Chapter 2
Geometry of Gauge Fields

The geometrization program of field theories has already established a remarkable tradition in modern physics. So far, this approach has centered on the gravitational fields whose intricate structures have found a sound and convincing consolidation in Einstein's theory of general relativity. The prevalence of his theoretical explanation is accounted for by the universality of this interaction (Eötvös–Dicke experiment; see MTW, p. 1050). Maxwell's theory of electromagnetism as an almost archetypical[1] model of a gauge theory (WEYL 1928) is also in harmony with this geometric paradigm. Later on, it was shown by YANG & MILLS (1954) that a theory invariant with respect to local rotations acting on the internal space of isotopic spin may have a related geometric interpretation. If one were to search ab initio for a nonlinear generalization of Maxwell's theory, three conceptually basic assumptions would have to be clarified:

(i) The idealization of the *spacetime continuum* must be expounded as the precondition of any field theory.
(ii) The notion of the physical field magnitudes that are attached to a point of the spacetime manifold, whether are of electromagnetic, gravitational, internal, or even quantum-mechanical origin, has to be defined globally.
(iii) The principle of *action at close distances* requires the existence of a connection between different fields in order to make possible interaction and, in the wake of it, to ensure measurable physical processes.

It turned out that the precise mathematical framework for such constructions is to be found in the theory of fiber bundles. Roughly speaking, these theories deal with appropriate generalizations of the conventional Cartesian product of the "external" and "internal" spaces in question. It is not by chance that these were first formulated by mathematicians (see, e.g., STEENROD 1951) in order to solve global topological problems. At the latest, it was the study of WU & YANG (1975) that made abundantly

[1]We are adopting here the striking and useful coinage of WEINBERG (1977).

E.W. Mielke, *Geometrodynamics of Gauge Fields*,
Mathematical Physics Studies, DOI 10.1007/978-3-319-29734-7_2

clear that it is exactly this property of the bundle theory that accounts for its being an ideal organ for the analytical description of interacting fields within the gauge-theoretic concept (YANG 1977).

There exists an extensive literature on fiber bundles. On the one hand, there are works that lead to a profound theoretical consolidation of differential geometry (KOBAYASHI & NOMIZU, Vols. I and II, 1963, 1969; hereinafter referred to as KN I and KN II respectively). On the other hand, there are studies (LUBKIN 1963; TRAUTMAN 1970; MAYER & DRECHSLER 1977; DANIEL & VIALLET 1980) that mainly try to work out a mathematically precise basis for the theory of gauge fields as presented in those treaties, which are of an outspokenly physically oriented nature (see, e.g., the reviews of ABERS & LEE 1973; WEINBERG 1974; 1977; TAYLOR 1979; O'RAIFEARTAIGH 1979; CHENG & LI 1984). Consequently, the present study can make use of the elegant and concise calculus of differential forms, and thus develop the geometric aspects of Yang–Mills theories almost exclusively on a deductive basis. Furthermore, we try to establish a general theoretical framework that allows not only the incorporation of the theory of gravity into this concept, but subsequently also that of the extended geometrodynamics. "Gauge invariance …has the character of general relativity… and can certainly only be understood with reference to it" (WEYL 1929). It is therefore advisable to present the gauge theories of particle physics from the beginning in a curved spacetime.

2.1 Differentiable Manifolds

The theory of general relativity as well as gauge theories in their classical, i.e., unquantized, form are based on the concept of a continuous spacetime, which, however, may comprise curvature and possibly a nontrivial topology, too. And this despite the fact that the hypothesis of continuity can no longer be taken as self-evident, as it was, due to historical limitations, in classical mechanics, but has to be modified with respect to the principles of quantum mechanics (compare, e.g., PENROSE 1968; WILSON 1976; HELLER & STARUSZKIEWICZ 1981; FRIEDBERG & LEE 1984).

At first, the illustrative notion of smooth (curved) surfaces will be generalized into the more abstract notion of a manifold, agreeing with RIEMANN (1854). The latter is an entity that is *locally* similar to the n-dimensional Euclidean space. For the sake of a more precise explanation, a *topological space* M is postulated, i.e., a set with a notion of neighborhood. This space should be equipped with *coordinates*. To achieve this, we consider injective (reversible) mappings $\mathrm{x} : \mathrm{M} \to \mathbb{R}^{\mathrm{n}}$, whose range extends to an open subset of the usual Euclidean space E^{n}. Such a map together with its domain $\mathrm{U_i}$ is called an n-dimensional chart. By the projection π_i onto the single axes of the coordinate system of $\mathbb{R}^{\mathrm{n}}$, the *local coordinates* $\mathrm{x}_i = \pi_i \circ \mathrm{x}$, $i = 1, \dots, \mathrm{n}$ (see Fig. 2.1), come into existence.

In general, one chart does not suffice to cover a set completely. (As is well known, at least two charts are necessary for the stereographic projection of the n-dimensional sphere S^{n}). However, it is possible to construct a collection of charts of identical

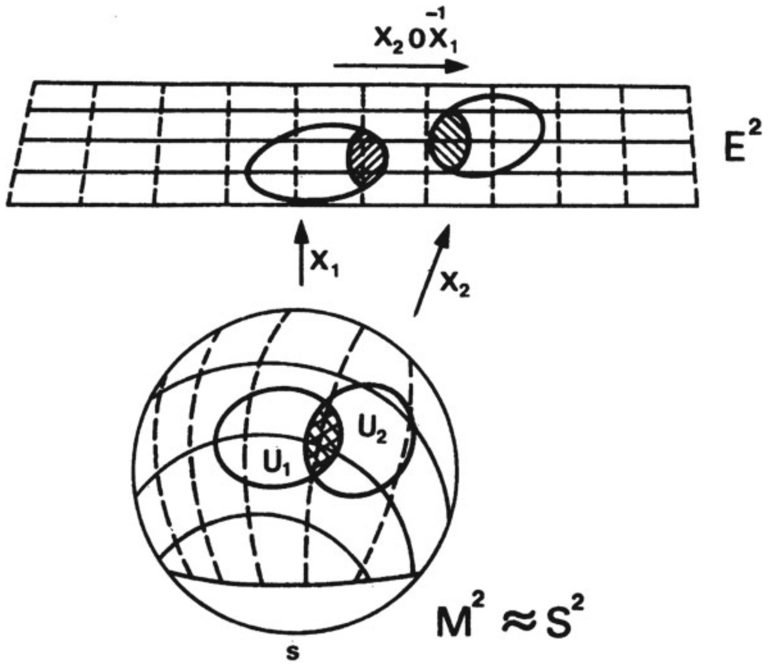

Fig. 2.1 Coordinates of the sphere obtained by stereographic projection

dimensions so that the set-theoretic union of their domains $U_i \subset M$ cover M completely. In order to achieve this, it is reasonable to demand compatibility, which means that the change of coordinates (transition function in the theory of the fiber bundles)

$$x_2 \circ \overset{-1}{x}_1 : \mathbb{R} \rightarrow \mathbb{R} \tag{2.1.1}$$

is a diffeomorphism in the area of the intersection of their domains. Such mappings form an infinite-dimensional group with respect to composition, i.e., the group $\mathscr{D}(M)$ of differential coordinate transformations (an analysis of the mathematical complex structure of the Lie group $\mathscr{D}(M)$ is, for instance, to be found in OMORI (1973).

A collection (U_i, x_i) of compatible charts is called a C^∞-atlas of M. (The common assumption is that these transformations are r times continuously differentiable, i.e., of class C^r). This atlas again can be extended unequivocally to a *complete* one, which in turn determines a differential structure of dimension n.

Definition A topological (locally connected) Hausdorff space M endowed with a C^∞-structure of dimension n is termed a *differentiable manifold* (or for the sake of brevity, "manifold").

It has to be pointed out that the notion of a manifold has been defined here intrinsically, and thus does not make use of the possible embedding into a Euclidean space of a higher-dimensional order. According to a theorem of Janet and Cartan, locally isometric embeddings of analytic Riemannian manifolds into $E^{n(n+1)/2}$ are possible (cf. KN II, p. 354), but the required embedding spaces would cause problems for a physical interpretation.

