BOUNDARY ELEMENT METHOD

Theoretical Formulation, Numerical Approximation, and Parallel Implementation

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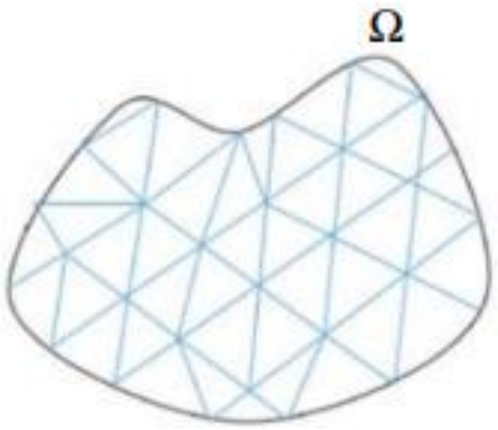
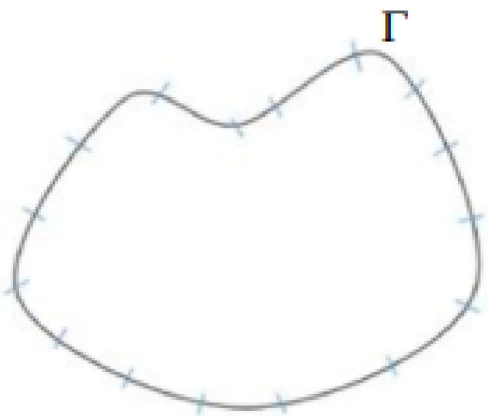
WHAT IS BEM?





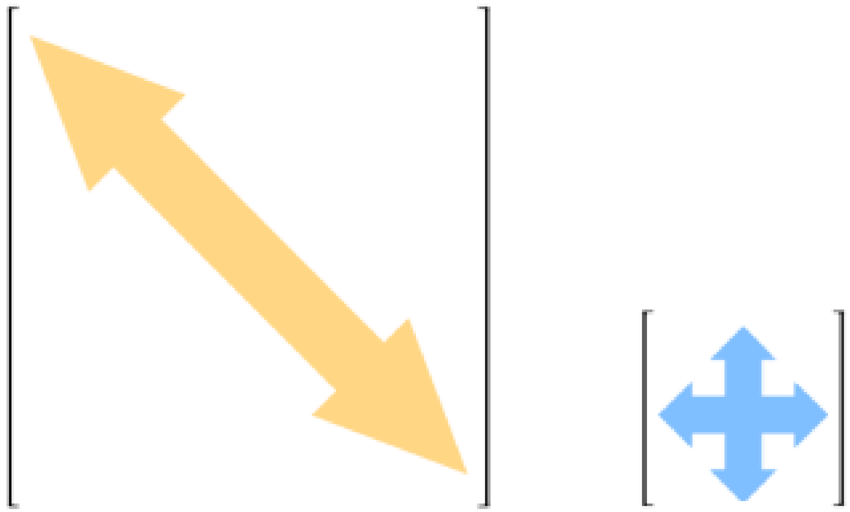
$$\Delta \phi = 0$$

WHY BEM?

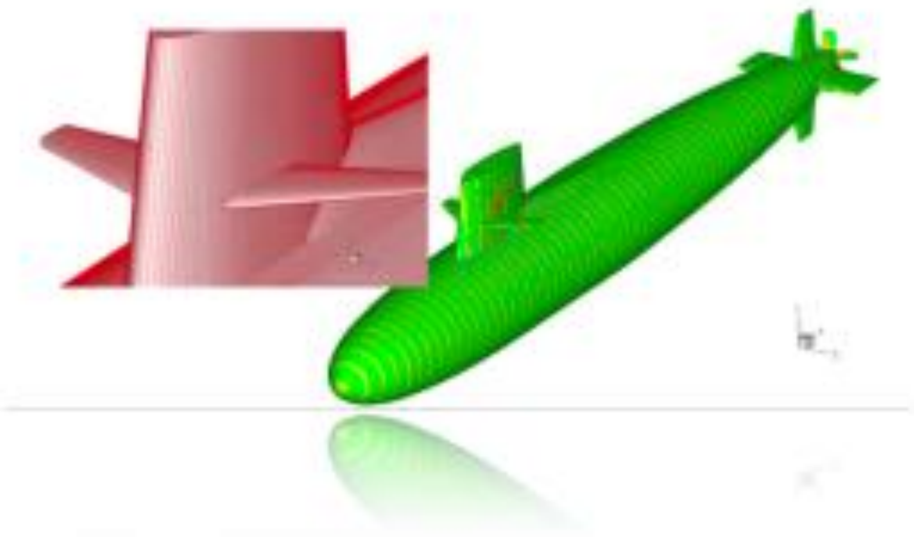
We already have FEM

- ✅ the solution is expressed as a continuous mathematical formula (BIE)
- ✅ for infinite domains, the problem is formulated simply as an exterior one
- ✅ significant reduction in the number of degrees of freedom of the numerical model

