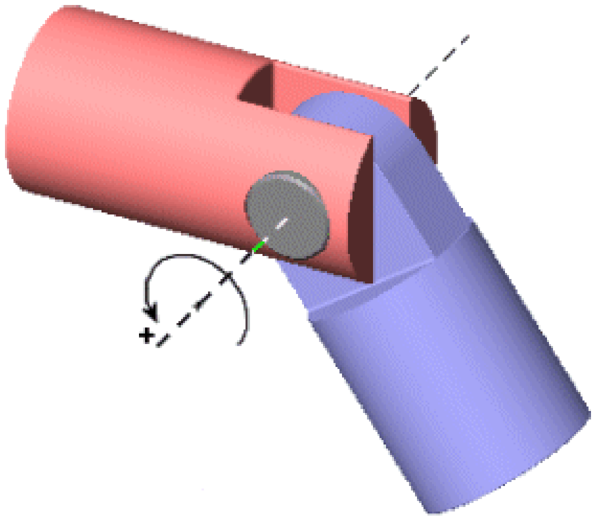
CS 3630

Robot Kinematics:

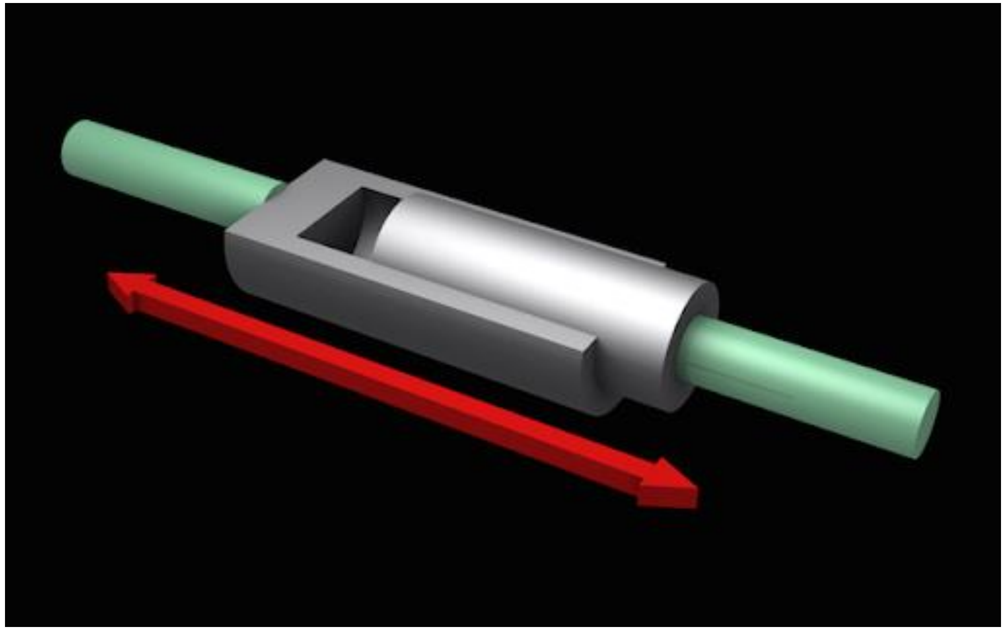
Planar Arms

Robot Arms

- A robot arm (aka serial link manipulator) consists of a series of rigid links, connected by joints (motors), each of which has a single degree of freedom.
  - Revolute Joint: Single degree of freedom is rotation about an axis.
  - Prismatic joint: Single degree of freedom is translation along an axis.



Revolute Joint

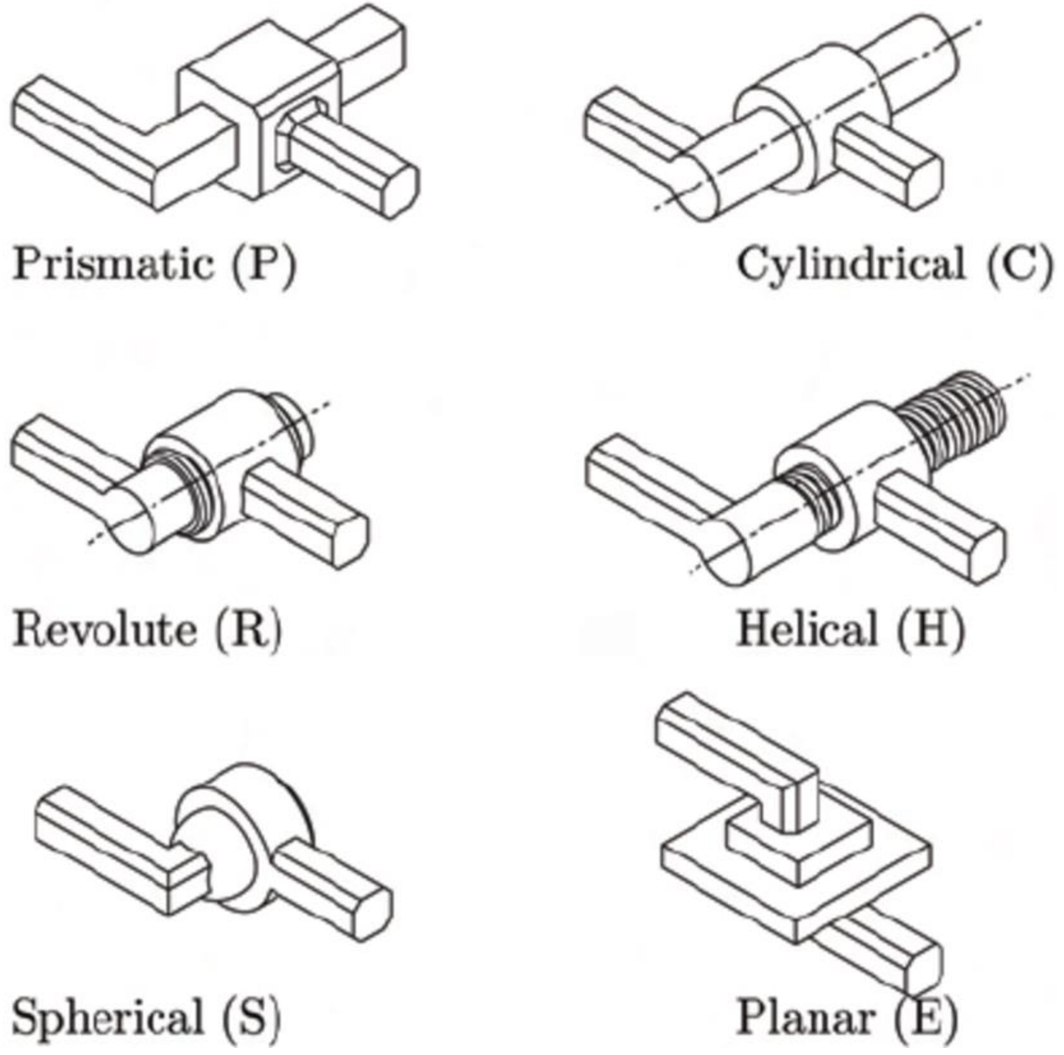


Prismatic Joint

Other Types of Joints

There are several types of joint that have more than one degree of freedom – but we do not consider those in this class.