A global isometric embedding theorem for a compact manifold into Euclidean dimension $N = n(3n + 11)/2$ was proven by NASH (1956); for a noncompact one, the extravagant dimension $N = n(n + 1)(3n + 11)/2$ results; cf. CHEN (2000) for a survey. Later, in the smooth case, FROLOV (2006) considered the embedding of the surface of a rotating Kerr–Newman black hole into $\mathbb{E}^4$.

For the present, local physics without interaction is applied only to the *tangent space* $T_m(M)$ attached to a point $m \in M$. Contrary to the given illustration (see Fig. 2.2), it is to be accentuated that the construction of this space should depend only on the structure of the manifold and avoid an embedding of any kind. In order to achieve this, a curve s(t) on the manifold is chosen that passes through a point $m \in M$. As an auxiliary device, an arbitrary smooth (i.e., C^∞-differentiable) function f is considered. Its derivative

$$e^{\mathrm{f}} := \left.\frac{df(s(t))}{dt}\right|_{t_o=\overset{-1}{s}(m)} = \frac{\partial \mathrm{f}}{\partial x^i} \cdot \frac{dx^i(s(t))}{dt} \tag{2.1.2}$$

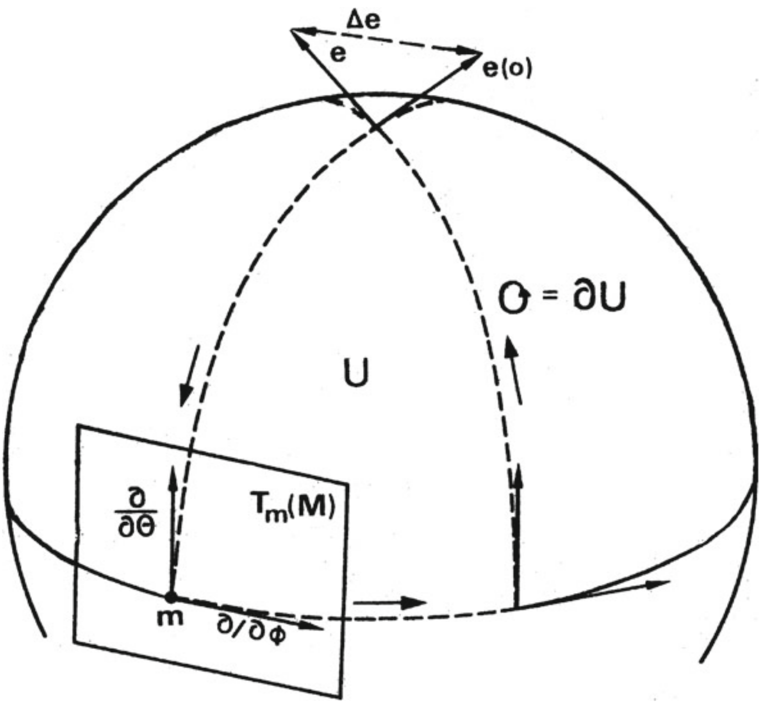

Fig. 2.2 Tangent vectors concerning holonomic coordinates on the sphere

along the direction of the curve, considered a mapping of the algebra of the differentiable functions f to the space $\mathbb{R}$ of real numbers, defines the *tangent vector* e(m) or "velocity vector" at m (see MILNOR & STASHEFF 1974, p. 5). Concerning a given rigid basis of $\mathbb{R}^n$, the n-dimensional tangent space $T_m(M)$ is spanned by the linearly independent vectors $e_\alpha(m), \alpha = 1, \ldots, n$. It follows from (2.1.2) that the partial derivates $\partial_i := \partial/\partial x^i$ constitute a local basis for $T_m(M)$ with respect to a *holonomic* coordinate system. By definition, the relation $[e_i(m), e_j(m)] = [\partial_i, \partial_j] = 0$ is satisfied. Accordingly, the tangent basis for the "comoving" anholonomic frame (Cartan's "repère mobile") can be transcribed as follows:

$$e_\alpha(m) = e^i_{.\alpha}(m)\, \partial_i. \tag{2.1.3}$$

The occurring nonsingular matrices $e^i_\alpha(m)$ depend on the point in question and are consequently called concomitant "n-Beine"[2] or *tetrads* (four dimensions). The latter were introduced by EINSTEIN (1928) in order to formulate a theory of gravity complying with the hypothesis of teleparallelism. By the union of all of the manifold's tangential spaces, the so-called *tangent bundle* is brought into existence:

$$T(M) = \bigcup_{m\in M} T_m(M). \tag{2.1.4}$$

Herein we find a first example of a bundle in terms of a modern mathematical concept, which has to be specified in the following paragraphs of the present study.

2.2 Tensor Fields and Exterior Forms

In addition to the tangent bundle T(M), a so-called *dual* bundle $T^*(M) = \bigcup_{m\in M} T^*_m(M)$ consisting of all possible cotangent spaces $T^*_m(M)$ on the manifold can be constructed. As is to be shown, these are represented by the canonical differential forms of first degree.

Let $\vartheta^\alpha, \alpha = 1, \ldots, n$, be a basis of $T^*_m(M)$. Then the "duality"[3] requires that ϑ^α be orthonormal to a basis of $T_m(M)$ with respect to the natural interior product

$$e_\beta \rfloor \vartheta^\alpha = \delta^\alpha{}_\beta. \tag{2.2.1}$$

With respect to the holonomic coordinate system, a natural basis of $T^*_m(M)$ is given by dx^i. In terms of the rigid basis of E^n, the 1-forms

$$\vartheta = \vartheta^\alpha P_\alpha \quad \in \quad C^\infty(T^*(M)), \tag{2.2.2}$$

[2] Gell-Mann's German term "Vielbeinfeld" is even more to the point.

[3] This duality should not be confused with the duality of exterior forms, as introduced later on.

which are also called Pfaffian forms, can be expressed in general by

$$\vartheta^{\alpha} = \mathrm{E}_{j}^{\alpha}(m)\, dx^{j}. \tag{2.2.3}$$

The latter constitute a so-called *anholonomic* (dual) basis, in as much as the exterior derivative $\mathrm{d}\vartheta^{\alpha}$, in contrast to that of dx^{j}, is generally different from zero. Following (2.2.1), $\mathrm{E}^{\alpha}_{\cdot\ j}$ can be understood as the dual tetrad field "reciprocal" to e^{i}_{β} (cf. (2.1.3)). These spacetime-dependent matrices are related to each other by

$$E_{i}^{\alpha}\ e^{i}_{\ .\beta} = \delta^{\alpha}{}_{\beta}. \tag{2.2.4}$$

By repeated tensorial multiplication of tangent and cotangent spaces with themselves, *tensor bundles* of generic, co- and contravariant degrees (p,q) will be obtained:

$$T_{p}^{q}(M) := \bigcup_{m\in M} \otimes^{p} T_{m}^{*}(M)\ \otimes^{q}\ T_{m}(M). \tag{2.2.5}$$

Completely symmetric or antisymmetric products are denoted by $\otimes_{\mathrm{s,a}}$ or, even more commonly, by the symbols $\vee$ and $\wedge$, respectively. Note that a tensor is a geometric object that is defined independently from the choice of coordinates. Relative to a basis of the tensor space, a tensor may, however, be locally expanded as follows:

$$T = T_{\alpha_1\cdots\alpha_p}^{\beta_1\cdots\beta_q}(m)\vartheta^{\alpha_1}\otimes\cdots\otimes\vartheta^{\alpha_p}\otimes e_{\beta_1}\otimes\cdots\otimes e_{\beta_q}\in C^{\infty}(T_{p}^{q}(M)). \tag{2.2.6}$$

The quantities $\mathrm{T}_{\alpha_1\cdots\alpha_\mathrm{p}}^{\beta_1\cdots\beta_\mathrm{q}}(\mathrm{m})$ are called its p covariant and q contravariant components, and $\mathrm{T}_{\mathrm{p}}^{\mathrm{q}}(\mathrm{M})$ may be regarded as a bundle associated with L(M) having the manifold M as a base, the tensor representation $\mathrm{D}^{(\mathrm{p,q})}$ of GL(n, $\mathbb{R}$), i.e., $\rho^{(\mathrm{p,q})}(\mathrm{GL}(\mathrm{n},\mathbb{R}))$ as a structure group and the (Cartesian) product $\otimes^{\mathrm{p}*}\mathbb{R}^{\mathrm{n}}\otimes{}^{\mathrm{q}}\mathbb{R}^{\mathrm{n}}$ as a typical fiber.

