SNB/HEP/99-03-01
March 1999

TOPOLOGICAL ASPECTS IN NON-ABELIAN GAUGE THEORY

R. P. Malik *

S. N. Bose National Centre for Basic Sciences,
Block-JD, Sector III, Salt Lake, Calcutta- 700 091, India.

ABSTRACT

We discuss the BRST cohomology and exhibit a connection between the Hodge decomposition theorem and the topological properties of a two dimensional free non-Abelian gauge theory having no interaction with matter fields. The topological nature of this theory is encoded in the vanishing of the Laplacian operator when equations of motion are exploited. We obtain two sets of topological invariants with respect to BRST and co-BRST charges on the two dimensional manifold and show that the Lagrangian density of the theory can be expressed as the sum of terms that are BRST- and co-BRST invariants.

*On leave of absence from Bogoliubov Laboratory of Theoretical Physics, JINR-Dubna, Moscow, Russia.
e-mail addresses: malik@boson.bose.res.in AND malik@thsun1.jinr.dubna.su

1 Introduction

The local gauge theories, endowed with the first class constraints [1,2], play a key role in the understanding of the basic interactions of nature (except quantum gravity). For the Becchi-Rouet-Stora-Tyutin (BRST) quantization of such a class of theories, the local gauge symmetry transformations of the classical theories are traded with the quantum gauge BRST symmetry transformations which are generated by the conserved ($\dot{Q}_B = 0$) and nilpotent ($Q_B^2 = 0$) BRST charge Q_B [3-5]. In particular, the importance of the BRST formalism comes to its full glory in the context of the covariant canonical quantization of the non-Abelian gauge theory where unitarity and gauge invariance both are respected together. The presence of the first class constraints of the original theories is found to be encoded in the subsidiary condition $Q_B|phys>= 0$ which implies that the physical states are annihilated by these constraints. The above two properties, i.e., the physical state condition $Q_B|phys>= 0$ and the nilpotency of the BRST charge $Q_B^2 = 0$, are the two key requirements to define the cohomological aspects of the BRST formalism. The inclusion of the BRST symmetry in the Batalin-Vilkovisky formalism (see, e.g., [6,7]), the discussion of the second class constraints in its framework [8], its indispensible use in the topological field theories [9-11] and string theories, etc., have enriched the physical and mathematical aspects of the BRST formalism to a fairly high degree of sophistication.

One of the most celebrated theorems in the mathematical aspects of the de Rham cohomology is the Hodge decomposition theorem defined on a compact manifold [12-14]. This theorem states that any arbitrary form can be written as the sum of a harmonic form, an exact form and a co-exact form. In principle, the cohomology can be defined w.r.t. the exterior derivative $d(d^2 = 0)$ and/or w.r.t. the dual extrerior derivative $\delta(\delta^2 = 0)$ where $\delta = \pm * d*$ is the Hodge dual of d. The operation of d on any arbitrary form increases the degree of the form by one whereas the operation of δ reduces it by one. In the cohomological description of BRST formalism, the conserved and nilpotent BRST charge (which generates a nilpotent quantum gauge symmetry) is identified with the exterior derivative d in any arbitrary dimensions of spacetime. It would be, therefore, an interesting endeavour to obtain the analogues of δ and the Laplacian $\Delta = (d + \delta)^2 = d\delta + \delta d$ in the language of the nilpotent (for δ), local, covariant and continuous symmetry properties of a given Lagrangian density. Some attempts [15] have been made towards this goal in four dimensions of spacetime but the symmetry transformations turn out to be nonlocal and noncovariant. In the covariant formulation, the symmetry transformations turn out to be even non-nilpotent and they become nilpotent only under certain specific restrictions [16].

The central theme of the present paper is to express the Hodge decomposition theorem in terms of the local and conserved charges corresponding to the analogues of d, δ and Δ of differential geometry and establish a connection with the topological nature of the two $(1 + 1)$ dimensional non-Abelian gauge theory having no interaction with matter fields. We generalise our works for the free Abelian $U(1)$ gauge theory in two dimensions

(2D) [17,18] and show that the Laplacian operator for the free non-Abelain gauge theory (without any interaction with matter fields) too, goes to zero when equations of motion are exploited. This happens due to the fact that both the physical degrees of freedom of the non-Abelian gauge boson are gauged away by the presence of the nilpotent BRST- and co(dual)- BRST symmetries in the theory. Thus, theory becomes topological in nature (see, e.g., Ref. [11]). Mathematically, the Lagrangian density of the theory can be expressed as the sum of BRST- and co-BRST invariant parts and, therefore, it bears an outlook similar to the Witten type theories [10]. The topological nature of this theory is confirmed by the existence of two sets of topological invariants on the 2D compact manifold. In Sec. 2, we set up the notations and give a concise description of the BRST formalism for the free D-dimensional non-Abelian gauge theory. This is followed by the derivation of the (anti)dual BRST symmetries and corresponding charges in Sec. 3. In Sec. 4, we obtain the symmetries that are generated by the analogue of the Laplacian(Casimir) operator and derive the full BRST algebra. Sec. 5 is devoted to the discussion of Hodge decomposition theorem and the derivation of the topological invariants. Finally, we make some concluding remarks and point out some directions that can be pursued in the future.

