Price Competition in Spectrum Markets: How Accurate is the Continuous Prices Approximation?

Aditya MVS, Abhishek Raghuvanshi and Gaurav S. Kasbekar

***Abstract*—Dynamic Spectrum Access technology enables two types of users to operate on a channel– primary users, which have prioritized access to the channel and secondary users, which can use the channel when it is not in use by the primaries. We consider a scenario in which multiple primaries own bandwidth in a large region (e.g., a state), which is divided into smaller locations (e.g., towns). A primary that has unused bandwidth in a time slot would like to lease it out to secondaries at a set of mutually non-interfering locations in return for a fee. This results in price competition among the primaries. In prior work, this price competition has only been studied under the approximation, made for analytical tractability, that the price of each primary takes values from a continuous set. However, in practice, the set of available prices is discrete. In this paper, we investigate the fundamental question of how the behaviour of the players involved in the price competition changes when this continuity assumption is removed. Our analysis reveals several important differences between the games with continuous and discrete price sets. For example, in the game at a single location, no pure strategy Nash equilibrium (NE) exists in the game with continuous price sets, whereas a pure strategy NE may exist in the game with discrete price sets. Also, multiple symmetric NE exist in the game with discrete price sets in contrast to the game with continuous price sets, where a unique NE exists. However, we show that as the number of available prices becomes large in the discrete prices game, the strategies of the primaries under every symmetric NE converge to the unique NE strategy of the game with continuous price sets.**

I. Introduction

The last decade has seen a tremendous growth in the use of wireless devices, thus increasing the demand for spectrum. Traditionally, a static spectrum allocation policy has been used, where network operators have *exclusive* spectrum rights. This has created an artificial scarcity of spectrum wherein most of the usable radio spectrum is allocated, but under-utilised [1]. *Dynamic spectrum access* (DSA) technology [2] has been proposed as a solution for the more efficient use of spectrum. This technology enables two types of users to operate on a channel– *primary users*, which have prioritised access to the channel and *secondary users*, which can use the channel when it is not in use by the primaries [2].

We consider a scenario in which multiple primaries own bandwidth in a large region (e.g., a state), which is divided into smaller locations (e.g., towns). Time is divided into slots of equal duration. During each slot, a primary can lease his unused (free) channels to secondaries for the duration of that slot. At each location, secondaries lease bandwidth from the primaries who set the lowest prices. This results in a *price competition* among the primaries to lease their free channels. This is similar to the classic *Bertrand price competition* [3], wherein a few firms compete among themselves to sell their goods to customers. However, there are several important differences between Bertrand price competition and that in a DSA market. Specifically, a primary may or may not have a free channel in a given slot and hence a primary that *has* a free channel does not know how many other primaries are selling their channels in the slot. He will unnecessarily get a low revenue if he sells his bandwidth at a low price when only a few primaries have free channels since his channel would have been bought even if he set a higher price. Conversely, his free channel may remain unsold if he chooses a high price when a large number of primaries have free channels. The other important difference between Bertrand price competition and that in a DSA market is that radio spectrum allows *spatial reuse*: the same band can be simultaneously used at multiple locations provided these locations are far apart; however, transmissions at neighboring locations interfere with each. Thus, each primary must jointly select a set of mutually non-interfering locations at which to offer bandwidth as well as the price at each location in the set. We formulate the above price competition between the primaries as a game and seek *Nash equilibria* (NE) [3] in it.

Spectrum pricing games have been studied in [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Price competition between secondaries in which a single operator senses and leases the free channels is studied as a Stackelberg game in [10]. Price competition between multiple primaries with uncertain bandwidth availability is studied in [9], [10], [12], [14]. In [14], the quality of bands is taken into account in addition to uncertainty in spectrum availability. NE in the spectrum pricing game are studied in [9], [12], [14]. However, in all the above papers, it is assumed that each player chooses a price from a *continuous set*, e.g., an interval $[a, b]$, where a and b are real numbers. The continuity of the price set is assumed as an approximation to simplify the mathematical analysis. However, note that in practice, a player can only choose a price from a *discrete set* (e.g., multiples of one cent). This restricts the number of prices a player can choose to a finite number. In this paper we investigate the fundamental question of how the behaviour of the players involved in the price competition changes when this continuity assumption is removed. To the best of our knowledge *this work is the first to investigate the effects of the continuity assumption on the NE of a spectrum pricing game*.