The (pseudo-)Riemannian metric on M is one of the most important examples not only for differential geometry, but also for gauge theories in curved spacetime. Formally, this metric can be defined as a covariant symmetric tensor field of degree (2, 0):

$$ds^{2} = g_{\alpha\beta}\vartheta^{\alpha}\otimes_{s}\vartheta^{\beta} =: g_{ij}\, dx^{i}\otimes_{s}\ dx^{j}. \tag{2.2.7}$$

This "metrical groundform" (Weyl 1923) or square of the line element determines the scale of the manifold. Due to the postulated symmetry of the components $\mathrm{g}_{\mu\,\nu}$, it is always possible, by a linear transformation of the main axes, to rearrange the metric locally as follows:

$$ds^{2} = g_{ij}\, dx^{i}\otimes_{s} dx^{j} \overset{(*)}{=} -\sum_{i=1}^{s}(\overset{\circ}{\vartheta}{}^{i})^{2} + \sum_{j=s+1}^{n}(\overset{\circ}{\vartheta}{}^{j})^{2}. \tag{2.2.8}$$

If these signatures occur in the same characteristic manner throughout all points of the manifold, it is generally called a *pseudo-Riemannian manifold of signature s*; cf. SAKHAROV (1984).

In the geometric description not only of gauge fields but of gravitational fields as well, those cases are prevalent in which one makes use only of an irreducible subspace of $T_p^q(M)$, i.e., the space of *completely antisymmetric* covariant tensor fields of degree (p, 0). Usually, $\wedge := \otimes_a$ is an abbreviation for the antisymmetrized tensor product, while cross sections of this special tensor bundle are called (alternating) differential form of degree p:

$$\alpha^{(p)} = \frac{1}{p!} A_{\alpha_1 \cdots \alpha_p} \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} \in C^\infty(\wedge^p T^*(M)). \tag{2.2.9}$$

The mathematically elegant calculus of exterior (alternating) differential forms, essentially developed by Poincaré and E. Cartan, is based on this. As far as we know, this calculus was first applied to physics by MISNER & WHEELER (1957) in order to achieve both a more concise reformulation of Maxwell's theory of electromagnetism and an incorporation of that theory into an "already unified field theory" of electromagnetism and gravitation. The standard reference book concerning gravitation (MTW 1973) gives an instructive account with respect to this mathematical tool. Five years later, HOWE & TUCKER (1978) rewrote the SU(2)-gauge theory in the language of differential forms and especially drew attention to the subtleties of the real Minkowski space.

2.3 Fiber Bundles as an Enlarged Geometric Arena

Atomic spectra can be described quantum-mechanically by the Schrödinger theory, which is based on a detailed knowledge of the *dynamics* of the microscopic system. Subsequently, a deepened interpretation of the atomic and nuclear phenomena was achieved by *group-theoretic* methods (WEYL 1928; WIGNER 1931). In the subnuclear domain of particles, we are almost completely dependent on a classification of particle properties according to such group-theoretic criteria. WIGNER's analysis (1939, 1957) of the representations of the transformation group in flat spacetime, i.e., the *Poincaré group*, is an outstanding example that yields the well-known invariant characterization of particles in terms of *mass* and *spin*. Moreover, the overwhelming number of "excited states" of stable particles that have been discovered recently has certainly received a thorough and satisfying classification by the assumption of "internal symmetries." These have been exemplified by the hypothesis of isotopic-spin invariance HEISENBERG (1932) or, for instance, by the hypothesis of the unitary group SU(3) of GELL- MANN & NEEMAN (1964).

Within the framework of quantum field theory (QFT, BJORKEN & DRELL 1964), the most crucial problem is to find a geometric relation between the spacetime symmetries and the postulated internal symmetries that leads to a natural concept of interaction. It is to be remembered that in QFT, an elementary particle is represented by a complex field with several components, or to be more precise, by an array of rays in a Hilbert space $\mathscr{H}$, which is to be transformed according to an irreducible unitary representation of the Poincaré group. From the geometric point of view, it is assumed that φ at a point m is equal to the value φ(m) in a complex vector space V_m. Additionally, it can be postulated that a vector space of this kind is attached to each point of the spacetime manifold. Similarly to the presentation of a tangent space, it is required, by a global point of view, to take into consideration a *bundle* of such abstract spaces of "particle attributes." In order to realize internal symmetries, the vector spaces V have to be generalized in such a way that they can function as representation spaces for the internal group G. Accordingly, these "internal rotations" are seen as operating pointwise on the spacetime manifold. It is only to be considered that under these circumstances, at least a transformed description of the same physical reality is achieved that is represented by matter fields.

Intuitive concepts, rather, like these have their precise counterpart in the theory of fiber bundles. As has already been mentioned, these are appropriate generalizations of the familiar Cartesian product of the spaces under consideration. In particular, it involves the group manifold associated with the internal symmetries and the spacetime continuum. However, this generalization is qualified to take *globally* nontrivial topological structures into consideration.

Definition A *fiber bundle* (F, M, π) consists of two C^∞-differentiable manifolds F and M and a smooth surjective mapping

$$\pi : F \to M. \tag{2.3.1}$$

For obvious reasons (see Fig. 2.3), F is called the *total space*, M the *base space*, and π the *projection*. The closed submanifold $F_m := \overset{-1}{\pi}(m)$ will be called a *typical fiber*, for in contrast to more general bundles, these are all isomorphic in the case of fiber bundles. Furthermore, (F, M, π) is required to be locally trivial, i.e., each

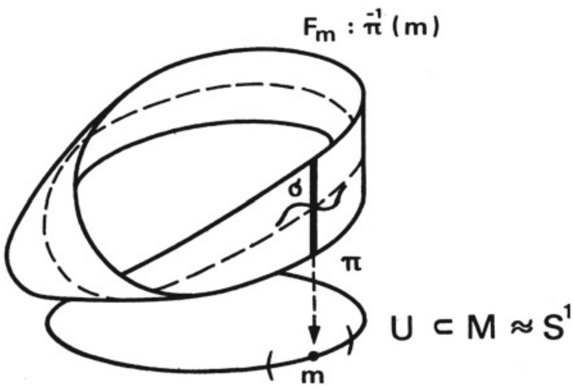

Fig. 2.3 Möbius strip as an example of a globally nontrivial fiber bundle

point m ∈ M has a neighborhood U such that $\overset{-1}{\pi}(U) \approx U \times F_m$ is isomorphic to the product bundle $(U \times F_m, U, \pi_u)$. The mapping

$$\sigma : U \to F \qquad \text{where} \qquad \pi \circ \sigma = \text{id} \tag{2.3.2}$$

defines a local *cross section* through the bundle. In the more physically oriented literature, this is better known as a choice of a “local gauge.”

The most striking example of a global nontrivial fiber bundle is the Möbius strip. In this case, the (multiply connected) circle is considered a base space, whereas a one-dimensional real vector space, e.g., the unit interval [0, 1], represents the typical fiber. This fiber is subjected to such a twist in the total space that opposite points of the interval can be identified after a full revolution.

The basic idea of the very concept of gauge invariance, however, is that interacting fields (e.g., gauge fields) at any given point of the spacetime can be varied by local “internal rotations” (e.g., by local isospin transformations), but that this results only in an equivalent description of the same physical reality, as far as matter fields are concerned. In order to heighten the precision of this notion, the internal symmetry group G (usually assumed to be continuous) and the spacetime manifold are considered as a whole and extended into a single *enlarged geometric arena*, the principal fiber bundle.

Definition Provided P and G are manifolds and the structure group G is a Lie group,then the collection P(M, G, π, δ) is called a *principal fiber bundle* if

(i) (P, M, π) is a fiber bundle with typical fiber G;
(ii) (P, G, δ) is a G-manifold (whereby G is acting on P from the right);
(iii) P is *locally* trivial, which means that every point of M has a neighborhood U ⊂ M together with an isomorphism $\iota : U \times G \to \overset{-1}{\pi}(U)$ for which the following property holds:

$$\iota(m, g_1 g_2) = \iota(m, g_1) \cdot g_2, \qquad m \in U, g \in G. \tag{2.3.3}$$

A manifold is called a *G-manifold* (P, G, δ) if G acts on it as a free transformation group either from the left or from the right:

$$\delta : \left\{ \begin{array}{ccccc|c} G & \times & P & \to & P & \\ \Psi & & \Psi & & \Psi & e \cdot p_\circ = p_\circ \in P \\ g_p & \cdot & p_\circ & = & p & \end{array} \right\}. \tag{2.3.4}$$

This fixes the transformations of the total space P if it is subjected to the action of the symmetry group G. (If the action of G were also a transitive one, i.e., if there always existed a g that would provide a relation $p_2 = g\, p_1$ for given p_1 and p_2, then G considered as a manifold would be isomorphic to the base M.)