2 Preliminary: BRST symmetries

Let us begin with the BRST invariant Lagrangian density ($\mathcal{L}_B$) for the D-dimensional free non-Abelian gauge theory (having no interaction with matter fields) in the Feynman gauge

$$\mathcal{L}_B = -\tfrac{1}{4} F^{\mu\nu a} F^a_{\mu\nu} + B^a (\partial \cdot A)^a + \tfrac{1}{2} B^a B^a - i \partial_\mu \bar{C}^a D^\mu C^a, \tag{2.1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ is the field strength tensor obtained from the group valued gauge connection A^a_μ, B^a is the group valued auxiliary field, $(\bar{C}^a)C^a$ are the (anti)ghost fields $((\bar{C}^a)^2 = (C^a)^2 = 0)$ and the covariant derivative is defined as: $D_\mu C^a = \partial_\mu C^a + g f^{abc} A^b_\mu C^c$, the D-dimensional flat Minkowski spacetime indices are $\mu, \nu ..... = 0, 1, 2, ... D-1$, the group indices $a, b, c, ... = 1, 2, 3, ...$ correspond to the compact Lie gauge group, g is the coupling constant and structure constants f^{abc} are chosen to be totally antisymmetric in a, b, c (see, e.g., [19]). The above Lagrangian density respects $(\delta_B \mathcal{L}_B = \eta \partial_\mu [B^a D^\mu C^a])$ off-shell nilpotent $(\delta_B^2 = 0)$ BRST symmetry transformations

$$\begin{array}{lll} \delta_B A^a_\mu = \eta D_\mu C^a, & \delta_B C^a = -\frac{\eta g}{2} f^{abc} C^b C^c, & \delta_B B^a = 0, \\ \delta_B F^a_{\mu\nu} = \eta g f^{abc} F^b_{\mu\nu} C^c, & \delta_B \bar{C}^a = +i\eta B^a, & \delta_B (\partial \cdot A)^a = \eta \partial_\mu D^\mu C^a, \end{array} \tag{2.2}$$

where η is an anticommuting $(\eta C^a = -C^a \eta, \eta \bar{C}^a = -\bar{C}^a \eta)$ spacetime independent transformation parameter. The on-shell $(\partial_\mu D^\mu C^a = 0)$ nilpotent $(\delta_b^2 = 0)$ BRST transformations

$$\begin{array}{lll} \delta_b A^a_\mu = \eta \, D_\mu \, C^a, & \delta_b F^a_{\mu\nu} = \eta g f^{abc} F^b_{\mu\nu} C^c, & \delta_b \bar{C}^a = -i\eta (\partial \cdot A)^a, \\ \delta_b C^a = -\frac{\eta g}{2} f^{abc} C^b C^c, & \delta_b (\partial \cdot A)^a = \eta \, \partial_\mu D^\mu C^a, & \delta_b (D_\mu C^a) = 0, \end{array} \tag{2.3}$$

can be derived from (2.2) by using the equation of motion $B^a = -(\partial \cdot A)^a$ and they leave the following Lagrangian density:

$$\mathcal{L}_b = -\tfrac{1}{4} F^{\mu\nu a} F^a_{\mu\nu} - \tfrac{1}{2} (\partial \cdot A)^a (\partial \cdot A)^a - i \partial_\mu \bar{C}^a D^\mu \, C^a, \tag{2.4}$$

quasi-invariant because $\delta_b \mathcal{L}_b = -\eta \partial_\mu [(\partial \cdot A)^a D^\mu C^a]$. These symmetries lead to the following expression for the conserved and nilpotent BRST charge ($Q_{(B,b)}$) (see, e.g., [3-5]):

$$\begin{aligned} Q_{(B,b)} &= \int d^{D-1}x \; [\; B^a D_0 C^a - \dot{B}^a C^a + \tfrac{1}{2} i g f^{abc} \; \dot{\bar{C}}^a \; C^b \; C^c \;], \\ &\equiv \int d^{D-1}x \; [\; \partial_0 (\partial \cdot A)^a C^a - (\partial \cdot A)^a D_0 C^a + \tfrac{1}{2} i g f^{abc} \; \dot{\bar{C}}^a \; C^b \; C^c \;]. \end{aligned} \tag{2.5}$$

The continuous global symmetry invariance of the total action under the transformations: $C^a \to e^{-\lambda} C^a, \bar{C}^a \to e^{\lambda} \bar{C}^a, A^a_\mu \to A^a_\mu, B^a \to B^a$, (where λ is a global parameter), leads to the derivation of the conserved ghost charge (Q_g)

$$Q_g = -i \int d^{(D-1)}x \; [\; C^a \; \dot{\bar{C}}^a + \bar{C}^a \; D_0 C^a \;]. \tag{2.6}$$

The derivation of the anti-BRST charge in non-Abelian gauge theory is more involved. In fact, one introduces another auxiliary field $\bar{B}^a$ in (2.1) for this purpose:

$$\mathcal{L}_{\bar{B}} = -\tfrac{1}{4} F^{\mu\nu a} F^a_{\mu\nu} + B^a (\partial \cdot A)^a + \tfrac{1}{2}(B^a B^a + \bar{B}^a \bar{B}^a) - i \partial_\mu \bar{C}^a D^\mu C^a, \tag{2.7a}$$

$$\mathcal{L}_{\bar{B}} = -\tfrac{1}{4} F^{\mu\nu a} F^a_{\mu\nu} - \bar{B}^a (\partial \cdot A)^a + \tfrac{1}{2}(B^a B^a + \bar{B}^a \bar{B}^a) - i D_\mu \bar{C}^a \partial^\mu C^a, \tag{2.7b}$$

where the auxiliary fields are restricted to satisfy [20]

$$B^a + \bar{B}^a = i g \; f^{abc} \; C^b \; \bar{C}^c. \tag{2.8}$$

Under the BRST transformations, it is interesting to note that $\bar{B}^a$ transforms as: $\delta_B \bar{B}^a = \eta g f^{abc} \bar{B}^b C^c$. The following off-shell nilpotent ($\delta_{AB}^2 = 0$) anti-BRST (δ_{AB}) transformations

