# Non-Abelian Gauge Theories with Spontaneous Symmetry Breaking : Higgs Mechanism

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**Abstract**

Goldstone theorem implies that theories with spontaneous symmetry breaking contain at least as many massless scalar fields as there are broken generators; the problem is that those massless bosons do not seem to correspond to physical particles. Similarly, non-Abelian gauge theories appear to be restrained to the description of massless gauge fields, whereas a theory of the weak interaction would require massive gauge fields. However, when a gauge theory is combined with a spontaneous symmetry breaking, the two problems solve themselves in a very elegant way. In this review, we present the fundamentals of non-Abelian gauge theories, and underline their inability to deal with massive vector fields without compromising the gauge invariance. The Georgi-Glashow model will serve us as a simple example in which this flaw is cured by the effect of the Higgs mechanism. We will finally analyse some general features of theories in which the Higgs mechanism takes place.

# 1 Non-Abelian Gauge Theories

## 1.1 Gauge principle

In electrodynamics, the gauge principle provides a method to transform a Lagrangian that is invariant with respect to global symmetry from the $U(1)$ group (Abelian) into a Lagrangian that is invariant with respect to local symmetry, or gauge-invariant. It consists in replacing all conventional derivatives $\partial_\mu$ by covariant derivatives $D_\mu = \partial_\mu + ieA_\mu$ and adding a kinetic energy term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$:

$$\mathcal{L}\big(\partial_\mu\varphi(x), \varphi(x)\big) \to \mathcal{L}\big(D_\mu\varphi(x), \varphi(x)\big) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

$A_\mu$ is called the gauge field, and the strength tensor $F_{\mu\nu}$ is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The choice of the kinetic energy term for the gauge field is unique up to normalization, provided that it is required to be (*i*) independent of the matter field $\varphi$, (*ii*) Lorentz-invariant, (*iii*) gauge-invariant, and (*iv*) quadratic in the first derivatives of the gauge field.

In the first part of this review we will generalize the gauge principle to non-Abelian groups, which has first been worked out by C.N. Yang and R.L. Mills in 1954 [10]. The motivation is the belief that the weak and even the strong interactions can be derived from non-Abelian gauge theories. Since we shall be interested only in Lagrangians with kinetic energy terms for the matter fields which are positive definite and group-invariant, and since only compact groups leave positive definite forms invariant, it will be sufficient to limit our considerations to compact groups.

## 1.2 Non-Abelian gauge transformations

The specificity of a non-Abelian Lie group is that its elements do not commute. In particular, its generators $t^a$, which form a basis for the vector space of infinitesimal transformations (called the Lie algebra), have the following commutation rule:

$$[t^a, t^b] = C^{abc}t^c. \tag{1}$$

The structure constants $C^{abc}$ are antisymmetric in the first two indices and real. The normalization is usually chosen as

$$Tr(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{2}$$

Within this orthonormal basis, the structure constants are antisymmetric with respect to all three indices. An element $\omega$ of the group can be expressed near unity in the form $\omega = exp(\theta^a t^a)$, where $\theta^a$ are the parameters of the transformation.

Let $\varphi(x)$ be a field that transforms covariantly in a given representation $T(\omega)$. This means that under a transformation we get

$$\varphi(x) \to \varphi'(x) = T(\omega)\varphi(x).$$

Since any representation of a compact group is equivalent to a unitary representation, we take $T(\omega)$ to be unitary without loss of generality. We assume that the Lagrangian $\mathcal{L}$ depends only on the field $\varphi(x)$ and the derivative $\partial_\mu\varphi(x)$:

$$\mathcal{L} = \mathcal{L}\big(\varphi(x), \partial_\mu\varphi(x)\big).$$

If the group element $\omega$ is independent of the spacetime coordinates (global symmetry), the derivation of the transformed field is equivalent to the transformation of the derived field:

$$\partial_\mu T(\omega)\varphi(x) = T(\omega)\partial_\mu\varphi(x).$$

Thus the field $\varphi$ and its derivative transform in the same way. By the unitarity of the representation, scalar products like $(\varphi,\varphi)$, $(\partial_\mu\varphi,\partial_\mu\varphi)$ or $(\varphi,\partial_\mu\varphi)$ are invariant under global transformation of the non-Abelian group. Any Lagrangian constructed out of such scalar products is globally invariant:

$$\mathcal{L}\big(\varphi(x),\partial_\mu\varphi(x)\big) = \mathcal{L}\big(T(\omega)\varphi, T(\omega)\partial_\mu\varphi\big).$$

**Covariant derivative** Our aim is the construction of a theory that is invariant under local transformations as well:

$$\boxed{\varphi(x) \to \varphi'(x) = T(\omega(x))\varphi(x),}$$

where $\omega(x) = exp(\theta^a(x)t^a)$ depend on the spacetime coordinates. The problem here is that the derivative of the field does not transform homogeneously:

$$\partial_\mu\varphi(x) \to \partial_\mu\varphi'(x) = T(\omega(x))\partial_\mu\varphi(x) + \big(\partial_\mu T(\omega(x))\big)\varphi(x),$$

and so a term like $(\partial_\mu\varphi,\partial_\mu\varphi)$ in a globally invariant Lagrangian will not be locally invariant. In analogy with electrodynamics, the idea is to replace conventional derivatives $\partial_\mu$ by covariant derivatives $D_\mu$, which by definition transforms like $\varphi(x)$:

$$\boxed{D_\mu\varphi(x) \to (D_\mu\varphi(x))' = T(\omega(x))D_\mu\varphi(x).} \tag{3}$$