- ❌ coefficient matrices fully populated and non-symmetric (then you can go with high orders)
- ❌ the problem has to be linear
- ❌ we need the fundamental solution ($-\Delta\phi = \delta_{\mathbf{x}}$)

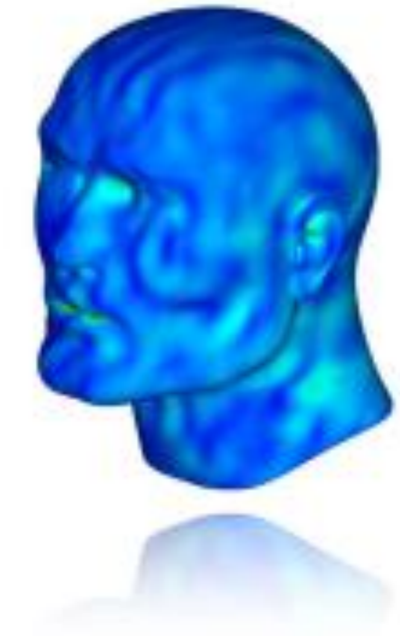


WHERE IS BEM?



BEM model of the Skipjack submarine impinged upon
by an incident wave in the direction (1, 0, -1).

WHERE IS BEM?



BEM mesh and sound-pressure plots for a human-head model.

HOW IS BEM?

1. Fundamental solution: $-\Delta G = \delta_{\mathbf{x}} \overset{\mathbb{R}^3}{\rightsquigarrow} G(\mathbf{x},\mathbf{y}) = \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \quad \forall\, \mathbf{y} \in \mathbb{R}^3$
2. Laplace equation: $\Delta\phi = 0$ + mixed boundary conditions
3. Integrate by parts twice:

$$0 = \int_\Omega \Delta\phi\ \varphi\ \mathrm{d}\Omega = \int_\Gamma \frac{\partial\phi}{\partial n}\ \varphi\ \mathrm{d}S - \int_\Omega \nabla\phi\ \nabla\varphi\ \mathrm{d}\Omega$$

$$0 = \int_\Gamma \frac{\partial\phi}{\partial n}\ \varphi\ \mathrm{d}S - \int_\Gamma \frac{\partial\varphi}{\partial n}\ \phi\ \mathrm{d}S + \int_\Omega \Delta\varphi\ \phi\ \mathrm{d}\Omega$$

$$\int_\Omega -\Delta\varphi\ \phi\ \mathrm{d}\Omega = \int_\Gamma \frac{\partial\phi}{\partial n}\ \varphi\ \mathrm{d}S - \int_\Gamma \frac{\partial\varphi}{\partial n}\ \phi\ \mathrm{d}S$$

HOW BEM?

4. Choosing $\varphi \equiv G$:

$$\underbrace{\int_{\Omega} \underbrace{-\Delta G(\mathbf{x}, \mathbf{y})}_{\delta_{\mathbf{x}}} \; \phi(\mathbf{y}) \; \mathrm{d}\Omega(\mathbf{y})}_{\phi(\mathbf{x})} = \int_{\Gamma} \left[\frac{\partial \phi}{\partial n}(\mathbf{y}) \; G(\mathbf{x}, \mathbf{y}) - \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y}) \; \phi(\mathbf{y}) \right] \mathrm{d}S(\mathbf{y})$$

5. Let $\mathbf{x}$ lie on the boundary:

$$\alpha(\mathbf{x})\phi(\mathbf{x}) = \int_{\Gamma} \frac{\partial \phi}{\partial n}(\mathbf{y}) \; G(\mathbf{x}, \mathbf{y}) \; \mathrm{d}S(\mathbf{y}) - \int_{\Gamma} \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y}) \; \phi(\mathbf{y}) \; \mathrm{d}S(\mathbf{y}) \qquad \forall \, \mathbf{x} \in \Gamma$$

HOW BEM?