$$\begin{array}{lll} \delta_{AB} A^a_\mu = \eta D_\mu \bar{C}^a, & \delta_{AB} \bar{C}^a = -\frac{\eta g}{2} f^{abc} \bar{C}^b \bar{C}^c, & \delta_{AB} \bar{B}^a = 0, \\ \delta_{AB} F^a_{\mu\nu} = \eta g f^{abc} F^b_{\mu\nu} \bar{C}^c, & \delta_{AB} C^a = + i \eta \bar{B}^a, & \delta_{AB} B^a = \eta g f^{abc} B^b \bar{C}^c, \end{array} \tag{2.9}$$

leave the Lagrangian density (2.7b) quasi-invariant ($\delta_{AB} \mathcal{L}_{\bar{B}} = -\eta \partial_\mu [\bar{B}^a D^\mu \bar{C}^a]$). The above transformations (2.9) are generated by the nilpotent and conserved anti-BRST charge

$$Q_{AB} \;=\; \int d^{D-1}x \; [\; \dot{\bar{B}}^a \bar{C}^a - \bar{B}^a D_0 \bar{C}^a - \tfrac{1}{2} i g f^{abc} \dot{C}^a \bar{C}^b \bar{C}^c \;]. \tag{2.10}$$

Together, the above three conserved charges satisfy:

$$\begin{array}{l} \{Q_B, Q_B\} \;=\; \{Q_{AB}, Q_{AB}\} \;=\; 0, \\ \{Q_B, Q_{AB}\} = Q_B Q_{AB} \;+\; Q_{AB} Q_B = 0, \\ i[Q_g, Q_B] = + Q_B, \quad i[Q_g, Q_{AB}] = - Q_{AB}, \end{array} \tag{2.11}$$

where the basic canonical (anti)commutators for the BRST invariant Lagrangian densities have been exploited. It can be seen that the specific combinations of transformations: $(\delta_B \delta_{AB} + \delta_{AB} \delta_B)$ acting on any field generate no transformation at all (as $\{Q_B, Q_{AB}\} = 0$). In particular, it can be checked that $\{\delta_B, \delta_{AB}\} A^a_\mu = 0$ is obeyed if and only if the restriction

(2.8) is satisfield. For the non-Abelian compact Lie algebra we have considered, the anti-commutator of the BRST- and anti-BRST charges is zero †. Thus, the anti-BRST charge (Q_{AB}) is not the analogue of the dual exterior derivative (δ).

3 Dual BRST symmetries

We consider here a two $(1+1)$ dimensional non-Abelian gauge theory and discuss dual BRST- and anti-dual BRST symmetries. The Lagrangian density (2.4) in 2D ‡

$$\mathcal{L}_b = \tfrac{1}{2} E^a E^a - \tfrac{1}{2} (\partial \cdot A)^a (\partial \cdot A)^a - i \partial_\mu \bar{C}^a D^\mu C^a, \tag{3.1}$$

remains quasi-invariant ($\delta_d \mathcal{L}_b = \eta \partial_\mu [E^a \partial^\mu \bar{C}^a]$) under the following on-shell ($D_\mu \partial^\mu \bar{C}^a = 0$) nilpotent ($\delta_d^2 = 0$) symmetry transformations

$$\begin{array}{llll} \delta_d A^a_\mu = -\eta \varepsilon_{\mu\nu} \partial^\nu \bar{C}^a, & \delta_d \bar{C}^a = 0, & \delta_d C^a = -i \eta E^a, & \delta_d (D_\mu \partial^\mu \bar{C}^a) = 0, \\ \delta_d (\partial \cdot A)^a = 0, & \delta_d E^a = \eta D_\mu \partial^\mu \bar{C}^a, & \delta_d F^a_{\mu\nu} = \eta (\varepsilon_{\mu\rho} D_\nu - \varepsilon_{\nu\rho} D_\mu) \partial^\rho \bar{C}^a. & \end{array} \tag{3.2}$$

We christen this symmetry as the dual BRST symmetry by taking analogy with the Abelian gauge theory where, like the above transformations, it is the gauge-fixing term $(\partial \cdot A)^a$ that remains invariant [17,18]. At this stage, it is essential to pin-point some of the differences and similarities between the BRST- and dual BRST symmetries in Abelian $U(1)$ gauge theory and the same in the context of non-Abelian gauge theory. In the Abelian theory, the gauge-fixing term $\delta A = (\partial \cdot A)$ with $\delta = \pm * d *$ is the Hodge dual of the two-form $F = dA$ which is the electric field E in 2D. This is not the case, however, for the non-Abelian gauge theory because the field strength tensor $F^a_{\mu\nu}$ contains a self-interacting term $g f^{abc} A^b_\mu A^c_\nu$ which is not present in the two-form $F = dA$ of the Abelian gauge theory. Under the BRST transformations in the Abelian gauge theory, it is the two-form $F = d\,A$ that remains invariant ($\delta_B F_{\mu\nu} = 0$) but for the non-Abelian gauge theory the field strength tensor transforms: $\delta_B F^a_{\mu\nu} = \eta g f^{abc} F^b_{\mu\nu} C^c$. It is the total kinetic energy term $(-\frac{1}{4} F^{\mu\nu a}\, F^a_{\mu\nu})$, however, that remains invariant in both kinds of gauge theories. Similarly, the dual BRST symmetry corresponds to a symmetry in which the gauge-fixing term remains invariant. The analogue of the Lagrangian density (2.1) can be written for the two-dimensional case by introducing one more auxiliary field $\mathcal{B}^a$ as:

$$\mathcal{L}_\mathcal{B} = \mathcal{B}^a E^a - \tfrac{1}{2} \mathcal{B}^a \mathcal{B}^a + B^a (\partial \cdot A)^a + \tfrac{1}{2} B^a B^a - i \partial_\mu \bar{C}^a D^\mu C^a. \tag{3.3}$$

This Lagrangian density respects the off-shell nilpotent ($\delta_D^2 = 0$) dual BRST symmetry δ_D as well as the off-shell nilpotent ($\delta_B^2 = 0$) BRST symmetry δ_B. These symmetries, for the above Lagrangian density, are juxtaposed as

† In a recent work [21], it has been pointed out that the cohomologically higher order BRST- and anti-BRST operators do not anticommute and their anticommutator leads to the definition of a cohomologically higher order Laplacian operator.