The covariant derivative is constructed out of the combination of the conventional derivative with a set of gauge fields $A_\mu^a(x)$. There must be one gauge field for each generator $t^a$ of the symmetry group.[1] The covariant derivative is defined by

$$D_\mu\varphi(x) = (\partial_\mu + gA_\mu^a(x)T^a)\varphi(x),$$

with $T^a$ the generators in the representation $T$:

$$T^a = T(t^a).$$

With the introduction of the matrix field

$$A_\mu(x) = A_\mu^a(x)t^a, \tag{4}$$

we can rewrite this definition as

$$\boxed{D_\mu\varphi(x) = \big(\partial_\mu + gT(A_\mu(x))\big)\varphi(x).} \tag{5}$$

The arbitrary constant $g$ is the coupling constant. Note that the covariant derivative depends on the transformation property of the field on which it acts, since $A_\mu$ is in the same representation as the field $\varphi$.

[1]For example, the dimension of the Lie algebra of $SU(n)$, and thus the number of its generators, is $n^2-1$: indeed, an arbitrary complex $n\times n$ matrix has $2n^2$ elements but the two conditions of anti-hermicity and nullity of the trace leave only $n^2-1$ independent parameters. And so $SU(2)$ has 3 generators, $SU(5)$ 24 and so on.

The transformation rule for the field $A_\mu(x)$ is determined by the requirement that $D_\mu\varphi$ transforms like $\varphi$. The left part of (3)

$$D'_\mu\varphi' = \big(\partial_\mu + gT(A'_\mu)\big)\varphi' = T(\omega)\partial_\mu\varphi + (\partial_\mu T(\omega))\varphi + gT(A'_\mu)T(\omega)\varphi$$

must be equal to its right part

$$T(\omega)D_\mu\varphi = T(\omega)\partial_\mu\varphi + gT(\omega)T(A_\mu)\varphi.$$

The field $\varphi$ being arbitrary, the validity of this equality does not depend on it, and the requirement becomes

$$T(A'_\mu) = T(\omega)T(A_\mu)T(\omega)^{-1} - \frac{1}{g}(\partial_\mu T(\omega))T(\omega)^{-1}.$$

Using the consistency of the representation $T$ with the group operations and with the operations in the Lie algebra, we obtain

$$T(A'_\mu) = T(\omega A_\mu \omega^{-1}) - \frac{1}{g}T\big((\partial_\mu\omega)\omega^{-1}\big).$$

Since this is true for any representation, the transformation rule for $A_\mu$ reads finally

$$\boxed{A_\mu \to A'_\mu = \omega A_\mu \omega^{-1} + \tfrac{1}{g}\omega\partial_\mu\omega^{-1},} \tag{6}$$

where we have used the identity $\partial_\mu(\omega\omega^{-1}) = (\partial_\mu\omega)\omega^{-1} + \omega\partial_\mu\omega^{-1} = 0$.

We have determined the form and the transformation properties of the non-Abelian covariant derivative, so that we are able to perform the first step of the generalized gauge principle:

$$\mathcal{L}\big(\partial_\mu\varphi(x), \varphi(x)\big) \to \mathcal{L}\big(D_\mu\varphi(x), \varphi(x)\big).$$

**Kinetic energy term for $A_\mu$** Let us now construct the kinetic energy term for the vector field $A_\mu$. By analogy to electrodynamics, we want to construct it out of a strength tensor $F_{\mu\nu}(x)$ transforming in the following way:

$$\boxed{F_{\mu\nu} \to F'_{\mu\nu} = \omega F_{\mu\nu}\omega^{-1}.} \tag{7}$$

However, $F_{\mu\nu}$ cannot have in the non-Abelian case the same form as in the Abelian case, because it would not have the required transformation property. We see indeed from (6) that

$$\begin{aligned} \partial_\mu A'_\nu - \partial_\nu A'_\mu = {} & \omega(\partial_\mu A_\nu - \partial_\nu A_\mu)\omega^{-1} \\ & + \big[(\partial_\mu\omega)A_\nu\omega^{-1} + \omega A_\nu \partial_\mu\omega^{-1} \\ & + \frac{1}{g}\partial_\mu\omega\partial_\nu\omega^{-1} + \frac{1}{g}\omega\partial_\mu\partial_\nu\omega^{-1} - (\mu \leftrightarrow \nu)\big], \end{aligned} \tag{8}$$

where $(\mu \leftrightarrow \nu)$ stands for the repetition of the terms in the brackets but with the Lorentz indices switched. The terms with second derivatives cancell out, but the other terms in the brackets prevent $F_{\mu\nu}$ from transforming homogeneously.

To eliminate those terms we add to the definition of the strength tensor in electrodynamics a term containing the commutator $[A_\mu, A_\nu]$:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu].} \tag{9}$$

Notice that this addition is compatible with the antisymmetry of $F_{\mu\nu}$ and its belonging to the Lie algebra. So defined, the non-Abelian strength tensor meets the requirement (7).