In this paper, we study the NE in a scenario where multiple primaries, each of which may or may not have free bandwidth in a given slot in a region, which is divided into multiple locations, sell their free bandwidth to secondaries at individual locations, when the number of prices the players can choose is finite (as in practice). The same model was studied in [9], but under the approximation that players choose prices from a continuous set. In Section II, we formulate the price competition between the primaries as a game for the case when there is *a single location*. In Section III, the main results obtained for the single location game in [9] are summarized. In Section IV, we analyse the NE of the single location game (with a finite number of prices) for the case when there are two primaries and one secondary. This game, though simple, reveals several important differences between the games with continuous and discrete price sets. For example, no pure strategy NE exists in the game with continuous price sets, whereas a *pure strategy NE may exist* in the game with discrete price sets. Also, in the game with discrete price sets, there exist *multiple symmetric*

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NE. Moreover, the *expected payoff* that each primary gets under the symmetric NE may be *different under different NE*. This is in sharp contrast to the game with continuous price sets in [9], where there is a unique symmetric NE. However, in Section V, we analyse the price competition between the primaries at a single location when there are an arbitrary number of primaries and secondaries and show that, as the number of available prices becomes large, the *strategies of the primaries under every symmetric NE converge pointwise to the unique symmetric NE strategy of the game with continuous price sets*. In Section VI, we extend the analysis in Section V to the multiple locations case. The results in Sections IV, V, and VI show that although, as is consistent with intuition, the equilibrium behaviour of the players in the game with discrete prices is similar to that in the game with continuous prices when the number of prices is large, it is significantly different when the number of prices is small; thus caution must be exercised while using the continuous prices approximation in the context of price competition in spectrum markets. Also, the results in Sections V and VI provide a *formal justification for the continuous prices approximation*; to the best of our knowledge, our work is the first to provide such a justification for *any* spectrum pricing game. In Section VII, we study the infinitely repeated version of the single location game described in Section II via simulations. We find that when each primary independently uses the well-known *Softmax learning algorithm* [20] to adapt his strategy based on the payoffs it got in the past slots, the long run strategies of the players converge to a NE of the corresponding one-shot game only when there exists a pure strategy NE. However, when only mixed strategy NE exist in the one-shot game, the long run strategies do not converge to a NE. Finally we conclude the paper in Section VIII.

II. Network Model

Suppose there are $n \geq 2$ primaries and $k \geq 1$ secondaries in a location. Each primary owns one channel (one unit of bandwidth) and each secondary has a demand of one channel. Time is divided into slots and trade takes place at the beginning of each slot. In every slot, each primary has a free channel with probability (w.p.) $q \in (0,1)$. Each primary with a free channel selects a price at which to offer it to secondaries. Now, a primary that leases a channel to a secondary may incur some cost, *e.g.*, if the secondary uses some of the former's infrastructure. Let this cost be $c \geq 0$ for each primary. A primary does not sell his bandwidth below this price as he would incur a loss. We will also assume that there is a limit, say v, to the maximum price a primary can quote. This limit may be due to the following reasons [9]: (i) The spectrum regulator may impose this limit to prevent the primaries from charging excessively when they collude or when the number of primaries with free bandwidth is less than the number of secondaries. (ii) Each secondary may have a valuation of v for a channel and would not buy a channel for a price greater than v. Thus, if we denote primary i's price by p_i, then $c < p_i \leq v$.