These highly formalized constructions, however, yield only an “arena” for the representation of physical fields.

2.4 Associated Bundles and Physical Fields

In order to characterize *matter fields* within the framework of bundle theory, the notion of *vector bundles* being *associated* with P(M, G, π, δ) is also required. For this purpose, the fiber bundles V(M, F; G, P) associated with P(M, G, π, δ) are constructed as follows:

Let the typical fiber F be a manifold on which G acts from the right-hand side, i.e., on the product space P × F, an action from the right-hand side is defined by

$$\delta_g(p, \zeta) := (pg, g^{-1}\zeta) \subset P \times F, \quad g \in G. \tag{2.4.1}$$

The quotient space of P × F with regard to this action of the group G will be denoted by $V = P \times_G F$. The isomorphism $\overset{-1}{\pi}(U) \approx U \times G$ concerning the original domain of a neighborhood U, which results from the very construction of a fiber bundle, will induce the isomorphism $\overset{-1}{\pi}_V(U) \approx U \times F$ in the associated bundle. Provided that $\overset{-1}{\pi}_V(U)$ is an open submanifold of V, then V can be equipped with a differentiable structure as a whole.

Usually, in the cases of physical relevance it is not the structure group itself that occurs, but its linear representation $\rho : G \to GL(N, \mathbb{C})$, i.e., the Lie homomorphism of G into the general linear group of complex N × N matrices. If their representation space, the N-dimensional vector space $\mathbb{C}^N$ over the field of complex numbers, is considered a typical fiber, an associated bundle can be constructed from it,

$$V^\rho := V^\rho(M, \mathbb{C}^N, \rho(G) \subseteq GL(N, \mathbb{C}), P), \tag{2.4.2}$$

which will be referred to as a *complex vector bundle* on M. By taking the product with the complexified[4] cotangent bundle $T^*_{\mathbb{C}}(M)$, further vector bundles will come into existence on the spacetime manifold.

Physical fields on M are then to be considered p-forms of (representation) type ρ, i.e., the cross section

$$\phi^{(p)} := \varphi^p \otimes b \in C^\infty(\wedge^p T_{\mathbb{C}}{}^*(M) \otimes V^\rho), \quad b \in C^\infty(V^\rho). \tag{2.4.3}$$

If this space is endowed with a Hermitian inner product, then such fields are referred to as being of "charged type" (compare Mack 1981).

For $N \geq 1$, these fields form an infinite-dimensional vector space over $\mathbb{C}^N$. In (2.4.3), the bundle coordinates relative to the local basis fields $b = (b_A) : U \subset M \to V^\rho$ are denoted by

$$\varphi^{(p)} = \left\{\varphi^{(p)A} | A = 1, \ldots, N\right\}. \tag{2.4.4}$$

[4]For reasons of consistency, the cotangent bundle has itself to be complexified, since in physics, complex vector bundles are dealt with almost exclusively.

Let the Lie homomorphism $\rho : \mathrm{G} \to \mathrm{GL}(N, \mathbb{C})$ be explicitly given by the nonsingular complex $\mathrm{N} \times \mathrm{N}$ matrices $\rho_{\mathrm{A}}{}^{\mathrm{B}}(\mathrm{g}(\xi))$, where $\xi^{\mathrm{j}}(\mathrm{j} = 1, \ldots, \dim \mathrm{G})$ prescribes a parametrization of the Lie algebra. Then the *infinitesimal operators*

$$L_j := \left[\frac{\partial \rho_A{}^B(g(\xi))}{\partial \xi^j} \Big|_{g=e} \right] \in T_e(\rho(G)) \tag{2.4.5}$$

are constructed by means of the "derived" homomorphism of G. In the case of an identical representation, i.e., for $\rho = \mathrm{id}$, the infinitesimal generators $\mathrm{I_j}$ form a basis of the Lie algebra $\mathfrak{g}$ of G. As a representation of $\mathrm{I_j}$, the infinitesimal operators inherit the commutation relations from the Lie algebra, i.e.,

$$\left[L_i, L_j \right] = c_{ij}{}^k L_k. \tag{2.4.6}$$

These algebraic relations are determined by the structure constants $\mathrm{c_{ij}{}^k}$ of the Lie algebra $\mathfrak{g}$. "Gauge fields," unlike matter fields, are considered as cross sections of a vector bundle $\mathrm{V^{Ad}}$ respecting the adjoint representation of G as a structure group. Thus, they can be represented by the Lie-algebra-valued forms

$$\phi^{(p)} = \frac{1}{p!} \phi^j_{\alpha_1 \ldots \alpha_p} L_j \otimes \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} \in C^\infty \left(\wedge^p T^*_{\mathbb{C}}(M) \otimes V^{Ad} \right). \tag{2.4.7}$$

All rules of the calculus of exterior forms are valid, except that in the nonabelian case, the wedge product is no longer an alternating operation. In contrast, the commutator

$$\left[\phi^{(p)}, \psi^{(q)} \right] := \phi^{(p)} \wedge \psi^{(q)} - (-1)^{pq} \psi^{(q)} \wedge \phi^{(p)} = (-1)^{pq+1} \left[\psi^{(q)}, \phi^{(p)} \right] \tag{2.4.8}$$

of Lie-algebra-valued forms has this useful alternating property. Accordingly, the commutator of a form of even degree with itself is equal to zero on account of

$$\left[\phi^{(2k)}, \phi^{(2k)} \right] = - \left[\phi^{(2k)}, \phi^{(2k)} \right] = 0. \tag{2.4.9}$$

2.5 Connection and Covariant Derivative

Up to now, these bundle structures are still unconnected in the sense that the internal spaces that are thought of as being attached to the spacetime M cannot be "compared" in a differentiable manner along a given curve $\mathrm{s(t)} \subset \mathrm{M}$ in the base manifold. In order to make possible such a parallel displacement of the fibers (Levi- Civita 1926; KN I, p. 68), a *connection* is required, i.e., a kind of guiding field.

There are several equivalent definitions concerning the connection in a fiber bundle (compare, e.g., Eguchi et al. 1980). Such a connection is imprinted on a principal fiber bundle $\mathrm{P(M, G}, \pi, \delta)$ in a mathematically rather abstract way: Proceed from the cotangent bundle $\mathrm{T_p{}^*(P)}$ at the point $\mathrm{p} \in \mathrm{P}$ (see KN I, Chap. 1). Then decompose it

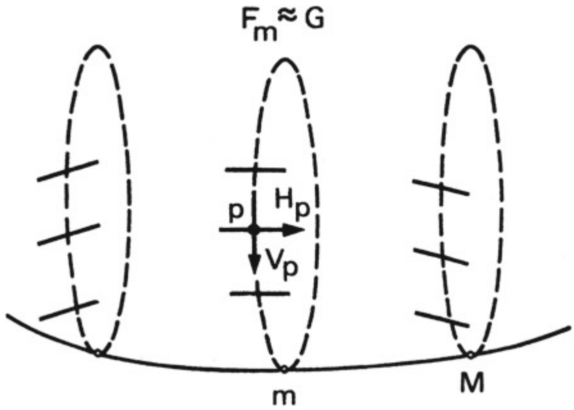

Fig. 2.4 Horizontal and vertical cotangent spaces of a principal fiber bundle

into the direct sum of horizontal and the vertical subspaces, i.e., $T_p{}^*(P) = H_p \oplus V_p$. Then consider an abstract parallel displacement in the vertical subspace (see the straight lines in Fig. 2.4) prescribed by

$$V_p{}^* = T_p{}^*(F_m(M)), \tag{2.5.1}$$

i.e., in the cotangent space of the bundle $F_m(M)$ of typical fibers on M.