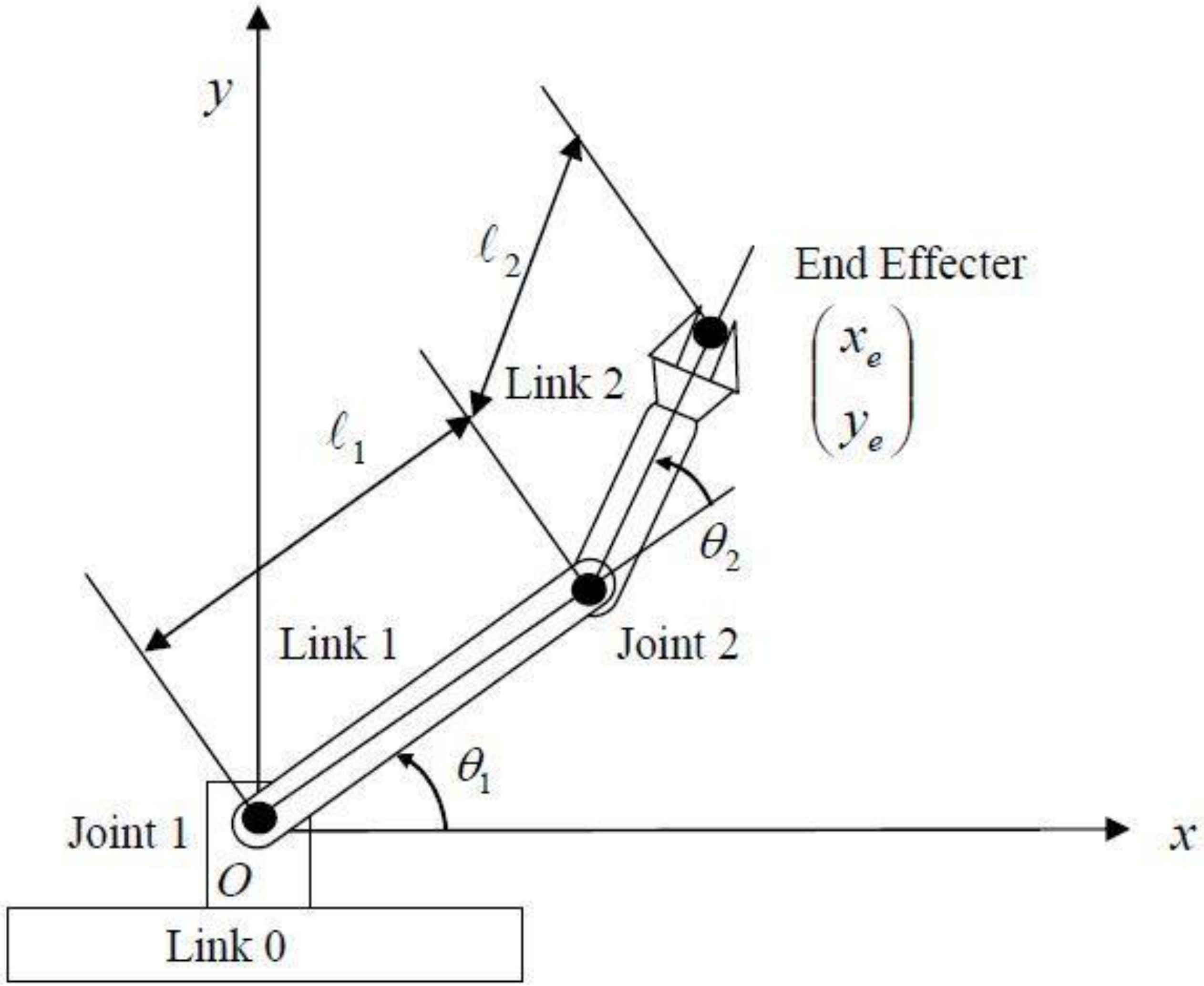
In fact, all of the higher degree-of-freedom joints can be described by combinations of one degree-of-freedom joints, so there is no need to explicitly consider these.



Describing Serial Link Arms

- Number the links in sequence.
- For a robot with n joints:
  - Base (which does not move) is Link 0.
  - End-effector (tool) is attached to Link n.
  - Joint i connects Link $i-1$ to Link i
  - We define the joint variable q_i for joint i as:

$$q_i = \begin{cases} \theta_i \text{ if joint } i \text{ is revolute} \\ d_i \text{ if joint } i \text{ is prismatic} \end{cases}$$



Two-link Planar Arm:

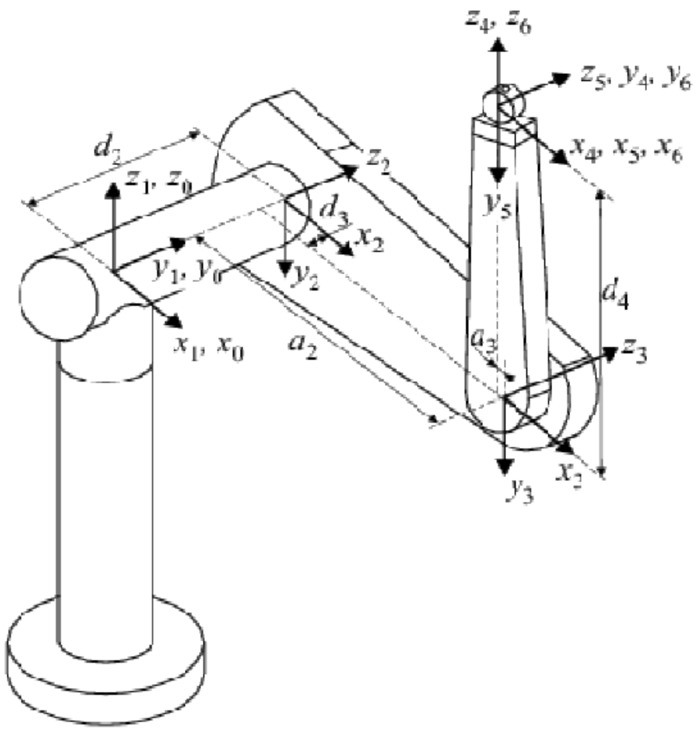
- $n = 2$,
- both links are always coplanar (no rotation out of the plane).
- $q_1 = \theta_1, \ q_2 = \theta_2$

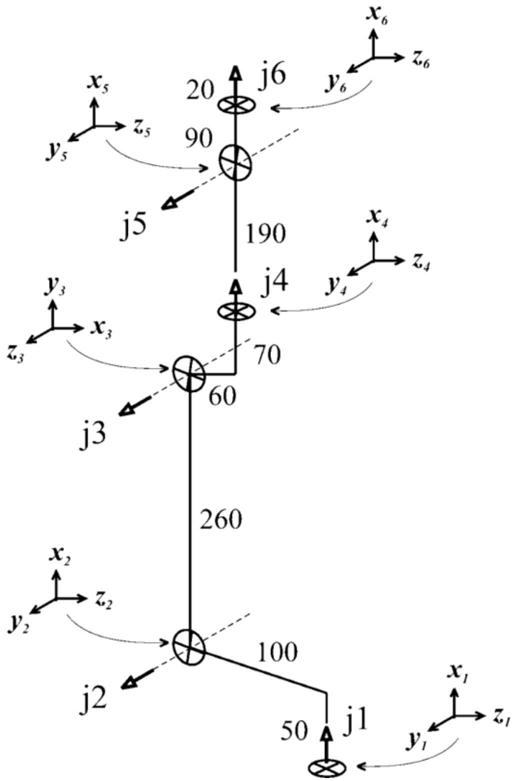
Manipulator Kinematics

- Kinematics describes the position and motion of a robot, without considering the forces required to cause the motion.
- Forward Kinematics: *Given the value for each joint variable, q_i, determine the position and orientation of the end-effector (gripper, tool) frame.*

<u>The basic idea:</u>

➢ Assign lots of coordinate frames, and express these frames in terms of the joint variables, q_i.





General Approach

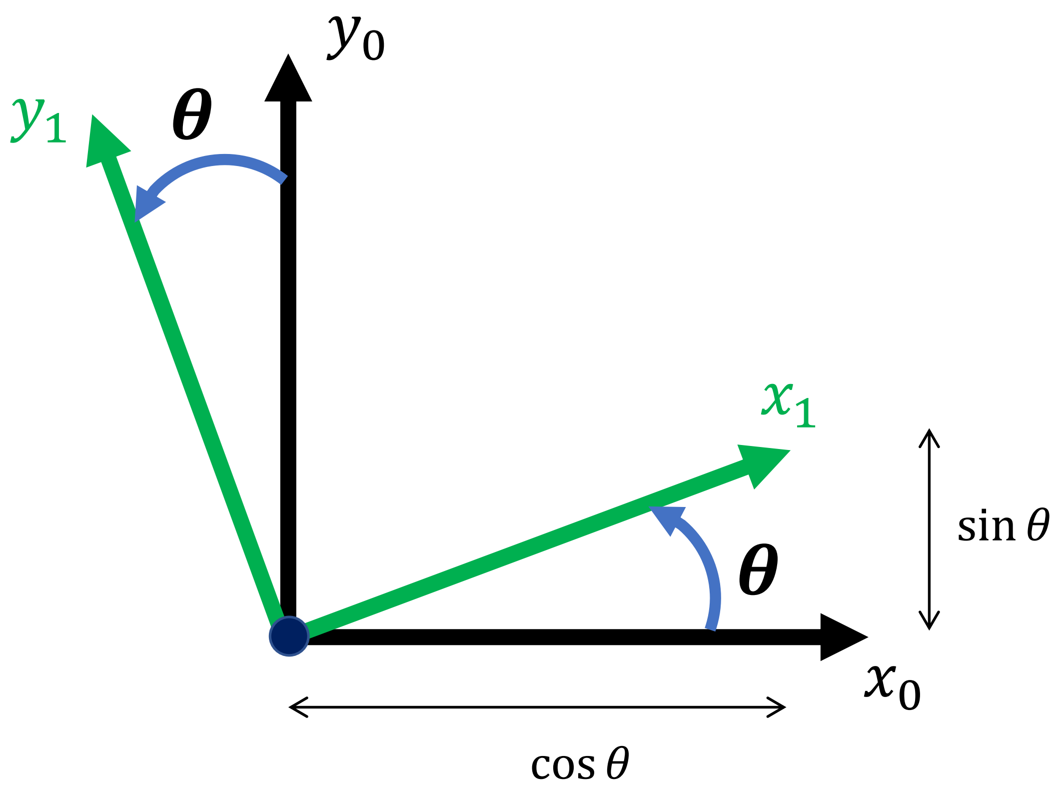
- Each link is a rigid body.
- We know how to describe the position and orientation of a rigid body:
  - Attach a coordinate frame to the body.
  - Specify the position and orientation of the coordinate frame relative to some reference frame.
- If two links, say link $i-1$ and link i are connected by a single joint, then the relationship between the two frames can be described by a homogeneous transformation matrix T_i^{i-1} which *will depend only on the value of the joint variable*!

