Diffusion/Elliptic Problems

Weak Formulation. Given V, $F : V \to \mathbb{R}$ functional and $a : V \times V \to \mathbb{R}$ bilinear form,

$$\boxed{\text{find } u \in V \ : \ a(u, v) = F(v) \quad \forall v \in V}$$

Lax-Milgram Lemma. Assume that:

1. V Hilbert space with $\|\cdot\|_V$ and $\langle \cdot, \cdot \rangle$
2. $F \in V'$, i.e. $|F(v)| \le \|F\|_{V'} \|v\|_V$
3. a cont., i.e. $|a(u, v)| \le M \|u\|_V \|v\|_V$
4. a coercive, i.e. $a(v, v) \ge \alpha \|v\|_V^2$

Then: $\exists !\ u$ sol. di WF, and $\|u\|_V \le \|F\|_{V'}/\alpha$

Galerkin Approximation. If you can build $V_h \subset V$ s.t. dim $V_h = N_h < \infty$ ($\Rightarrow V_h$ closed subspace), then WF becomes G:

$$\boxed{\text{find } u_h \in V_h \ : \ a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h}$$

- *well-posedness* follows from LM
- *stability* is the continuous dependence from data in LM
- *consistency* $\equiv$ **Galerkin Orthogonality**:

$$a(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h$$

- if we assume **space saturation**

$$\inf_{v_h \in V_h} \|v - v_h\|_V = 0 \quad \forall v \in V$$

then *convergence* $\equiv$ **Céa Lemma**:

$$\|u - u_h\|_V \le \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V$$

(Céa + space saturation $\equiv$ convergence)

Last but not least: Problem G is equivalent to the following linear system of equations:

$$\boxed{\text{find } \mathbf{u} \in \mathbb{R}^{N_h} \ : \ A\mathbf{u} = \mathbf{F}}$$

where $A \in \mathbb{R}^{N_h \times N_h}$, $\mathbf{F} \in \mathbb{R}^{N_h}$.

Proof. $V_h = \text{span}\left\{\phi_1, \ldots, \phi_{N_h}\right\}$ so

$$u_h(\mathbf{x}) = \sum_{j=1}^{N_h} U_j \phi_j(\mathbf{x}), \quad U_j \in \mathbb{R}\ \forall j$$

thus G becomes: Find $\{U_j\}_{j=1}^{N_h}$ s.t.

$$a\left(\sum_j U_j \phi_j, \phi_i\right) = F(\phi_i) \quad \forall\, i = 1 : N_h$$

$$\sum_j U_j\, a(\phi_j, \phi_i) = F(\phi_i) \quad \forall\, i = 1 : N_h$$

$$\sum_j A_{ij}\, U_j = F_i \quad \forall\, i = 1 : N_h$$

$$A\mathbf{U} = \mathbf{F}$$

Moral of the story: V_h must be chosen to ensure the saturation assumption and the computation of the integrals $A_{ij} = a(\phi_j, \phi_i)$ and $F_i = F(\phi_i)$. a

■ **The Finite Element Method.**