This will be achieved by the assertion of a 1-form $\omega(e) \in \mathscr{C}$ with values in the Lie algebra $\mathfrak{g}$ of the structure group G that is subject to the following conditions:

(i) With respect to a right-transformation of the tangent vectors of P, this 1-form transforms itself according to the inverse, adjoint representation of G:

$$\omega(eg) = g^{-1}\omega(e)g, \qquad e \in T(P). \tag{2.5.2}$$

(ii) For vertical elements $e_v = pdg \in V_p$, it will be mapped to the left-invariant element

$$\omega(p\,dg) = g^{-1}dg \qquad \in \quad T_e(G) \approx \mathfrak{g} \tag{2.5.3}$$

of the Lie algebra of G.

For the treatment of physical problems, the following equivalent approach to the concept of connections is more appropriate (Atiyah 1978). In this second approach, a connection in an associated vector bundle $\bigwedge^p T^*_{\mathbb{C}}(M) \otimes V^\rho$ is defined via the *covariant derivative*. This is analogous to the procedure known from classical differential geometry (see Milnor & Stasheff 1974). Consider a linear differential operator

$$D : C^\infty(V^\rho) \longrightarrow C^\infty(T^*_{\mathbb{C}}(M) \otimes V^\rho) \tag{2.5.4}$$

in the vector bundle V^ρ and regard it as a mapping of the space of sections of V^ρ onto those of the product bundle $T^*_{\mathbb{C}}(M) \otimes V^\rho$. This operator will be defined in such a way that it acts similarly to the "absolute differential," which was introduced by LEVI- CIVITA (1926), with respect to the bundle basis b.

To be more precise, this means that the result of the action of D on b can be expressed as a linear transformation of the selfsame basis of V^{ρ}, i.e.,

$$Db = \omega \otimes b, \quad b \in C^{\infty}(V^{\rho}). \tag{2.5.6}$$

(The called-for linearity is by no means as self-evident as it might seem. For a system of paths that generalize geodesics, *nonlinear connections* can be introduced into differential geometry, too (LAUGWITZ 1965, p. 190; cf. GOENNER 1984). Furthermore, for the cross sections of the vector-valued forms (2.4.3), the differential operator D is expected to have a natural *extension* that satisfies the *generalized Leibniz rule*:

$$D(\varphi^{(p)} \otimes b) = d\varphi^{(p)} \otimes b + (-1)^p \varphi^{(p)} \otimes Db. \tag{2.5.7}$$

(Attention should be drawn to the fact that the symbol D is used throughout our presentation, although the occurrence of varying representations ρ of the same group G is by no means excluded in the formulas.) Concerning the adjoint representation in the relevant case of a vector bundle V^{Ad}, (2.5.7) is converted into the more familiar relation

$$D(\varphi^{(p)} \otimes b) = \left\{ d\varphi^p + (-1)^p [\varphi^{(p)}, \omega] \right\} \otimes b. \tag{2.5.8}$$

The image of Db will be referred to as the *(G-)covariant exterior derivative* of b. Since D is a local operator, a global connection can be defined unequivocally by its restriction to a neighborhood U of M. Let $\mathrm{b_A}$, $\mathrm{A} = 1, \ldots, \mathrm{N}$ be a local basis for the cross section $\mathrm{V}^{\rho}_{\mathbb{U}}$ restricted to U. With respect to this local system of reference, the resulting effect of D on $\mathrm{b_A}$ can be expanded as follows:

$$Db_A = \omega_A{}^B b_B, A, B = 1, \ldots, N. \tag{2.5.9}$$

This is identical to (2.5.6) if given in the more abstract notation of matrices. The matrix $[\omega_{\mathrm{A}}{}^{\mathrm{B}}]$ acquires values in $\mathrm{T}^*_{\mathbb{C}}(\mathrm{M})$ and will consequently be called a *connection 1-form*, which means that it can be expanded locally as

$$\Gamma := \overset{*}{\sigma}\omega = \Gamma_\alpha{}^j L_j \otimes \vartheta^\alpha, \quad j = 1, \ldots, \dim G, \tag{2.5.10}$$

i.e., by a pullback via the cross section.

The coefficients $\Gamma^{\mathrm{j}}_{\alpha}(\mathrm{m})$ will turn out to be generalized gauge potentials as they occur in modern physical field theories. With respect to unitary structure groups U(f), these potentials will be denoted, as is usually done, by

$$A_\alpha{}^j(m) := \Gamma_\alpha{}^j(m)_{|G=U(f)} \tag{2.5.11}$$

2.6 Curvature

For differential forms in general, the identity dd $\equiv 0$ holds, which may be regarded as a counterpart to the homological relation $\partial\partial \equiv 0$. Figuratively speaking, this means that the "boundary of a boundary" of a manifold vanishes identically. In general, such a relation would not hold for the twofold *covariant* exterior derivative, for the departure from integrability is accounted for by the curvature of the bundle space, in analogy to the concepts of differential geometry:

$$D\,Db =: \Omega \otimes b. \tag{2.6.1}$$

It follows from

$$\begin{aligned} D\,Db = D(\omega \otimes b) &= d\omega \otimes b - \omega \wedge Db \\ &= (d\omega - \omega \wedge \omega) \otimes b \end{aligned} \tag{2.6.2}$$

that the curvature 2-form Ω satisfies the *second structure equation* of É. Cartan:

$$\boxed{\Omega = d\omega - \omega \wedge \omega = d\omega - \frac{1}{2}[\omega, \omega]}. \tag{2.6.3}$$

As was to be expected, Ω is a 2-form of type Ad G. Most often, its local presentation

$$\overset{*}{\sigma}\Omega = \frac{1}{2} F_{\alpha\,\beta}{}^{j} L_j \otimes \vartheta^{\alpha} \wedge \vartheta^{\beta} \tag{2.6.4}$$

is preferred in physical applications. Its components with two spacetime indices and one group index are called the field strengths of the gauge fields in question.

If we are using (2.5.10) in order to notate (2.6.3), it will become obvious that these field strengths are in absolute compliance with the familiar relations

$$F_{\alpha\beta}{}^{j} = \partial_{\alpha}\Gamma_{\beta}{}^{j} - \partial_{\beta}\Gamma_{\alpha}{}^{j} - c_{kl}{}^{j}\,\Gamma_{\alpha}{}^{k}\,\Gamma_{\beta}{}^{l}, \tag{2.6.5}$$

which are well known from the Yang–Mills theory. Concerning electromagnetism, which can be formulated in a fiber bundle with abelian structure group G $= U(1)$, the structure constants are identically zero. Consequently, in holonomic coordinates, the well-known relation

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{2.6.6}$$

holds.

The curvature of the bundle is interpreted geometrically, similar to the explanation that is offered in Riemannian geometry. For illustrative purposes, we consider the variation

$$\Delta b = b - b(\circ) = \oint (db)_{\mathrm{hor}}, \tag{2.6.7}$$

which in the local basis b of the bundle results from a parallel displacement of b along an *infinitesimal* closed curve $\circlearrowleft = \partial U \subset M$. The very concept of a connection guarantees that

$$(db)_{\text{hor}} = Db = \omega \otimes b \tag{2.6.8}$$

is valid, since it is only the *horizontal* part of this displacement that matters. This relation together with Stokes's theorem yields

$$\begin{aligned}\Delta b &= \int_{\partial U} \omega \otimes b = \int_U d(\omega \otimes b)_{\text{hor}} = \int_U D(\omega \otimes b) \\ &= \int_U \Omega \otimes b \simeq \Omega \otimes b\end{aligned} \tag{2.6.9}$$

as a measure of the *nonintegrability* of a parallel displacement along the infinitesimal closed curve ∂U. Figure 2.2 shows this for a parallel displacement of a pair of tangent vectors along a path constructed entirely of great circles of the sphere. The so-called holonomy group H(M, m) of a manifold is generated by linear transformations of $T_m(M)$ onto itself. These are generated by displacements of $e(m) \in T_m(M)$ along arbitrary curves that begin and end at m. According to the theorema egregium,

$$K = \lim_{U \to 0} \left(\int_U \Omega \otimes b \right) \Big/ \left(\int_U 1 \otimes b \right) \tag{2.6.10}$$

is the Gaussian or local curvature of a 2-dimensional surface (LAUGWITZ 1965; cf. SULANKE & WINTGEN 1972, p. 242).