‡ We adopt here the notations in which the 2D flat Minkowski metric is : $\eta_{\mu\nu} = (+1, -1)$ and $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_0 \partial_0 - \partial_1 \partial_1$, $\dot{\phi}^a = \partial_0 \phi^a$, $F^a_{01} = \partial_0 A^a_1 - \partial_1 A^a_0 + g f^{abc} A^b_0 A^c_1 = E^a = F^{10a}$, $\varepsilon_{01} = \varepsilon^{10} = +1$, $(\partial \cdot A)^a = \partial_0 A^a_0 - \partial_1 A^a_1$, $D_\mu \phi^a = \partial_\mu \phi^a + g f^{abc} A^b_\mu \phi^c$, $D_\mu (\phi^a \psi^b) = D_\mu \phi^a \psi^b + \phi^a D_\mu \psi^b$.

$$
\begin{array}{lcll}
\delta_D A^a_\mu & = & -\eta\varepsilon_{\mu\nu}\partial^\nu \bar{C}^a, & \delta_B A^a_\mu = \eta\ D_\mu C^a,\\
\delta_D \bar{C}^a & = & 0, & \delta_B \bar{C}^a = i\eta B^a,\\
\delta_D C^a & = & -i\eta \mathcal{B}^a, & \delta_B C^a = -\frac{\eta g}{2} f^{abc}\ C^b\ C^c,\\
\delta_D \mathcal{B}^a & = & 0, & \delta_B \mathcal{B}^a = \eta g f^{abc}\mathcal{B}^b C^c,\\
\delta_D (\partial\cdot A)^a & = & 0, & \delta_B (\partial\cdot A)^a = \eta\partial_\mu D^\mu C^a,\\
\delta_D B^a & = & 0, & \delta_B B^a = 0,\\
\delta_D E^a & = & \eta D_\mu \partial^\mu \bar{C}^a, & \delta_B E^a = \eta g f^{abc}\ E^b\ C^c,\\
\delta_D F^a_{\mu\nu} & = & \eta(\varepsilon_{\mu\rho}D_\nu - \varepsilon_{\nu\rho}D_\mu)\partial^\rho \bar{C}^a, & \delta_B F^a_{\mu\nu} = \eta g f^{abc} F^b_{\mu\nu} C^c.
\end{array}
\tag{3.4}
$$

Under the above dual BRST symmetry, the Lagrangian density transforms as: $\delta_D \mathcal{L}_\mathcal{B} = \eta\partial_\mu(\mathcal{B}^a\partial^\mu \bar{C}^a)$. The on-shell nilpotent symmetry transformations (3.2) lead to the Noether conserved current $J^\mu_d = [F^{\mu\alpha a} + \eta^{\mu\alpha}(\partial\cdot A)^a]\varepsilon_{\alpha\rho}\partial^\rho \bar{C}^a$ which ultimately leads to the dual BRST charge $Q_d = \int dx[E^a \dot{\bar{C}}^a - (\partial\cdot A)^a\partial_1 \bar{C}^a]$. Now using the partial integration and the equation of motion $D_0 E^a + \partial_1(\partial\cdot A)^a + igf^{abc}C^b\partial_1 \bar{C}^c = 0$, this charge can be expressed as:

$$
\begin{aligned}
Q_{(d,D)} &= \int dx[\ E^a \dot{\bar{C}}^a - D_0 E^a \bar{C}^a - igf^{abc}\bar{C}^a\partial_1 \bar{C}^b C^c\ ],\\
&\equiv \int dx[\ \mathcal{B}^a \dot{\bar{C}}^a - D_0 \mathcal{B}^a \bar{C}^a - igf^{abc}\bar{C}^a\partial_1 \bar{C}^b C^c\ ],
\end{aligned}
\tag{3.5}
$$

where the latter expression (for Q_D) has been obtained due to the validity of the equation of motion $E^a = \mathcal{B}^a$. It can be checked that under the following off-shell nilpotent ($\delta^2_{AD} = 0$) anti-dual BRST (δ_{AD}) transformations

$$
\begin{array}{llll}
\delta_{AD} A^a_\mu = -\eta\varepsilon_{\mu\nu}\partial^\nu C^a, & \delta_{AD} C^a = 0, & \delta_{AD} \bar{C}^a = +i\eta\mathcal{B}^a, & \delta_{AD}\mathcal{B}^a = 0,\\
\delta_{AD}(\partial\cdot A)^a = 0, & \delta_{AD} B^a = 0, & \delta_{AD}\bar{B}^a = 0, & \delta_{AD} E^a = \eta D_\mu\partial^\mu C^a,
\end{array}
\tag{3.6}
$$

the analogue of the Lagrangian density (2.7b) in 2D:

$$
\mathcal{L}_{\bar{\mathcal{B}}} = \mathcal{B}^a E^a - \tfrac{1}{2}\mathcal{B}^a\mathcal{B}^a - \bar{B}^a(\partial\cdot A)^a + \tfrac{1}{2}(B^a B^a + \bar{B}^a\bar{B}^a) - iD_\mu \bar{C}^a\partial^\mu C^a,
\tag{3.7}
$$

remains quasi-invariant because $\delta_{AD}\mathcal{L}_{\bar{\mathcal{B}}} = \eta\partial_\mu[\mathcal{B}^a\partial^\mu C^a]$. Using the Noether theorem, we obtain the conserved current as: $J^\mu_{AD} = F^{\mu\alpha a}\varepsilon_{\alpha\beta}\ \partial^\beta C^a + \varepsilon_{\alpha\beta}\partial^\beta C^a\eta^{\mu\alpha}\bar{B}^a$ which leads to the anti-dual BRST charge $Q_{AD} = \int dx\ [\ \mathcal{B}^a\dot{C}^a - \bar{B}^a\partial_1 C^a\ ]$. Using the partial integration and exploiting the equations of motion : $\partial_1\bar{B}^a = igf^{abc}\bar{C}^b\partial_1 C^c - D_0\mathcal{B}^a$, we obtain

$$
Q_{AD} = \int dx[\ \mathcal{B}^a\dot{C}^a - D_0\mathcal{B}^a C^a + igf^{abc}C^a\partial_1 C^b\bar{C}^c\ ].
\tag{3.8}
$$

It is straightforward to check that the above nilpotent and conserved charges $Q_r(r = B, AB, D, AD)$ are the generators of the transformations (2.2), (2.9), (3.4) and (3.6) because these transformations can be concisely expressed as $\delta_r\phi = -i\eta[\phi, Q_r]_\pm$ where $(+)-$ stands for the (anti)commutator corresponding to the generic field ϕ being (fermionic)bosonic.

4 Symmetries generated by the Casimir operator

It is evident that the conserved and nilpotent charges Q_B and Q_D are the fermionic symmetry generators corresponding to the transformations δ_B and δ_D for the Lagrangian density $\mathcal{L}_{\mathcal{B}}$ (cf. (3.3)). Their anticommutator $W = \{Q_B, Q_D\}$ will also generate a bosonic symmetry transformation $\delta_W = \{\delta_B, \delta_D\}$. The following transformations

$$\begin{array}{l} \delta_W \bar{C}^a = 0, \quad \delta_W C^a = 0, \quad \delta_W B^a = 0, \quad \delta_W \mathcal{B}^a = 0, \\ \delta_W A_\mu^a = \kappa[\, D_\mu \mathcal{B}^a + \varepsilon_{\mu\nu}\partial^\nu B^a - igf^{abc}\varepsilon_{\mu\nu}\partial^\nu \bar{C}^b C^c \,], \\ \delta_W E^a = -\kappa[\, D_\mu \partial^\mu B^a + \varepsilon^{\mu\nu} D_\mu D_\nu \mathcal{B}^a - igf^{abc} D_\mu(\partial^\mu \bar{C}^b C^c) \,], \\ \delta_W (\partial\cdot A)^a = \kappa[\, \partial_\mu D^\mu \mathcal{B}^a + igf^{abc}\varepsilon^{\mu\nu}\partial_\mu \bar{C}^b \partial_\nu C^c \,], \end{array} \tag{4.1}$$

(with bosonic transformation parameters $\kappa = -i\eta\eta'$) are indeed the symmetry transformations because the Lagrangian density $\mathcal{L}_{\mathcal{B}}$ transforms to a total derivative as:

$$\delta_W \mathcal{L}_{\mathcal{B}} = \kappa\partial_\mu[B^a D^\mu \mathcal{B}^a - \mathcal{B}^a \partial^\mu B^a + igf^{abc}(\mathcal{B}^a \partial^\mu \bar{C}^b + \varepsilon^{\mu\nu}\partial_\nu \bar{C}^a B^b) C^c]. \tag{4.2}$$

In the expression for the bosonic transformation parameter $\kappa = -i\eta\eta'$, the fermionic parameters η and η' correspond to the transformations generated by Q_B and Q_D. It will be noticed that, out of three basic fields, the ghost- and antighost fields do not transform under δ_W and the gauge boson field A_μ^a transforms to its own equation of motion: $D_\mu \mathcal{B}^a + \varepsilon_{\mu\nu}\partial^\nu B^a - igf^{abc}\varepsilon_{\mu\nu}\partial^\nu \bar{C}^b C^c (= 0)$. The generator of the above transformations (4.1) is a conserved charge W given by the following expression:

$$W = \int\, dx\; [\, \mathcal{B}^a \dot{B}^a - B^a D_0 \mathcal{B}^a - igf^{abc}(\mathcal{B}^a \dot{\bar{C}}^b - \partial_1 \bar{C}^a B^b) C^c \,]. \tag{4.4}$$