➢ ***Let's have a quick review of Homogeneous Transformations….***

Specifying Orientation in the Plane

Given two coordinate frames with a common origin, we describe the orientation of Frame 1 w.r.t. Frame 0 by:

Specifying the directions of x_1 and y_1 w.r.t. Frame 0 by projecting onto x_0 and y_0.



$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Notation: x_1^0 denotes the x-axis of Frame 1, specified w.r.t Frame 0.

$$y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

We obtain y_1^0 in the same way.

Rotation Matrices (rotation in the plane)

We combine these two vectors to obtain a ***rotation matrix***: $R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

All rotation matrices have certain properties:

1. The two columns are each unit vectors.
2. The two columns are orthogonal, i.e., $c_1 \cdot c_2 = 0$.
3. $\det R = +1$

$\boxed{\textit{For such matrices } R^{-1} = R^T}$

- The first two properties imply that the matrix R is ***orthogonal***.
- The third property implies that the matrix is ***special***! (After all, there are plenty of orthogonal matrices whose determinant is -1, not at all special.)

The collection of 2×2 rotation matrices is called the ***Special Orthogonal Group of order 2***, or, more commonly $\boldsymbol{SO(2)}$.

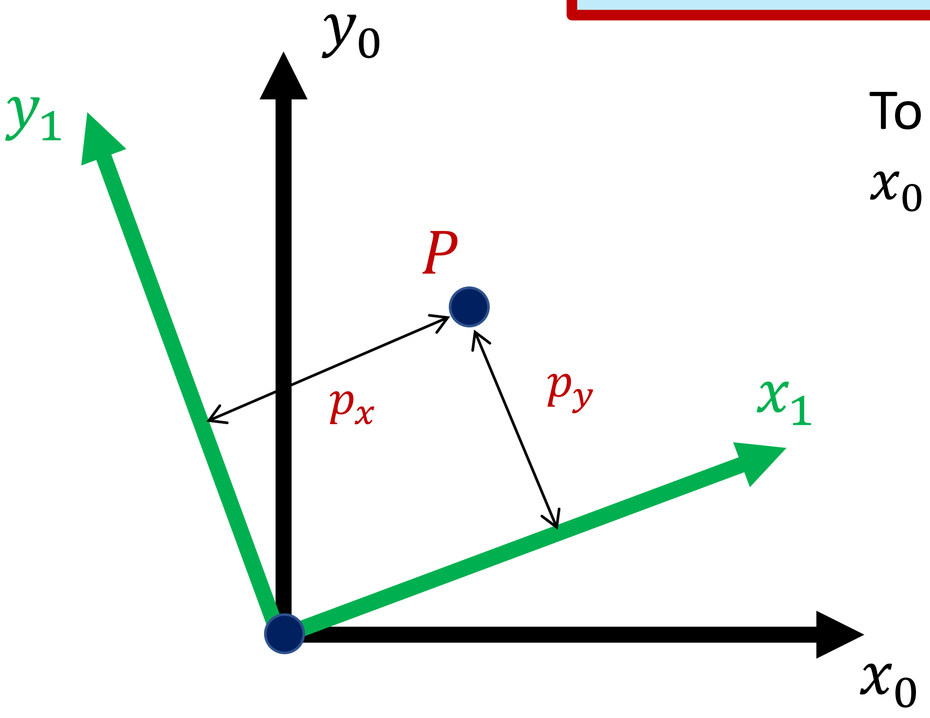
This concept generalizes to $\boldsymbol{SO(n)}$ for $n \times n$ rotation matrices.

Coordinate Transformations (rotation only)

Suppose a point P is rigidly attached to coordinate Frame 1, with coordinates given by $P^1 = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$.

We can express the location of the point P in terms of its coordinates

$$P = p_x x_1 + p_y y_1$$



To obtain the coordinates of P w.r.t. Frame 0, we project P onto the x_0 and y_0 axes:

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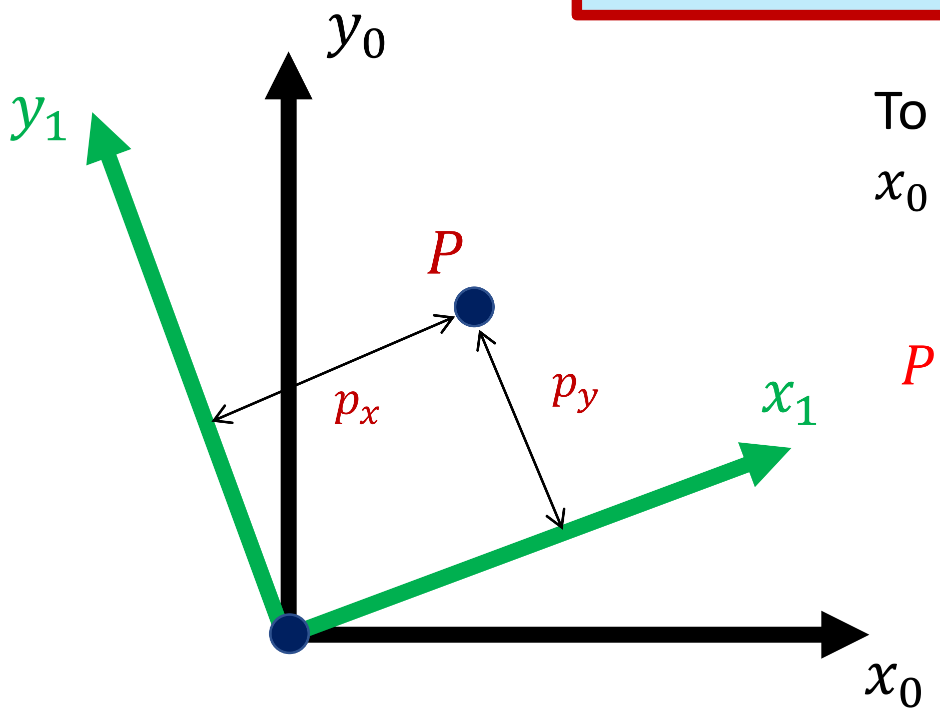

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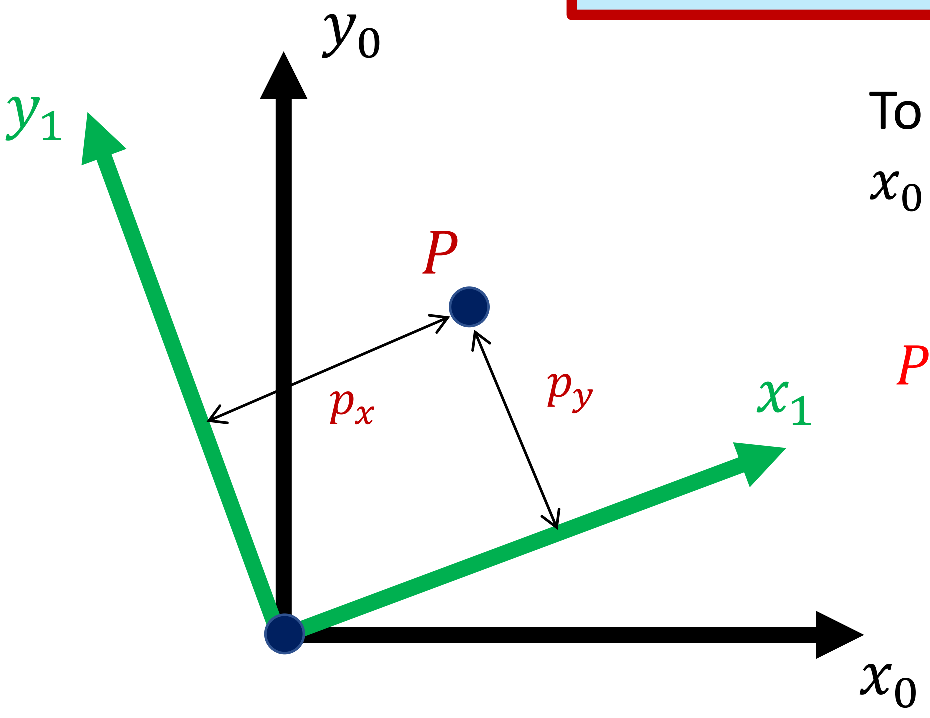