6. Collocation method:

$$\alpha(\mathbf{x})\phi(\mathbf{x}) = \int_\Gamma \frac{\partial \phi}{\partial n}(\mathbf{y})\ G(\mathbf{x},\mathbf{y})\ \mathrm{d}S(\mathbf{y}) - \int_\Gamma \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y})\ \phi(\mathbf{y})\ \mathrm{d}S(\mathbf{y}) \qquad \forall\,\mathbf{x} \in \Gamma$$

$$\Downarrow$$

$$(\alpha + N)\hat{\phi} - D\hat{\gamma} = 0$$

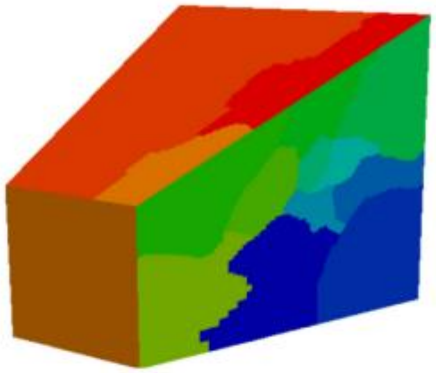
$$\Downarrow$$

$$At = b$$

BOTTLENECK OF STRAIGHTFORWARD IMPLEMENTATION

Function	Time (s)
Assemble cycle	68.44
Solvetime	5.741
Post process (Gradient Recovery)	0.1645
Total time	78.6

Test case on a single CPU.



Truncated pyramid

***Matrix assembly** is the major part of the overall program computationally demanding.*
The parallelization is the strategy to solve the bottleneck.

SIMPLEST PARALLELISATION STRATEGY

- *Full MPI, each row can be indipendently computed, so that these are distribuited among different processors.*

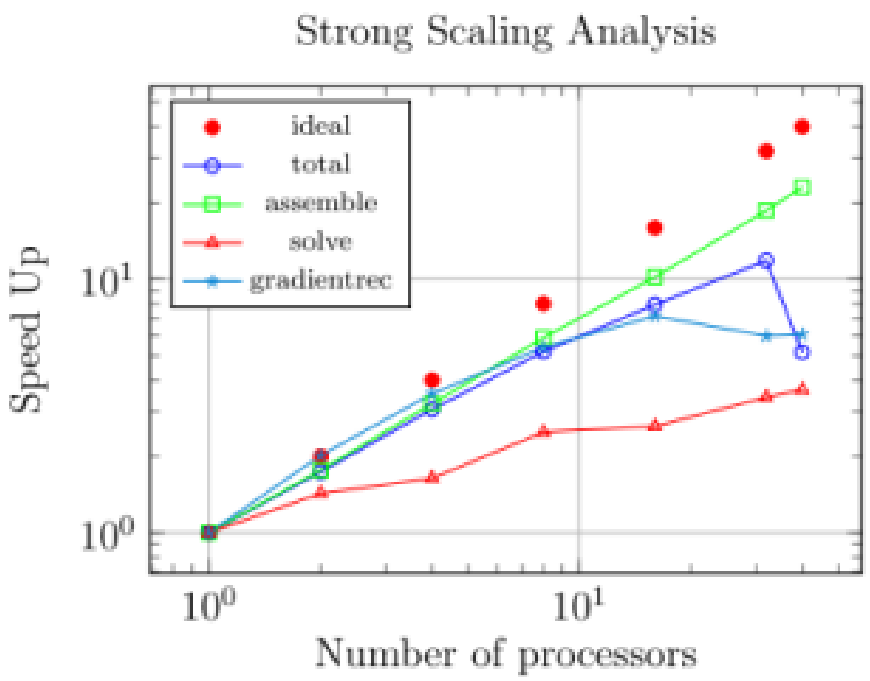
REMARK: a proper work balance among processors is of paromount importance