To show it, we compute the transformation rule for the new term:

$$\begin{aligned} g[A'_\mu, A'_\nu] &= g\omega[A_\mu, A_\nu]\omega^{-1} \\ &+ \big[\omega(\partial_\mu\omega^{-1})\omega A_\nu\omega^{-1} - \omega A_\nu\omega^{-1}\omega(\partial_\mu\omega^{-1}) \\ &+ \frac{1}{g}\omega(\partial_\mu\omega^{-1})\omega(\partial_\nu\omega^{-1}) - (\mu \leftrightarrow \nu)\big]. \end{aligned} \tag{10}$$

The terms in the brackets are equal to the undesirable terms in (8), except for their signs:

$$\begin{aligned} \omega(\partial_\mu\omega^{-1})\omega A_\nu\omega^{-1} &= -(\partial_\mu\omega)A_\nu\omega^{-1}, \\ -\omega A_\nu\omega^{-1}\omega(\partial_\mu\omega^{-1}) &= -\omega A_\nu\partial_\mu\omega^{-1}, \\ \frac{1}{g}\omega(\partial_\mu\omega^{-1})\omega(\partial_\nu\omega^{-1}) = -\frac{1}{g}(\partial_\mu\omega)\omega^{-1}\omega(\partial_\nu\omega^{-1}) &= -\frac{1}{g}\partial_\mu\omega\partial_\nu\omega. \end{aligned}$$

In consequence, the non-Abelian strength tensor defined in (9) transforms as required.

Reminding the definition of the matrix field $A_\mu = A^a_\mu t^a$, we can write

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A^a_\nu t^a - \partial_\nu A^a_\mu t^a + g[A^b_\mu t^b, A^c_\nu t^c] \\ &= \partial_\mu A^a_\nu t^a - \partial_\nu A^a_\mu t^a + gA^b_\mu A^c_\nu[t^b, t^c] \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc}A^b_\mu A^c_\nu)t^a, \end{aligned}$$

where we have used the commutation rule (1) and the antisymmetry of the structure constant $C^{abc}$. And thus the components of $F_{\mu\nu}$ in the basis formed by the generators are

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc}A^b_\mu A^c_\nu. \tag{11}$$

The only thing left to do to achieve the generalization of the gauge principle is to construct a kinetic energy term for $A_\mu$ that would be, as in electrodynamics, $(i)$ independent of the matter field $\varphi$, $(ii)$ Lorentz-invariant, $(iii)$ gauge-invariant, and $(iv)$ quadratic in the first derivatives of the gauge field. The trace of the product of the strength tensor with itself $Tr(F_{\mu\nu}F^{\mu\nu})$ satisfies all those conditions. In particular, the gauge-invariance is provided by the cyclicity of the trace:

$$Tr(F_{\mu\nu}F^{\mu\nu}) \to Tr(\omega F_{\mu\nu}F_{\mu\nu}\omega^{-1}) = Tr(F_{\mu\nu}F^{\mu\nu}).$$

## 1.3 Gauge-invariant Lagrangian

We have reached our initial goal: we are now in possession of a model that is invariant with respect to local transformations from a non-Abelian symmetry group. The Lagrangian of this model may contain the following gauge-invariant quantities:

$$(D_\mu\varphi, D^\mu\varphi),$$

$$(\varphi, \varphi),$$

and

$$Tr(F_{\mu\nu}F^{\mu\nu}).$$

For example the Lagrangian may look like this:[2]

$$\mathcal{L} = (D_\mu\varphi, D^\mu\varphi) - m^2\varphi^2 - \lambda\varphi^4 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}.$$

It is clear that the gauge principle is a universal procedure, which is independent of the form of the Lagrangian, and of the nature of the matter fields (*i.e.* of the representations to which they belong). It is important to note that the gauge principle not only extends a global to a local symmetry, but determines uniquely the interaction of the gauge field - with itself (through the kinetic energy term) and with the matter fields (through the covariant derivative). Thus it determines not only the symmetry, but also the dynamics.

**Gauge field masses** An serious defect of the gauge-invariant Lagrangian obtained is that it resists the introduction of mass terms for the gauge fields. Indeed, if we introduce a mass term like

$$\frac{1}{2}m^2 A_\mu A^\mu,$$

the gauge invariance would be destroyed, since this term is manifestly not invariant with respect to the transformation (6) of the gauge fields. This is very problematic, because if non-Abelian gauge theories are to be applied to the description of physical interactions, the gauge fields must be identified with observable vector fields, and yet the only massless vector field which has been observed is the photon. To be consistent with experiment, gauge theories need massive gauge fields.

How then can masses be introduced without destroying the gauge invariance of the Lagrangian? The answer is that mass terms are induced by spontaneously broken symmetry.

[2] The terms of order greater than quartic in the fields are generally not considered because in quantum theory they turn out to be non-renormalizable.

# 2 Higgs Mechanism

It has been shown by J. GOLDSTONE that theories with spontaneous symmetry breaking imply the existence of massless particles [3]; however, massless particles are not observed in reality. On the other hand, we have seen that non-Abelian gauge theories seem to be only suited to describe massless vector fields, while the gauge bosons of the weak interaction are massive. We owe to the scottish physicist P.W. HIGGS the discovery that the combination of these two problematic theories leads surprisingly to a most elegant resolution [4]. We will familiarize ourselves with the Higgs mechanism thank to a simple example.

## 2.1 The Georgi-Glashow model

As an illustration we shall study a model proposed in 1972 by H. GEORGI and S.L. GLASHOW from Harvard as a candidate for the theory of weak interaction [2].

