq I > 0$ in Ω and I is Lipschitz on $\bar{\Omega}$. We will not recall these assumptions below. Observe that we expect to have $I = 0$ on $\partial\Omega$ and in fact $\partial u/\partial n = -\infty$ on $\partial\Omega$ where n denotes the unit outward normal to $\partial\Omega$. Of course, this plays the role of a boundary condition but an extremely singular one.

We next recall the definition of viscosity solutions (see [4]) of general equations of the form

$$F(x, u, \nabla u) = 0 \quad \text{in } D \tag{14}$$

where $u \in C(\bar{D})$, $F \in C(\bar{D} \times \mathbb{R} \times \mathbb{R}^N)$. Of course, (13) is a special case of (14) with the choice $F(x, t, p) = F(x, p) = (1 + |p|^2)^{-1/2} - I(x)$. Let us also remark that the definition we recall will be applied with $D = \Omega$ or with $D = \bar{\Omega}$ (the difference being important in order to take into account boundary conditions).

We say that u is a viscosity subsolution (resp. supersolution) of (14) in D if we have for all $\varphi \in C^1(\bar{D})$ the following property: at each local in D maximum (resp. minimum) point x^0 of $u - \varphi$, we have

$$F(x^0, u(x^0), \nabla\varphi(x^0)) \leqq 0 \tag{15}$$

(resp.

$$F(x^0, u(x^0), \nabla\varphi(x^0)) \geqq 0) \tag{16}$$

Finally, u is a viscosity solution of (14) if u is both a viscosity sub- and supersolution of (14).

Remark 1. We deduce immediately from the definition that u is a viscosity solution of (13) if and only if $w = -u$ is a viscosity solution of

$$I\sqrt{1 + |\nabla w|^2} = 1 \quad \text{in } \Omega \tag{17}$$

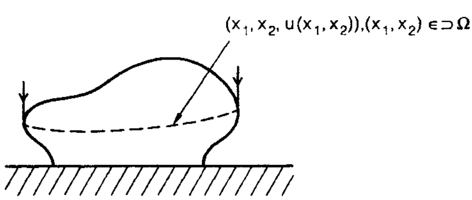

Fig. 1.

or equivalently

$$|\nabla w| = (I^{-2} - 1)^{1/2} \quad \text{in } \Omega \,. \tag{18}$$

Next we go back to our original motivation and we observe that if $I = 0$ and "$\partial u/\partial n = -\infty$", then, for each $\varphi \in C^1(\bar{\Omega})$, $u - \varphi$ cannot reach a local maximum at $x^0 \in \partial\Omega$. Therefore, by defect, u is automatically a subsolution of (13) in $\bar{\Omega}$. Let us also observe that if $u \in C(\bar{\Omega})$ is a viscosity subsolution of (13) in $\bar{\Omega}$ and $I = 0$ on $\partial\Omega$ then the inequality (15) cannot hold at a point $x^0 \in \partial\Omega$. Therefore, we recover the fact (automatically) that, for each $\varphi \in C^1(\bar{\Omega})$, $u - \varphi$ cannot have a local maximum at a point on $\partial\Omega$.

This is precisely how we are going to incorporate the boundary condition corresponding to apparent contours. The fact that a "subsolution up to $\partial\Omega$" (or a supersolution . . .) could be one way of formulating a boundary condition was first discovered by Soner [20] in the study of optimal deterministic control problems with state-constraints (see also [2, 3, . . .]). Before we formulate general existence and uniqueness results, we wish to recall here that nonuniqueness is possible even for C^1 solutions and, mathematically, this is due to the fact that (13) does not contain terms involving u and that I may achieve the value 1 (see [19] and the analysis below for more details). This is why we will consider first a model case when $\{I = 1\} = \{\bar{x}\}$ where $\bar{x} \in \Omega$. We shall consider later on the general case when $\{I = 1\}$ is an arbitrary compact set in Ω. To give one example of nonunique solutions, we just consider a one-dimensional version of our problem (this is the case when u does not depend on x_2 . . .). It is quite clear that the two shapes of Fig. 2 yield the same I.

We thus begin with the

Theorem 1. *We assume that* $\{I = 1\} = \{\bar{x}\}$ *where* $\bar{x} \in \Omega$.

i) *Let* $u, v \in C(\bar{\Omega})$ *be viscosity solutions of* (13) *in* Ω *and subsolutions of* (13) *in* $\bar{\Omega}$. *Then,* $u - v$ *is constant on* $\bar{\Omega}$.

ii) *If* I *satisfies*

(19) $\quad (\inf\{I(x)/d(x) = t\})^{-1}$ *is integrable at* $t = 0^+$, *where* $d(x) = \operatorname{dist}(x, \partial\Omega)$,

then there exists a unique viscosity solution $u \in C(\bar{\Omega})$ *of* (13) *in* Ω *which is a viscosity subsolution of* (13) *in* $\bar{\Omega}$ *and which satisfies* $u(\bar{x}) = 0$.

Remark 2. It is easily seen on simple (essentially one-dimensional) examples that a condition like (19) is needed in order to ensure the existence of a continuous solution on $\bar{\Omega}$ (in particular of a solution remaining finite on $\partial\Omega$).

Remark 3. The above problem (see also Remark 1) is closely related to the study of "ergodic state-constraints problems" in Capuzzo-Dolcetta and Lions [2] with two additional difficulties and one simplification due to the specificity of the problem.

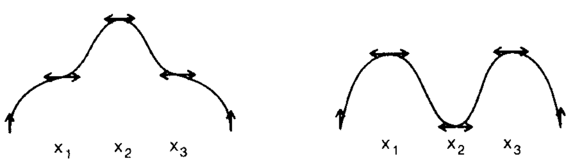

Fig. 2.

Indeed, we allow here in (18) a right-hand side which may blow-up at the boundary (and it does in the original Shape-from-Shading problem above!). In addition, for such "ergodic" problems, the uniqueness of solutions up to the normalization constant is not true in general; only, the "average cost" which is constant is known to be unique. Here, and this is due to the fact that $I(\bar{x}) = 1$, this constant is 0 and we are able to show the uniqueness of the solution. Let us also point out to understand the relationships with the results of [2] that in part ii) of the Theorem 1, $-u$ is a solution of (17) or (18) in Ω as observed in Remark 1 and $-u$ is a supersolution of (17) or (18) in $\bar{\Omega}$. In the event that $I = 0$ at some points of $\partial\Omega$, this only means that no local (in $\bar{\Omega}$) minima of $(-u) - \varphi$ can occur at those points for any $\varphi \in C^1(\Omega)$.

Remark 4. Of course, if u is a solution of (13), then $u + C$ is also a solution of (13).

Remark 5. In view of the observations made before Theorem 1, any C^1 solution u of (13) satisfying $\partial u/\partial n = -\infty$ on $\partial\Omega$ is a viscosity solution in Ω and subsolution in $\bar{\Omega}$ of (13).

Remark 6. The fact that $\{I = 1\} = \{\bar{x}\}$ is a very severe restriction on the shape since it means in particular that, if u is C^1 in order to simplify the discussion, then u has a single stationary (critical) point on Ω namely $\bar{x}$.

Proof of Theorem 1. We begin with the proof of i). In view of Remark 4 above, we may assume without loss of generality that $u(\bar{x}) = v(\bar{x}) = 0$. We are going to show that $u - v \equiv 0$. In order to do so, we adapt, as in Rouy and Tourin [19], a trick originally due to Kružkov [9], adapted to viscosity solutions theory by Crandall and Lions [4], and written in its most general form in Ishii [7]. We introduce $\tilde{v} = \theta v$ where $\theta \in (0, 1)$ and we check easily that $\tilde{v}$ is a viscosity supersolution of

$$(20) \qquad (1 + |\nabla\tilde{v}|^2)^{-1/2} = I(\theta^2 + (1 - \theta^2)I^2)^{-1/2} > I \quad \text{in } \Omega - \{\bar{x}\} .$$

The verification of this claim is in fact trivial if we use Eq. (18) satisfied by $(-v)$.

We then claim that $u \leqq \tilde{v}$ in $\bar{\Omega}$. To this end, we introduce $w = -u$, $z = -v$ and $\tilde{z} = -\theta v$. Next we remark (see Remarks 1 and 3) that $\tilde{z}$ is a viscosity subsolution in $\bar{\Omega}$ of

$$(21) \qquad I(1 + |\nabla\tilde{z}|^2)^{1/2} = (\theta^2 + (1 - \theta^2)I^2)^{1/2} = 1 - r$$

where $r \in C(\bar{\Omega})$, $r(x) > 0$ in $\bar{\Omega} - \{\bar{x}\}$, $r(\bar{x}) = 0$.

Similarly, w is a viscosity supersolution in $\bar{\Omega}$ of (17). The strict sign of r on $\bar{\Omega} - \{\bar{x}\}$ allows to apply the proof of Soner [20] in order to obtain

$$\sup_{\bar{\Omega}} (\tilde{z} - w) = (\tilde{z} - w)(\bar{x}) = 0 .$$

Therefore, $\tilde{z} \leqq w$ in $\bar{\Omega}$ or $u \leqq \tilde{v} = \theta v$ in $\bar{\Omega}$ and we conclude letting θ go to 1.

We now turn to the proof of part ii) of Theorem 1. We choose $\varphi \in C^1(\bar{\Omega})$ (or Lipschitz) such that $\varphi > 0$ on $\partial\Omega$, $\varphi \geqq 0$ on $\bar{\Omega}$ and $\varphi \equiv 0$ in a neighborhood of $\bar{x}$. Then, for δ small enough, $I_\delta = I + \delta\varphi$ satisfies: $I_\delta > 0$ in $\bar{\Omega}$, $I_\delta < 1$ in $\bar{\Omega} - \{\bar{x}\}$ and $I_\delta(\bar{x}) = 1$. We next apply the results of [2] and we obtain the existence of $(\lambda_\delta, w_\delta) \in \mathbb{R} \times C(\bar{\Omega})$ such that w_δ is Lipschitz on Ω and is a viscosity subsolution in Ω and supersolution in $\bar{\Omega}$ *of*

$$|\nabla w_\delta| = (I_\delta^{-2} - 1)^{1/2} - \lambda_\delta$$

and (for instance) $w_\delta(\bar{x}) = 0$. In addition, since $(I_\delta^{-2} - 1)^{1/2} \geqq 0$ in $\bar{\Omega}$, we deduce from [2] the fact that $\lambda_\delta \geqq 0$. Then, we claim that $\lambda_\delta = 0$. Indeed, if $\lambda_\delta > 0$, then we would deduce for η small enough

$$|\nabla w_\delta| \leqq -\frac{\lambda_\delta}{2} \quad \text{on } \{|x - \bar{x}| < \eta\} .$$

This is clearly impossible and the contradiction shows that $\lambda_\delta = 0$.