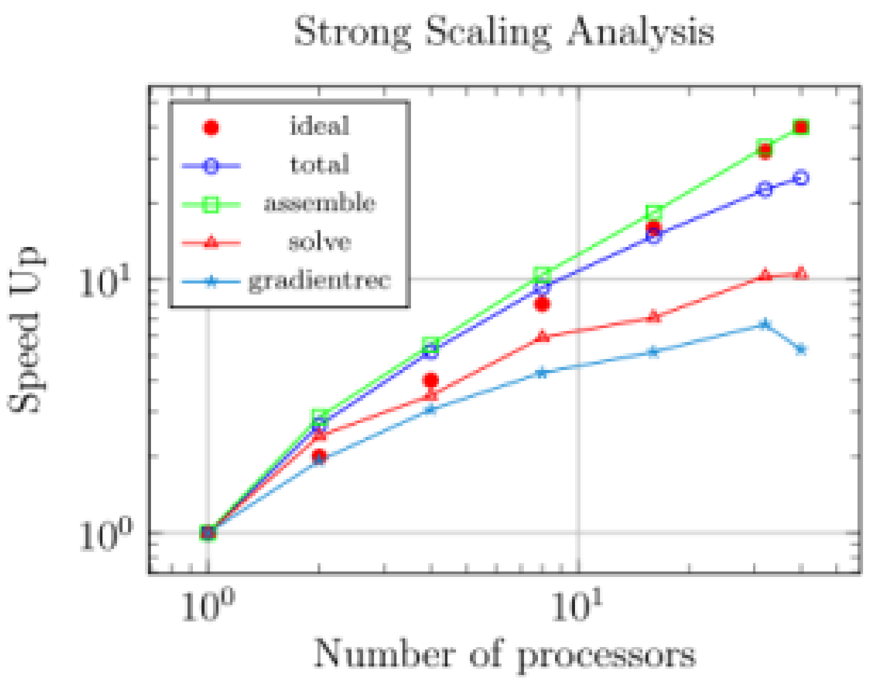
FEM vs BEM

Each matrix entry depends on information on the entire boundary of the domain, so every processor still needs access to the full discretisation of the boundary of the geometry

STRONG SCALABILITY

REMARK: the problem is solved using a parallel implementation of a preconditioned Generalised Minimal Residual (GMRES) solver





On the left, the analysis is carried out using 6534 degrees of freedom, while on the right, 25350

FAST MULTIPOLE METHOD

Computational cost reduction of assembling phase, from O(N^2) to O(N)

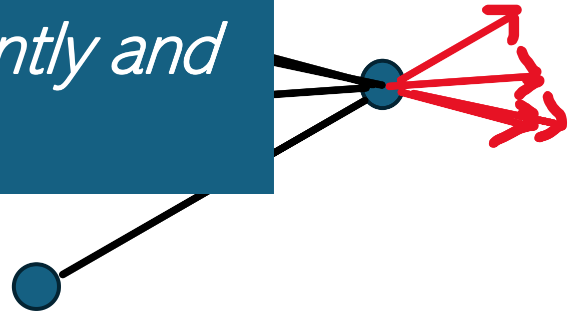
- *MAIN INGREDIENT:*

  Exploiting the fact
  Foundamental so
  reprensents the
  electromagnetic p

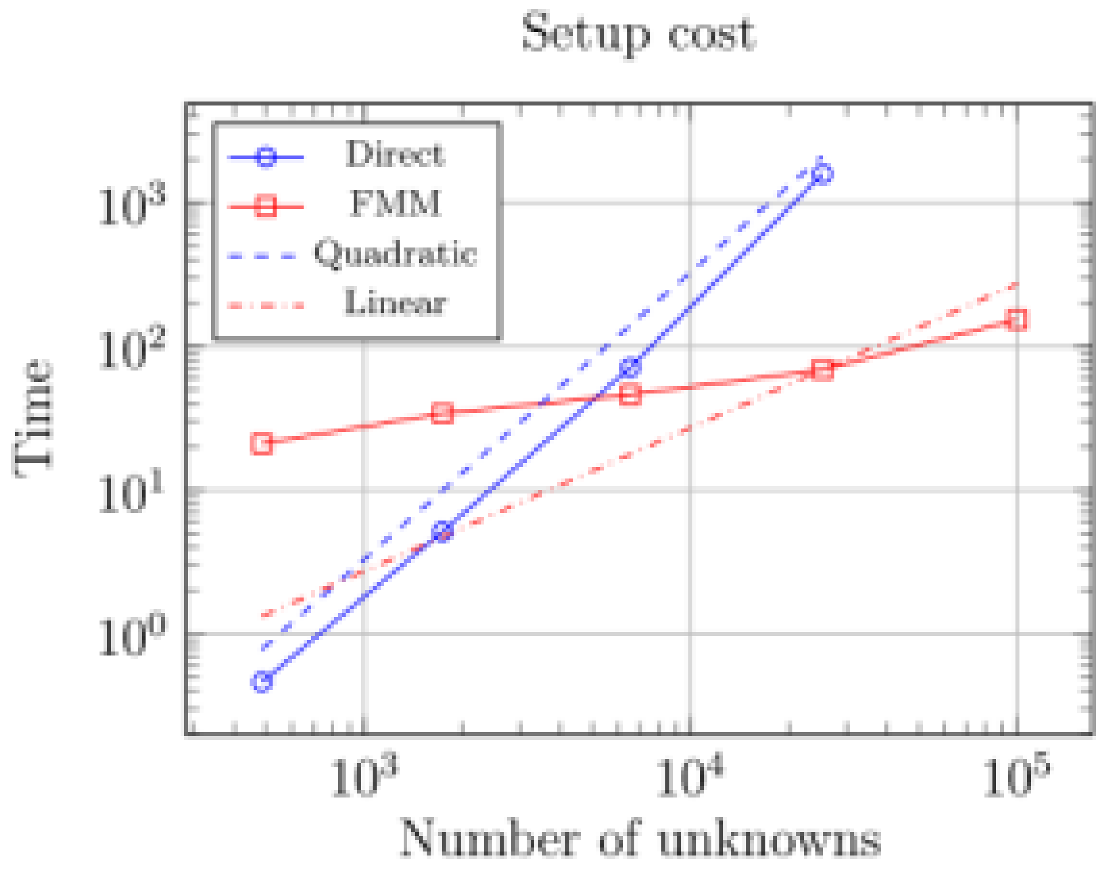
  Harmonic expansion
  of electromagnetic
  potential