**Adjoint representation** In this model, the gauge group is $SU(2)$ and the field $\varphi(x)$ is in the adjoint representation. As the space of the adjoint representation coincides with the Lie algebra of the symmetry group, we can write

$$\varphi(x) = \varphi^a(x)t^a,$$

where the $\varphi^a(x)$ are three real fields ($a = 1, 2, 3$); the $t^a = \frac{\tau^a}{2i}$ are the anti-Hermitian generators of $SU(2)$, which obey the commutation rule

$$[t^a, t^b] = \epsilon^{abc}t^c.$$

The linear operator $Ad(\omega)$ of the adjoint representation corresponding to the group element $\omega$ acts on a matrix $A$ belonging to the algebra as follows:

$$Ad(\omega)A = \omega A \omega^{-1}.$$

Thus $\varphi$ transforms as

$$\varphi \to \varphi' = Ad(\omega)\varphi = \omega\varphi\omega^{-1}.$$

The adjoint representation of a Lie algebra is such that for any elements $A, B$ of the algebra

$$Ad(A)B = [A, B].$$

Then the form of the covariant derivative in the adjoint representation follows from the definition (5):

$$\begin{aligned} D_\mu\varphi(x) &= \big(\partial_\mu + gT(A_\mu(x))\big)\varphi(x) \\ &= \partial_\mu\varphi + gAd(A_\mu)\varphi \\ &= \partial_\mu\varphi + g[A_\mu, \varphi]. \end{aligned}$$

If we express $\varphi$ and $A_\mu$ in components we get

$$D_\mu\varphi = \partial_\mu\varphi^a t^a + gA^b_\mu\varphi^c[t^b, t^c]$$

$$= (\partial_\mu \varphi^a + g\epsilon^{abc} A^b_\mu \varphi^c) t^a,$$

where we have used the antisymmetry of $\epsilon^{abc}$. Thus, $D_\mu \varphi$ is also an element of the Lie algebra, with components

$$D_\mu \varphi^a = \partial_\mu \varphi^a + g\epsilon^{abc} A^b_\mu \varphi^c,$$

and it transforms like $\varphi$ (by definition):

$$D_\mu \varphi \to (D_\mu \varphi)' = \omega (D_\mu \varphi) \omega^{-1}.$$

We have already met this kind of transformation when we talked about the strength tensor (see (7)), and we deduced from it that $Tr(F_{\mu\nu} F^{\mu\nu})$ was gauge-invariant. By analogy, the invariant quantities are here

$$Tr(D_\mu \varphi, D^\mu \varphi),$$

$$Tr(\varphi^2).$$

Having in mind the usual normalization (2) for the generators, we may write the Lagrangian of the Georgi-Glashow model in the following form:

$$\boxed{\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \tfrac{1}{2} (D_\mu \varphi)^a (D^\mu \varphi)^a - \tfrac{\lambda}{4} (\varphi^a \varphi^a - v^2)^2,} \tag{12}$$

where $v$ is a constant. The remarkable feature here is the form of the potential $V = \frac{\lambda}{4} (\varphi^a \varphi^a - v^2)^2$, which will lead to symmetry breaking.

**Field equations** To find the field equations for the gauge fields we begin by considering the variation of the action $S_A = \int d^4x (-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu})$ with respect to the real fields $A^a_\mu$. We find

$$\delta S_A = \int d^4x (-\frac{1}{2} \delta F^a_{\mu\nu} F^{a\mu\nu}),$$

where with (11)

$$\delta F^a_{\mu\nu} = \partial_\mu \delta A^a_\nu + g\epsilon^{abc} A^b_\mu \delta A^c_\nu - (\mu \leftrightarrow \nu).$$

But by the antisymmetry of $F^a_{\mu\nu}$ we have

$$F^a_{\mu\nu} = \frac{1}{2} (F^a_{\mu\nu} - F^a_{\nu\mu}),$$

and so

$$\delta F^a_{\mu\nu} = 2(\partial_\mu \delta A^a_\nu + g\epsilon^{abc} A^b_\mu \delta A^c_\nu).$$

Thus,

$$\delta S_A = \int d^4x \Big( \partial_\mu F^{a\mu\nu} + g\epsilon^{abc} A^b_\mu F^{c\mu\nu} \Big) \delta A^a_\nu,$$

where the first term has been submitted to integration by part and the indices of the second have been renamed. Without matter fields, the field equation for the gauge field would be

$$\partial_\mu F^{a\mu\nu} + g\epsilon^{abc} A^b_\mu F^{c\mu\nu} = 0,$$

or in a more compact form

$$(D_\mu F^{\mu\nu})^a = 0.$$

Together with the Bianchi identity $\epsilon^{\mu\nu\lambda\rho}(D_\nu F_{\lambda\rho})^a = 0$, this equation is analogous to the Maxwell equations in electrodynamics. We now variate the matter-part of the action with respect to $A^a_\mu$:

$$\begin{aligned}\delta S_\varphi &= \int d^4x \Big( (D^\mu\varphi)^a g\epsilon^{abc}\delta A^b_\mu \varphi^c \Big) \\ &= \int d^4x \Big( -g\epsilon^{abc}(D^\mu\varphi)^b\varphi^c \Big)\delta A^a_\mu.\end{aligned}$$

