

# 1 Zweikörperproblem

Abkürzungen

$$\vec{r}_1 \equiv \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \vec{r}_2 \equiv \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Lagrangefunktion

$$L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2) = \frac{m_1}{2}(\dot{\vec{r}}_1)^2 + \frac{m_2}{2}(\dot{\vec{r}}_2)^2 - V(|\vec{r}_2 - \vec{r}_1|)$$

Kanonische Impulse

$$\vec{p}_1 \equiv \begin{pmatrix} p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} = \nabla_{\dot{\vec{r}}_1} L \quad \vec{p}_2 \equiv \begin{pmatrix} p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} = \nabla_{\dot{\vec{r}}_2} L$$

$$\text{z.B. } p_{2x} = \frac{\partial L}{\partial \dot{x}_2}$$

Hamiltonfunktion

$$H(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} + V(|\vec{r}_2 - \vec{r}_1|)$$

Hamiltonsche Gleichungen

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q(t), p(t)) \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q(t), p(t))$$

mit

$$\{q_i\} = \{x_1, y_1, z_1, x_2, y_2, z_2\} \quad \{p_i\} = \{p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}\}$$

nehmen die Form an

$$\dot{\vec{r}}_1 = \nabla_{\vec{p}_1} H = \frac{\vec{p}_1}{m_1}$$

$$\dot{\vec{r}}_2 = \nabla_{\vec{p}_2} H = \frac{\vec{p}_2}{m_2}$$

$$\dot{\vec{p}}_1 = -\nabla_{\vec{r}_1} H = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \cdot V'(|\vec{r}_2 - \vec{r}_1|)$$

$$\dot{\vec{p}}_2 = -\nabla_{\vec{r}_2} H = -\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \cdot V'(|\vec{r}_2 - \vec{r}_1|)$$

mit der Ableitung

$$V'(x) \equiv \frac{\partial V(x)}{\partial x}$$

fehlt noch: Poissonklammern

$$\{f, g\}_{q,p} \equiv \{f, g\}_{Q,P}$$

## 2 Punkt-Transformation auf Differenz- und Schwerpunkt-Koordinaten

$$\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{p}_1 \\ \vec{p}_2 \end{pmatrix} \mapsto \begin{pmatrix} \vec{r} \\ \vec{R} \\ \vec{p} \\ \vec{P} \end{pmatrix}$$

$$\vec{r}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{r}_2 - \vec{r}_1 \quad \vec{R}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) \equiv \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{p}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) \equiv \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \frac{m_1 \vec{p}_2 - m_2 \vec{p}_1}{m_1 + m_2} \quad \vec{P}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) \equiv \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \vec{p}_1 + \vec{p}_2$$

umgekehrt

$$\vec{r}_1(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \vec{R} - \frac{m_2}{m_1 + m_2} \cdot \vec{r} \quad \vec{r}_2(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \vec{R} + \frac{m_1}{m_1 + m_2} \cdot \vec{r}$$

$$\vec{p}_1(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{m_1}{m_1 + m_2} \cdot \vec{P} - \vec{p} \quad \vec{p}_2(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{m_2}{m_1 + m_2} \cdot \vec{P} + \vec{p}$$

mit reduzierter Masse m und Gesamtmasse M

$$m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$M \equiv m_1 + m_2$$

Lagrangefunktion

$$L(\vec{r}, \vec{R}, \dot{\vec{r}}, \dot{\vec{R}}) = \frac{m}{2} (\dot{\vec{r}})^2 + \frac{M}{2} (\dot{\vec{R}})^2 - V(|\vec{r}|)$$

$$\vec{p} = \nabla_{\dot{\vec{r}}} L$$

$$\vec{P} = \nabla_{\dot{\vec{R}}} L$$

Hamiltonfunktion

$$H(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{\vec{p}^2}{2m} + \frac{\vec{P}^2}{2M} + V(|\vec{r}|)$$

Hamiltonsche Gleichungen mit

$$\{q_i\} = \{x, y, z, X, Y, Z\} \quad \{p_i\} = \{p_x, p_y, p_z, P_x, P_y, P_z\}$$

nehmen die Form an

$$\dot{\vec{r}} = \nabla_{\vec{p}} H = \frac{\vec{p}}{m}$$

$$\dot{\vec{R}} = \nabla_{\vec{P}} H = \frac{\vec{P}}{M}$$

$$\dot{\vec{p}} = -\nabla_{\vec{r}} H = -V'(|\vec{r}|) \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$\dot{\vec{P}} = -\nabla_{\vec{R}} H = \vec{0}$$

fehlt noch: Poisson-Klammern

Loesung

$$\vec{P} = \vec{P}(t=0) \Rightarrow \vec{R} = \frac{\vec{P}(t=0)}{M} + \vec{R}(t=0)$$

$$m \cdot \ddot{\vec{r}} = -V'(|\vec{r}|) \cdot \frac{\vec{r}}{|\vec{r}|} \quad (\text{allgemeiner} = -\nabla_{\vec{r}} \cdot V(\vec{r}))$$

### 3 Kanonische Transformation, die Orte und Impulse vertauscht

$$F_1 = \sum_i q_i Q_i$$

$$p_i = \frac{\partial F_1}{\partial q_i}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}$$

Erzeugende

$$F_1(\vec{r}_1, \vec{r}_2, \vec{R}_1, \vec{R}_2) = \vec{r}_1 \vec{R}_1 + \vec{r}_2 \vec{R}_2$$

$$\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{p}_1 \\ \vec{p}_2 \end{pmatrix} \mapsto \begin{pmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vec{P}_1 \\ \vec{P}_2 \end{pmatrix}$$

$$\vec{p}_1 = \vec{R}_1$$

$$\vec{p}_2 = \vec{R}_2$$

$$\vec{P}_1 = -\vec{r}_1$$

$$\vec{P}_2 = -\vec{r}_2$$

Hamiltonfunktion

$$H(\vec{R}_1, \vec{R}_2, \vec{P}_1, \vec{P}_2) = \frac{\vec{R}_1^2}{2m_1} + \frac{\vec{R}_2^2}{2m_2} + V(|\vec{P}_2 - \vec{P}_1|) \equiv \frac{1}{2}D_1\vec{R}_1^2 + \frac{1}{2}D_2\vec{R}_2^2 + V(|\vec{P}_2 - \vec{P}_1|)$$

mit Federkonstanten

$$D_1 \equiv \frac{1}{m_1} \quad D_2 \equiv \frac{1}{m_2}$$

Hamiltonsche Gleichungen mit

$$\{q_i\} = \{X_1, Y_1, Z_1, X_2, Y_2, Z_2\} \quad \{p_i\} = \{P_{1x}, P_{1y}, P_{1z}, P_{2x}, P_{2y}, P_{2z}\}$$

nehmen die Form an

$$\dot{\vec{R}}_1 = \nabla_{\vec{P}_1} H = -V'(|\vec{P}_2 - \vec{P}_1|) \cdot \frac{\vec{P}_2 - \vec{P}_1}{|\vec{P}_2 - \vec{P}_1|}$$

$$\dot{\vec{R}}_2 = \nabla_{\vec{P}_2} H = V'(|\vec{P}_2 - \vec{P}_1|) \cdot \frac{\vec{P}_2 - \vec{P}_1}{|\vec{P}_2 - \vec{P}_1|}$$

$$\dot{\vec{P}}_1 = -\nabla_{\vec{R}_1} H = D_1 \vec{R}_1$$

$$\dot{\vec{P}}_2 = -\nabla_{\vec{R}_2} H = D_2 \vec{R}_2$$

fehlt noch: Poisson-Klammern