Recall that in practice, there is only a finite set of prices a primary can choose from. Let this set be $\{a_1, a_2, \ldots, a_M\}$, where $a_i = c + \left(\frac{v-c}{M}\right) i$. Secondaries buy bandwidth from the primaries that set the lowest prices. Specifically, if Z primaries have free bandwidth, then the bandwidth of $min(Z,k)$[1] primaries gets sold. We model the above price competition between the primaries as a *game* [3] in which the actions of the primaries (players) are the prices they choose. Note that when primary i has free bandwidth, $p_i \in \{a_1, \ldots, a_M\}$; with a slight abuse of notation, we assume that $p_i = v+1$ when he does not have free bandwidth [2]. Also, let $a_{M+1} = v+1$. Next, recall that the *utility* or *payoff* represents the level of satisfaction of a player [16]. If a primary does not sell his bandwidth, his utility is defined to be 0. Let $u_i(p_1, \ldots, p_n)$ denote the utility of primary i when primary j selects price p_j, $j = 1, \ldots, n$. Consider primary 1 and let X_k denote the k'th lowest price among p_j, $j \in \{2, \ldots, n\}$. Since there are k secondaries, primary 1 sells his bandwidth w.p. 1 if $p_1 < X_k$ and does not sell his bandwidth if $p_1 > X_k$. If $p_1 = X_k$, then note that more than one primary chooses the price X_k. In this case, the tie is broken randomly [3]. Thus:

$$u_1(p_1, p_2, \ldots, p_n) = \begin{cases} p_1 - c, & \text{if} \quad X_k > p_1, \\ \frac{(k-l)(p_1-c)}{m}, & \text{if} \quad X_k = p_1, \\ 0, & \text{if} \quad X_k < p_1, \end{cases}$$

where in the second case, $l < k$ primaries choose a price less than X_k and m primaries (including primary 1) choose X_k. (Note that when primary 1 does not have free bandwidth, his utility is 0 even if $p_1 = X_k$.) The utilities of primaries $j = 2, \ldots, n$ are defined similarly.

Each primary i is allowed to randomly choose his price p_i using an arbitrary distribution function (d.f.) $\phi_i(\cdot)$. This d.f. is called the *strategy* [3] of primary i. The vector of strategies of all the players $(\phi_1(\cdot), \ldots, \phi_n(\cdot))$ is called the *strategy profile* [3]. Let $\phi_{-i}(\cdot) = (\phi_1(\cdot), \ldots, \phi_{i-1}(\cdot), \phi_{i+1}(\cdot), \ldots, \phi_n(\cdot))$ denote the strategy profile of all the players except player i. Also, let $E(u_i(\phi_1(\cdot), \ldots, \phi_n(\cdot)))$ denote the expected utility of primary i when the strategy profile used is $(\phi_1(\cdot), \ldots, \phi_n(\cdot))$. We will use the concept of NE, which is widely used as a solution concept in game theory [3]. A strategy profile $(\phi_1^*(\cdot), \ldots, \phi_n^*(\cdot))$ constitutes a NE [3] if $\forall i \in \{1, 2, \ldots, n\}$:

$$E(u_i(\phi_i^*(\cdot), \phi_{-i}^*(\cdot))) \geq E(u_i(\phi_i(\cdot), \phi_{-i}^*(\cdot))), \quad \forall \phi_i(\cdot).$$

When $n \leq k$, then clearly $p_i = v$ $\forall i = 1, \ldots, n$ is the unique NE of the game since by setting $p_i = v$, primary i can sell his bandwidth regardless of the choices made by the other primaries and also he gets the maximum possible payoff. So henceforth we assume that $n > k$. Note that the game described above is a finite *symmetric game* [3] since all the primaries have the same action sets (available prices), same utility functions and have free bandwidth with equal probability. We will seek *symmetric NE*, which is one in which $\phi_1(\cdot) = \phi_2(\cdot) = \ldots = \phi_n(\cdot) = \phi(\cdot)$ (say) [3]. Symmetric NE have been advocated as a solution concept for symmetric games by several game theorists [15], since in practice, it is challenging to implement a NE that is not symmetric. Also, it is shown in [15] that every finite symmetric game has atleast one symmetric mixed strategy NE.

III. Background

We now briefly summarise the results obtained in [9], which are for the model described in Section II with the difference that the price of a primary is allowed to be any *real number*

[1] $min(a,b)$, where $a, b \in \mathbb{R}$, denotes the minimum of a and b.

[2] As explained later, the expected payoff of a primary is a function of whether or not each of the other primaries has free bandwidth. This notation simplifies the exposition by eliminating the need to condition on whether they have free bandwidth. Also, the choice $p_i = v+1$ is arbitrary; any price above v can be chosen.