- *MAIN IDEA:*

FMM uses harmonic expansions to cluster long-distance interactions, allowing the computation of the potential to be performed efficiently and in a single operation.



FEM-BEM vs BEM (no Parallelisation)



Remark: The breakeven point is roughly located at 10^4 unknowns, the standard BEM is more competitive, this is due to the fact that in the FMM-BEM we have also to set up the octree partition of the domain

P-FMM-BEM CHALLENGES

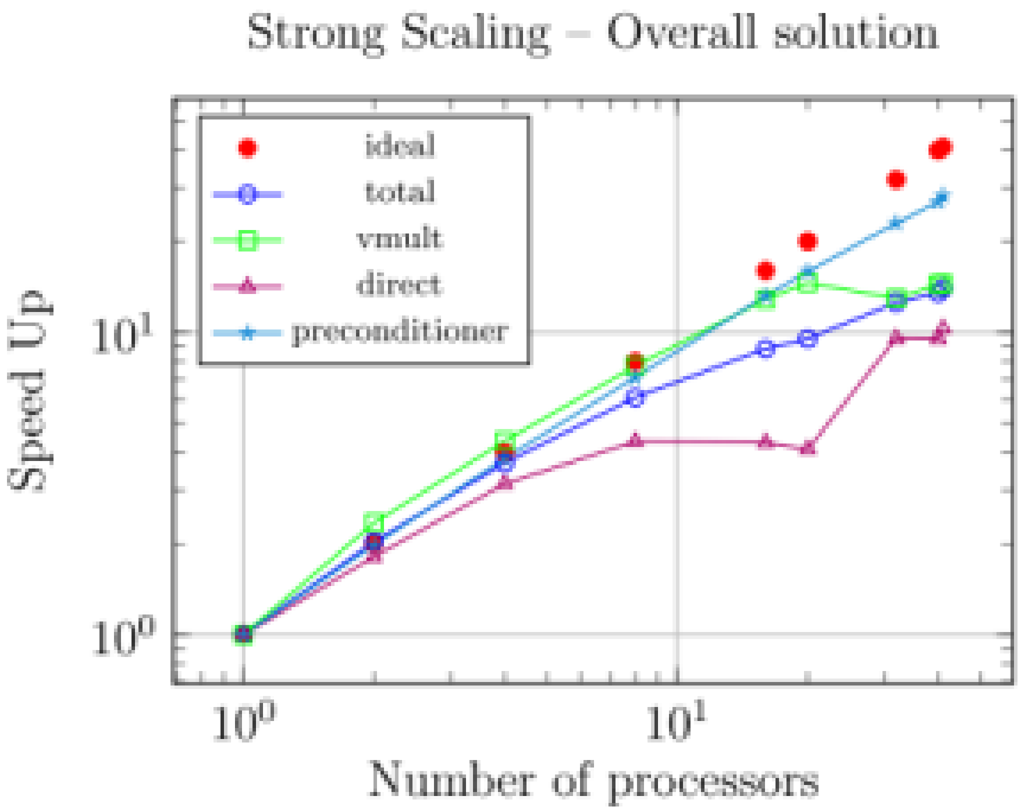
- *Complex geometry*
- *Local refinement*
- *Specific quadrature formulas (due to the weakly singular integrals)*
- *Domain decomposition*

$\Rightarrow$

OpenSource Libreries can't be used, and the so called Locally Essential Tree

THIS LED TO AN AD-HOC PARALLELISED IMPLEMENTATION

HYBRID TBB-MPI PARALLELISATION



Trade-off between computational efficiency and memory

This plot shows the scalability issue for the short range interaction that have to be computed directly

FINAL REMARK:
The preconditioner is not optimal, in this way the reasearch is still very active

Thank you for your attention