To obtain the coordinates of P w.r.t. Frame 0, we project P onto the x_0 and y_0 axes:

$$P^0 = \begin{bmatrix} P \cdot x_0 \\ P \cdot y_0 \end{bmatrix} = \begin{bmatrix} (p_x x_1 + p_y y_1) \cdot x_0 \\ (p_x x_1 + p_y y_1) \cdot y_0 \end{bmatrix} =$$

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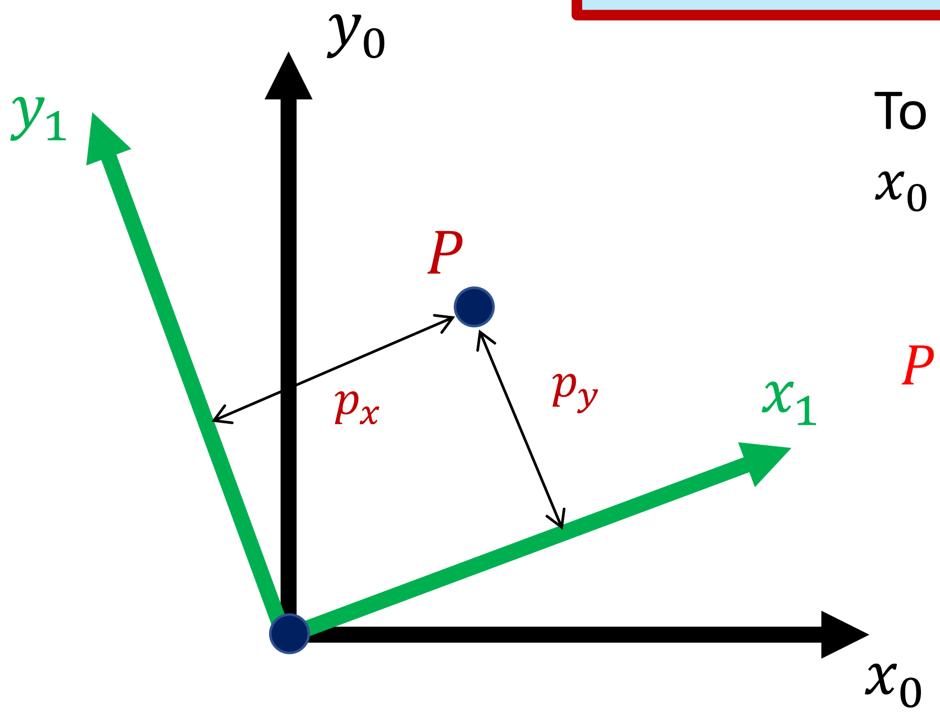

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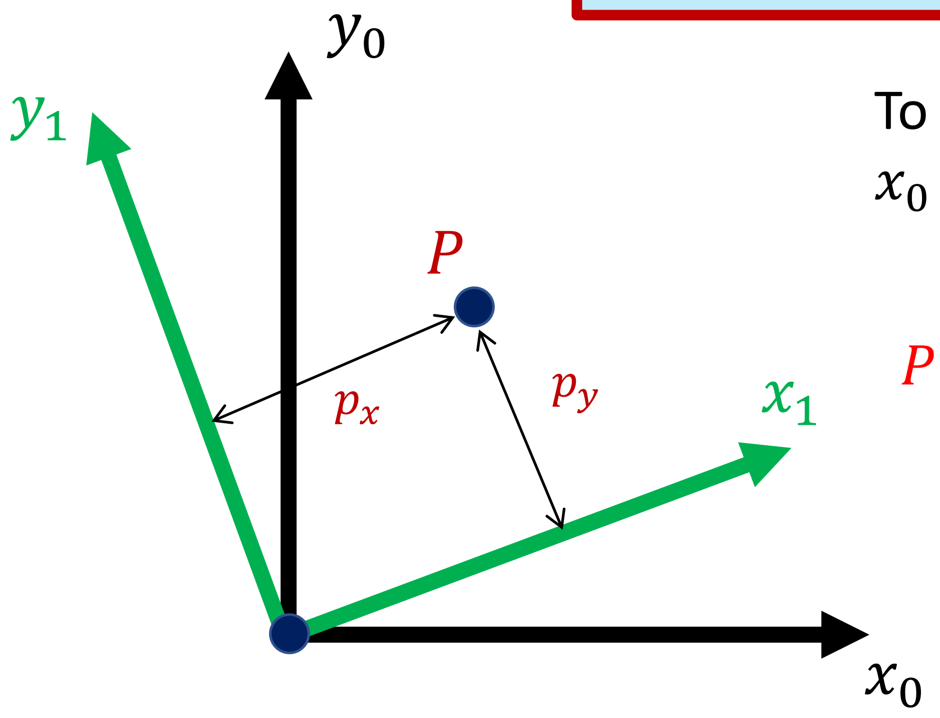

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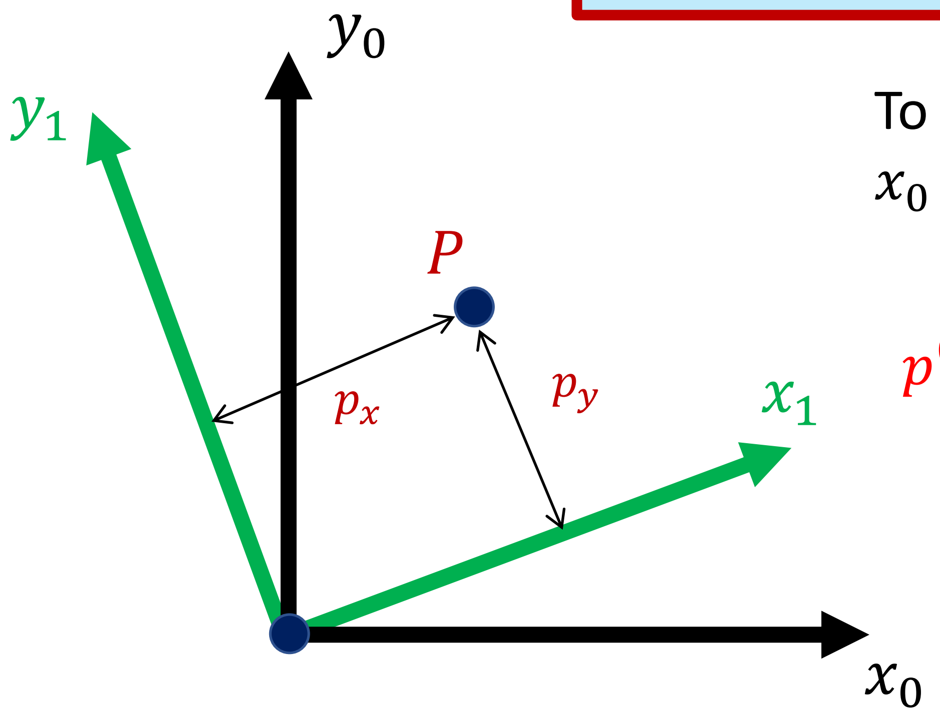