In view of the classical results on the stability of viscosity solutions (see [3]) we just have to check that w_δ and thus $u_\delta = -w_\delta$ are uniformly equicontinuous on $\bar{\Omega}$ and to let δ go to 0^+. This is in fact rather straightforward since $|\nabla w_\delta|$ is bounded, uniformly in δ, on compact subsets of Ω. In order to conclude, we observe that we have

$$|\nabla w_\delta| \leqq (I^{-2} - 1)^{1/2} \leqq \varphi(d) \quad \text{in } \Omega \tag{22}$$

where $\varphi(t) = ((\inf\{I(x)/d(x) = t\})^{-2} - 1)^{1/2}$. Next, since $\partial\Omega$ is smooth, we may write, in a neighbourhood of $\partial\Omega$, $x = x' - d(x)n(x')$ for a unique $x' \in \partial\Omega$ that depends continuously upon x (at least). Then, the assumption (19) combined with (22) yields

$$|w_\delta(x) - w_\delta(x')| \leqq \int_0^{d(x)} \varphi(t)\, dt . \tag{23}$$

Hence, we deduce from the uniform Lipschitz bounds on w_δ inside Ω the fact that w_δ is, uniformly in δ, uniformly continuous on $\partial\Omega$. And this fact combined with (23) allows to conclude that w_δ is, uniformly in δ, uniformly continuous in a neighbourhood of $\partial\Omega$. We may then conclude. $\square$

Next, we state without proof the extension to the case when $\{I = 1\}$ is an arbitrary compact set in Ω. In order to simplify a bit the presentation, we will assume in the result that follows that $I = 0$ on $\partial\Omega$. This implies in particular that there exist an at most countable family J and a family of disjoint compact connected components $(K_i)_{i \in J}$ such that $\{I = 1\} = \bigcup_{i \in J} K_i \subset K$, for some compact set K in Ω. We next introduce a distance between each K_i and K_j (directly inspired by the work by Lions [10])

$$L(i,j) = \inf\left\{ \int_0^1 F(x(s))\, ds / x \in C^1([0,1]; \Omega),\ x(0) \in K_i,\ x(1) \in K_j,\ |\dot{x}(s)| \leqq 1,\ s \in [0,1] \right\}, \tag{24}$$

where $F(y) = (I(y)^{-2} - 1)^{1/2}$ in Ω. Since F vanishes on each $K_i (i \in J)$, we could as well specify $x(0)$ in K_i and $x(1)$ in K_j without modifying the infimum. We shall introduce also for $x \in \Omega$, $i \in J$

$$L_i(x) = \inf\left\{ \int_0^1 F(x(s))\, ds / x \in C^1([0,1]; \Omega),\ x(0) = x,\ x(1) \in K_i \right\} . \tag{25}$$

And we observe that $L_i(x)$ could be equivalently defined by specifying $x(1)$ arbitrarily in K_i, that $L_i(x) = L(i,j)$, if $x \in K_j$ and that L_i vanishes on K_i. Combining the arguments in [10] and the proof of Theorem 1, one can show that L_i is locally Lipschitz in Ω and continuous on $\bar{\Omega}$.

Theorem 2. i) *Let* $u \in C(\bar{\Omega})$ *be a viscosity solution in* Ω *and subsolution in* $\bar{\Omega}$ *of* (13). *Then,* u *is constant on each* K_i ($i \in J$) *and we denote this constant by* $u(K_i)$. *And we have*

$$u(x) = \sup_{i \in J} \{u(K_i) - L_i(x)\}, \quad \forall x \in \Omega . \tag{26}$$

In particular, $|u(K_i) - u(K_j)| \leqq L(i,j)(\forall i \neq j \in J)$.

ii) *Let* $\alpha_i \in \mathbb{R}$ ($i \in J$) *satisfy*

$$|\alpha_i - \alpha_j| \leqq L(i,j)(\forall i \neq j \in J) . \tag{27}$$

Then, there exists a unique viscosity solution $u \in C(\bar{\Omega})$ *of* (13) *in* Ω *which is a viscosity subsolution of* (13) *in* $\bar{\Omega}$ *and which satisfies:* $u(K_i) = \alpha_i$.

Furthermore, if $u \in C^1(\Omega)$,*we have*

$$\forall i \in J, \quad \exists j \in J, \quad j \neq i, \quad |u(K_i) - u(K_j)| = L(i,j) . \tag{28}$$

Remark 7. We refer to [10] and [19] for similar characterizations of C^1 solutions as extremal points of all possible viscosity solutions. This is why we shall not prove this part of the theorem. The rest is essentially a combination of arguments given in [10] and of the proof of Theorem 1.

In the rest of the paper, we shall see in each situation nonuniqueness phenomena associated to zones (points or compact subsets) where I achieves a particular (maximal) value. The extension made in Theorem 2 from Theorem 1 is typical of the additional losses of uniqueness created when the number of such zones exceeds one: *we need then to specify*, up to a normalization constant, *the differences of u between those zones* and among all possible solutions C^1 solutions are extremal. This is why, in all the rest of the paper, we shall always consider the particular case of a single "singular" point $\bar{x}$ where I achieves a particular (maximal) value.

We next want to conclude this section by considering another natural case for the Shape-from-Shading problem whose mathematical treatment is very much similar to the one made above. This is the case when $\partial\Omega$ (still assumed to be smooth . . .) is the two-dimensional (x_1, x_2) projection of a level curve for u or in other words u is constant on $\partial\Omega$. Of course, we can treat this assumption as a Dirichlet boundary condition as it was done in [19]. Then, we can always normalize all solutions to be 0 on $\partial\Omega$. In addition, we assume, for the reasons detailed above, that $0 < I \leqq 1$ on $\bar{\Omega}$, $I(\bar{x}) = 1$, $I(x) < 1$ on $\bar{\Omega} - \{\bar{x}\}$ for some $\bar{x} \in \Omega$. And, one can show (see [10]) that, if

$$\bar{L} = \inf \left\{ \int_0^1 F(x(s))\, ds / x \in C^1([0,1]; \bar{\Omega}),\ x(0) = \bar{x},\ x(1) \in \partial\Omega \right\}$$

($\bar{L} > 0$), for each $\alpha \in [-\bar{L}, +\bar{L}]$ there exists a unique viscosity solution $u \in C(\bar{\Omega})$ of (13) in Ω such that $u|_{\partial\Omega} \equiv 0$ and $u(\bar{x}) = \alpha$. Furthermore, if $u \in C^1(\bar{\Omega})$ then $|\alpha| = \bar{L}$.

We want to present here a different mathematical formulation which relies upon a natural geometrical assumption on the shape. Indeed, we shall assume that I is Lipschitz continuous on $\bar{\Omega}$, $0 < I \leqq 1$ in Ω, $I(\bar{x}) = 1$ for some $\bar{x} \in \Omega$, $I(x) < 1$ in $\bar{\Omega} - \{\bar{x}\}$ and that $\partial\Omega$ is a level curve of u or equivalently that u is constant on $\partial\Omega$. The geometrical assumption we want to make is that the surface is "oriented upwards" in $\bar{\Omega}$ i.e. that $\partial u/\partial n \leqq 0$ on $\partial\Omega$ (possibly infinite at some points). Thus, if we want to recover such a protruding object, we first need to analyse its implications on the boundary conditions in the case when u is C^1 (except at the points of $\partial\Omega$ that correspond to apparent contours i.e. where I vanishes). Indeed, from Eq. (13) and the fact that u is constant on $\partial\Omega$, we deduce

$$\left(1+\left(\frac{\partial u}{\partial n}\right)^2\right)^{-1/2} = I \quad \text{on } \partial\Omega \text{ (or on } \partial\Omega - \{I = 0\} \ldots)$$

which, combined with $\partial u/\partial n \leqq 0$, identifies completely $\partial u/\partial n$.

Next, we observe that if $u - \varphi$ achieves a maximum (locally in $\bar{\Omega}$) at a point x^0 of $\partial\Omega$ where $\varphi \in C^1(\bar{\Omega})$ (and thus $I(x^0) > 0$), then on one hand $\nabla\varphi(x^0) = \frac{\partial\varphi}{\partial n}(x^0)\cdot n(x^0)$ since u is constant on $\partial\Omega$. And, on the other hand, $\frac{\partial\varphi}{\partial n}(x^0) \leqq \frac{\partial u}{\partial n}(x^0) \leqq 0$. Therefore, we have at x^0

$$(1 + |\nabla\varphi|^2)^{-1/2} \leqq I\ .$$

In other words, u is still a viscosity subsolution of (13) in $\bar{\Omega}$. Thus, *u is uniquely determined* (up to a trivial normalization i.e. up to a constant) since *Theorem* 1 *still applies to this situation* — this is why, in fact, we did not assume $I = 0$ in Theorem 1.

3. Grazing light edges

In this section, we wish to study in the model case of a single oblique light source the boundary conditions corresponding to one type of edges for a shadow. Indeed, a shadow region will be bounded by two edges (or boundaries) namely the boundary where the light rays are grazing and where "the shadow begins" and the boundary of the projected shadow i.e. where "the shadow ends". In this section, we shall be concerned with the first one and we want to study the boundary condition that must correspond to it, boundary condition that will have to be satisfied on the two-dimensional (x_1, x_2) projection of the grazing light edge. Let us denote by γ_0 this two-dimensional projection. We assume that γ_0 is a closed smooth curve and of course $\gamma_0 \subset \partial\Omega$. In any realistic situation, we cannot expect to have $\gamma_0 = \partial\Omega$ and $\partial\Omega$ will be split into two (essentially distinct except for some isolated intersection points) parts namely γ_0 and Γ. On the rest of the boundary i.e. Γ, we expect another natural boundary behaviour like the ones we studied in the previous section i.e. Γ is a piece of the two-dimensional (x_1, x_2) projection of an apparent contour (or of a level curve of u).

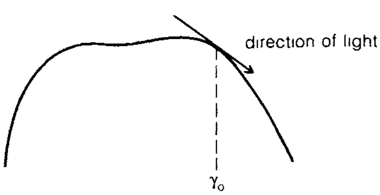

Fig. 3.

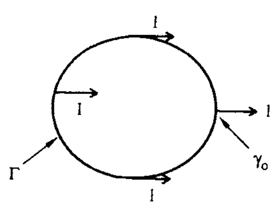

Fig. 4.

Therefore, Eq. (8) will be set in $\Omega \subset \mathbb{R}^2$ where $\partial\Omega = \Gamma \cup \gamma_0$ where Γ and γ_0 are two closed smooth curves that generically meet at two points in such a way that $\partial\Omega$ is of class $C^{1,1}$ but not C^2 in general. In addition, let $l = (\alpha, \beta) \in \mathbb{R}^2$ ($(-\alpha, -\beta, \gamma) \in S^2_+$ is the light source) so that $|l|^2 + \gamma^2 = 1$ where $\gamma > 0$; then, $l \cdot n < 0$ on $\Gamma - (\Gamma \cap \gamma_0)$, $l \cdot n > 0$ on $\gamma_0 - (\Gamma \cap \gamma_0)$ and thus $l \cdot n = 0$ on $\Gamma \cap \gamma_0$ (see Fig. 4).