As the variation of the total action $S = S_A + S_\varphi$ must vanish, we get the following equation for the gauge fields:

$$\boxed{(D_\mu F^{\mu\nu})^a = g\epsilon^{abc}(D^\nu\varphi)^b\varphi^c.}$$

The variation of the action with respect to the fields $\varphi^a$ gives

$$\begin{aligned}\delta S &= \int d^4x \Big( (D^\mu\varphi)^a(\partial_\mu\delta\varphi^a + g\epsilon^{abc}A^b_\mu\delta\varphi^c) - \lambda(\varphi^b\varphi^b - v^2)\varphi^a\delta\varphi^a \Big) \\ &= \int d^4x \Big( -(\partial_\mu(D^\mu\varphi)^a + g\epsilon^{abc}A^b_\mu(D^\mu\varphi)^b - \lambda(\varphi^b\varphi^b - v^2)\varphi^a \Big)\delta\varphi^a.\end{aligned}$$

The field equation for the matter fields is then:

$$\boxed{D_\mu(D^\mu\varphi)^a = -\lambda(\varphi^b\varphi^b - v^2)\varphi^a.}$$

The energy functional is easily computed by first fixing the gauge at $A_\mu = 0$. We obtain

$$\begin{aligned}E &= \int d^3x T_{00} = \int d^3x \Big( \partial_0 A^a_i \partial_0 A^a_i + \partial_0\varphi^a\partial_0\varphi^a - \mathcal{L} \Big) \\ &= \int d^3x \Big( \frac{1}{2}\partial_0 A^a_i\partial_0 A^a_i + \frac{1}{4}F^a_{ij} + \frac{1}{2}\partial_0\varphi^a\partial_0\varphi^a + \frac{1}{2}(D_i\varphi)^a(D_i\varphi)^a + V \Big).\end{aligned}$$

As the energy does not depend on the choice of gauge, we can restore the generality, taking care to make the necessary changes to preserve the gauge invariance. The energy functional for the Georgi-Glashow model is finally given by

$$\begin{aligned}E = \int d^3x \Big( &\frac{1}{2}F^a_{0i}F^a_{0i} + \frac{1}{4}F^a_{ij}F^a_{ij} \\ &+\frac{1}{2}(D_0\varphi)^a(D_0\varphi)^a + \frac{1}{2}(D_i\varphi)^a(D_i\varphi)^a \\ &+ \frac{\lambda}{4}(\varphi^a\varphi^a - v^2)^2 \Big). \end{aligned} \tag{13}$$

## 2.2 Vacuum

The vacuum, or ground state, is the configuration of the fields $A_\mu(x)$ and $\varphi(x)$ that minimizes the energy. The first two terms in (13) are minimal when the electric and magnetic fields are equal to zero, *i.e.* $A_\mu$ is a pure gauge:

$$A_\mu = \frac{1}{g}\omega\partial\omega^{-1}.$$

The third and fourth terms are minimal when $D_\mu\varphi = 0$, which means that the field $\varphi(x)$ is covariantly constant:

$$\varphi(x) = \omega(x)\varphi_{(v)}\omega^{-1}(x),$$

where $\varphi_{(v)}$ is a constant column. The value of $\varphi_{(v)}$ is determined by the minimization of the potential energy term, which gives

$$\varphi^a\varphi^a = v^2.$$

Thus $\varphi(x)$ may be associated to a constant vector of norm $v$ in the configuration space, pointing from the origin to an arbitrary orientation.

We have obtained a family of gauge-equivalent vacua, among which we have to choose one. For simplicity, we choose the ground state to be

$$\varphi^1_{(v)} = \varphi^2_{(v)} = 0,$$

$$\varphi^3_{(v)} = v \tag{14}$$

and

$$A_\mu^{(v)} = 0.$$

The non-zero value of the vacuum is the source of asymmetry. Indeed the chosen vacuum is not invariant with respect to rotations around the first and second axis (see Figure 1). Yet, it conserves the invariance with respect to rotations around the third axis, which means that a subgroup of the original symmetry group $SU(2)$ remains unbroken: it is the $U(1)$ group (the symmetry group of electrodynamics). That is an example of partial symmetry breaking.

In case of gauge symmetry however the symmetry is not really broken, since the chosen vacuum can always be gauge-transformed to any other, and the Lagrangian written in terms of deviations of the fields from the vacuum values is still gauge-invariant. This is why some authors rather talk about "hidden" symmetry.

## 2.3 Particle spectrum

In practice, it is not possible to observe a field in the ground state directly, since any observation has to do with changes of physical quantities in space and time. But the non-triviality of the ground state leads to meaningful consequences for the small excitations around it, which in quantum theory correspond to elementary particles. The study of the form of the Lagrangian (12) for small perturbations around the vacuum will tell us of the spectrum of the particles in the Georgi-Glashow model.

Figure 1: Configuration space. The chosen vacuum $\varphi_{(v)}$ is not invariant under the action of the generators $t^1$ and $t^2$.

**Mass terms** We are particularially interested in quadratic terms appearing in the Lagrangian. The reason for it is that if the action for some real scalar field $\psi(x)$ has the form

$$S = \int d^4x \Big( \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{m^2}{2} \psi^2 \Big),$$

then the field equation $(\Box + m^2)\psi$ corresponds to the Klein-Gordon equation for a massive scalar field. For a vector field $B_\mu(x)$, the action leading to the massive Klein-Gordon equation has the form

$$S = \int d^4x \Big( -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{m^2}{2} B_\mu B^\mu \Big),$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Indeed, the field equation is given by

$$\partial^\mu B_{\mu\nu} + m^2 B_\nu = 0;$$

differentiating and using the antisymmetry of $B_{\mu\nu}$, we obtain $\partial_\nu B_\nu = 0$. With this result and the definition of $B_{\mu\nu}$, we get $\partial^\mu \partial_\mu B_\nu - \partial^\mu \partial_\nu B_\mu + m^2 B_\nu = 0$, and then, as expected,

$$(\Box + m^2) B_\nu = 0.$$

The reasonings for complex fields are analogous. Therefore, every field possessing in addition to a kinetic energy term a quadratic term in the Lagrangian has a mass.