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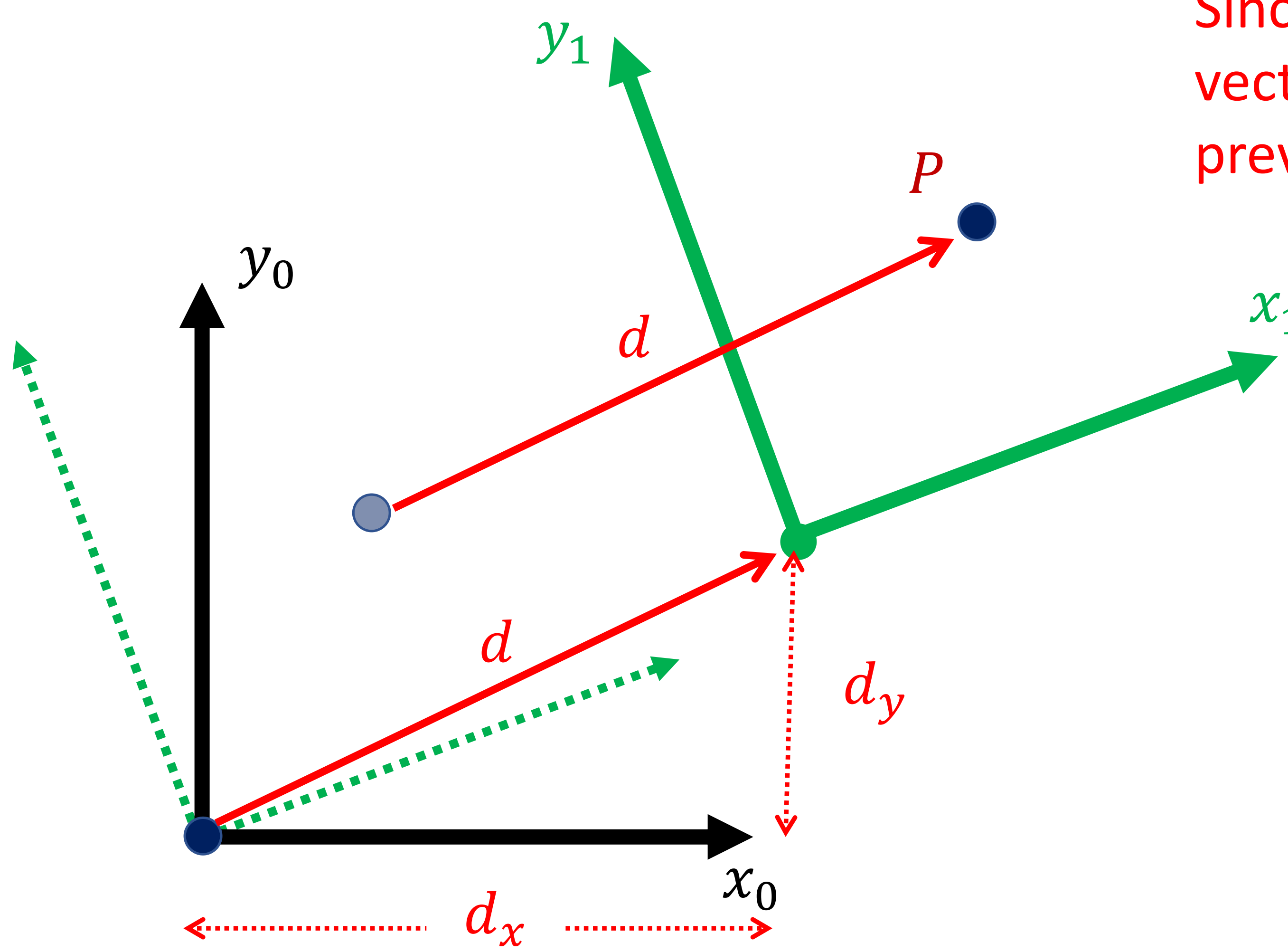
$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \boldsymbol{R_1^0}\, \boldsymbol{P^1}$$

$$\boxed{\boldsymbol{P^0} = \boldsymbol{R_1^0}\, \boldsymbol{P^1}}$$

Specifying Pose in the Plane

Suppose we now translate Frame 1 (*no new rotatation*).
What are the coordinates of P w.r.t. Frame 0?



Since we merely translated P by a fixed vector d, simply add the offset to our previous result!

$$\boldsymbol{P^0} = \boldsymbol{R_1^0}\,\boldsymbol{P^1} + \boldsymbol{d^0}$$

$$\boldsymbol{d^0} = \begin{bmatrix} \boldsymbol{d_x} \\ \boldsymbol{d_y} \end{bmatrix}$$

Homogeneous Transformations

We can simplify the equation for coordinate transformations by augmenting the vectors and matrices with an extra row:

This is just our eqn from the previous page

$$\boxed{\begin{bmatrix} \boldsymbol{P}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 \boldsymbol{P}^1 + \boldsymbol{d}^0 \\ 1 \end{bmatrix}} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{d}^0 \\ 0_2 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^1 \\ 1 \end{bmatrix}$$

in which $0_2 = [0 \quad 0]$

The set of matrices of the form $\begin{bmatrix} R & d \\ 0_n & 1 \end{bmatrix}$, where $R \in SO(n)$ and $d \in \mathbb{R}^n$ is called the ***Special Euclidean Group of order n***, or $SE(n)$.

Homogeneous Transformations

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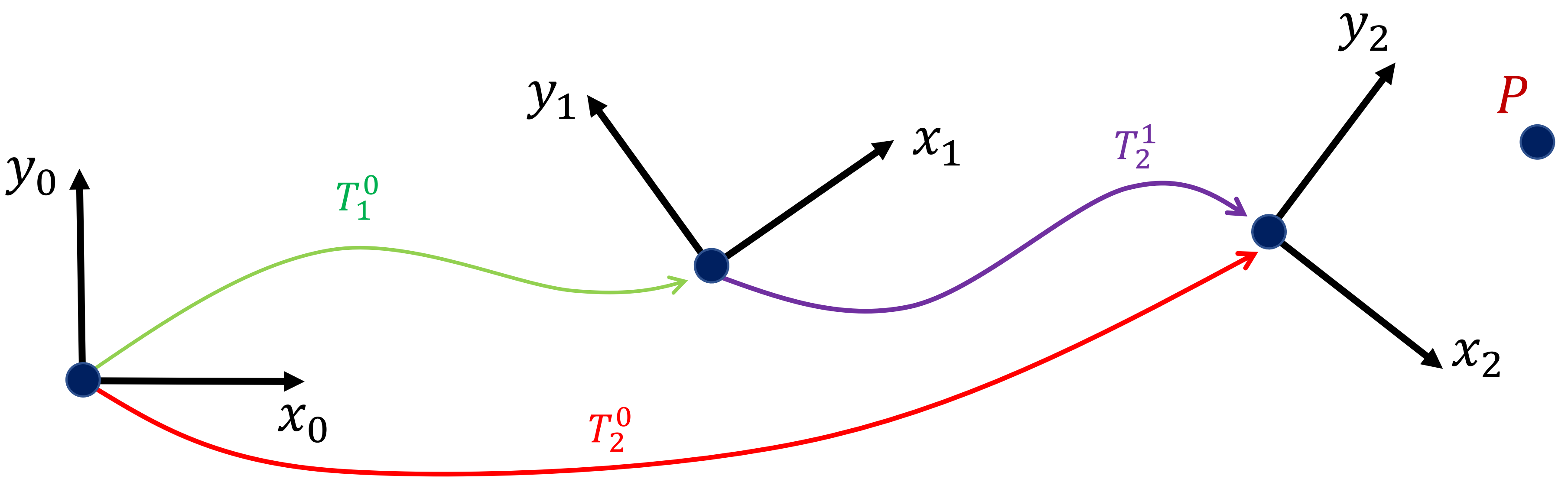
$$\begin{bmatrix} \boldsymbol{P}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 \boldsymbol{P}^1 + \boldsymbol{d}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{d}^0 \\ 0_2 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^1 \\ 1 \end{bmatrix}$$

$$\tilde{P}^0 = \begin{bmatrix} \boldsymbol{P}^0 \\ 1 \end{bmatrix}, \tilde{P}^1 = \begin{bmatrix} \boldsymbol{P}^1 \\ 1 \end{bmatrix}$$

$$\tilde{P}^0 = T_1^0 \tilde{P}^1$$

- **$\mathbf{T}_1^0$ is called a homogeneous transformation matrix**
- **$\widetilde{\mathbf{P}}^0$ are the homogeneous coordinates for $\mathbf{P}^0$**