Let us recall the equation to be satisfied in Ω:

$$\frac{l \cdot \nabla u + \gamma}{\sqrt{1 + |\nabla u|^2}} = I \quad \text{in } \Omega . \tag{29}$$

And we assume in all that follows that $0 < I \leqq 1$ in Ω, I is Lipschitz on $\bar{\Omega}$. For the reasons explained at the end of the preceding section, we shall consider only the model case when $I < 1$ in $\Omega - \{\bar{x}\}$, $I(\bar{x}) = 1$ for some $\bar{x} \in \Omega$.

Remark 8. The main modification of the facts shown in the preceding section concerning these "nonuniqueness" zones where $I = 1$ is that solutions are no more constant on such zones. Indeed, for C^1 solutions of (29) or even for arbitrary $C(\bar{\Omega})$ viscosity solutions of (29) in Ω, we can show that if K denotes a connected compact subset of Ω on which $I \equiv 1$, then,

$$\nabla u = \frac{l}{\gamma} \quad \text{on } K$$

And thus if $\bar{x} \in K$, then we have:

$$u(x) = \frac{l}{\gamma} \cdot (x - \bar{x}) + u(\bar{x}), \quad \forall x \in K .$$

We next want to discuss the boundary conditions we shall impose on Γ and γ_0. We begin with γ_0. Recall that γ_0 is the two-dimensional (x_1, x_2) projection of the grazing light edges. Therefore, if the surface is smooth (say C^1) and thus if $u \in C^1(\Omega \cup \gamma_0)$ we expect to find $I = 0$ on γ_0 and

$$l \cdot \nabla u + \gamma = 0 \quad \text{on } \gamma_0 . \tag{30}$$

Notice this condition is in fact very much analogous to a Neumann or more precisely to an oblique derivative boundary condition since $l \cdot n \geqq 0$ on γ_0 and $l \cdot n > 0$ on $\gamma_0 - (\Gamma \cap \gamma_0)$. The oblique derivative boundary condition only degenerates at (the two points) $\Gamma \cap \gamma_0$. This leads (following [12], see also [3, 8]) to the following viscosity formulation on γ_0: $u \in C(\bar{\Omega})$ is a viscosity subsolution (resp. supersolution) of (29) and (30) if, for all $\varphi \in C^1(\bar{\Omega})$, at each local maximum point (resp. minimum point) x^0 of $u - \varphi$ in $\bar{\Omega}$ we have, when $x^0 \in \Omega \cup \gamma_0$,

$$l \cdot \nabla\varphi(x^0) + \gamma \leqq 0 \quad \text{if } x^0 \in \gamma_0, \frac{l \cdot \nabla\varphi(x^0) + \gamma}{\sqrt{1 + |\nabla\varphi(x^0)|^2}} \leqq I(x^0) \text{ if } x^0 \in \Omega \tag{31}$$

(resp.

$$l \cdot \nabla\varphi(x^0) + \gamma \geqq 0 \quad \text{if } x^0 \in \gamma_0, \frac{l \cdot \nabla\varphi(x^0) + \gamma}{\sqrt{1 + |\nabla\varphi(x^0)|^2}} \geqq I(x^0) \text{ if } x^0 \in \Omega) . \tag{32}$$

And $u \in C(\bar{\Omega})$ is a viscosity solution of (29) and (30) if it is both a viscosity subsolution and supersolution of (29) and (30).

Remark 9. In fact, this is not the usual viscosity formulation of the boundary condition (30) which should be instead of (31) (for example, (32) is modified in a similar way replacing min by max)

$$\min\left(l \cdot \nabla\varphi(x^0) + \gamma, \frac{l \cdot \nabla\varphi(x^0) + \gamma}{\sqrt{1 + |\nabla\varphi(x^0)|^2}} - I(x^0)\right) \leqq 0 .$$

But since $I = 0$ on γ_0, this inequality turns out to be equivalent to (31). Notice for the same reason that $u \in C(\bar{\Omega})$ is a viscosity solution of (29) and (30) if and only if u is a viscosity solution of (29) in $\Omega \cup \gamma_0$. We decided to use a (sort of) Neumann type boundary condition since it seems more natural in view of the intuition we have of grazing light rays.

We now turn to the boundary condition on Γ. First of all, if Γ corresponds to an apparent contour (i.e. if Γ is the two-dimensional (x_1, x_2) projection of an apparent contour) we expect to find on Γ:

$$\frac{\partial u}{\partial n} = -\infty$$

and thus $u - \varphi$ cannot have a maximum point in $\bar{\Omega}$ at a point $x^0 \in \Gamma$. And, in particular, we impose, as in the preceding section, that u is a viscosity subsolution of (29) in $\Omega \cup \Gamma$. Let us observe that all this means in particular that $I = -(l \cdot n)$ on Γ (notice that we recover $I = 0$ on $\Gamma \cap \gamma_0$).

Also, as in the preceding section, the fact that u is a viscosity subsolution of (29) in $\Omega \cup \Gamma$ is still true in another case of interest namely when Γ is a piece of a level curve of u. In that case, for reasons that will be clear below, we need to impose $I \geqq -(l \cdot n)$ on Γ and we shall concentrate, as in the preceding section on a "protruding object". We expect to have in that case

$$\frac{(l \cdot n)\dfrac{\partial u}{\partial n} + \gamma}{\sqrt{1 + \left(\dfrac{\partial u}{\partial n}\right)^2}} = I \tag{33}$$

and since the function $\dfrac{(l \cdot n)t + \gamma}{\sqrt{1 + t^2}}$ is increasing for $t \leqq \dfrac{l \cdot n}{\gamma}$ ($\leqq 0$ on Γ), we claim that when $I \geqq -(l \cdot n)$ on Γ, $\dfrac{\partial u}{\partial n} \leqq \dfrac{l \cdot n}{\gamma}$ on Γ (and Γ is a piece of a level curve of u) then u is indeed a viscosity subsolution of (29) on $\Omega \cup \Gamma$. This fact is elementary, since if $u - \varphi$ achieves a local maximum in $\bar{\Omega}$ at $x^0 \in \Gamma$ for some $\varphi \in C^1(\bar{\Omega})$, then

$$\nabla\varphi(x^0) = \frac{\partial \varphi}{\partial n}(x^0) n(x^0) \qquad \text{and} \qquad \frac{\partial \varphi}{\partial n}(x^0) \leqq \frac{\partial u}{\partial n}(x^0),$$

therefore, we have

$$\frac{l \cdot \nabla\varphi(x^0) + \gamma}{\sqrt{1 + |\nabla\varphi(x^0)|^2}} = \frac{l \cdot \dfrac{\partial \varphi}{\partial n}(x^0) + \gamma}{\sqrt{1 + \left|\dfrac{\partial \varphi}{\partial n}(x^0)\right|^2}} \leqq \frac{l \cdot \dfrac{\partial u}{\partial n}(x^0) + \gamma}{\sqrt{1 + \left|\dfrac{\partial u}{\partial n}(x^0)\right|^2}} = I(x^0),$$

and our claim is proven.

This is why, in conclusion, we shall work with $u \in C(\bar{\Omega})$ viscosity solution of (29), (30) and subsolution of (29) on $\Omega \cup \Gamma$.

The main (a posteriori) reason for such a definition of solutions is given by the following rather striking uniqueness result.

Theorem 3. *Let $u, v \in C(\bar{\Omega})$ be viscosity solutions of (29), (30) and subsolution of (29) on $\Omega \cup \Gamma$. Then, $u - v$ is constant on $\bar{\Omega}$.*

Remark 10. Let us recall that we are assuming $I < 1 \in \bar{\Omega} - \{\bar{x}\}$, $I(\bar{x}) = 1$ for some $\bar{x} \in \Omega$.

Remark 11. Let us point out, as we did several times before, that if u is a solution (in the sense of the above theorem) then $u + C$ is also a solution for any $C \in \mathbb{R}$. One way to normalize solutions is to prescribe arbitrarily u at $\bar{x}$.

Proof of Theorem 3. As in the proof of Theorem 1, we are going to work in fact with $w = -u$, $z = -v$ which are viscosity solutions in $\Omega \cup \gamma_0$ and supersolutions in $\Omega \cup \Gamma$ of

$$I(1 + |\nabla\Psi|^2)^{1/2} + l \cdot \nabla\Psi = \gamma\,. \tag{34}$$

Also, we may assume, in view of Remark 11 above, without loss of generality that $u(\bar{x}) = v(\bar{x}) = 0$.

We next consider (as in [19]), for $\theta \in (0, 1)$, $\tilde{w} = \theta w - (1 - \theta) l \cdot (x - \bar{x}) \gamma^{-1}$.

Notice that $\tilde{w}(\bar{x}) = z(\bar{x}) = 0$. Also since $(p \to I\sqrt{1 + |p|^2} + l \cdot p)$ is convex, we deduce easily that $\tilde{w}$ is a viscosity subsolution in $\Omega \cup \gamma_0$ of

$$I(1 + |\nabla\Psi|^2)^{1/2} + l \cdot \nabla\Psi = \gamma - r \tag{35}$$

where $r = \dfrac{1 - I}{\gamma} > 0$ in $\bar{\Omega} - \{\bar{x}\}$, $r(\bar{x}) = 0$. Recall that $|l|^2 + \gamma^2 = 1$.

This allows, by a straightforward adaptation of the results by Soner [20] (see also [2]), to deduce that

$$\sup_{\bar{\Omega}} (\tilde{w} - z) = (\tilde{w} - z)(\bar{x}) = 0\,.$$

Therefore, $\tilde{w} \leqq z$ and letting θ go to 1 we deduce $w \leqq z$ in $\bar{\Omega}$ i.e. $u \geqq v$ in $\bar{\Omega}$. And we conclude exchanging u and v.

Remark 12. We used in this proof, as in proof of Theorem 1, the fact that the solution u of Theorem 3 is such that $w = -u$ is a viscosity solution in $\Omega \cup \gamma_0$ and a viscosity supersolution in $\Omega \cup \Gamma$ of (34). This is easy to check by a simple use of the definitions, once we recall Remark 9 which explains why a viscosity solution of (29), (30) is a viscosity solution of (29) in $\Omega \cup \gamma_0$.