Analogously, the *nonintegrability* of the parallel displacement of the bundle basis is measured by the integral (2.6.9). This accounts not only for the Riemannian curvature of the base space M, but also for the "internal" curvature derived from the prescribed connection in the vector bundle that has been dealt with so far. For the covariant derivative, a relation similar to dd ≡ 0 occurs at a higher degree of differentiation only:

$$DDDb \equiv 0. \tag{2.6.11}$$

This relation implies the (second) Bianchi identity

$$\boxed{D\Omega \equiv 0} \tag{2.6.12}$$

for the curvature 2-form. The proof is obtained by writing out (2.6.11) explicitly by inserting the structure equation (2.6.3) repeatedly:

$$
\begin{aligned}
DDDb &= D(\Omega \otimes b) = (d\Omega + [\Omega, \omega]) \otimes b \\
&= (d\Omega + \Omega \wedge \omega - \omega \wedge \Omega) \otimes b \\
&= (dd\omega - d\omega \wedge \omega + \omega \wedge d\omega \\
&\qquad + d\omega \wedge \omega - \omega \wedge \omega \wedge \omega \\
&\qquad + \omega \wedge \omega \wedge \omega - \omega \wedge d\omega) \otimes b \equiv 0.
\end{aligned}
\tag{2.6.13}
$$

On the other hand, this derivation suggests relating the Bianchi identity to the homological identity $\partial\partial \equiv 0$. A very stimulating discussion of these far-reaching theorems of differential topology are to be found in the standard reference book on gravitation (MTW, Chap. 15).

2.7 Gauge Transformations

Einstein's theory of general relativity is founded firmly on the following basic principle as far as its axiomatic argument is concerned: "Natural laws are to be expressed by equations that are covariant under the group of continuous coordinate transformations" (EINSTEIN 1949, p. 69).

It is not by chance that we have formulated our approach to gauge theories in terms of differential forms: these transform themselves covariantly with respect to the group $\mathscr{D}$(M) of coordinate transformation. Additionally, it has been suggested by the empirical occurrence of internal symmetries that use should be made of the principal fiber bundle P(M, G, π, δ) as an "enlarged geometric arena." Thus it is to be expected that additional transformations of P play a part in gauge theories similar to that of the group $\mathscr{D}$(M) of diffeomorphisms in GR. In particular, those diffeomorphisms

$$G(p) : P \to P, \quad G(p) \in \mathscr{G}_p \tag{2.7.1}$$

of a principal fiber bundle P(M, G, π, δ) should be taken into consideration, which are subject to the following conditions (ATIYAH 1978):

(i) G(p) is equivariant, i.e.,

$$G(g\,p) = g\,G(p), \quad g \in G, \quad p \in P; \tag{2.7.2}$$

(ii) G(p) preserves each fiber $\mathrm{F_m} = \overset{-1}{\pi}$(m), i.e., acts trivially on the base space,

$$\pi \circ G = \pi. \tag{2.7.3}$$

These diffeomorphisms generate *inner automorphisms* of P. In terms of composition, they constitute an infinite-dimensional group, the group $\mathscr{G}_\mathrm{P}$ of *local gauge transformations* (which are of the so-called second kind). This group can be identified with the group of smooth cross sections of the product bundle of P and G under the adjoint action of G (Bourguignon & Lawson 1981):

$$\mathscr{G}_p \approx C^\infty(P \times_{\mathrm{Ad}} G). \tag{2.7.4}$$

The exponential mapping $\exp : \mathfrak{g} \to \mathrm{G}$ from the Lie algebra $\mathfrak{g}$ into the structure group G of P induces a corresponding mapping within $P \times_{\mathrm{Ad}} \mathrm{G}$. As a result, each element of the group of gauge transformations can be expressed locally in the following form:

$$G(p) = \exp i\,\theta^k(m) L_k \;\in\; \mathscr{G}_p. \tag{2.7.5}$$

Here $\theta^k(m) \in C^\infty(M)$ denote real functions on the base space. Gauge transformations within an associate vector bundle $\mathrm{V}^\rho(\mathrm{M}, \mathbb{C}^{\mathrm{N}}, \mathrm{GL}(\mathrm{N}, \mathbb{C}), \mathrm{P})$, i.e., elements of $\mathscr{G}_{\mathrm{V}}$, are represented by the same expression to the extent that L^{k} is to be considered an infinitesimal operator with respect to a representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{N}, \mathbb{C})$.

For simplicity's sake, let us consider a physical system that is determined by a 0-form, i.e., a scalar of representation type ρ:

$$\phi = \varphi \otimes b = {}^{G^{-1}}\varphi \otimes {}^{G}b \qquad \in\; C^\infty(V^\rho). \tag{2.7.6}$$

An equivalent local description of the same system will be obtained if the physical system, represented by the so-called bundle coordinates φ, is subjected to the *active* gauge transformation

$$\varphi \to {}^{G^{-1}}\varphi := G^{-1}\varphi \qquad \in\; C^\infty(M, \mathbb{C}),\, G \in \mathscr{G}_V, \tag{2.7.7}$$

whereas the local basis of the sections suffers from a *passive* transformation:

$$b \to {}^{G}b := Gb \qquad \in\; C^\infty(V^\rho). \tag{2.7.8}$$

It has to be emphasized that the bundle coordinates $\overline{\varphi}$ of the vector bundle constructed on the conjugate ("Dirac adjoint") representation $\overline{\rho}$ transforms according to

$$\overline{\varphi} \to {}^{G^{-1}}\overline{\varphi} := \overline{\varphi}G \qquad \in\; C^\infty(V^{\overline{s}}). \tag{2.7.9}$$

As has already been indicated in (2.7.6), the total effect of these transformations is physically unobservable. Thus the bundle theory provides us automatically with a sensible instruction for a "recalibration" or gauging of the matter fields. This rule is completely equivalent to the formalism developed by Utiyama (1956, 1980).

2.8 Topological Invariants

Solutions of source-free gauge field equations do not have a merely local meaning. Some of them may even have a global extension to the whole base space. For a classification of the configuration spaces of such global solutions, the mathematics of the fiber bundles is mandatory and not to be considered as only instrumental. The characterization will be achieved via invariant polynomials of the curvature of the bundles in question.

Let V(M, $\mathbb{C}^N$, GL(N, $\mathbb{C}$), P) be a complex vector bundle that is associated with the "geometric arena" P(M, G, π, δ) of the particular Yang–Mills-type model. If we consider the determinant of the curvature of the former bundle, we obtain

$$\det\left(1+\frac{i}{2\pi}\Omega\right)=\overset{*}{\pi}(1+\gamma_1+\cdots+\gamma_m) \tag{2.8.1}$$

as an invariant polynomial. As can be shown, it is decomposable into gauge-invariant 2k-forms γ_k on P whose inverse images after the projection on the base space M read as follows:

$$\overset{*}{\pi}(\gamma_k)=\frac{(-1)^k}{(2\pi i)^k k!}\mathrm{Tr}(\Omega\wedge\cdots\wedge\Omega). \tag{2.8.2}$$

The number of curvature 2-forms Ω in the exterior product is k $\leq n/2$. From the Bianchi identity (2.6.12) and its projection onto a 2k-form on M (KN II, Chap. XII), it follows that the γ_k are *closed* exterior forms, i.e.,

$$\begin{aligned}\overset{*}{\pi}(d\gamma_k)&=d\overset{*}{\pi}(\gamma_k)=D\overset{*}{\pi}(\gamma_k)\\&=\frac{(-1)^k}{(2\pi i)^k(k-1)!}\mathrm{Tr}(D\Omega\wedge\Omega\wedge\cdots\wedge\Omega)=0.\end{aligned} \tag{2.8.3}$$

In that case, they are known to determine so-called cohomology classes. The structure of the latter is not determined by the particular choice of the connection ω, but depends solely on the bundle structure of P(M, G, π, δ) or its associated vector bundle V. In other words, the 2k-forms correspond to the *characteristic classes* of V. More precisely (see: KN II, Theorem 3.1), it can be stated that the abstractly defined k^{th} Chern class c_k(V) of a complex vector bundle V is represented by the closed 2k-form γ_k, as given above.