[3] For example, suppose $k-1$ primaries choose a price less than X_k, 2 primaries (including primary 1) choose the price X_k and the rest choose a price more than X_k. Then, primary 1 sells his bandwidth w.p. $\frac{1}{2}$.

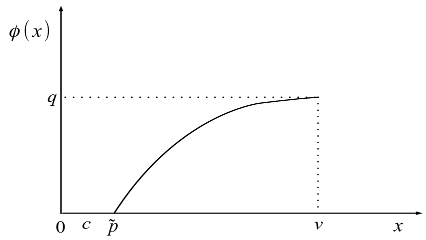

Fig. 1. $\phi(x)$ is continuous and strictly increasing in the interval $[\tilde{p}, v]$.

in the interval $(c, v]$. It was proved in [9] that this game has no pure strategy NE [4]. Let:

$$w(q, n) = \sum_{i=k}^{n-1} \binom{n-1}{i} q^i (1-q)^{n-1-i}. \tag{1}$$

Since each primary independently has unused bandwidth w.p. q, $w(q, n)$ is the probability that k or more out of $n-1$ primaries have unused bandwidth. Let:

$$\tilde{p} = v - w(q, n)(v - c) \tag{2}$$

and note that $c < \tilde{p} < v$. It was shown in [9] that in the above price competition game, there is a unique symmetric NE. Also, in this NE, each primary selects his price randomly using a d.f. $\phi(\cdot)$, whose *support set* [5] is the interval $[\tilde{p}, v]$. Moreover, $\phi(\cdot)$ is continuous and strictly increasing on $[\tilde{p}, v]$ as illustrated in Fig. 1. The function $\phi(\cdot)$ can be computed as follows [9]. Let X_k be as in Section II and $F(\cdot)$ denote the d.f. of X_k. It was shown in [9] that under the unique symmetric NE:

$$F(x) = \begin{cases} 0, & x \leq \tilde{p}, \\ \frac{x-\tilde{p}}{x-c}, & \tilde{p} < x \leq v. \end{cases} \tag{3}$$

Also, the symmetric NE price selection d.f. $\phi(\cdot)$ is the unique solution of the following equation [9]:

$$F(x) = \sum_{i=k}^{n-1} \binom{n-1}{i} (\phi(x))^i (1-\phi(x))^{n-1-i}. \tag{4}$$

Note that since $\phi(\cdot)$ is strictly increasing on $[\tilde{p}, v]$, prices in every sub-interval of the interval $[\tilde{p}, v]$ are played with positive probability by each primary in this NE; also, prices in $(c, \tilde{p})$ are *not* played. The utility of each primary under the above symmetric NE was shown to be [9]:

$$E(u_1(\phi(\cdot), \phi_{-1}(\cdot))) = \tilde{p} - c = u_{max} \quad \text{(say)}. \tag{5}$$

Next, consider the model with discrete price sets described in Section II and let $\phi(\cdot)$ be as defined in that section. We state a lemma [16], which provides necessary and sufficient conditions for $\phi(\cdot)$ to constitute the price selection strategy of each primary under a symmetric NE, and which we will extensively use in the following sections. Let S be the support set of the d.f. $\phi(\cdot)$, *i.e.*, the subset of prices from $\{a_1, \ldots, a_M\}$ that are selected with positive probabilities under $\phi(\cdot)$, and let $S^c = \{a_1, \ldots, a_M\} \setminus S$.

Lemma 1: The d.f. $\phi(\cdot)$ constitutes the price selection strategy of each primary under a symmetric NE iff:

$$E(u_1(a_l, p_2, \ldots, p_n)) = E(u_1(a_m, p_2, \ldots, p_n)), \; \forall a_l, a_m \in S,$$
$$E(u_1(a_l, p_2, \ldots, p_n)) \geq E(u_1(a'_l, p_2, \ldots, p_n)), \; \forall a_l \in S, a'_l \in S^c.$$

Lemma 1 states that the expected payoffs that primary 1 gets at all prices that it plays with positive probability under the symmetric NE are equal, and are $\geq$ the payoff at each price that it does not play under the NE.