**Small perturbations** Excitations of the vector fields are described by $A_\mu(x)$ itself and excitations of the matter fields by three real scalar fields $\xi^a(x)$. The small perturbations around the vacuum (14) are given by $A_\mu(x)$ and $\varphi(x)$, with

$$\begin{aligned}\varphi^1(x) &= \xi^1(x),\\ \varphi^2(x) &= \xi^2(x),\\ \varphi^3(x) &= v + \xi^3(x).\end{aligned} \tag{15}$$

In the Lagrangian for these perturbative fields, the first term reduces to quadratic order to

$$-\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu},$$

where $\mathcal{F}^a_{\mu\nu}$ is the strength tensor of electrodynamics: $\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$. To linear order, the covariant derivative becomes

$$\begin{aligned}\begin{pmatrix} (D_\mu\varphi)^1 \\ (D_\mu\varphi)^2 \\ (D_\mu\varphi)^3 \end{pmatrix}^{lin} &= \begin{pmatrix} \partial_\mu\xi^1 \\ \partial_\mu\xi^2 \\ \partial_\mu\xi^3 \end{pmatrix} + g\begin{pmatrix} \epsilon^{1bc}A^b_\mu\varphi^c \\ \epsilon^{2bc}A^b_\mu\varphi^c \\ \epsilon^{3bc}A^b_\mu\varphi^c \end{pmatrix}\\ &= \begin{pmatrix} \partial_\mu\xi^1 + gvA^2_\mu \\ \partial_\mu\xi^2 - gvA^1_\mu \\ \partial_\mu\xi^3 \end{pmatrix}\end{aligned}$$

The potential energy term for the perturbations is given by

$$V = \frac{\lambda}{4}\big((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + 2\xi^3 v\big)^2.$$

To quadratic order only one term remains:

$$V^{quad} = \lambda v^2(\xi^3)^2.$$

The quadratic part of the Lagrangian written in terms of the perturbations around the vacuum is then given by

$$\begin{aligned}\mathcal{L}^{quad} = &-\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu} + \frac{1}{2}(\partial_\mu\xi^3)^2 - \lambda v^2(\xi^3)^2\\ &+\frac{1}{2}(\partial_\mu\xi^1)^2 + \frac{1}{2}(\partial_\mu\xi^2)^2\\ &+\frac{g^2v^2}{2}(A^1_\mu)^2 + \frac{g^2v^2}{2}(A^2_\mu)^2\\ &+ 2gvA^2_\mu\partial^\mu\xi^1 - 2gvA^1_\mu\partial^\mu\xi^2.\end{aligned} \tag{16}$$

The fields $A^1_\mu, A^2_\mu$ and $\xi^1, \xi^2$ have gotten mixed up in a way whose interpretation is not immediately apparent. Without the two last terms, we could have concluded that the vector fields $A^1_\mu$ and $A^2_\mu$ have masses equal to $gv$ and that the scalar fields $\xi^1$ and $\xi^2$ are massless.

In the case of global symmetry, in which the vector potential could be "switched off", the spectrum would agree with the predictions of Goldstone theorem: the massless fields would correspond to the broken generators $t^1$ and $t^2$, and the massive field $\xi^3$ to the unbroken $t^3$. But this spectrum disagrees with observation.

Figure 2: The $(\varphi^1, \varphi^3)$-plane in configuration space. A small deviation $\xi^1$ from the vacuum $\varphi_{(}v)$ in the $\varphi^1$ direction is induced by the generator $t^2$.

**Unitary gauge** To discover the particle spectrum, we recall that the Lagrangian is invariant under gauge transformations. The idea is to use this gauge freedom to get rid of the undesirable fields $\xi^1$ and $\xi^2$.

Which gauge transformation should we use? We notice that the small deviations of the fields $\varphi^1$ and $\varphi^2$ could be obtained by rotations around the second and first axis respectively (see Figure 1); those rotations correspond to the generators $t^2$ and $t^1$ of $SU(2)$. The answer is then obvious: the gauge transformation that makes the fields $\xi^1$ and $\xi^2$ disappear will just be the inverse rotation. We understand from Figure 2 that deviations from the ground state in the $\varphi^1$ and $\varphi^2$ directions of angles $\alpha^1 = \frac{\xi^1}{v}$ and $\alpha^2 = \frac{\xi^2}{v}$ respectively are induced by the following gauge transformation:

$$\omega(x) = exp(\alpha^1 t^2 - \alpha^2 t^1).$$

By the application of the transformation $\omega^{-1}$ to the generic perturbations (15), we obtain gauge-equivalent perturbations in which the fields $\xi^1$ and $\xi^2$ do not appear:

$$\varphi \to \omega^{-1}\varphi\omega = \begin{pmatrix} 0 \\ 0 \\ v + \xi^3 \end{pmatrix}.$$

Note that the excitation $\xi^3$ is not induced by the generator $t^3$, which leaves the vacuum $\varphi_{(v)}$ invariant, and so cannot be eliminated by gauge transformation.