Composition of Transformations



From our previous results, we know:

$$\tilde{P}^0 = T_1^0 \tilde{P}^1$$
$$\tilde{P}^1 = T_2^1 \tilde{P}^2$$

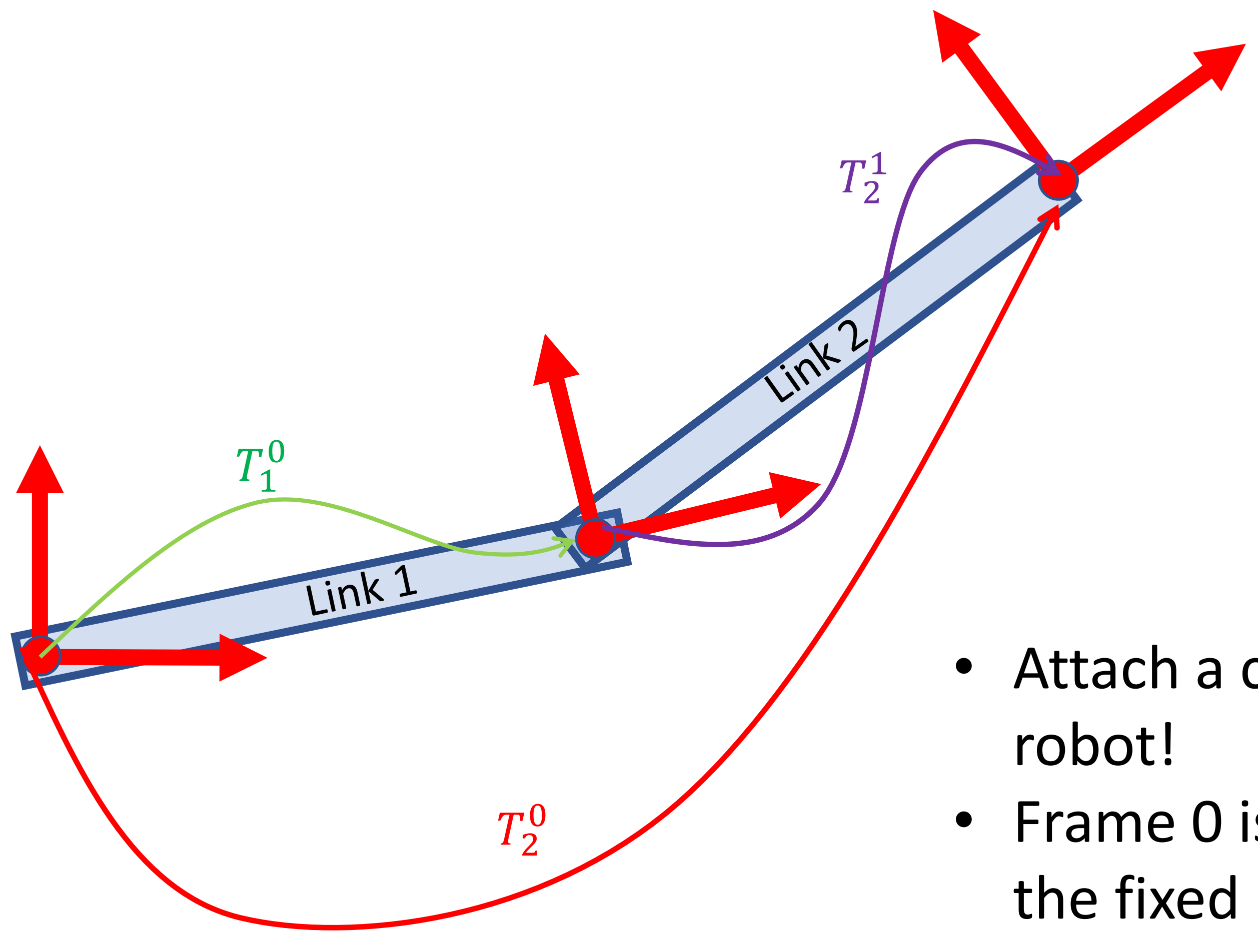
$$\tilde{P}^0 = T_1^0 T_2^1 \tilde{P}^2$$

But we also know: $\tilde{P}^0 = T_2^0 \tilde{P}^2$

$$T_2^0 = T_1^0 T_2^1$$

This is the composition law for homogeneous transformations.

What about robot arms??

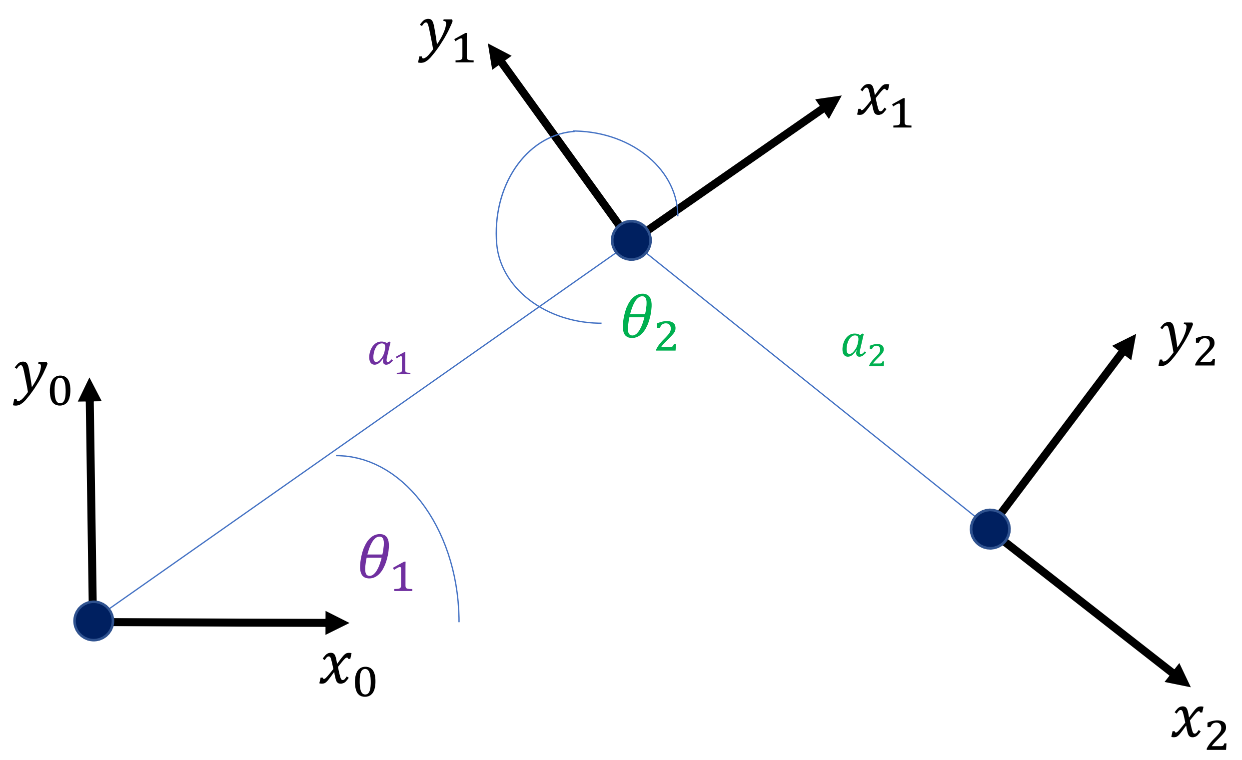


- Attach a coordinate frame to each link of the robot!
- Frame 0 is attached to Link 0, which is merely the fixed mounting point to the environment.
- Now, the trick is to express T_i^{i-1} as a function of θ_i

A special case

Suppose the axis x_i is collinear with the origin of Frame $i-1$:

- x_1 is collinear with the origin of Frame 0
- x_2 is collinear with the origin of Frame 1



$$T_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & a_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & a_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Use this to simplify link coordinate frames!