IV. Price Competition in the Special Case $n = 2$, $k = 1$

We now analyse the special case $n = 2$ and $k = 1$ of the game described in Section II. Our analysis reveals the following important differences between the NE in the games with discrete price sets and continuous price sets:
1) No pure strategy NE exists in the game with continuous price sets, whereas a *pure strategy NE may exist* in the game with discrete price sets.
2) There exist *multiple symmetric NE* in the game with discrete price sets unlike in the continuous price set case where there exists a unique symmetric NE. Moreover, the *expected payoff that each primary gets under the symmetric NE may be different under different NE.*
3) Recall from Section III that in the continuous price set case, for each value of $q \in (0, 1)$, under the unique symmetric NE, the support set of the price selection strategy, $\phi(\cdot)$, of each primary is the set $[\tilde{p}, v]$, i.e., *all the available prices above a threshold* ($\tilde{p}$). However, in the discrete prices case, this type of symmetric NE, i.e., one in which the support set of each primary is the set, $\{a_{M-P}, ..., a_{M-1}, a_M\}$, of all the available prices above a threshold, *exists only for some values of* $q \in (0, 1)$ *and not for others and no matter how large* M *is, there are certain values of* q *where this type of NE does not exist*. Fig. 2 shows, for an example, the set of values of q where this type of NE exists when $n = 2$ and $k = 1$.

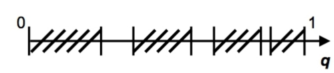

Fig. 2. The shaded area represents the set of values of q where a symmetric NE in which the support set is the set of all available prices above a threshold exists.

Note that the game with $n = 2$, $k = 1$, and a finite number of prices for each player to choose from, is a finite symmetric game and thus has atleast one symmetric NE [15]. Suppose in this NE each primary selects a price from $\{a_1, a_2, .., a_M, a_{M+1}\}$ using the probability mass function (PMF) $R(\cdot)$. Then, $R(\cdot)$ satisfies the following equations:

$$\sum_{i=1}^{M} R(a_i) = q, \tag{6a}$$

$$R(a_{M+1}) = 1 - q. \tag{6b}$$

Now, primary 1 sells his free bandwidth w.p. 1 when $p_1 < p_2$ and w.p. $\frac{1}{2}$ when $p_1 = p_2$. So his expected utility is:

$$E(u_1(p_1, p_2)) = (p_1 - c)\left[P(p_2 > p_1) + \frac{P(p_1 = p_2)}{2}\right], \tag{7}$$

where $P(A)$ denotes the probability of event A. Fix q and suppose the PMF $R(\cdot)$ has support set $\{a_{i_1}, a_{i_2}, ..., a_{i_m}\}$,

[4] A pure strategy NE is one in which each primary plays a single price w.p. 1 [3].

[5] Recall that the support set of a d.f. is the smallest closed set whose complement has probability 0 under the d.f. [18]. Also, since we defined $p_i = v+1$ if primary i does not have free bandwidth, the price $v+1$ is always in the support set of the price selection strategy of a primary. However, we ignore it throughout the paper since we are concerned with the price selection strategies of primaries that have free bandwidth.

TABLE I
SUPPORT SETS OF SYMMETRIC NE AT DIFFERENT VALUES OF q FOR $M = 4$

Support Set	Valid q
$\{a_4\}$	$(0, 0.5]$
$\{a_3\}$	$[0.4, 0.67]$
$\{a_2\}$	$[0.67, 1]$
$\{a_1\}$	$[0.86, 1]$
$\{a_3, a_4\}$	$[0.4, 0.5]$
$\{a_2, a_4\}$	$[0.67, 0.75]$
$\{a_1, a_4\}$	$[0.86, 0.9]$
$\{a_2, a_3\}$	$[0.57, 0.67]$
$\{a_1, a_3\}$	$[0.84, 0.89]$
$\{a_1, a_2\}$	$[0.8, 1]$
$\{a_2, a_3, a_4\}$	$[0.57, 0.75]$
$\{a_1, a_3, a_4\}$	$[0.84, 0.9]$
$\{a_1, a_2, a_4\}$	$[0.8, 0.875]$
$\{a_1, a_2, a_3\}$	$[0.82, 0.89]$
$\{a_1, a_2, a_3, a_4\}$	$[0.82, 0.875]$