In order to obtain the characteristic classes of a real vector bundle $V^{\mathbb{R}}$ over M with the typical fiber $\mathbb{R}^N$, it will be enlarged to a complex vector bundle V with the typical fiber $\mathbb{C}^N$. The latter arises from the complexification of each fiber of $V^{\mathbb{R}}$. Then the so-called k^{th} Pontryagin class of $V^{\mathbb{R}}$ is defined by

$$p_k(V^{\mathbb{R}})=(-1)^k c_{2k}(V). \tag{2.8.4}$$

In the physically important case of a four-dimensional compact "spacetime" of the Euclidean signature s = 0, we find locally, cf. DANIEL & VIALLET (1980),

$$\overset{*}{\pi}(\gamma_1) = -\frac{1}{2\pi i}\mathrm{Tr}\Omega \quad (= 0) \tag{2.8.5}$$

$$\overset{*}{\pi}(\gamma_2) = -\frac{1}{8\pi^2}\{\mathrm{Tr}(\Omega \wedge \Omega) - \mathrm{Tr}(\Omega) \wedge \mathrm{Tr}(\Omega)\} \tag{2.8.6}$$

$$\left(= -\frac{1}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}{}^{j}F_{\mu\nu j}\sqrt{|g|}d^4x\right)$$

$$\overset{*}{\pi}(\gamma_k) = 0 \quad \text{for} \quad k \geq 3. \tag{2.8.7}$$

It is typical for the principal fiber bundles $\mathrm{P}(\mathrm{S}^4, \mathrm{SU(f)}, \pi, \delta)$, which are to be found in Yang–Mills theories, that they have special Lie groups as structure groups, i.e., those for which det G = 1 holds. Inasmuch as the trace of the Lie-algebra-valued curvature 2-form Ω vanishes in such instances, the gauge-invariant 4-form $\mathrm{Tr}(\Omega \wedge \Omega)$ suffices for a complete characterization of the configuration space. Any consideration of the general dynamics of the Yang–Mills gauge fields should therefore incorporate this form into the Lagrangian formalism.

The integration of the second Chern class $\overset{*}{\pi}(\gamma_2)$ over the base space yields a characteristic number,[5] which due to (2.8.1) is termed the Chern index:

$$c_2(M) = -\int_M \overset{*}{\pi}(\gamma_2) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(\Omega \wedge \Omega). \tag{2.8.8}$$

In the literature of physics, this topological invariant is often called the "Pontryagin index"; see JACKIW (1977). Concerning bundles with structure group G = SU(f), however, it is mathematically more precise to call it the Chern index, in order to preserve the term Pontryagin index for the classification of real associated vector bundles (MAYER & DRECHSLER 1977). Nevertheless, the denotation of BELAVIN et al. (1975) is correct, since in their paper, the isospin group SU(2) has been enlarged to the real structure group $\widetilde{\mathrm{SO}}(4) \approx \mathrm{SU}(2) \otimes \mathrm{SU}(2)$, due to a special isomorphism (HELGASON 1962).

The actual calculation of this index is firmly based on the fact that the projection of the closed form γ_2 onto M is even an exact one. This is a property that it shares with all exterior forms representing characteristic classes (CHERN & WHITE 1976). In this particular case, the relation

$$\mathrm{Tr}(\Omega \wedge \Omega) = d\,\mathrm{Tr}\left(\omega \wedge \Omega + \frac{1}{3}\omega \wedge \omega \wedge \omega\right) \tag{2.8.9}$$

[5] Excepting the case of the meron solutions, this will usually be an integer; see DE ALFARO et al. (1979)

holds. As for the proof, it has to be remembered first that the exterior derivative d commutes with the linear operator Tr of forming the trace. Then, the right-hand side of (2.8.9) yields the following chain of equations:

$$\begin{aligned}\mathrm{Tr}\left\{d\left(\omega\wedge d\omega-\frac{2}{3}\omega\wedge\omega\wedge\omega\right)\right\} &= \mathrm{Tr}\{d\omega\wedge d\omega-\omega\wedge dd\omega\\ &-\frac{2}{3}d\omega\wedge\omega\wedge\omega+\frac{2}{3}\omega\wedge d\omega\wedge\omega-\frac{2}{3}\omega\wedge\omega\wedge d\omega\}\\ &=\mathrm{Tr}\{d\omega\wedge d\omega-d\omega\wedge\omega\wedge\omega-\omega\wedge\omega\wedge d\omega\}\\ &=\mathrm{Tr}\{(d\omega-\omega\wedge\omega)\wedge(d\omega-\omega\wedge\omega)\}=\mathrm{Tr}(\Omega\wedge\Omega).\end{aligned} \tag{2.8.10}$$

(Since they are under trace, these forms can be treated as ordinary exterior forms, although otherwise, they are to be considered as Lie-algebra-valued differential forms.) It is especially the last step that makes use of the identity

$$\mathrm{Tr}\{\omega\wedge(\omega\wedge\omega\wedge\omega)\}=-\mathrm{Tr}\{(\omega\wedge\omega\wedge\omega)\wedge\omega\}=0. \tag{2.8.11}$$

Thus by the application of Stokes's theorem, the Chern index can be determined by the following integral over the boundary of M:

$$\begin{aligned}c_2(M)&=\frac{1}{8\pi^2}\int\limits_M d\mathrm{Tr}\left(\omega\wedge\Omega+\frac{1}{3}\omega\wedge\omega\wedge\omega\right)\\ &=\frac{1}{8\pi^2}\int\limits_{\partial M}\mathrm{Tr}\left(\omega\wedge d\omega-\frac{2}{3}\omega\wedge\omega\wedge\omega\right).\end{aligned} \tag{2.8.12}$$

For a further evaluation of this latter term, more information concerning the asymptotic behavior of the gauge fields is needed. If the dynamics of a Yang–Mills gauge theory is determined by a Lagrangian 4-form, it can be shown by an analysis of the action integral

$$S_{\mathrm{YM}}=\int\limits_M L_{\mathrm{YM}} \tag{2.8.13}$$

that the field strengths $\mathrm{F}_{\alpha\beta}{}^{\mathrm{j}}$ (components of the curvature form) have to vanish faster than $|\mathrm{x}|^{-2}$ at infinity, since otherwise, the integral (2.8.13) would not exist. In order to guarantee its finiteness, it is sufficient to postulate that the solutions behave asymptotically as "pure" ("fake" according to UTIYAMA 1980) gauge fields:

$$\overset{\infty}{\omega}:=-G^{-1}dG, \tag{2.8.14}$$

i.e., that the relation

$$\omega\sim\overset{\infty}{\omega} \tag{2.8.15}$$

should hold for $|x| \to \infty$. Here $\overset{\infty}{\omega}$ denotes a Maurer–Cartan connection, i.e., a left-invariant $\mathfrak{g}$-valued 1-form that satisfies the equation

$$\begin{aligned} &d\overset{\infty}{\omega} - \overset{\infty}{\omega} \wedge \overset{\infty}{\omega} \\ &= -d(G^{-1}) \wedge dG - G^{-1}ddG - (G^{-1}dG) \wedge (G^{-1}dG) \\ &= -d(G^{-1}) \wedge dG + G^{-1}Gd(G^{-1}) \wedge dG = 0 \end{aligned} \tag{2.8.16}$$

(see KN I, p. 41; however, with opposite sign conventions). Consequently, the curvature, derived from a pure gauge field, has to vanish:

$$\Omega(\overset{\infty}{\omega}) = 0. \tag{2.8.17}$$

If the boundary ∂M is chosen in such a way that only a pure gauge connection $\overset{\infty}{\omega}$ exists there, we get from (2.8.12) the relation

$$c_2(M) = \frac{1}{24\pi^2} \int_{\partial M} \mathrm{Tr}(\overset{\infty}{\omega} \wedge \overset{\infty}{\omega} \wedge \overset{\infty}{\omega}), \tag{2.8.18}$$

in full compliance with (2.8.15); cf. JACKIW (1980). If $\mathrm{G} = \mathrm{SU}(2)$ serves as a structure group, the 3-dimensional integral reduces itself to the (invariant) Haar integral over the 3-dimensional Lie group SU(2), regarded as a manifold. If ∂M is taken to be S^3 topologically, then it can be shown that $\mathrm{c}_2(\mathrm{M})$ determines the "mapping degree" or the *winding number*[6] of the mapping of S^3 in $\mathrm{S}^3 \approx \mathrm{SU}(2)$. These considerations explain why in the physics literature, the Chern index is regarded as the "quantum number" of the *topological charge*. It is of considerable importance for the classification of the so-called instanton solutions. Before focusing on these configurations, the works of ATIYAH & JONES (1978), ATIYAH (1979), which deal with further aspects of such global solutions, should be mentioned for those readers who prefer a more mathematically oriented approach.