In this particular gauge the quadratic Lagrangian has the more pleasant form

$$\begin{aligned} \mathcal{L}^{quad} &= -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu} \\ &+\frac{1}{2}(\partial_\mu \xi^3)^2 - \lambda v^2 (\xi^3)^2 \\ &+ \frac{g^2 v^2}{2}(A^1_\mu)^2 + \frac{g^2 v^2}{2}(A^2_\mu)^2. \end{aligned} \tag{17}$$

The unphysical Goldstone fields have been gauged away, so to speak. This gauge, in which only the physical fields appear in the Lagrangian, is called the unitary gauge.

**Interpretation** The Lagrangian (17) contains no terms coupling different fields, so that the spectrum can be simply read off quadratic terms. We identify the following fields:

- one scalar field $\xi^3$ with mass $m_H = \sqrt{2\lambda}v$,
- two massive vector fields $A^1_\mu$ and $A^2_\mu$ with mass $m_V = gv$,
- one massless vector field $A^3_\mu$ corresponding to the unbroken generator $t^3$.

The two scalar fields that would have been massless Goldstone fields in a theory with global symmetry have disappeared and two of the vector fields have become massive. It is as if the vector fields had "eaten up" the would-be Goldstone fields and acquired a mass. The remaining massless gauge field $A^3_\mu$ is identified with the gauge field of electrodynamics, that is the photon. Applied to non-Abelian gauge theories, the Higgs mechanism appears as a plausible solution to the problem of the gauge fields masses, which are then generated through spontaneous symmetry breaking; on the same occasion, the unphysical Goldstone fields can be evacuated by gauge transformation.

# 3 Discussion

**Spontaneous symmetry breaking** A necessary condition for the Higgs mechanism to take place is the non-triviality of the vacuum expectation value. We have seen that this is related to the specific form of the potential energy term in the Lagrangian. Let us analyse this relation in greater details, starting from a generic potential:

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda\varphi^4,$$

with the constant $\lambda > 0$. The ground state is the configuration that minimizes the potential energy and so must be a solution of:

$$\frac{\partial V}{\partial \varphi} = m^2\varphi + \lambda\varphi^3 = 0.$$

Two cases are to be distinguished, caracterized by the sign of the parameter $m^2$.

If $m^2 \geq 0$, the potential energy has a unique extremum at $\varphi_{(v)} = 0$. This vacuum does not break the symmetry and so the particle spectrum is straightforward: it is composed of the small excitations around the vacuum.

Now in the case where $-m^2 = \mu^2 > 0$, the potential has three extrema (see Figure 3). $\varphi = 0$ is a local maximum and depicts an unstable configuration. There are two new solutions which minimize the potential energy: $\varphi_{(v)} = \pm\frac{\mu}{\sqrt{\lambda}}$. The choice of any one of the two corresponding configurations breaks (or hide) the symmetry of the Lagrangian.[3] The particle spectrum is modified by the effect of the Higgs mechanism: to each broken generator corresponds a gauge field that became massive by the absorption of a Goldstone field. If the symmetry is

[3]It should be mentionned that at high temperature, the vacuum expectation value $v$ is neglectable, so that the symmetry is restored.

Figure 3: Supercritical bifurcation. The variation of the parameter $\mu^2$ leads to a qualitative change of the configuration $\varphi_{(v)}$ that minimizes the potential energy $V$: for $\mu^2 \leq 0$ $V$ has a unique minimum at $\varphi_{(v)} = 0$, but for $\mu^2 > 0$ both $V(\pm v)$ are minima, whereas $V(0)$ is a local maximum (unstable configuration).

only partially broken, there are also as many massless gauge fields as unbroken generators.

Thus we see that by the variation of a parameter of the theory, the particle spectrum can undergo dramatic modifications. This kind of bifurcation phenomenon is very frequent in physics, and takes place for example in superconductivity (Ginzburg-Landau potential).

The Georgi-Glashow model studied in section 2 is an illustration of the case $\mu^2 > 0$; here $v$ is equal to $\frac{\mu}{\sqrt{\lambda}}$ and the minimum of the potential has been set to zero by the addition of the constant term $\frac{\mu^4}{4\lambda}$, which does not modifiy the equations of motion.

**Degrees of freedom** The Goldstone fields do not purely vanish from the theory, but are in fact responsible for the longitudinal components of the massive gauge fields. In the Georgi-Glashow model, we can understand it if we rewrite the Lagrangian (16) as

$$\begin{aligned}\mathcal{L}^{quad} &= -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}_a^{\mu\nu} + \frac{1}{2}(\partial_\mu\xi^3)^2 - \lambda v^2(\xi^3)^2 \\ &+\frac{g^2v^2}{2}(A^2_\mu + \frac{1}{gv}\partial_\mu\xi^1)^2 \\ &+\frac{g^2v^2}{2}(A^1_\mu - \frac{1}{gv}\partial_\mu\xi^2)^2,\end{aligned}$$

and change the field variables in the following way

$$B^1_\mu = A^1_\mu - \frac{1}{gv}\partial_\mu\xi^2$$

$$B^2_\mu = A^2_\mu + \frac{1}{gv}\partial_\mu\xi^1.$$

We then obtain a theory with the same spectrum that the one we found by fixing the gauge. The massive vector fields $B^\alpha_\mu$ ($\alpha = 1, 2$) are composed of the massless vector fields $A^\alpha_\mu$, which have two transversal degrees of freedom, and a term containing the derivatives of the Goldstone fields. If we consider the fields $\xi^\alpha$ as plane waves, we see that their derivatives are proportional to the wave vectors $k^\alpha_\mu$, and therefore the second terms in the expression of the fields $B^\alpha_\mu$ are longitudinal components. The massive gauge fields have thus three degrees of freedom each.