where $i_j \in \{1, 2, ..., M\}$ and $i_1 < i_2 < ... < i_m$. Primaries select $a_{i_{m+1}} = v + 1$ if they do not have a free channel. By Lemma 1, we get $\forall\ i_j, i_l \in \{i_1, i_2, ..., i_m\}$:

$$E(u_1(a_{i_j}, p_2)) = E(u_1(a_{i_l}, p_2)), \tag{8a}$$

$$E(u_1(a_{i_j}, p_2)) \geq E(u_1(a_i, p_2)) \quad \forall i \in \{1, 2, ..., M\}. \tag{8b}$$

By (7), the utility of primary 1 when he chooses price a_{i_m} is:

$$\begin{aligned} E(u_1(a_{i_m}, p_2)) &= (a_{i_m} - c)[P(p_2 > a_{i_m}) + \frac{P(p_2 = a_{i_m})}{2}] \\ &= \frac{v - c}{M} i_m [R(a_{i_{m+1}}) + \frac{R(a_{i_m})}{2}] \\ &= \frac{v - c}{M} i_m [1 - q + \frac{R(a_{i_m})}{2}] \text{ (by (6b))} \end{aligned} \tag{9}$$

Similarly we can write for $j = 1, .., m - 1$:

$$E(u_1(a_{i_j}, p_2)) = \frac{v - c}{M} i_j [\sum_{l=j+1}^{m+1} R(a_{i_l}) + \frac{R(a_{i_j})}{2}]. \tag{10}$$

By (8a), we get $E(u_1(a_{i_j}, p_2)) = E(u_1(a_{i_m}, p_2))\ \forall\ j = 1, .., m - 1$. This gives us a set of $m - 1$ linear equations (one for each $j = 1, .., m - 1$) with m unknowns $(R(a_{i_1}), ..., R(a_{i_m}))$. These $m-1$ linear equations along with (6a) result in m linear equations with m unknowns. By solving these linear equations, we get the following expressions for $R(a_{i_j})$, $j = 1, 2, .., m$.

Case (i): When m is even:

$$\begin{aligned} R(a_{i_m}) &= -2 + \frac{(2Q - 1)}{Q - 1} q, \\ R(a_{i_{2l+1}}) &= 2 - \frac{2Q_{i_{2l+1}}}{Q - 1} q, \\ R(a_{i_{2l}}) &= -2 + \frac{2Q_{i_{2l}}}{Q - 1} q, \end{aligned} \tag{11}$$

where $l = 0, 1, .., \frac{m}{2} - 1$, $Q = \frac{i_m}{i_1} - \frac{i_m}{i_2} + ... + \frac{i_m}{i_{m-1}}$, $Q_{i_{2l}} = \frac{i_m}{i_1} - \frac{i_m}{i_2} + ... + \frac{i_m}{i_{2l-1}} - \frac{i_m}{2 i_{2l}}$ and $Q_{i_{2l+1}} = \frac{i_m}{i_1} - \frac{i_m}{i_2} + ... - \frac{i_m}{i_{2l}} + \frac{i_m}{2 i_{2l+1}}$. The expressions for the case when m is odd are relegated to our technical report [17] due to space constraints.

For the special case $n = 2$, $k = 1$ and $M = 4$, Table I provides an *exhaustive* list of all possible support sets, $\{a_{i_1}, a_{i_2}, ..., a_{i_m}\}$, of a symmetric NE price selection PMF $R(\cdot)$ in the first column, and the set of all values of q for which symmetric NE with these support sets exist in the second column. For example, consider the fifth entry: $\{a_3, a_4\}$ constitutes the support set of a symmetric NE price selection PMF $R(\cdot)$ for $q \in [0.4, 0.5]$. The table is obtained by calculating $R(\cdot)$ for every possible combination of prices, $\{a_{i_1}, a_{i_2}, ..., a_{i_m}\}$, as support set and noting that only the combinations which satisfy (8b) constitute a valid support set. Table I provides the following insights. First, recall from Section III that in case of continuous price sets, no pure strategy NE exists. However, the first four entries in Table I show that a *pure strategy NE may exist* with discrete prices. Second, recall from Section III that in case of continuous price sets, there is a unique symmetric NE for a given value of q. In contrast, there are *multiple symmetric NE* in case of discrete prices. For example at $q = 0.5$, $\{a_4\}$, $\{a_3\}$ and $\{a_3, a_4\}$ all constitute support sets of symmetric NE price selection PMFs $R(\cdot)$. Third, it is easy to check that the expected payoff that each primary gets may also be different under the different symmetric NE for a given value of q. In the above example, the expected payoffs under the three symmetric NE at $q = 0.5$ are $\frac{3}{4}(v - c)$, $\frac{9}{16}(v - c)$ and $\frac{3}{4}(v - c)$ respectively.