We would like to conclude this section by an observation which can be useful for practical applications. Indeed, it is often the case that the measurement of I is noisy and produces minor errors. In particular, numerically, the boundary γ_0 may have been selected in such a way that I does not exactly vanish there so we only have $I \geqq 0$ on γ_0. In that case, keeping all the other information, we have to investigate the uniqueness of viscosity solutions of (29) in Ω, that are subsolutions of (29) in $\Omega \cup \Gamma$ and which satisfy (30) in viscosity sense i.e. in the sense of Remark 9.

The same uniqueness result as in Theorem 3 holds in that case since the formulation of Remark 9 implies that such a solution is automatically a viscosity subsolution of (29) in $\Omega \cup \gamma_0$ (since $I \geqq 0$) and thus in $\bar{\Omega}$. Therefore, we simply observe that if u, v are viscosity solutions of (29) in Ω and subsolutions of (29) in $\bar{\Omega}$, then $u - v$ is constant in $\bar{\Omega}$ (we still assume that I achieves the value 1 at one point $\bar{x} \in \Omega$, $I \geqq 0$ in Ω and that I is Lipschitz). And this statement covers the above case. In fact, it also covers the situation given in Theorem 3 since a viscosity solution of (29), (30) is also a viscosity subsolution of (29) in $\Omega \cup \gamma_0$.

A direct proof of the uniqueness is also possible by using and combining the methods of proofs in [2, 20] for state constraints problems and in [8, 12] for oblique derivative boundary conditions. The only advantage of this proof is that it covers cases where, for reasons similar to the ones mentioned above, we have instead of (30)

$$l \cdot \nabla\varphi + \gamma = e \quad \text{on } \gamma_0$$

where e is, say, of class C^1 on γ_0.

4. Shadow edges

We next want to consider another type of boundaries (the last one!) and thus of boundary conditions. To explain the relevance of this third type, we shall consider the model case of a single light source $(-\alpha, -\beta, \gamma) \in S^2_+$ and we shall allow now the shape (or surface) to be such that a shadow forms. This clearly means in fact the formation of two edges: one was studied in the preceding section and corresponds to the curve in the surface where the light rays are grazing. The other one is the border of the projected shadow or in other words the curve where the "shadow ends". Geometrically (and mathematically) two slightly different cases may occur. In fact, many other cases are possible that are somewhat nongeneric, but that can be analysed in a similar manner like an accumulation of shadows . . . These two cases correspond to the possibility of the two curves described above to meet at one or two points. Of course, we have to consider the two-dimensional (x_1, x_2) projection of these curves and the model situations we wish to analyse are represented by Fig. 5 and 6.

We shall begin with the *first case* i.e. when the projections of the grazing light edge and the projected shadow edge do not meet. In that case we have two distinct regions Ω and ϑ where Eq. (29) holds and $I > 0$ on $\Omega \cup \vartheta$. These two regions are

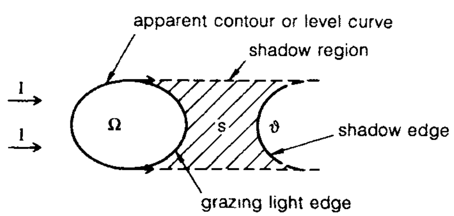

Fig. 5.

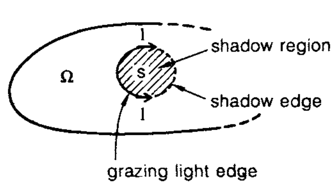

Fig. 6.

separated by the shadow (region) S where no light is received and where, of course, there is no way to reconstruct the surface. In the region Ω, we can directly analyse the problem as we did in the preceding section and typically, $\partial\Omega$ consists of two curves meeting at two points: the projection of the grazing light edge (denoted by γ_0 in the preceding section) and the projection of the apparent contour or level curve (denoted by Γ in the preceding section). Of course, we can always assume to know more about u on Γ and for instance we can also simply prescribe u on Γ. In all the cases, after a possible normalization of u, we construct u in $\bar{\Omega}$ (up to some classified indetermination if $\{I = 1\}$ consists of several connected components) as it was explained in the preceding sections.

We may now investigate the reconstruction of u in $\bar{\vartheta}$. Again, geometrically and mathematically many situations are possible for the various types of boundaries since we could have on some part of $\partial\vartheta$ essentially all the possibilities we already investigated and we could find inside ϑ the situation mentioned in Fig. 6. In order to keep the ideas clear, we shall mention only two situations.

In both cases, $\partial\vartheta$ consists of the union of two smooth closed ($C^{1,1}$ for instance) meeting at two points that we denote by γ_0' and Γ'. γ_0' is the shadow edge and Γ' may be, typically, either an apparent contour (or a level curve) as this is the case in Fig. 7 or a union of three (typically but it may be only two . . .) smooth curves, two of which being apparent contours or level curves and the last one being the projection of grazing light edges, as this is the case in Fig. 8. As we saw in the preceding sections, we always deduce that u is a viscosity solution of (29) in ϑ and subsolution of (29) in $\vartheta \cup \Gamma'$ (even if it is slightly better to write on the grazing light edge a "Neumann" boundary condition, in which case the result below also remains true. We thus have to explain the boundary condition on γ_0'

But, if γ_0' is the shadow edge, this means that, for each $x \in \gamma_0'$, there exists a unique (generically, otherwise simple adaptations have to be performed) point, denoted by T_x, on γ_0 (the grazing light edge, part of the boundary of Ω) such that

$$x - T_x = \lambda l \quad \text{for some } \lambda > 0, \qquad u(x) = u(T_x) - \lambda\gamma \,. \tag{36}$$

But this condition implies that u is given on γ_0'! We may then conclude easily in view of the following proposition, where we assume to simplify the presentation that $I < 1$ on $\vartheta \cup \Gamma'$ — otherwise the uniqueness holds modulo the prescription of

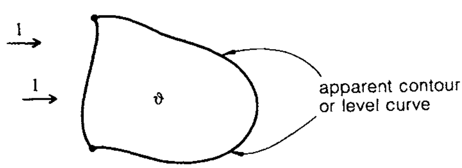

Fig. 7.

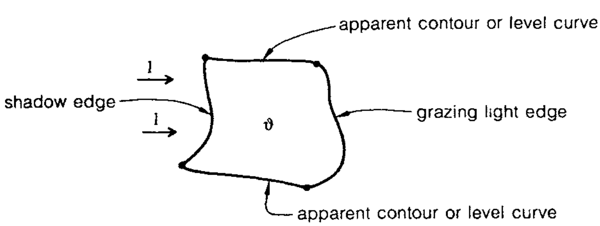

Fig. 8.

u on the various zones where $\{I = 1\}$ and C^1 solutions are extremal ..., see Sect. 2.

Proposition 1. *Let* $u, v \in C(\bar{\vartheta})$ *be viscosity solutions of* (29) *in* ϑ *and subsolutions of* (29) *in* $\vartheta \cup \Gamma'$. *We assume that* I *is Lipschitz on* ϑ, *and that* $0 \leqq I \leqq 1$ *in* $\vartheta \cup \Gamma'$. *Then we have*

$$\sup_{\bar{\vartheta}} (u - v) = \sup_{\gamma'_0} (u - v) , \tag{37}$$

Remark 13. We skip the proof of this result since the proof follows the same method as the proof of Theorem 3, translating the above assumption as the formulation (34) for $-u$, $-v$ and introducing the same perturbation $\tilde{w} = -\left(\theta u + \dfrac{1-\theta}{\gamma} l \cdot x\right)$.

We may turn to the second case namely the one illustrated by Fig. 6, i.e. when the grazing light edge γ_0 and the shadow edge γ'_0 meet typically at two points. Each of these edges may be assumed to be smooth but the resulting regions (the interior one and the exterior one) are not necessarily smooth (they are often $C^{1,1}$ but not always ...). All these possible singularities reflect geometrical singularities (due to apparent contours, projections ...) and we refer to Arnold [1] for a closely related study of similar singularities. A few examples of possible situations are shown in Figs. 9–11.

However, in most situations (and again more complicated possibilities can be analysed by suitable modifications), the following properties are satisfied

$$\gamma_0 \cap \gamma'_0 = \{x_1, x_2\} \quad \text{with } x_1 \neq x_2 \tag{38}$$

$$l \cdot n < 0 \quad \text{on } \gamma_0 - \{x_1, x_2\}, \qquad l \cdot n > 0 \quad \text{on } \gamma'_0 - \{x_1, x_2\} \tag{39}$$

$$l^\perp \cdot x \in (l^\perp \cdot x_1, l^\perp \cdot x_2) \quad \text{for } x \in (\gamma_0 \cup \gamma'_0) - \{x_1, x_2\} \tag{40}$$

where $l^\perp$ is a unit orthogonal vector to l such that $l^\perp \cdot x_1 < l^\perp \cdot x_2$.

Notice that the domain enclosed by γ_0 and γ'_0 that we denote by S is the projection of the region occupied by the shadow and, as in the analysis of the previous case, no reconstruction of u inside S can be expected. We shall work thus in the exterior domain Ω as illustrated by Fig. 6 and, in order to do so, we have to specify what happens "after the end of the shadow γ'_0 when we follow the direction l". Of course, new shadows could occur (or apparent contours or level curves ...). Again, to simplify the presentation and to keep the ideas clear, we consider the following model cases of Figs. 12 and 13.

These cases correspond to situations described respectively in Sects. 2 and 3. And we already saw there that we could summarize these situations by considering viscosity solutions of (29) in Ω which are subsolutions of (29) in $\Omega \cup \Gamma$, where Γ is the part of the boundary of Ω which is distinct from $\gamma_0 \cup \gamma'_0$ (i.e. the "exterior" boundary) and we assume Γ to be smooth ($C^{1,1}$ for instance). Let us recall once

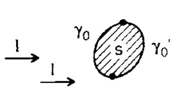

Fig. 9.

Fig. 10.

Fig. 11.

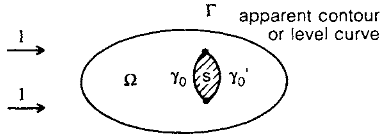

Fig. 12.

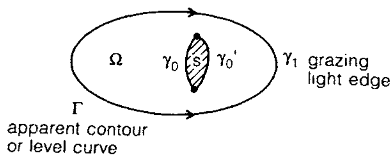

Fig. 13.

more that in the case where "Γ contains a grazing light edge", it would be "better" to use an oblique derivative boundary condition on that part even if, mathematically (if no measurement errors are made), it implies the above assumption as we saw in the preceding section.