We can check that the Higgs mechanism preserves the number of degrees of freedom: generic perturbations count three massive scalar fields (with one degrees of freedom each) and three massless vector fields (with two degrees of freedom each), so a total of 9 degrees of freedom. On the other hand, the particles spectrum obtained contains one massive scalar field, two massive vector fields (with three degrees of freedom), and one massless vector field, for a total of 9 degrees of freedom too.

**The $W^+_\mu$ and $W^-_\mu$ bosons** Let us determine the interaciton of the fields with the electromagnetic field. Instead of the two real fields $A^1_\mu$ and $A^2_\mu$, it is more convenient to consider a single complex field $W^+_\mu$ and its conjugate $W^-_\mu$ defined by

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu).$$

The Lagrangian to quadratic order becomes

$$\begin{aligned}\mathcal{L}^{quad} = &-\frac{1}{2}\mathcal{W}^+{}_{\mu\nu}\mathcal{W}^{-\mu\nu} - \frac{1}{4}\mathcal{F}^3_{\mu\nu}\mathcal{F}^{3\mu\nu}\\ &+\frac{1}{2}(\partial_\mu\xi^3)^2 - \lambda v^2(\xi^3)^2\\ &+g^2v^2W^+_\mu W^-_\mu,\end{aligned}$$

where we have noted

$$\mathcal{W}^\pm{}_{\mu\nu} = \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu.$$

The mass of the $W^\pm_\mu$ field is $m_W = gv$.

The unbroken subgroup $U(1)_{em}$ consists of the transformations with gauge function

$$\omega(x) = e^{\alpha(x)\frac{\tau^3}{2i}}.$$

It acts on $A_\mu = A^a_\mu \frac{\tau^a}{2i}$ according to the rule (6):

$$\begin{aligned}A'_\mu &= e^{\alpha\frac{\tau^3}{2i}}(A^1_\mu\frac{\tau^1}{2i} + A^2_\mu\frac{\tau^2}{2i} + A^3_\mu\frac{\tau^3}{2i})e^{-\alpha\frac{\tau^3}{2i}}\\ &+\frac{1}{g}e^{\alpha\frac{\tau^3}{2i}}\partial_\mu e^{-\alpha\frac{\tau^3}{2i}}\\ &= (A^1_\mu cos\alpha - A^2_\mu sin\alpha))\frac{\tau^1}{2i}\\ &+(A^2_\mu cos\alpha + A^1_\mu sin\alpha))\frac{\tau^2}{2i}\end{aligned}$$

$$+(A^3_\mu - \frac{1}{g}\partial_\mu\alpha)\frac{\tau^3}{2i}.$$

The transformation rules for the fields $A^a_\mu$ under the action of the $U(1)_{em}$ group are then:

$$A^1_\mu \to A'^1_\mu = A^1_\mu cos\alpha - A^2_\mu sin\alpha$$

$$A^2_\mu \to A'^2_\mu = A^2_\mu cos\alpha + A^1_\mu sin\alpha$$

$$A^3_\mu \to A'^3_\mu = A^3_\mu - \frac{1}{g}\partial_\mu\alpha.$$

This result was to be expected, as the generator $\frac{\tau^3}{2i}$ corresponds to rotations around the third axis in configuration space. The transformation rules for the $W^\pm_\mu$ fields can now be deduced:

$$\frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu) \to \frac{1}{\sqrt{2}}\big(A^1_\mu(cos\alpha \pm isin\alpha) \pm iA^2_\mu(cos\alpha \pm isin\alpha)\big),$$

and thus,

$$W^\pm_\mu \to W'^\pm_\mu = e^{\pm i\alpha}W^\pm_\mu.$$

This implies that the fields $W^+_\mu$ and $W^-_\mu$ have electric charges $+1$ and $-1$ respectively. They are thus valid candidates to describe the exchange bosons $W^\pm$ of weak interactions.

Note that the Higgs field $\xi^3$ does not transform under the $U(1)_{em}$ gauge group, and is then electrically neutral.

However, it was not the Georgi-Glashow model but a slightly more complicated one elaborated independently by S. Weinberg and A. Salam that proved successfull in the description of weak interaction. In this model, the gauge group is $SU(2) \times U(1)$ and the matter field is a doublet of complex scalar fields transforming in the fundamental representation. The Higgs mechanism gives a mass to three gauge bosons: the $W^\pm$ and the electrically neutral $Z^0$, whereas one gauge field remains massless and is therefore identified with the photon. This theory allows then the unification of the electromagnetic and weak interactions, for which Glashow, Weinberg and Salam won the Nobel Prize in 1979. The gauge bosons of the weak interaction were detected in 1983 at the LEP in CERN; the experimental values of their masses are $m_W = 80$ GeV and $m_Z = 91$ GeV, in good agreement with the theoretical calculations. Yet the massive scalar field predicted by the theory, known as the Higgs boson ($\xi^3$ in our notation), is still to be discovered.

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