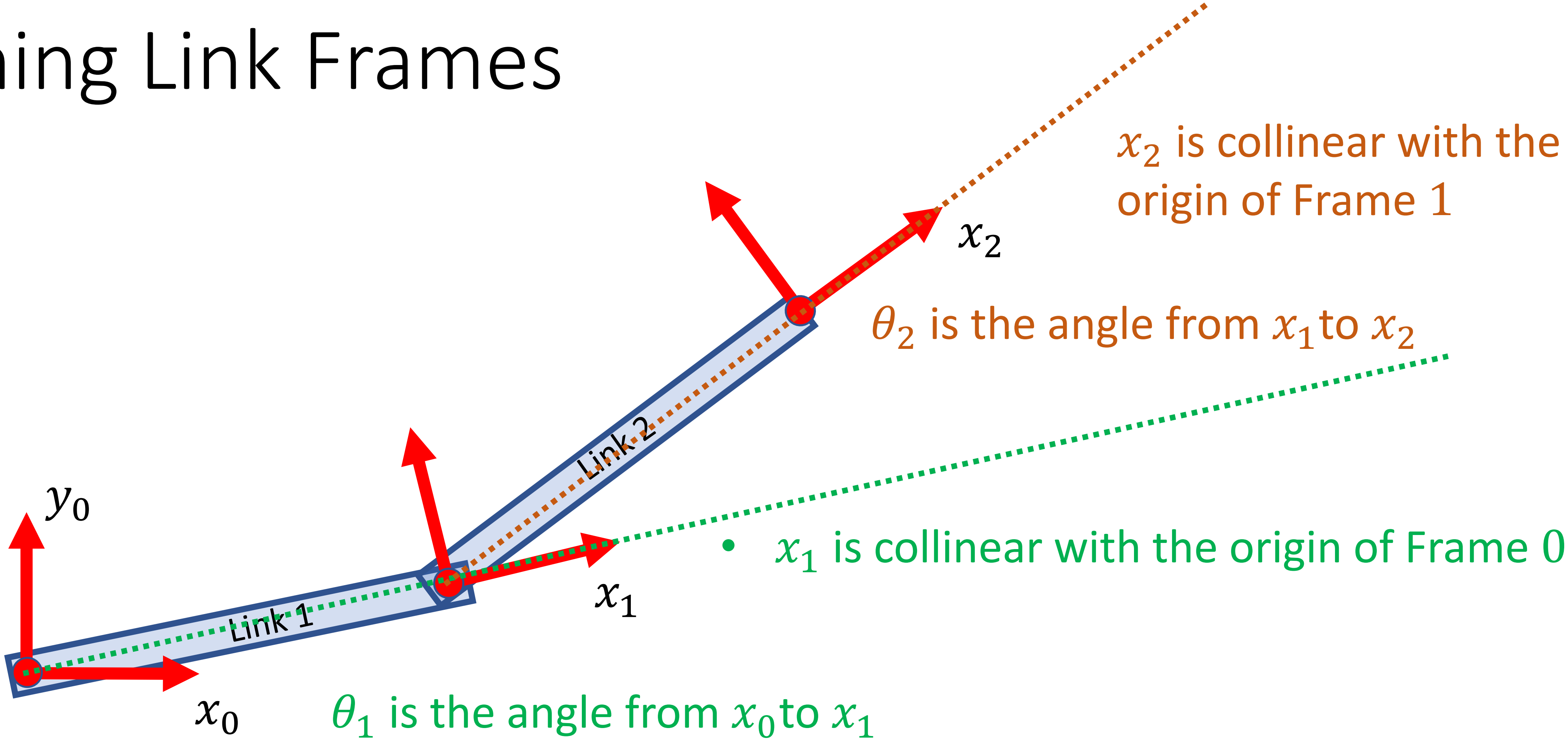
$$T_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & a_i\sin\theta_i \\ 0 & 0 & 1 \end{bmatrix}$$

Assigning Coordinate Frames to Links

- Frame 0 (the base frame) has its origin at the center of Joint 1 (on the axis of rotation).
- Frame i is ***rigidly attached*** to Link i, and has it's origin at the center of Joint $i-1$.
- The x_i-axis is collinear with the origin of Frame $i-1$.
- The link length, a_i is the distance between the origins of Frames i and $i-1$.
- The homogeneous transformation that relates adjacent frames is given by:

$$T_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & a_i\sin\theta_i \\ 0 & 0 & 1 \end{bmatrix}$$

Assigning Link Frames



The Forward Kinematic Map

- The forward kinematic map gives the position and orientation of the end-effector frame as a function of the joint variables:

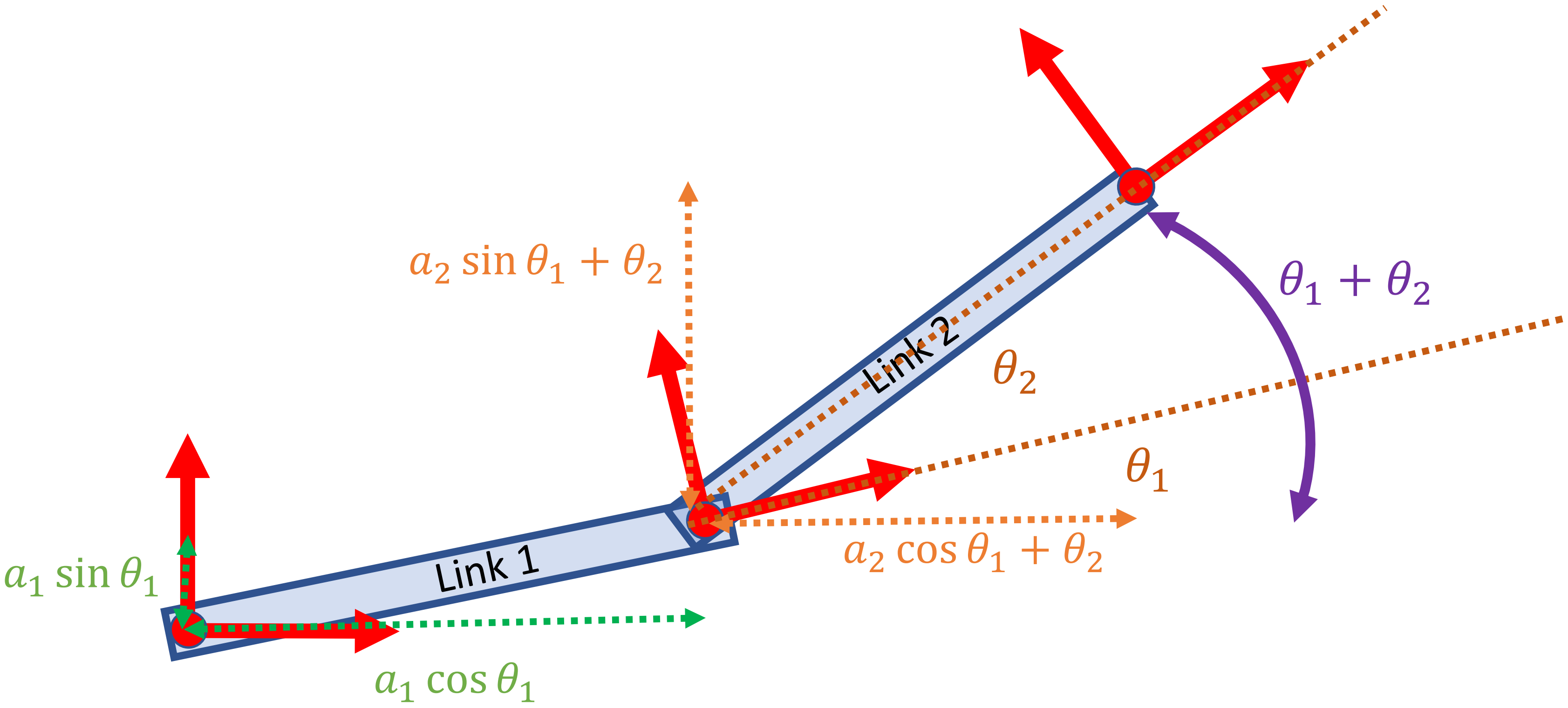
$$T_n^0 = F(q_1, \dots, q_n)$$

- For the two-link planar arm, we have

$$T_2^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & a_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & a_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Simple Geometry…