Once more, in order to simplify, we shall assume that I is Lipschitz in Ω, $0 \leqq I < 1$ in $\bar{\Omega} - \{\bar{x}\}, I\{\bar{x}\} = 1$ for some $\bar{x} \in \Omega$. We next want to explain the boundary conditions on γ_0 and γ'_0. Since γ_0 is the projection of grazing light edge, we should use the boundary condition (30) as explained in Sect. 3. Exactly for the same reasons as above, we simply require the solutions u to be a viscosity subsolution of (29) on γ_0 as well and thus on $\Omega \cup \Gamma \cup \gamma_0$. And we assume that $I = 0$ on γ_0. Next on γ'_0, we observe that, as in the case treated above, we have

$$\forall x \in \gamma'_0 - \{x_1, x_2\}, \qquad \exists ! T_x \in \gamma_0 - \{x_1, x_2\}, \qquad x - T_x = \lambda l \quad \text{for some } \lambda > 0 \tag{41}$$

and

$$\forall x \in \gamma'_0 - \{x_1, x_2\}, \qquad u(x) = u(T_x) - \lambda \gamma\,. \tag{42}$$

This will define the boundary condition on γ'_0. One thus sees that it is a nonlocal (nonstandard) boundary condition. However, as we should expect in view of the general argument presented in the Introduction, this boundary condition "preserves order" and is "compatible with the maximum principle". This fact will be quite clear from the proof below. Let us only mention at this stage that for a representation of solutions of linear or nonlinear first-order equations in terms of ordinary differential equations or controlled (or games . . .) ones, this would mean that when the process reaches $\gamma'_0(- \{x_1, x_2\})$, it immediately jumps from x to T_x, while one must add "a cost" $\lambda\gamma$ to the representation . . .

We can now prove the

Theorem 4. *Let $u, v \in C(\bar{\Omega})$ be viscosity solutions of* (29) *in Ω and of* (42) *on γ'_0, and viscosity subsolutions of* (29) *in $\Omega \cup \gamma_0 \cup \Gamma$. We assume that u, v are Lipschitz continuous near x_1 and x_2. Then $u - v$ is constant on $\bar{\Omega}$.*

Remark 14. As in all cases above, if u is a solution, $u + C$ is a solution for any $C \in \mathbb{R}$.

Remark 15. The assumption that u, v are Lipschitz continuous at x_1 and x_2 is the simplest one to state that allows us to work at the difficult points x_1 and x_2. In particular, we can modify the proof presented below in such a way that we need to assume only $u, v \in C(\bar{\Omega})$ and for x sufficiently close to x_1 and x_2

$$\Pi_i = \left\{ \varepsilon_i l^{\perp} \cdot (x - x_i) \leqq C_i \left(\frac{l}{|l|} \cdot (x - x_i) \right)^2 \right\} \subset \Omega \tag{43}$$

$$\text{for some } C_i > C_i^0, \text{ for } i = 1, 2$$

where $\varepsilon_1 = +1$, $\varepsilon_2 = -1$ and where $C_i^0 = \overline{\lim_{y \to x_i}} \frac{I(y)}{|y - x_i|}$ $(i = 1, 2)$. Recall that $I(x_i) = 0$ for $i = 1, 2$, $I \geqq 0$ and I is Lipschitz so that $0 \leqq C_i^0 < +\infty$ $(i = 1, 2)$.

In particular, if I is C^1 then, since we have $I \geqq 0$, $I = 0$ at x_i we should expect $\nabla I(x_i) = 0$ and thus $C_i^0 = 0$. In that case, (43) is automatically satisfied as soon as γ_0 admits l as a tangent direction at (x_i, x_2) in a nondegenerate way. The condition (43) is illustrated by Fig. 14.

Even if such a condition is somewhat more natural, it involves technical modifications of the argument introduced below and we prefer to skip it.

Proof of Theorem 4. We shall use the formulation (34) with $w = -u$, $z = -v \in C(\bar{\Omega})$. Both w and z are viscosity solutions in Ω and supersolutions in $\Omega \cup \Gamma \cup \gamma_0$ of (34). Next, w and z satisfy instead of (42)

$$\forall x \in \gamma_0' - \{x_1, x_2\}, \qquad \varphi(x) = \varphi(T_x) + \lambda\gamma . \tag{44}$$

As in all the proofs made above, we introduce $\tilde{w} = \theta w - (1 - \theta)\, l/\gamma \cdot (x - \bar{x})$ for $\theta \in (0, 1)$. Again, as we did several times above, we can assume without loss of generality that $u(\bar{x}) = v(\bar{x}) = 0$ in view of Remark 14. Therefore, we have $w(\bar{x}) = \tilde{w}(\bar{x}) = z(\bar{x}) = 0$ and we thus only have to show that $\tilde{w} \leqq z$ in $\bar{\Omega}$ in order to conclude letting θ go to 1.

We also deduce, from the argument used several times above, that $\tilde{w}$ is a viscosity subsolution in $\Omega \cup \gamma_0 \cup \Gamma$ of

$$I\sqrt{1 + |\nabla\tilde{w}|^2} + l \cdot \nabla\tilde{w} = \gamma - r \tag{45}$$

where $r > 0$ in $\Omega \cup \gamma_0 \cup \Gamma - \{\bar{x}\}$, $r(\bar{x}) = 0$, $r \in C(\bar{\Omega})$.

Using the uniqueness argument of [2, 20], we deduce, from this information

$$\sup_{\bar{\Omega}}(\tilde{w} - z) = \max\{(\tilde{w} - z)(\bar{x}), \sup_{\gamma_0'}(\tilde{w} - z)\} = \sup_{\gamma_0'}(\tilde{w} - z)^+ . \tag{46}$$

Next, we argue by contradiction and we assume that $\sup_{\bar{\Omega}}(\tilde{w} - z) > 0$ so that this supremum is achieved at a point $\hat{x} \in \gamma_0'$.

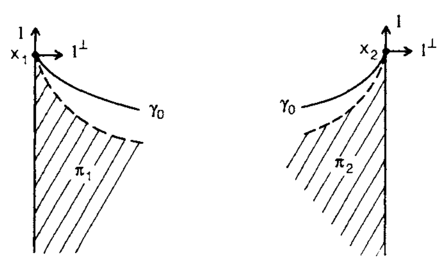

Fig. 14.

We then claim that $\hat{x} = x_1$ or x_2. Indeed, if this were not the case, we would have $\hat{x} \in \gamma'_0 - \{x_1, x_2\}$. Then, using (44), we find

$$(\tilde{w} - z)(\hat{x}) = (\tilde{w} - z)(T_{\hat{x}}) - (1 - \theta)\lambda\gamma - (1 - \theta)\frac{\lambda}{\gamma}|l|^2$$

$$= (\tilde{w} - z)(T_{\hat{x}}) - (1 - \theta)\frac{\lambda}{\gamma}.$$

And we reach a contradiction with the definition of $\hat{x}$. The contradiction proves our claim.

Next, we want to show by a local argument that x_1 or x_2 cannot be a maximum point of $\tilde{w} - z$. We will show this fact for x_1, the proof being exactly the same for x_2. In order to simplify the presentation, we make a translation and a rotation so that $x_1 = 0$, $l^\perp = e_1$ and $l = |l|e_2$ where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. Next, let $\delta > 0$ be such that z is Lipschitz in $B_\delta \cap \Omega$ where B_δ is the ball centered at 0, of radius δ. We denote by L a Lipschitz constant. Next, let $\underline{r} = \inf\{r(y)/y \in B_\delta \cap \Omega\} > 0$. We deduce from (34) and (45) that $\tilde{w}$ (resp. z) is a viscosity subsolution (resp. supersolution) of

$$|l|\partial_2 \tilde{w} = \gamma - \underline{r} \quad \text{in } B_\delta \cap \Omega \tag{47}$$

(resp.

$$|l|\partial_2 z = \gamma - I(1 + L^2)^{1/2} \text{ in } B_\delta \cap \Omega). \tag{48}$$

We denote here by ∂_2 the partial derivation in the e_2 direction ($= l$ direction). This last inequality uses standard manipulations in viscosity solutions theory (see for example [4]).

We may now conclude recalling first that $I(0) = 0$ (and I is continuous). Therefore, for $\varepsilon \in (0, \delta)$ small enough, we find that z is a viscosity supersolution of

$$|l|\partial_2 z = \gamma - \tfrac{1}{2}\underline{r} \quad \text{in } B_\delta \cap \Omega. \tag{49}$$

Next, we observe that (47) and (49) imply by classical results on viscosity solutions (already shown in [4]) the following inequalities

$$\begin{cases} \tilde{w}(x) - \tilde{w}(x - te_2) \leqq \dfrac{1}{|l|}(\gamma - \underline{r})t \\[2ex] z(x) - z(x - te_2) \geqq \dfrac{1}{|l|}(\gamma - \tfrac{1}{2}\underline{r})t \end{cases} \tag{50}$$

if $\{x - se_2/s \in [0, t]\} \subset B_\varepsilon \cap \Omega$.

In particular, if we choose $x = 0(=x_1)$, we observe that $-te_2 \in B_\varepsilon \cap \Omega$ for all $t > 0$ small enough, so that we deduce from (50) for $t > 0$ small enough

$$(\tilde{w} - z)(0) \leqq (\tilde{w} - z)(-te_2) - \frac{1}{2|l|}\underline{r}t.$$

And we reach a contradiction with the definition of $x_1 = 0$.

The contradiction completes the proof of Theorem 4. □

5. Multiple and distributed light sources

We want to give in this section some elements of a general study of the reconstruction of a shape illuminated by an arbitrary "number" of light sources that are all distant. As explained in the Introduction, we thus have to study Eq. (5) set in (for example) a smooth domain Ω of $\mathbb{R}^2$. To simplify the presentation, we shall consider only the model case when there are no shadow i.e. $I > 0$ in Ω and Eq. (5) is then replaced by (9) and when $\partial\Omega$ is a projection (on the (x_1, x_2) plane) of an apparent contour as we did in Sects. 2 and 3. We could study as well different situations by adapting some of the arguments presented in the preceding sections. And we shall come back on more general cases of Eq. (5) in a future publication. This is why we shall assume throughout this section in particular that I is Lipschitz in Ω, $I > 0$ in Ω. The fact that we are assuming that $\partial\Omega$ is the projection of an apparent contour implies in fact that "$\partial u/\partial n = -\infty$" on $\partial\Omega$ and thus, for each $x \in \partial\Omega$, $K_x = \{\omega \in S^2_+/\omega' \cdot n(x) \geqq 0\}$ and

$$I(x) = \int_{K_x} \omega' \cdot n(x)\, d\mu(\omega)\,. \tag{51}$$

Similarly, if $\partial\Omega$ is the projection of a level curve (see also Sects. 2 and 3) and if we assume that Ω is "protruding", we would replace (51) by assuming that I is larger than the right-hand side of (51). In fact, in the analysis made below, we shall not need the equality (51) (or an inequality . . .) and thus we will not assume it. Also, observe that the facts that $\partial\Omega$ is the projection of an apparent contour and that $I > 0$ in Ω (and thus near $\partial\Omega$) imply in particular the following property of μ: for all $n \in S^1(=\{p \in \mathbb{R}^2, |p| = 1\})$, $\mu(\{\omega \in S^2_+, n \cdot \omega' \geqq 0\}) > 0$. But again, as we chose this "boundary condition on $\partial\Omega$" just as an example of the mathematical results that can be obtained, we shall ignore here such implicit requirements which would not be necessary for other kinds of "boundary behaviours".

