1 Zweikoerperproblem

Abkuerzungen

$$\vec{r_1} \equiv \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\vec{r_2} \equiv \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Lagrangefunktion

$$L(\vec{r_1}, \vec{r_2}, \dot{\vec{r_1}}, \dot{\vec{r_2}}) = \frac{m_1}{2}(\dot{\vec{r_1}})^2 + \frac{m_2}{2}(\dot{\vec{r_2}})^2 - V(|\vec{r_2} - \vec{r_1}|)$$

Kanonische Impulse

$$\vec{p_1} \equiv \begin{pmatrix} p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} = \nabla_{\dot{\vec{r_1}}} L$$

$$\vec{p_2} \equiv \begin{pmatrix} p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} = \nabla_{\dot{\vec{r_2}}} L$$

z.B. $p_{1x} = \frac{\partial L}{\partial \dot{x_1}}$ Hamiltonfunktion

$$H(\vec{r_1}, \vec{r_2}, \vec{p_1}, \vec{p_2}) = \frac{\vec{p_1}^{\,2}}{2m} + \frac{\vec{p_2}^{\,2}}{2m} + V(|\vec{r_2} - \vec{r_1}|)$$

Hamiltonsche Gleichungen

$$\dot{r_1} = \nabla_{\vec{p_1}} H$$

$$\dot{r_2} = \nabla_{\vec{p_2}} H$$

$$\dot{p_1} = -\nabla_{\vec{r_1}} H$$

$$\dot{p_2} = -\nabla_{\vec{r_2}} H$$

oder

$$\frac{\partial H}{\partial p_i}(q(t), p(t)) \ , \quad \dot{p_i} = -\frac{\partial H}{\partial q_i}(q(t), p(t))$$

mit

$$q = \{x_1, y_1, z_1, x_2, y_2, z_2\} \ , \quad p = \{p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}\}$$

fehlt noch: Poissonklammern

$$\{f, g\}_{q,p} \equiv \{f, g\}_{Q,P}$$

2 Transformation auf Differenz- und Schwerpunkt-Koordinaten

$$\begin{pmatrix} \vec{r_1} \\ \vec{r_2} \\ \vec{p_1} \\ \vec{p_2} \end{pmatrix} \mapsto \begin{pmatrix} \vec{r} \\ \vec{R} \\ \vec{p} \\ \vec{P} \end{pmatrix}$$

$$\vec{r}(\vec{r_1}, \vec{r_2}, \vec{p_1}, \vec{p_2}) \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{r_2} - \vec{r_1}$$

$$\vec{R}(\vec{r_1}, \vec{r_2}, \vec{p_1}, \vec{p_2}) \equiv \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

$$\vec{p}(\vec{r_1}, \vec{r_2}, \vec{p_1}, \vec{p_2}) \equiv \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \frac{1}{m_1 + m_2}(m_1\vec{p_2} - m_2\vec{p_1})$$

$$\vec{P}(\vec{r_1}, \vec{r_2}, \vec{p_1}, \vec{p_2}) \equiv \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \vec{p_1} + \vec{p_2}$$

umgekehrt

$$m := \frac{m_1 m_2}{m_1 + m_2}$$

$$M := m_1 + m_2$$

$$\vec{r_1}(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \vec{R} - \frac{m_2}{m_1 + m_2}\vec{r}$$

$$\vec{r_2}(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r}$$

$$\vec{p_1}(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{m_1}{m_1 + m_2}\vec{P} - \vec{p}$$

$$\vec{p_2}(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{m_2}{m_1 + m_2}\vec{P} + \vec{p}$$

$$L(\vec{r}, \vec{R}, \dot{\vec{r}}, \dot{\vec{R}}) = \frac{m}{2}(\dot{\vec{r}})^2 + \frac{M}{2}(\dot{\vec{R}})^2 - V(|\vec{r}|)$$

$$\vec{p} = \nabla_{\dot{\vec{r}}} L$$

$$\vec{P} = \nabla_{\dot{\vec{R}}} L$$

$$H(\vec{r}, \vec{R}, \vec{p}, \vec{P}) = \frac{\vec{p}^2}{2m} + \frac{\vec{P}^2}{2M} + V(|\vec{r}|)$$

Hamiltonsche Gleichungen

$$\dot{r} = \nabla_{\vec{p}} H$$

$$\dot{R} = \nabla_{\vec{P}} H$$
$$\dot{p} = -\nabla_{\vec{r}} H$$
$$\dot{P} = -\nabla_{\vec{R}} H$$

oder

$$\frac{\partial H}{\partial p_i}(q(t), p(t))\ , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q(t), p(t))$$

mit

$$q = \{x, y, z, X, Y, Z\}\ , \quad p = \{p_x, p_y, p_z, p_X, p_Y, p_Z\}$$

fehlt noch: Poisson-Klammern und Probe rechnen!!

3 Kanonische Transformation, die Orte und Impulse vertauscht

$$F_1 = \sum_i q_i Q_i$$
$$p_i = \frac{\partial F_1}{\partial q_i}$$
$$P_i = -\frac{\partial F_1}{\partial Q_i}$$

Erzeugende

$$F_1(\vec{r_1}, \vec{r_2}, \vec{R_1}, \vec{R_2}) = \vec{r_1}\vec{R_1} + \vec{r_2}\vec{R_2}$$
$$\begin{pmatrix} \vec{r_1} \\ \vec{r_2} \\ \vec{p_1} \\ \vec{p_2} \end{pmatrix} \mapsto \begin{pmatrix} \vec{R_1} \\ \vec{R_2} \\ \vec{P_1} \\ \vec{P_2} \end{pmatrix}$$
$$\vec{p_1} = \vec{R_1}$$
$$\vec{p_2} = \vec{R_2}$$
$$\vec{P_1} = -\vec{r_1}$$
$$\vec{P_2} = -\vec{r_2}$$

$$H(\vec{R_1}, \vec{R_2}, \vec{P_1}, \vec{P_2}) = \frac{\vec{R_1}^{\,2}}{2m_1} + \frac{\vec{R_2}^{\,2}}{2m_2} + V(\left|\vec{P_2} - \vec{P_1}\right|) := \frac{1}{2} D_1 \vec{R_1}^{\,2} + \frac{1}{2} D_2 \vec{R_2}^{\,2} + V(\left|\vec{P_2} - \vec{P_1}\right|)$$

Hamiltonsche Gleichungen

$$\dot{R}_1 = \nabla_{\vec{P_1}} H$$
$$\dot{R}_2 = \nabla_{\vec{P_2}} H$$

$$\dot{P}_1 = -\nabla_{\vec{R}_1} H$$

$$\dot{P}_2 = -\nabla_{\vec{R}_2} H$$

oder

$$\frac{\partial H}{\partial p_i}(q(t), p(t)) \ , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q(t), p(t))$$

mit

$$q = \{X_1, Y_1, Z_1, X_2, Y_2, Z_2\} \ , \quad p = \{p_{X1}, p_{Y1}, p_{Z1}, p_{X2}, p_{Y2}, p_{Z2}\}$$

fehlt noch: Poisson-Klammern und Probe rechnen!!