Since in the BRST formalism, there are two more conserved and nilpotent charges Q_{AB} and Q_{AD}, the anticommutator of these two can also define operator W. Concentrating on the Lagrangian density (3.7), we can check that the anticommutator $\{\delta_{AB}, \delta_{AD}\}$ leads to the following bosonic transformation δ_W (with bosonic transformation parameter $\kappa = -i\eta\eta'$)

$$\begin{array}{l} \delta_W \bar{C}^a = 0, \quad \delta_W C^a = 0, \quad \delta_W B^a = 0, \quad \delta_W \mathcal{B}^a = 0, \quad \delta_W \bar{B}^a = 0, \\ \delta_W A_\mu^a = \kappa\; [\, -D_\mu \mathcal{B}^a + \varepsilon_{\mu\nu}\partial^\nu \bar{B}^a - igf^{abc}\varepsilon_{\mu\nu}\partial^\nu C^b \bar{C}^c \,], \\ \delta_W E^a = -\kappa\; [\, D_\mu \partial^\mu \bar{B}^a - \varepsilon^{\mu\nu} D_\mu D_\nu \mathcal{B}^a - igf^{abc} D_\mu(\partial^\mu C^b \bar{C}^c) \,], \\ \delta_W (\partial\cdot A)^a = -\kappa\; [\, \partial_\mu D^\mu \mathcal{B}^a + igf^{abc}\varepsilon^{\mu\nu}\partial_\mu \bar{C}^b \partial_\nu C^c \,], \end{array} \tag{4.5}$$

where η and η' are the anticommuting transformation parameters corresponding to δ_{AB} and δ_{AD} respectively. It is straightforward to check that the Lagrangian density (3.7) undergoes the following change under the transformations (4.5)

$$\delta_W \mathcal{L}_{\bar{B}} = \kappa\; \partial_\mu\; [\bar{B}^a D^\mu \mathcal{B}^a - \mathcal{B}^a \partial^\mu \bar{B}^a + igf^{abc}(\mathcal{B}^a \partial^\mu C^b + \bar{B}^a \varepsilon^{\mu\nu}\partial_\nu C^b)\bar{C}^c \,], \tag{4.6}$$

where use has been made of the identity

$$igf^{abc}\mathcal{B}^a D_\mu(\partial^\mu C^b \bar{C}^c) - igf^{abc}\partial^\mu C^a D_\mu \mathcal{B}^b \bar{C}^c = \partial_\mu[igf^{abc}\mathcal{B}^a \partial^\mu C^b \bar{C}^c].$$

The generator of the symmetry transformations (4.5) is

$$W = \int\, dx\; [\, \mathcal{B}^a \dot{\bar{B}}^a - \bar{B}^a D_0 \mathcal{B}^a + igf^{abc}(\mathcal{B}^a \dot{C}^b - \bar{B}^a \partial_1 C^b)\bar{C}^c \,]. \tag{4.7}$$

Both the expressions for the conserved charge W are equivalent because they differ by a total space derivative when equations of motion for the Lagrangian densities are used.

There are other simpler ways to obtain the expression for the analogue of the Laplacian operator W. For instance, the symmetries (2.2), (2.9), (3.4) and (3.6) alone can be exploited for the derivation of W. Since Q_r (r = B, AB, D, AD) are the generators of all these transformations, it can be seen that the following transformations

$$\begin{array}{rcl} \delta_B Q_D &=& -i\eta \, \{Q_D, Q_B\} \;=\; -i\eta W \;\equiv\; \delta_D Q_B, \\ \delta_{AB} Q_{AD} &=& -i\eta \, \{Q_{AD}, Q_{AB}\} = \; -i\eta W \equiv \delta_{AD} Q_{AB}, \end{array} \tag{4.8}$$

also lead to the derivation of W. Furthermore, these expressions for W can also be obtained from the anticommutators $\{Q_B, Q_D\}$ or $\{Q_{AD}, Q_{AB}\}$ by directly exploiting the basic canonical (anti)commutators for (3.3) and (3.7) which are juxtaposed as

$$\begin{array}{ll} [A_0^a(x,t), B^b(y,t)] = i\delta^{ab}\delta(x-y), & [A_0^a(x,t), \bar{B}^b(y,t)] = -i\delta^{ab}\delta(x-y), \\ [A_1^a(x,t), \mathcal{B}^b(y,t)] = i\delta^{ab}\delta(x-y), & [A_1^a(x,t), \mathcal{B}^b(y,t)] \; = i\,\delta^{ab}\,\delta(x-y), \\ \{C^a(x,t), \dot{\bar{C}}^b(y,t)\} = \delta^{ab}\delta(x-y), & \{C^a(x,t), D_0\bar{C}^b(y,t)\} = \delta^{ab}\delta(x-y), \\ \{\bar{C}^a(x,t), D_0 C^b(y,t)\} = -\delta^{ab}\delta(x-y), & \{\bar{C}^a(x,t), \dot{C}^b(y,t)\} = -\delta^{ab}\delta(x-y), \end{array} \tag{4.9}$$

and all the rest of the (anti)commutators turn out to be zero. The above canonical (anti)commutators lead to the derivation of the following extended BRST algebra

$$\begin{array}{l} [W, Q_k] = 0, k = B, D, AB, AD, g, \\ Q_B^2 \;=\; Q_{AB}^2 \;=\; Q_D^2 \;=\; Q_{AD}^2 \;= 0, \\ \{Q_B, Q_D\} \;=\; \{Q_{AB}, Q_{AD}\} \;=\; W, \\ i[Q_g, Q_B] = +Q_B, \qquad i[Q_g, Q_{AB}] = -Q_{AB}, \\ i[Q_g, Q_D] = -Q_D, \qquad i[Q_g, Q_{AD}] = +Q_{AD}, \end{array} \tag{4.10}$$

which is constituted by six conserved charges corresponding to six symmetries present in the theory and all the rest of the (anti)commutators turn out to be zero. It is clear now that the operator W is the analogue of the Laplacian operator and is the Casimir operator for the whole algebra. It can be also seen that the ghost number for Q_B and Q_{AD} is $+1$ and that of Q_{AB} and Q_D is -1. Now given a state $|\psi>$ in the quantum Hilbert space with ghost number n (i.e., $iQ_g|\psi>= n|\psi>$), it is straightforward to check that:

$$\begin{array}{rcl} iQ_g \; Q_B|\psi> &=& (n+1) \; Q_B|\psi>, \\ iQ_g \; Q_D|\psi> &=& (n-1) \; Q_D|\psi>, \\ iQ_g \; W \; |\psi> &=& n \; W \; |\psi> . \end{array} \tag{4.11}$$

The above equation shows that the ghost number of the BRST exact state is one higher (and that of the co-BRST exact state is one lower) than the original state. This property is similar to the operation of an exterior derivative (and a dual exterior derivative) on a given differential form. Thus, the geometrical quantities d, δ, Δ find their identifications in the language of symmetry properties that are generated by Q_B, Q_D and W.