Next, we recall that μ is a probability measure on $S^2_+ = \{\omega \in \mathbb{R}^3, |\omega| = 1, \omega_3 > 0\}$ which accounts for a distribution of source lights. For instance, a convex combination of Dirac masses on S^2_+ corresponds to a finite number of point light sources located at the points defining the masses. But, μ can also be an absolutely continuous measure on S^2_+ corresponding, in some sense, to a "large and continuous" source of light.

In order to explain the (partial) results we have shown on Eq. (5), we shall consider a few simple cases. These cases rely upon the following condition

$$\forall x \in \Omega, \quad \{\omega \in \mathrm{Supp}(\mu), \omega_3 - \omega' \cdot \nabla u(x) \geqq 0\} \subset K \tag{52}$$

where

$$K = \{\omega \in \mathrm{Supp}(\mu)/u(x + t\omega') < u(x) + t\omega_3 \text{ for } t > 0\}$$

and $\mathrm{Supp}(\mu)$ denotes the support of μ.

If this condition holds, we replace (5) (see Example 3) by

$$\int_{S^2_+} (\omega_3 - \omega' \cdot \nabla u(x))^+ (1 + |\nabla u|^2)^{-1/2}\, d\mu(\omega) = I(x) \quad \text{in } \Omega\,. \tag{53}$$

Notice that (52) holds if u is *concave* but it can hold for more general shapes . . . Clearly, (53) is an equation of the form: $F(\nabla u) = I(x)$ in Ω where F is bounded, Lipschitz on $\mathbb{R}^2$ and given by

$$F(p) = (1 + |p|^2)^{-1/2} \left\{ \int_{S_+^2} (\omega_3 - \omega' \cdot p)^+ \, d\mu(\omega) \right\}. \tag{54}$$

Even if we add boundary conditions like the prescription of u on $\partial\Omega$ (say $u = 0$ on $\partial\Omega$), the uniqueness of viscosity (or C^1) solutions is not automatic. For general F, counterexamples exist (see [4] for instance) and the uniqueness is highly dependent on the geometry of F.

Let us begin our discussion with an easy case namely the geometrical situation where $\omega_3 - \omega' \cdot \nabla u(x) \geqq 0$ if $\omega \in \mathrm{Supp}(\mu)$, for all $x \in \Omega$ or in other words where every point of the surface $(x, u(x))$ is illuminated by any "source" ω (in the support of μ of course). The simplest way to formulate this assumption is to require

$$\forall x \in \Omega, \quad \forall \omega \in \mathrm{Supp}(\mu), \qquad u(y) \leqq u(x) + t\omega_3 \quad \text{if } y \in \Omega, \qquad y = x + t\omega', \quad t > 0\,. \tag{55}$$

In that case, (53) reduces to

$$(\bar{\omega}_3 - \bar{\omega}' \cdot \nabla u(x))(1 + |\nabla u(x)|^2)^{-1/2} = I(x) \quad \text{in } \Omega \tag{56}$$

and F can be replaced by $(\bar{\omega}_3 - \bar{\omega}' \cdot p)(1 + |p|^2)^{-1/2}$. Here and everywhere below, we denote by $\bar{\omega}_i = \int \omega_i d\mu(\omega)$ $(i = 1, 2, 3)$ and $\bar{\omega}' = (\bar{\omega}_1, \bar{\omega}_2)$. Clearly, the problem is thus equivalent to the problem involving a single point source of light located at $\hat{\omega} = \bar{\omega}/|\bar{\omega}|$, replacing I by $I/||\bar{\omega}|$. In that case, we do not need to make a new analysis and we only observe that the "nonuniqueness zones" i.e. the zones where $I = 1$ become the zones where $I = |\bar{\omega}|$ and that $I \leqq |\bar{\omega}|$ on Ω.

But, if we do not make the assumption (55), the analysis of Eq. (53) is much more delicate. We first want to show by a counterexample that some conditions on μ are necessary. To simplify the presentation, we present an example which is one-dimensional (i.e. where $\Omega = (-1, +1)$) and where we prescribe u on $\partial\Omega$ to be 0 for instance. In fact, the one-dimensional case corresponds to a two-dimensional surface u which is independent of x_2, illuminated by sources of light $\omega \in S_+^2$ such that $\omega_2 = 0$. Then we find on $(-1, +1)$

$$\int (\omega_3 - \omega_1 u')^+ \, d\mu(\omega)(1 + (u')^2)^{-1/2} = I\,. \tag{57}$$

Then, we are going to choose (for instance): $\mu = \frac{1}{2}\delta_{\omega^1} + \frac{1}{2}\delta_{\omega^2}$ where

$$\omega^1 = (\sqrt{1 - h^2}, 0, h) \qquad \text{and} \qquad \omega^2 = (-\sqrt{1 - h^2}, 0, h)$$

with $0 < h < 1$. Therefore, $F(u') = F(|u'|)$ and

$$F(t) = \begin{cases} h(1 + t^2)^{-1/2} & \text{if } 0 \leqq |t| \leqq h(1 - h^2)^{-1/2} \\ \dfrac{1}{2} \dfrac{h + \sqrt{1 - h^2}|t|}{\sqrt{1 + t^2}} & \text{if } |t| \geqq \dfrac{h}{\sqrt{1 - h^2}}\,. \end{cases}$$

We then pick $h \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$ so that $h < (1 - h^2)^{1/2} < 2h(1 - h^2)^{1/2}$. And we observe that F is decreasing for $t \in [0, h(1 - h^2)^{-1/2}]$, increasing for $t \in [h(1 - h^2)^{-1/2}, (1 - h^2)^{1/2}h^{-1}]$ and finally decreasing to $\frac{1}{2}(1 - h^2)^{1/2}$ on $[(1 - h^2)^{1/2}h^{-1}, +\infty)$.

Observing that

$$F(h(1-h^2)^{-1/2}) = h(1-h^2)^{1/2} > \tfrac{1}{2}(1-h^2)^{1/2} ,$$

we deduce that there exists $\beta > (1-h^2)^{1/2}h^{-1}$ such that $F(\beta) = h(1-h^2)^{1/2}$. We then set $I \equiv h(1-h^2)^{1/2}$ (notice that $I < h = |\bar{\omega}|$) and $u_1(x) = -\alpha[|x|-1]$, $u_2(x) = -\beta[|x|-1]$ on $[-1,+1]$ where $\alpha = h(1-h^2)^{-1/2} < \beta$ (so that $u_1 \not\equiv u_2$). These choices show a new nonuniqueness phenomenon, as we claimed above. Indeed, we claim that u_1 and u_2 are viscosity solutions of: $F(u') = I$ in $(-1,+1)$ while of course $u_1(\pm 1) = u_2(\pm 1) = 0$. This claim is easy to check observing that $F(\alpha) = F(\beta) = I$ and $F(t) \geqq I$ if $|t| \leqq \alpha$ or if $|t| \leqq \beta$.

We next present a uniqueness result for Eq. (53). We first observe that necessarily we have $I \leqq |\bar{\omega}|$ and, exactly as in the case mentioned above, multiple solutions are to be expected if $\{I = |\bar{\omega}|\}$ is composed of at least two connected components. This is why, as we did all throughout this paper (essentially), we assume that $I < |\bar{\omega}|$ on $\bar{\Omega} - \{\bar{x}\}$ and $I(\bar{x}) = |\bar{\omega}|$ for some $\bar{x} \in \Omega$.

As we just saw above, we need an assumption on F (and this means in some undirect way an assumption on μ)

(58)
$$\forall \theta \in (0,1), \quad \exists \nu = \nu(\theta) \in (0,1) \quad \text{such that } \forall p \in \mathbb{R}^2 \quad \text{with } F(p) > 0 ,$$
$$F\left(\theta p - (1-\theta)\frac{\bar{\omega}'}{\bar{\omega}_3}\right) \geqq F(p) + \nu(\theta)(|\bar{\omega}| - F(p))\left(1 + \left|\theta p - (1-\theta)\frac{\bar{\omega}'}{\bar{\omega}_3}\right|^2\right)^{-1/2}$$

(recall that $0 \leqq F(p) \leqq |\bar{\omega}|$ for all $p \in \mathbb{R}^2$).

With this assumption, that we shall discuss below, we can prove by the same arguments as the ones we used several times in this paper the following

Theorem 5. *Let $u, v \in C(\bar{\Omega})$ be viscosity solutions in Ω and subsolution in $\bar{\Omega}$ of* (53). *Then, with the above assumptions, $u - v$ is constant on $\bar{\Omega}$.*

Sketch of proof. Let $G(p) = \int_{S^2_+} (\omega_3 + \omega' \cdot p)^+ \, d\mu(\omega)$. Then, $w = -u$ and $z = -v$ are viscosity solutions in Ω and supersolutions in $\bar{\Omega}$ of

$$I\sqrt{1 + |\nabla \Psi|^2} - G(\nabla \Psi) = 0 . \tag{59}$$

In addition, $\tilde{w} = \theta w + (1-\theta)\dfrac{\bar{\omega}'}{\bar{\omega}_3} \cdot (x - \bar{x})$ is a viscosity subsolution of

$$I\sqrt{1 + |\nabla \tilde{w}|^2} - G(\nabla \tilde{w}) \leqq -\nu(\theta)(|\bar{\omega}| - I) . \tag{60}$$

We may then conclude as we did several times before. □

We conclude this section with a few examples. Of course, the assumption (58) holds in the case of a single light source since we remark that $F(p) > 0$ implies that

$$F(p) = \frac{\bar{\omega}_3 - \bar{\omega}' \cdot p}{(1 + |p|^2)^{1/2}} \qquad \text{and} \qquad \bar{\omega} = \omega .$$

And we are back to the situation we studied in the preceding sections, where, in fact, we use (58) that holds with $\nu(\theta) = \min(1, (1-\theta)\bar{\omega}/\bar{\omega}_3)$.