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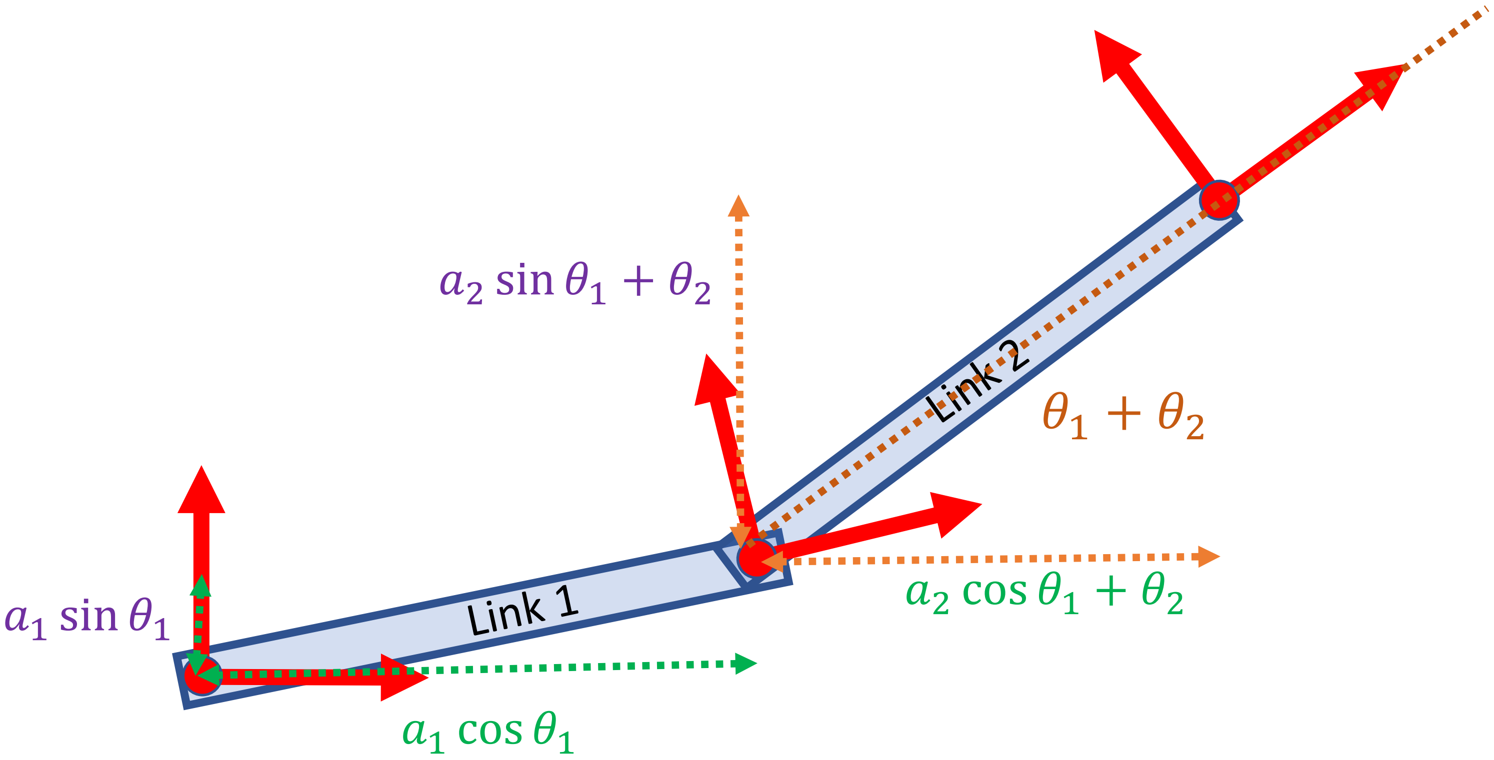
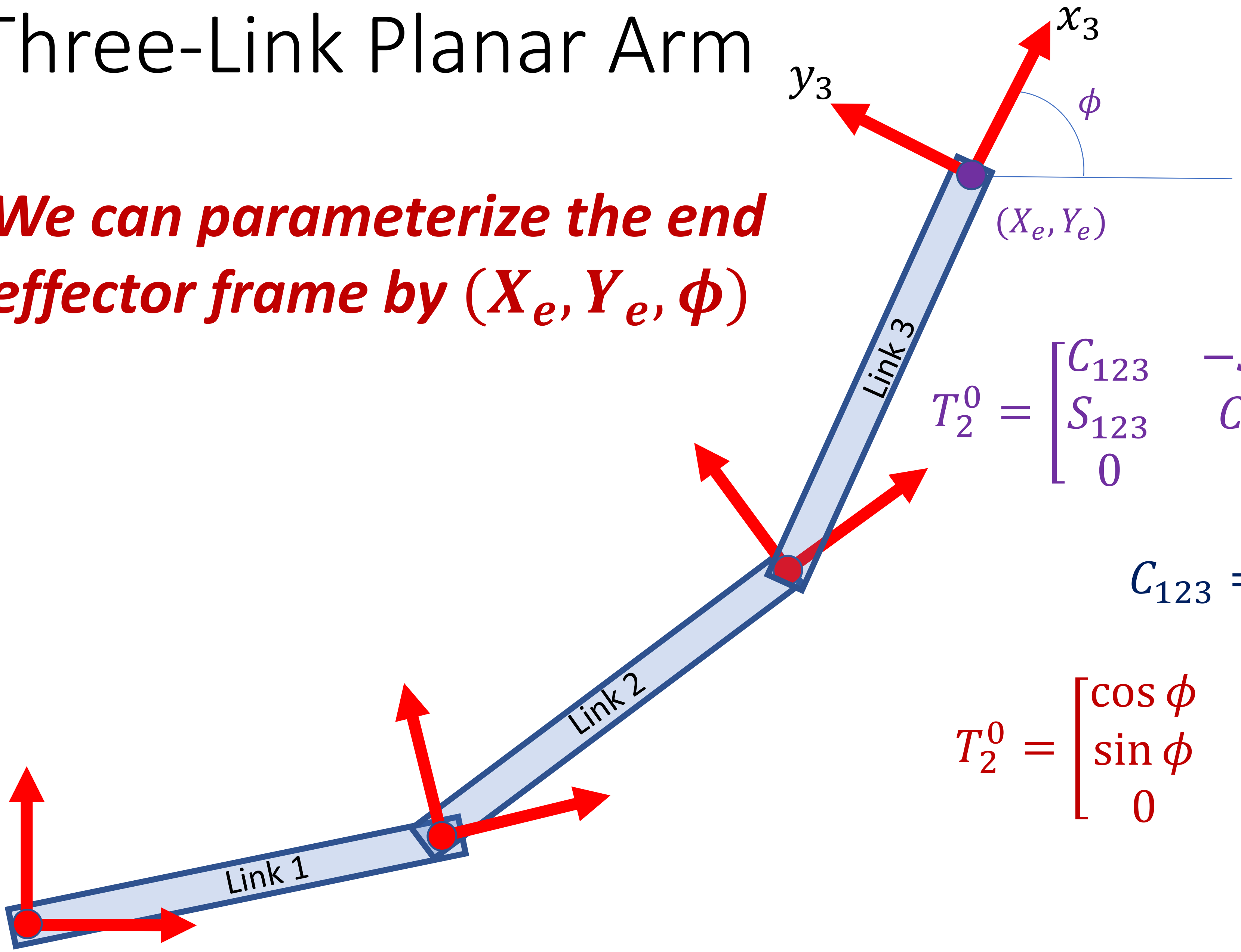


$$T_2^0 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_1 \cos \theta_1 + a_2 \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_1 \sin \theta_1 + a_2 \sin(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Three-Link Planar Arm



We can parameterize the end effector frame by (X_e, Y_e, ϕ)

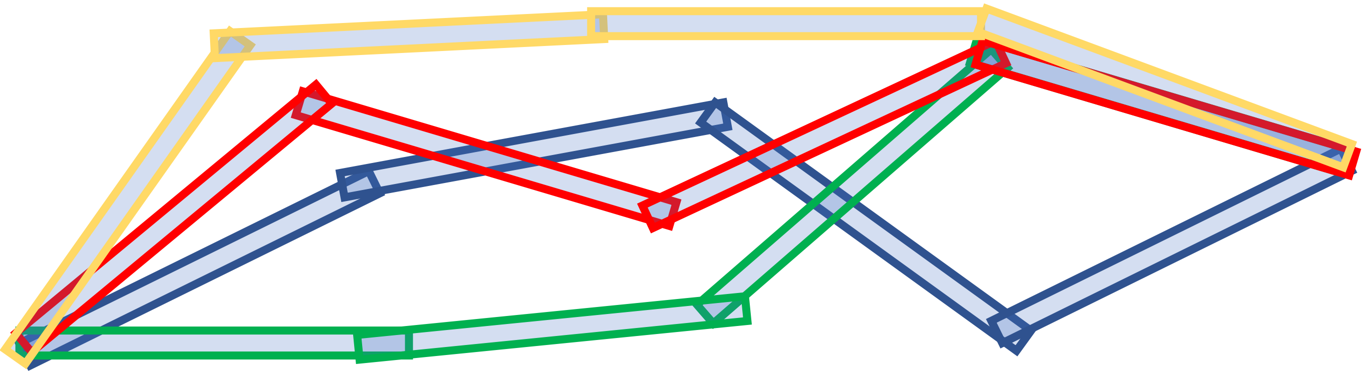
$$T_2^0 = \begin{bmatrix} C_{123} & -S_{123} & a_1C_1 + a_2C_{12} + a_3C_{123} \\ S_{123} & C_{123} & a_1S_1 + a_2S_{12} + a_3S_{123} \\ 0 & 0 & 1 \end{bmatrix}$$

$C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, etc.

$$T_2^0 = \begin{bmatrix} \cos\phi & -\sin\phi & X_e \\ \sin\phi & \cos\phi & Y_e \\ 0 & 0 & 1 \end{bmatrix}$$

About the Forward Kinematic Map

- For the two-link arm, we can **position** the end-effector origin anywhere in the arm's workspace: two inputs (θ_1, θ_2) and two "outputs" (X_e, Y_e).
- For the three-link arm, we can position the end-effector origin anywhere in the arm's workspace, ***<u>and</u>*** we can choose the orientation of the frame: three inputs $(\theta_1, \theta_2, \theta_3)$ and three "outputs" (X_e, Y_e, ϕ).
- Suppose we had a four-link arm?
  - Infinitely may ways to achieve a desired end-effector configuration (X_e, Y_e, ϕ).



More General Robot Arms

- With a bit of work, this can be generalized to arbitrary robot arms.
- We shall not do this bit of work in CS3630.