References

Abers ES, Lee BW (1973) Gauge theories. Phys Rept 9:1

Atiyah MF (1979) Geometry of Yang-Mills fields. Lezioni Fermiane, Accademia Nazionale dei Lincei Scuola Normale Superione, Pisa

Atiyah MF, Jones JDS (1978) Topological aspects of Yang-Mills theory. Commun Math Phys 61(2):97–118

[6] Speaking more generally, the n^{th}-homotopy group of the n-dimensional sphere S^{n} is given by the group $\mathbb{Z}$ of integers, i.e., by $\pi_n(S^n) = \mathbb{Z}$. As such, it determines the winding number of the mapping $\mathrm{S}^{\mathrm{n}} \to \mathrm{S}^{\mathrm{n}}$.

Atiyah MF, Hitchin NJ, Singer IM (1978) Self-duality in four-dimensional Riemannian geometry. Proceedings of the Royal Society of London a: mathematical, physical and engineering sciences 362:425–461
Belavin AA, Polyakov AM, Schwartz AS, Tyupkin YS (1975) Pseudoparticle solutions of the Yang-Mills equations. Phys Lett B 59(1):85–87
Bjorken JD, Drell SD (1964) Relativistic quantum mechanics. McGraw-Hill, San Francisco
Bourguignon JP, Lawson HB (1981) Stability and isolation phenomena for Yang-Mills fields. Commun Math Phys 79(2):189–230
Chen BY (2000) Riemannian submanifolds. Handb Differ Geom 1:187–418
Cheng TP, Li LF (1984) Gauge theory of elementary particle physics. Clarendon Press, Oxford
Chern SS, White J (1976) Duality properties of characteristic forms. Inventiones mathematicae 35(1):285–297
Daniel M, Viallet CM (1980) The geometrical setting of gauge theories of the Yang-Mills type. Rev Mod Phys 52(1):175
de Alfaro V, Fubini S, Furlan G (1979) Gauge theories and strong gravity, Il Nuovo Cimento A 50(4):523–554
Eguchi T, Gilkey PB, Hanson AJ (1980) Gravitation, gauge theories and differential geometry. Phys Rep 66(6):213–393
Einstein A (1928) Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus, Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität, Sitzungsber. Preuss. Akad. Wiss. (Berlin), Phys. math. Kl. 217; 224
Einstein A (1949) Autobiographisches. Albert Einstein: Philosopher-Scientist. Cambridge University Press, Cambridge, pp 1–95
Friedberg R, Lee TD (1984) Derivation of Regge's action from Einstein's theory of general relativity. Nucl Phys B 242(1):145–166
Frolov VP (2006) Embedding of the Kerr-Newman black hole surface in Euclidean space. Phys Rev D 73(6):064021
Gell-Mann M, Neeman Y (1964) The eightfold way, WA Benjamin, Inc
Goenner HF (1984) Theories of gravitation with nonminimal coupling of matter and the gravitational field. Found Phys 4(9):865–881
Heisenberg W (1932) Über den Bau der Atomkerne. I. Z. Phys 77(1)
Helgason S (1962) Differential geometry and symmetric spaces. Academic Press, New York
Heller M, Staruszkiewicz A (1981) Fuzzy space-time. Zeitschrift für Naturforschung A 36(6):609–610
Howe PS, Tucker RW (1978) An approach to SU (2) gauge fields in Minkowski space-time. Nucl Phys B 138(1):73–84
Jackiw R (1977) Quantum meaning of classical field theory. Rev Mod Phys 49(3):681
Jackiw R (1980) Introduction to the Yang-Mills quantum theory. Rev Mod Phys 52(4):661
Kobayashi S, Nomizu K (1963) Foundations of differential geometry, vol 1. Interscience, New York
Kobayashi S, Nomizu K (1969) Foundations of differential geometry, vol 2. Interscience, New York
Laugwitz D (1965) Differential and Riemannian geometry. Academic Press, New York
Levi-Civita T (1926) The absolute differential calculus. Blackie & Son, London
Lubkin E (1963) Geometric definition of gauge invariance. Ann Phys 23(2):233–283
Mack G (1981) Physical principles, geometrical aspects, and locality properties of gauge field theories. Fortschr Phys 29(4):135–185
Mayer ME, Drechsler W (1977) Fiber bundle techniques in gauge theories. Lecture notes in physics, vol. 67, Springer
Milnor JW, Stasheff JD (1974) Characteristic classes, vol 76. Princeton University Press, Princeton
Misner C, Wheeler J (1957) Classical physics as geometry: gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space. Ann Phys 2(1957):525–603
Misner C, Thorne K, Wheeler J (1973) Gravitation. Freeman and Co, San Francisco (quoted as MTW)
Nash J (1956) The imbedding problem for Riemannian manifolds. Ann Math 63(1):20–63

Omori H (1973) Groups of diffeomorphisms and their subgroups. Trans Am Math Soc 179:85–122
O'Raifeartaigh L (1979) Hidden gauge symmetry. Rep Prog Phys 42(1):159
Penrose R (1968) Structure of space-time. In: DeWitt CM, Wheeler JA (eds) published in Battelle rencontres, 1967 lectures in mathematics and physics, Benjamin, New York, pp 121–235
Riemann B (1854) Habilitationschrift: Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. In: Weber H (ed) The collected works of Bernhard Riemann. Dover Publications (1953)
Sakharov AD (1984) Cosmological transitions with changes in the signature of the metric. Sov Phys JETP 60:214
Steenrod NE (1951) The topology of fibre bundles, vol 14. Princeton University Press, Princeton
Sulanke R, Wintgen P (1972) Differentialgeometrie und Faserbündel. Birkhäuser, Basel
Taylor JC (1979) Gauge theories of weak interactions. Cambridge University Press
Trautman A (1970) Fibre bundles associated with space-time. Rep Math Phys 1(1):29–62
Utiyama R (1956) Invariant theoretical interpretation of interaction. Phys Rev 101(5):1597
Utiyama R (1980) Introduction to the theory of general gauge fields. Progress Theoret Phys 64(6):2207–2221
Weinberg S (1974) Recent progress in gauge theories of the weak, electromagnetic and strong interactions. Rev Mod Phys 46(2):255
Weinberg S (1977) The future of unified gauge theories. Phys Today 30:42
Weyl H, (1923) Raum, Zeit, Materie: Vorlesungen über allgemeine Relativitätstheorie, Nachdruck der 5. Auflage Wiss. Buchgesellschaft, Darmstadt, (1961) English edition: Space-Time-Matter. Dover, N.Y, p 1952
Weyl H (1928) Gruppentheorie und Quantenmechanik. Verlag von S, Hirzel, Leipzig
Weyl H (1929) Gravitation and the electron. Proc Natl Acad Sci 15(4):323–334
Wigner EP (1931) Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren. Vieweg, Braunschweig
Wigner EP (1939) On unitary representations of the inhomogeneous Lorentz group. Ann Math 40:149–204
Wigner EP (1957) Relativistic invariance and quantum phenomena. Rev Mod Phys 29(3):255
Wilson KG (1976) Quarks on a lattice, or, the colored string model. Phys Rept 23:331
Wu TT, Yang CN (1975) Concept of nonintegrable phase factors and global formulation of gauge fields. Phys Rev D 12(12):3845
Yang CN, Mills RL (1954) Conservation of isotopic spin and isotopic gauge invariance. Phys Rev 96(1):191
Yang CN (1977). Magnetic monopoles, fiber bundles, and gauge fields, in Five Decades of Weak Interactions-Marshak Festschrift, Vol 294. Annals of the New York Academy of Sciences, New York, p. 86

http://www.springer.com/978-3-319-29732-3

Geometrodynamics of Gauge Fields
On the Geometry of Yang-Mills and Gravitational Gauge Theories
Mielke, E.W.
2017, XVII, 373 p. 18 illus., 8 illus. in color., Hardcover
ISBN: 978-3-319-29732-3