In order to present really new examples, we consider only the case of a "symmetric" measure μ. More precisely, we assume that μ is invariant under all rotations around the $(0,0,1)$ axis. In that case, we immediately observe that $\bar{\omega}=(0,0,\bar{\omega}_3)$ and that $F(p)$ is spherically symmetric on $\mathbb{R}^2$. We first claim that (58) holds if

$$\operatorname{Supp}(\mu) \subset \left\{ \omega \in S^2_+ / \omega_3 \geqq \frac{1}{\sqrt{2}} \right\}.$$

Indeed, we observe that

$$F(p) = \frac{1}{2(1+|p|^2)^{1/2}} \int (\omega_3 - |p|\omega_1)^+ + (\omega_3 + |p|\omega_1)^+ \, d\mu(\omega).$$

Then, by the analysis made in the one-dimensional counterexample above, we deduce that for each $\omega \in \operatorname{Supp}(\mu)$, since $\omega_3 \geqq |\omega_1|$, $\frac{1}{2}(1+|p|^2)^{-1/2}\{(\omega_3 - \omega_1|p|)^+ + (\omega_3+\omega_1|p|)^+\}$ is strictly decreasing with respect to $|p|$ (and goes to $|\omega_1|$ as $|p|$ goes to $+\infty$). From this fact, it is straightforward to deduce that (58) holds.

Another example is the case when μ is the uniform probability measure on S^2_+. In that case, we find using spherical coordinates: $\omega_2 = \sin\varphi \left(\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right)$, $\omega_1 = \cos\varphi \sin\theta$, $\omega_3 = \cos\varphi\cos\theta \left(\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right)$ on S^2_+.

$$\begin{aligned} F(p) &= \frac{1}{2\pi} \int_{-\pi/2}^{+\pi/2} \frac{(\cos\theta - \sin\theta|p|)^+}{\sqrt{1+|p|^2}} \, d\theta \int_{-\pi/2}^{+\pi/2} \cos^2\varphi \, d\varphi \\ &= \frac{1}{4} \int_{-\pi/2}^{+\pi/2} (\cos(\theta+\theta_0))^+ \, d\theta \end{aligned}$$

where $\theta_0 \in \left[0, \frac{\pi}{2}\right]$ is defined by $\cos\theta_0 = \dfrac{1}{\sqrt{1+|p|^2}}$, $\sin\theta_0 = \dfrac{|p|}{\sqrt{1+|p|^2}}$.

Therefore, we deduce

(61) $$F(p) = \frac{1}{4} \int_{-\pi/2}^{(\pi/2)-\theta_0} \cos(\theta+\theta_0)\, d\theta$$

(62) $$= \frac{1}{4}\left(1 - \sin\left(-\frac{\pi}{2} + \theta_0\right)\right) = \frac{1}{4}\left(1 + \frac{1}{\sqrt{1+|p|^2}}\right).$$

In that case, we have $|\bar{\omega}| = \frac{1}{2}$ and we check easily that (58) holds since we have, for all $\theta \in (0,1)$,

$$F(\theta p) \geqq F(p) + \frac{1-\theta^2}{2}\left(\tfrac{1}{2} - F(p)\right)(1+|\theta p|^2)^{-1/2}$$

for all $p \in \mathbb{R}^2$.

6. Numerical experiments

We present in this section various numerical experiments illustrating the uniqueness results and the characterization of smooth solutions obtained in Rouy and

Tourin [19] and in this paper. We shall not detail the algorithms and the discretization we used since they are variants and adaptations of the ones introduced in [19]. Let us only recall that they rely upon either optimal control considerations or the theory of monotone schemes. And we only had to adapt the

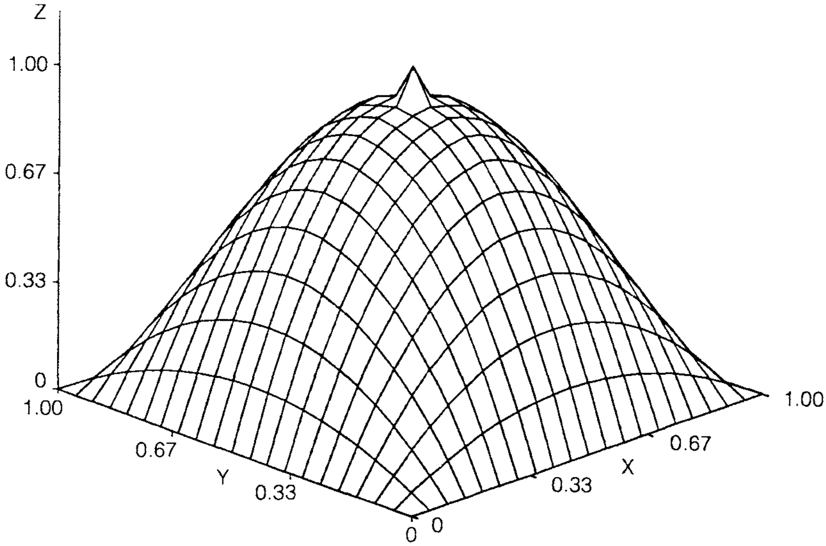

Fig. 15. 21 ×21 points, $\varepsilon = 2.9\ 10^{-2}$, $n = 27$.

$$I = \frac{1}{\sqrt{1 + (16y(1-y)(1-2x))^2 + (16x(1-x)(1-2y))^2}} \qquad u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 1$$

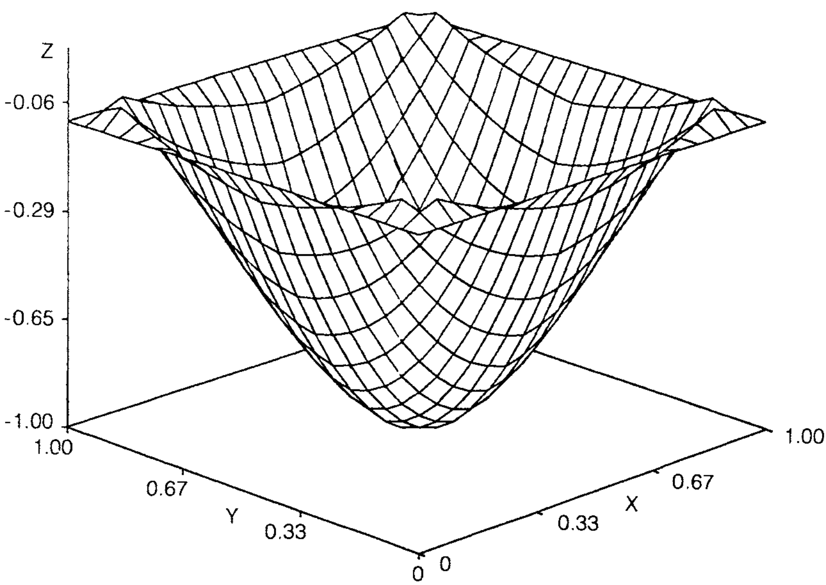

Fig. 16. 21 ×21 points, $\varepsilon = 6.9\ 10^{-2}$, $n = 19$. I as in. Fig. 15. $\quad u(\frac{1}{2}, \frac{1}{2}) = -1$

method of [19] in order to incorporate the various boundary conditions described here. Let us also mention that we are using the acceleration method proposed by Osher and Rouy [13] (as in [14, 19]).

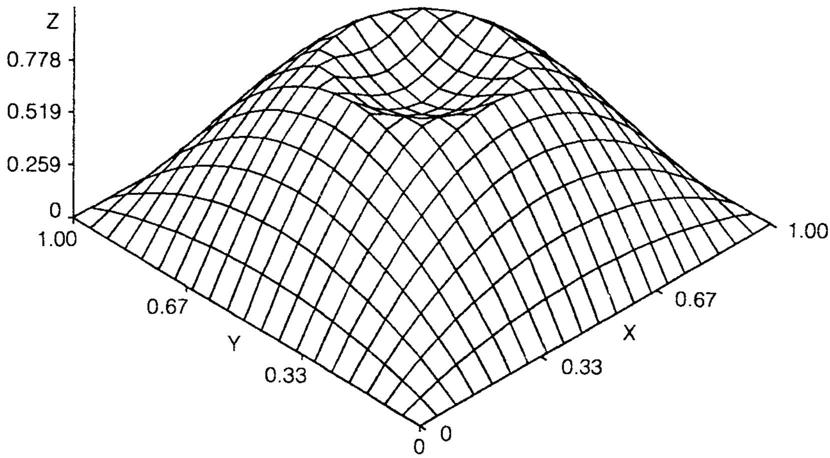

Fig. 17. 21 ×21 points, $\varepsilon = 4.6\ 10^{-2}$, $n = 14$. I as in Fig. 15. $u\left(\frac{1}{2},\frac{1}{2}\right) = \frac{47}{81}$

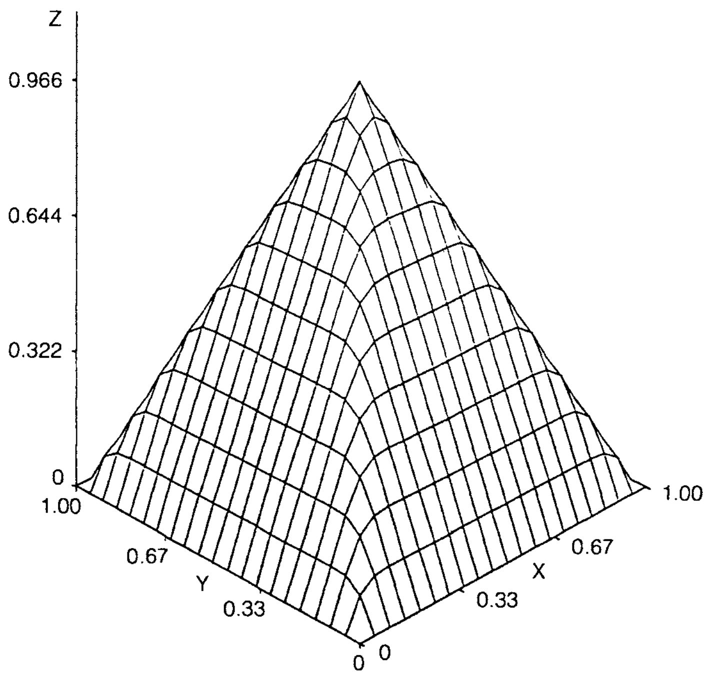

Fig. 18. 21 ×21 points, $\varepsilon = 1.1\ 10^{-2}$, $n = 10$. $I = \frac{1}{\sqrt{5}}$

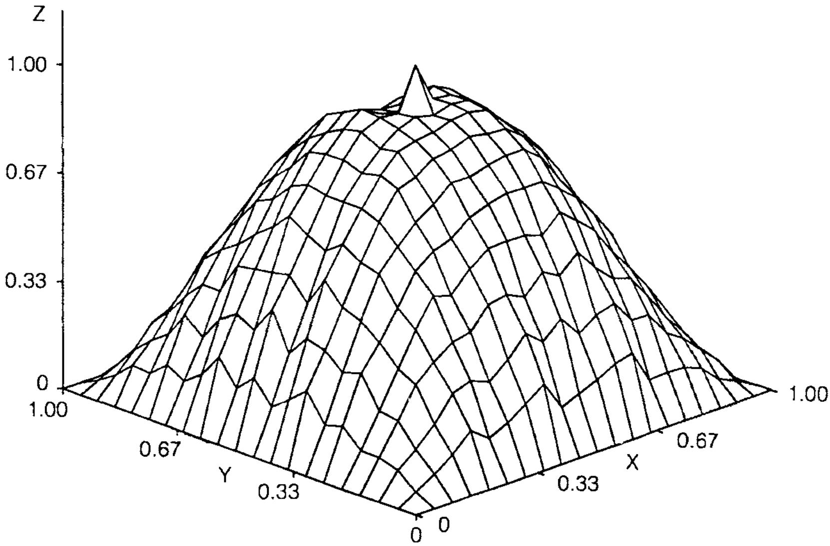

Fig. 19. Uniform noise on I: 10%, $\varepsilon = 3.1\ 10^{-2}$, $n = 43$. I and $u(\frac{1}{2}, \frac{1}{2})$ as in Fig. 15