5 Hodge decomposition theorem and topological invariants

A close look at the extended BRST algebra (4.10) and the considerations of the ghost numbers for the BRST- and co-BRST exact states and harmonic state (cf. (4.11)) allows one to implement the Hodge decomposition theorem in its full glory on any arbitrary state of the quantum Hilbert space (see, e.g., [4], [5], [13])

$$|\psi >_n = |\omega >_n + Q_B|\, \theta >_{n-1} + Q_D\, |\chi >_{n+1}\, . \tag{5.1}$$

The above equation implies that any arbitrary state $|\psi >_n$ with ghost number n can be decomposed into a harmonic state $|\omega >_n$ $(W|\omega >_n = 0, Q_B|\omega >_n = 0, Q_D|\omega >_n = 0)$, a BRST exact state $Q_B|\theta >_{n-1}$ and a dual-BRST exact state $Q_D|\chi >_{n+1}$. In fact, this equation is the analogue of the mathematical statement on a compact manifold that any arbitrary p-form f_p can be written as the sum of a harmonic form ω_p $(\Delta\omega_p = 0, d\omega_p = 0, \delta\omega_p = 0)$, an exact form dg_{p-1} and a co-exact form δh_{p+1} due to the Hodge decomposition theorem:

$$f_p = \omega_p + dg_{p-1} + \delta h_{p+1}.$$

Thus, the ghost number of a state in the quantum Hilbert space plays the same role as the degree of a differential form defined on a compact manifold. It will be noticed that the BRST cohomology can be defined either w.r.t. Q_B or Q_D or w.r.t. both. To refine the BRST cohomology, however, we have to choose a representative state from the total states of (5.1) as a physical state. The harmonic states $|\omega >$ are very special for a given physical theory because they are finite in number (see, e.g., [12]). Let us define our physical state as the harmonic state (i.e., $|phys >= |\omega >$). By definition, such a state would satisfy

$$W|phys >= 0, \qquad Q_B\, |phys >= 0, \qquad Q_D\, |phys >= 0. \tag{5.2}$$

In our earlier work on $U(1)$ gauge theory [17,18], it has been shown that both the physical degrees of a single photon state in 2D can be gauged away by the subsidiary conditions: $Q_B|phys >= 0, Q_D|phys >= 0$ alone. Thus, the operation of W on this physical photon state becomes superfluous. In fact, the Laplacian operator goes to zero when the equations of motion are exploited and the theory becomes topological in nature. In an analogous manner, it turns out that both the expressions for W in (4.4) and (4.7) become

$$\begin{array}{lll} W &=& \int dx\, \frac{d}{dx}\, [\, \frac{1}{2}\, \mathcal{B}^a \mathcal{B}^a - \frac{1}{2}\, B^a B^a\,] \rightarrow 0 \qquad \text{as} \quad x \rightarrow \pm\, \infty, \\ &\equiv& \int dx\, \frac{d}{dx}\, [\, \frac{1}{2}\, \mathcal{B}^a \mathcal{B}^a - \frac{1}{2}\, \bar{B}^a \bar{B}^a\,] \rightarrow 0 \qquad \text{as} \quad x \rightarrow \pm\, \infty, \end{array} \tag{5.3}$$

as a consequence of the equation of motion for the gauge boson field from (3.3)

$$\begin{array}{lll} D_0 \mathcal{B}^a - \partial_1 B^a + ig f^{abc}\, \partial_1 \bar{C}^b\, C^c &=& 0, \\ D_1 \mathcal{B}^a - \partial_0 B^a + ig f^{abc}\, \partial_0 \bar{C}^b\, C^c &=& 0, \end{array} \tag{5.4a}$$

and the same from Lagrangian density (3.7)

$$\begin{array}{rcl} D_0\mathcal{B}^a + \partial_1 \bar{B}^a - igf^{abc}\bar{C}^b\partial_1 C^c &=& 0, \\ D_1\mathcal{B}^a + \partial_0 B^a - igf^{abc}\bar{C}^b\partial_0 C^c &=& 0. \end{array} \tag{5.4b}$$

It will be noticed that both the expressions for W in (5.3) are equivalent because $B^a\partial_1 B^a = \bar{B}^a\partial_1\bar{B}^a - \partial_1 X$ where $X = igf^{abc}\bar{B}^a C^b\bar{C}^c + \frac{1}{2}g^2 f^{abc}f^{amn}C^b\bar{C}^c C^m\bar{C}^n$. The vanishing of the operator W in (5.3) is the reflection of the fact that there are no physical degrees of freedom left in the theory (as the BRST and co-BRST symmetries gauge away both the degrees of freedom of gauge boson in 2D) and it becomes topological in nature [11]. This situation should be contrasted with the interacting $U(1)$ gauge theory where Dirac fermions couple to the gauge field. As it turns out, the Laplacian operator does not go to zero even on the on-shell [22] because of the presence of the fermionic degrees of freedom in the theory.

