Practice Problems from assignment 3

4.6: a) $\frac{1}{\sqrt{n}}$; We will pick our $N_0 \in \mathbb{N}$ st. $\forall n \geq N_0$, (and letting ϵ be some arbitrary positive number), $\sqrt{n} > \frac{1}{\epsilon} \Rightarrow \frac{1}{\sqrt{n}} < \epsilon \Rightarrow |\frac{1}{\sqrt{n}} - 0| < \epsilon$. So, we can see that the sequence converges to 0 for the specified N_0

b)$\frac{2n+1}{n+1}$; We will start by simplifying our expression in terms of a single n so that we can pick an n st. when we write out the proof it agrees with our intuition that this should converge to 2.

$$|\frac{2n+1}{n+1} - 2| = |\frac{2n+1-2(n+1)}{n+1}| = |\frac{-1}{n+1}| < \epsilon$$

From this it is not hard to see we should select an n st.

$$n > \frac{1}{\epsilon} - 1$$

So, letting ϵ be some arbitrary positive number, we will choose an $N_0 \in \mathbb{N}$ st. $\forall n > N_0, n > \frac{1}{\epsilon} - 1$ Then, $n + 1 > \frac{1}{\epsilon} \Rightarrow \frac{1}{n+1} < \epsilon \Rightarrow |\frac{-1}{n+1}| < \epsilon \Rightarrow |\frac{2n+1-2n-2}{n+1}| < \epsilon \Rightarrow |\frac{2n+1}{n+1} - 2| < \epsilon$

4.8: By the theorem proved in the first lesson on sequences, we know that a convergent series is bounded. So it is safe to say that any convergent sequence has both a supremum and an infimum. Let $x = \sup((a_n)_{n=1}^{\infty})$, and let $y = \inf((a_n)_{n=1}^{\infty})$. If $(a_n)_{n=1}^{\infty}$ is truly convergent, then ($x \vee y \in (a_n)_{n=1}^{\infty} \Leftrightarrow$ the sequence contains a smallest or largest term).

Let's assume $\neg(x \vee y \in (a_n)_{n=1}^{\infty})$ This implies that there is a subsequence of our sequence that continually approaches the infimum, and another that continually approaches the supremum. But this just implies $(a_n)_{n=1}^{\infty} < x \wedge (a_n)_{n=1}^{\infty} > y, \forall n \in \mathbb{N}$

Since we know that the series converges to a single value, if it approaches the infimum and supremum, then the infimum must be the supremum; $y = x$, and, by definition of infimum and supremum this implies that the sequence must be st. $\forall a_n \in (a_n)_{n=1}^{\infty}, y = a_n = x$. Which contradicts our assumption ■

4.9:

Example st. $a_n - b_n \to 0$ but $\frac{a_n}{b_n}$ does not tend to 1:

consider: $(a_n)_{n=1}^{\infty} := \{a_n = \frac{-1}{n}^n\}$ and $(b_n)_{n=1}^{\infty} := \{b_n = \frac{1}{n}\}$. Or really any sequence in which they both approach 0 but one's values oscillate about 0. Or where both oscillate about but are out of phase by some non integer

multiple of 2π i.e. $\frac{\pi}{2}$, where $(a_n)_{n=1}^{\infty} := \{a_n = (sin(\frac{n\pi}{2} + \frac{\pi}{4}))^{-n}\}$ and $(b_n)_{n=1}^{\infty}$ $:= \{b_n = (sin(\frac{n\pi}{2} - \frac{\pi}{4}))^{-n}\}$
Also if $(a_n)_{n=1}^{\infty}$ is just 0 for every term and $(b_n)_{n=1}^{\infty}$ converges to 0 works as well.

Example st. $\frac{a_n}{b_n}$ tends to 1 but $\neg(a_n - b_n \to 0)$:
Consider: $(a_n)_{n=1}^{\infty} := \{a_n = n + 1\}$ and $(b_n)_{n=1}^{\infty} := \{b_n = n\}$.

Problem 4.10:
If (a_n) is convergent, then that means that for all but some finite $n \in \mathbb{N}$,
$|a_n - L| < \epsilon \Rightarrow L - \epsilon \ < |an| < L + \epsilon \Rightarrow L - \epsilon \ < ||an|| < L + \epsilon \Rightarrow ||an| - L| < \epsilon$
So, the absolute value of the sequence converges to L as well ■

In general, this logic does not work in reverse. Consider the sequence $(a_n)_{n=1}^{\infty}$ $:= \{a_n = (\frac{1}{n} + 3)(1)^{-n}\}$ The limit does not converge to anything, however its abs. value converges to 3.
In one specific case it does work: if $L = 0$, proved in class

Problem 4.12:
If $(a_n) \to a > 0$, then for all but some finite $n \in \mathbb{N}$,
$|a_n - a| < \epsilon \Rightarrow a - \epsilon \ < |a_n| < a + \epsilon \to \sqrt{a - \epsilon} < \sqrt{|a_n|} < \sqrt{a + \epsilon}$
(not sure if this is valid??)
Since we left ϵ arbitrary we can say that $\sqrt{a \pm \epsilon} = \sqrt{a} \pm \epsilon_2$
Where ϵ_2 is some new epsilon. From this,

$$\sqrt{a - \epsilon} < \sqrt{|a_n|} < \sqrt{a + \epsilon} \Rightarrow \sqrt{a} - \epsilon_2 < \sqrt{|a_n|} < \sqrt{a} + \epsilon_2 \Rightarrow |\sqrt{a_n} - \sqrt{a}| < \epsilon_2$$

This whole above method is not valid because you can't use arbitrary epsilon to prove convergence to a specific value!!!!
This part is valid:
If $(a_n) \to a > 0$, then for all but some finite $n \in \mathbb{N}$,

$$|a_n - a| < \epsilon\sqrt{a} \Rightarrow |\sqrt{a_n} - \sqrt{a}| = \frac{|a_n + a|}{\sqrt{a_n} + \sqrt{a}} < \frac{|a_n + a|}{\sqrt{a}} < \frac{\epsilon\sqrt{a}}{\sqrt{a}} = \epsilon$$

■

Problem 4.13:
An example of a sequence that does not converge but who's mean does is simply the sequence $(a_n)_{n=1}^{\infty} := \{a_n = n\}$