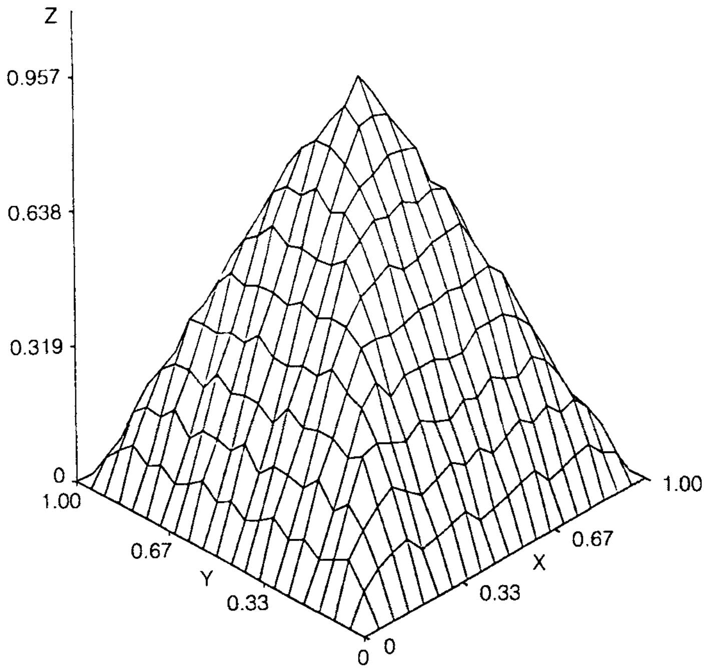

Fig. 20. Uniform noise on I: 10%, $\varepsilon = 1.2\ 10^{-2}$, $n = 14$. I as in Fig. 18

We next describe the experiments presented below. Each time we give the final numerical result and we indicate the number of grid points, the numerical error (L^1 error) and the number of iterations. Also, we give the exact value of I.

6.1 Figures 15–17. We prescribed here $u = 0$ on $\partial\Omega$ and as explained in [19] and recalled here, we expect (at most) two C^1 solutions since I achieves the value 1 only at one point $\bar{x}$ and a continuum of viscosity solutions parametrized by their value at the point $\bar{x}$. The C^1 solutions are respectively the maximum and the minimum solutions. On those three experiments, we indicate the prescription of the solution at $\bar{x}$.

6.2 Figure 18. This experiment illustrates the reconstruction of a shape having one singularity but where I does not achieve the value 1 so that we obtain a simple uniqueness result prescribing for instance that $\partial\Omega$ is a level set of u.

6.3 Figures 19 and 20. These experiments show that our reconstruction is robust since I has been perturbed by a significant amount of noise (about 10%) and the essential shape is recovered.

6.4 Figures 21 and 22. In those two experiments, $\partial\Omega$ is a level set of u and I achieves the value 1 at 5 points. Among all solutions, C^1 solutions are extremal and those experiments show the reconstruction of all four of them (the last two are obtained by changing u into $-u$).

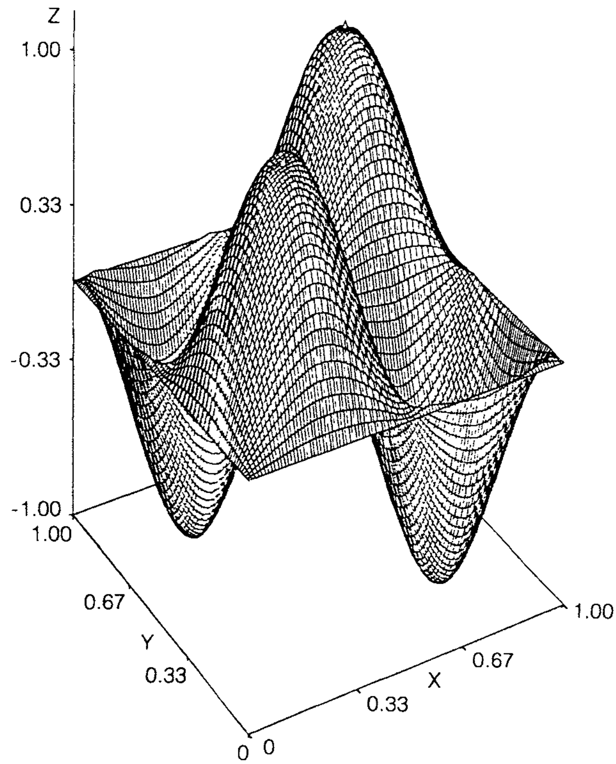

Fig. 21. 101 ×101 points, $\varepsilon = 2.6\ 10^{-2}$, $n = 255$.

$$I = \frac{1}{\sqrt{1 + (2\pi \sin(2\pi y)\cos(2\pi x))^2 + (2\pi \sin(2\pi x)\cos(2\pi y))^2}}$$

$$u(\tfrac{1}{4}, \tfrac{1}{4}) = u(\tfrac{3}{4}, \tfrac{3}{4}) = 1, \quad u(\tfrac{1}{4}, \tfrac{3}{4}) = u(\tfrac{3}{4}, \tfrac{1}{4}) = -1, \quad u(\tfrac{1}{2}, \tfrac{1}{2}) = 0$$

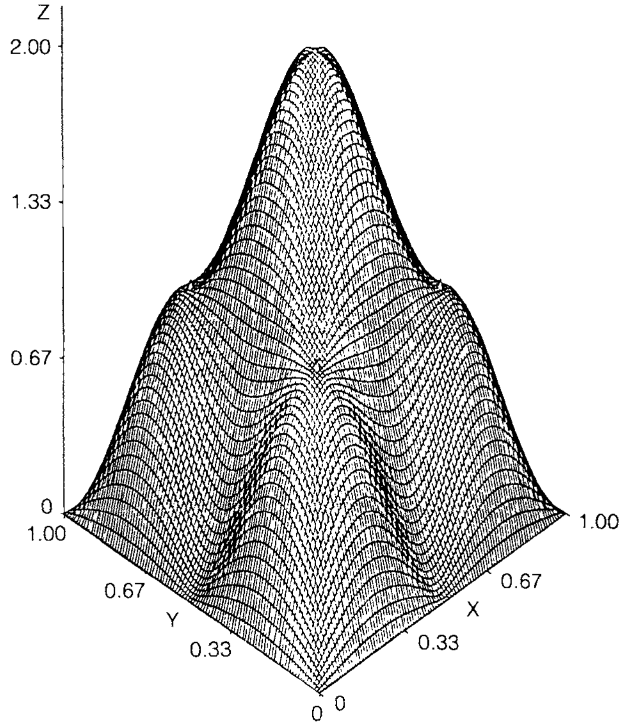

Fig. 22. 101 ×101 points, $n = 256$. I as in Fig. 21, maximal solution

$$u(\tfrac{1}{4}, \tfrac{1}{4}) = u(\tfrac{3}{4}, \tfrac{3}{4}) = u(\tfrac{1}{4}, \tfrac{3}{4}) = u(\tfrac{3}{4}, \tfrac{1}{4}) = 1, \qquad u(\tfrac{1}{2}, \tfrac{1}{2}) = 2$$

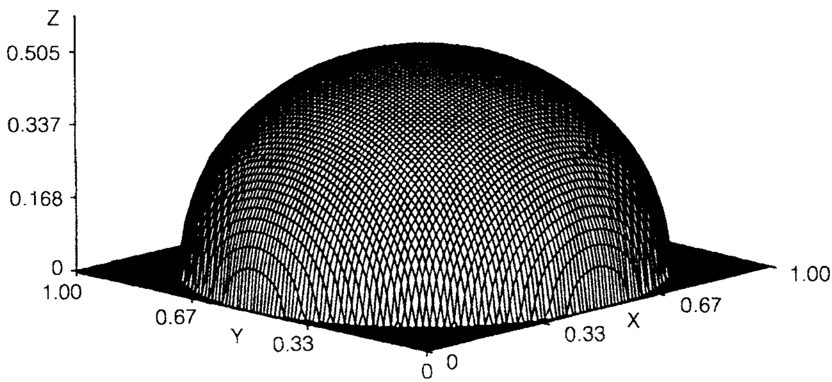

Fig. 23. 101 ×101 points, $\varepsilon = 3.1\ 10^{-3}$, $n = 166$. $I = 2\sqrt{\frac{1}{4} - (x - \frac{1}{2})^2 - (y - \frac{1}{2})^2}$

6.5 Figure 23. This experiment shows the reconstruction of a simple shape illuminated by a vertical light with a boundary condition corresponding to the projection of an apparent contour.

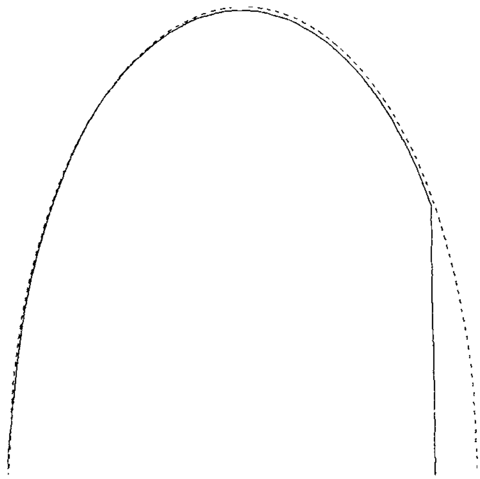

Fig. 24. 101 points, $\alpha = 0.6$, $\gamma = 0.8$, $\varepsilon = 9.7\ 10^{-2}$, $n = 4$.

$$I = -\tfrac{6}{5}(x - \tfrac{1}{2}) + \tfrac{8}{5}\sqrt{\tfrac{1}{4} - (x - \tfrac{1}{2})^2} \quad \text{if } x \leqq \tfrac{9}{10},\ I = 0 \text{ otherwise}$$

6.6 Figure 24. In a one-dimensional setting, we illustrate the recovery of a shape which is bounded on one side by an apparent contour and on the other side by a grazing light edge.

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