The topological nature of the theory is confirmed by the presence of the topological invariants on the 2D manifold. The two sets of these invariants w.r.t. both the conserved and nilpotent charges Q_B and Q_D are (see, e.g., [10],[11],[23])

$$I_k[C_k] = \oint_{C_k} V_k, \qquad J_k[C_k] = \oint_{C_k} W_k, \qquad k = 0, 1, 2 \tag{5.5}$$

where C_k are the k-dimensional homology cycles and the k-forms V_k and W_k are

$$\begin{array}{ll} V_0 = B^a\, C^a - \frac{ig}{2} f^{abc}\bar{C}^a C^b C^c, & W_0 = \mathcal{B}^a\, \bar{C}^a, \\ V_1 = [B^a A^a_\mu + iC^a D_\mu \bar{C}^a]\; dx^\mu, & W_1 = [\bar{C}^a \varepsilon_{\mu\rho}\partial^\rho C^a - i\, \mathcal{B}^a A^a_\mu]\; dx^\mu, \\ V_2 = i[A^a_\mu D_\nu \bar{C}^a - \bar{C}^a D_\mu A^a_\nu] dx^\mu \wedge dx^\nu, & W_2 = i[\varepsilon_{\mu\rho}\partial^\rho C^a A^a_\nu + \frac{C^a}{2}\varepsilon_{\mu\nu}(\partial\cdot A)^a] dx^\mu \wedge dx^\nu. \end{array} \tag{5.6}$$

It can be seen that V_0 and W_0 are BRST and co-BRST invariant respectively and V_2 and W_2 are closed and co-closed respectively. These invariants for $(k = 1, 2)$ obey [23], [10]

$$\begin{array}{rclc} \delta_B V_k &=& \eta\; d\; V_{k-1}, & d = dx^\mu\; \partial_\mu, \\ \delta_D W_k &=& \eta\; \delta\; W_{k-1}, & \delta = i\; dx^\mu\; \varepsilon_{\mu\nu}\; \partial^\nu, \end{array} \tag{5.7}$$

where d and δ are the exterior and dual-exterior derivatives on the 2D compact manifold. Unlike the $U(1)$ gauge theory [18] here there are no specific transformations that can relate I_k and J_k. Using the on-shell nilpotent transformations (2.3) and (3.2), it is interesting to verify that, modulo some total derivatives, the Lagrangian density (3.1) can be written as the sum of BRST- and co-BRST invariant parts:

$$\eta\mathcal{L}_b = \tfrac{1}{2}\;\delta_d\;[\; iE^a C^a\;] - \tfrac{1}{2}\;\delta_b\;[\; i(\partial\cdot A)^a\bar{C}^a\;]. \tag{5.8}$$

Using the fact that conserved and nilpotent charges Q_r (r = b, d) are the generator of transformations $\delta_r\phi = -i\eta[\phi, Q_r]_\pm$, where $(+)-$ stands for the (anti)commutator corresponding to the generic field ϕ being (fermionic)bosonic, it can be seen that the Lagrangian density (5.8) can be recast as: $\mathcal{L}_b = \{Q_d, S_1\} + \{Q_b, S_2\}$ for $S_1 = \frac{1}{2}E^a C^a$ and $S_2 = -\frac{1}{2}(\partial\cdot A)^a\bar{C}^a$. This demonstrates that the topological theory under consideration is similar in outlook as the Witten type theories [10]. There is a bit of difference, however. This is because of the fact that there are two conserved and nilpotent charges in our discussion whereas there exists only one conserved and nilpotent BRST charge in Ref. [10]. It is straightforward to check that the partition functions and expectation values of the BRST invariants, co-BRST invariants and the topological invariants are metric independent §. The main argument to show this fact in the framework of BRST cohomology is the requirement that $Q_b|phys>=0, Q_d|phys>=0$ (see, e.g., Ref. [11]) and the metric independence of the path integral measure (see, e.g., Ref. [23]).

6 Conclusions

We have shown that the nilpotent symmetry transformations under which the gauge-fixing term remains invariant ($\delta_D[(\partial \cdot A)^a (\partial \cdot A)^a] = 0$) is the dual BRST symmetry in contrast to the usual BRST symmetry under which the total kinetic energy term remains invariant ($\delta_B[F^{\mu\nu a} F^a_{\mu\nu}] = 0$). The anticommutator of these two symmetries corresponds to a symmetry (generated by the Laplacian(Casimir) operator) under which the ghost fields do not transform ($\delta_W C^a = \delta_W \bar{C}^a = 0$) and the gauge connection transforms to its own equation of motion. Thus, this symmetry becomes trivial on the on-shell. This triviality is connected with the topological nature of the 2D free non-Abelian gauge theory as the BRST- and co-BRST symmetries are good enough to gauge away both the degrees of freedom of the gauge boson. In fact, the Laplacian operator goes to zero when the equations of motion are exploited. The on-shell expression of the Laplacian operator encompasses the degrees of freedom left in the theory. In the case of $U(1)$ gauge field coupled to the Dirac fermion, it has been shown that the Laplacian operator does not go to zero even on the on-shell and its expression contains only the fermionic degrees of freedom [22]. Furthermore, it has been demonstrated that the co-BRST transformation on the $U(1)$ gauge field corresponds to the quantum chiral transformation on the Dirac fermions in 2D. This symmetry, therefore, might shed light on the ABJ anomaly in 2D and might provide clue to the consistency of the "anomalous" gauge theory in 2D (see, e.g., [24,25]). It will be interesting to study the BRST cohomolgy when non-Abelian gauge field is coupled to the matter fields and generalise these understandings to the case of gauge theories in physical four dimensions.

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§It will be noticed that we have taken here the flat Minkowski metric. However, our arguments and discussions are valid for any nontrivial metric. The metric independence of the path integral measure for the topological field theories has been shown in Ref. [23].

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