Homework 3

Due 3/29 11:59pm The homework must be uploaded to [LMS (https://lms.dgist.ac.kr)](https://lms.dgist.ac.kr)

Problem 1

1-1.

Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

1-2.

Find the best straight-line fit (least square) to the measurements

$$(t, b) = (-2, 4), (-1, 3), (0, 1), (2, 0)$$

for $C + Dt = b$ by using the projection of $b = (4, 3, 1, 0)$ onto the column space of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

Problem 2

2-1.

Apply the Gram-Schmidt process to

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form $A = QR$ where Q is the matrix whose column vectors are orthonormal basis and R is a upper-triangular matrix.

2-2.

Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

and find the least-square solution of $Ax = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

Problem 3

The **rank** of a matrix A if the number of pivots in the reduced row echelon form of A. It is also the dimension of the column space $C(A)$, which is the dimension of the row space $R(A)$. The fact that the dimension of $C(A)$ and $R(A)$ are equal is very important, and it is not at all obvious. We will discuss more detail throughout this problem.

3-1.

Let A be a $n \times m$ matrix and R be the reduced row echelon form of A. Let $v_1, \cdots, v_m$ be the column vectors of A and $w_1, \cdots, w_m$ be the column vectors of R.

$$A = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}, \quad R = \begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}$$

Show that a set of vectors $v_{j_1}, \cdots, v_{j_s}$ is linearly independent (with respectively, dependent) if and only if the set of vectors $w_{j_1}, \cdots, w_{j_s}$ is linear independent (or dependent).

Caution: The set $v_{j_1}, \cdots, v_{j_s}$ is a subset of the set of column vectors $v_1, \cdots, v_m$ ($s \le m$). Also, the indices of $v_{j_1}, \cdots, v_{j_s}$ and $w_{j_1}, \cdots, w_{j_s}$ coincide, and this is the main focus of the problem.

3-2.

Show that the dimension of $R(A)$ is equal to the number of pivots in R.

Conclusion (read and think)

Let us show that $\dim C(A) = \dim R(A)$. Let R be the reduced row echelon form of A and r be the number of pivots in R.

Let us show that $r = \dim C(A)$. Let $j_1, \cdots, j_r$ be the columns of pivots. By problem 3-1, the set of column vectors $v_{j_1}, \cdots, v_{j_r}$ is linearly independent. Moreover, we show that any column vector v_k of A is written as linear combination of $v_{j_1}, \cdots, v_{j_r}$. If v_k is one of $v_{j_1}, \cdots, v_{j_r}$, it is a linear combination of itself. If not, the set $v_k, v_{j_1}, \cdots, v_{j_r}$ is linearly dependent because the corresponding set $w_k, w_{j_1}, \cdots, w_{j_r}$ (column vectors in R) is linearly dependent. (The set $w_{j_1}, \cdots, w_{j_r}$ forms a canonical basis of $C(W)$.) By problem 3-1, the set $v_k, v_{j_1}, \cdots, v_{j_r}$ is also linearly dependent. That is, there are coefficients $c_k, c_{j_1}, \cdots, c_{j_r}$, which are not all zero, satisfying

$$c_k v_k + c_{j_1} v_{j_1} + \cdots + c_{j_r} v_{j_r} = 0.$$

Here, $c_k \neq 0$ because $v_{j_1}, \cdots, v_{j_r}$ is linearly independent. Thus v_k can be written as linear combination of $v_{j_1}, \cdots, v_{j_r}$.

By problem 3-2, we know that $r = \dim R(A)$. Thus we can conclude that $\dim C(A) = \dim R(A)$.

Remark

Later, the **rank** plays crucial role in determine whether the matrix is nonsingular. If a $n \times n$ matrix A has rank less than n, then A is singular, that is A has no inverse. If the matrix A is of rank n, in other words *full rank*, then A has its inverse. Moreover, a full rank matrix has nonzero determinant, and vice versa.