More importantly, the differences between the NE in the games with discrete and continuous price sets observed for $M = 4$ in the previous paragraph in fact hold *for every value of M, no matter how large.* Specifically, it is easy to verify using the above analysis that selection of the price a_M w.p. 1 by each primary that has a free channel constitutes a symmetric NE when $q \in (0, \frac{2}{M}]$. Hence, no matter how large the value of M is, there exists a pure strategy NE in the game with discrete price sets for certain values of q, in contrast to the game with continuous price sets, in which no pure strategy NE exists for any value of q. It can also be checked using the above analysis that $\{a_{M-1}, a_M\}$ constitutes the support set of a symmetric NE price selection PMF $R(\cdot)$ when $q \in \left[\frac{2}{M+1}, \frac{2}{M}\right]$. Hence, for $q \in \left[\frac{2}{M+1}, \frac{2}{M}\right]$, $\{a_M\}$ as well as $\{a_{M-1}, a_M\}$ constitute support sets of symmetric NE price selection PMFs. Thus, no matter how large the value of M is, there exist multiple symmetric NE in the game with discrete price sets for certain values of q, in contrast to the game with continuous price sets, in which there is a unique symmetric NE for every value of q.

The above observations show that *the actions (price selection strategies) taken by players as well as the rewards (expected payoffs) they get at equilibrium may differ substantially in the games with continuous and discrete price sets.*

Next, recall from Section III that for every value of $q \in (0, 1)$, the support set of the unique symmetric NE price selection strategy in the game with continuous price sets is of the form $[\tilde{p}, v]$. Hence, the support set is the set of all the available prices above a threshold ($\tilde{p}$). We are interested in symmetric NE with a similar support set in the game with discrete prices, i.e., a support set containing all the available prices above a threshold. Suppose the support set is $\{a_{M-P}, a_{M-P+1}, ..., a_{M-1}, a_M\}$; note that there are $P + 1$ consecutive prices in the support set. The expressions for the symmetric NE price selection PMF $R(\cdot)$ are obtained by substituting $i_1 = M - P, i_2 = M - P + 1, ..., i_P = M - 1, i_{P+1} = M$ in (11) when P is odd and in a similar set of equations provided in our technical report [17] when P is even. Now, since the prices $a_i, i = M - P, M - P + 1, ..., M$ are in the support set:

$$R(a_i) > 0,\ i = M - P, M - P + 1, ..., M. \tag{12}$$

Also, for $R(\cdot)$ to constitute a symmetric NE price selection strategy, (8b) must be satisfied. Let V^P denote the set of values of q for which inequalities (12) and (8b) are satisfied. It is proved in our technical report [17] that V^P is an open interval. Let L^P and U^P denote the lower and upper endpoints of the interval V^P respectively. It is proved in our technical report [17] that for $P \in \{0, 1, 2, ..\}$, $U^P < L^{P+1}$.

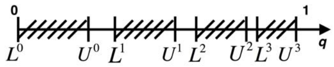

Fig. 3. The figure illustrates L^P and U^P for $P = 0, 1, 2, 3$ on the q line.

Fig. 3 shows an example of the endpoints L^P and U^P. It follows that for certain values of q (e.g., for the values in $[U^0, L^1], [U^1, L^2]$ and $[U^2, L^3]$ in Fig. 3), there *does not exist a symmetric NE whose price selection strategy support set is the set of all the available prices above a certain threshold* (a_{M-P}). Also, surprisingly, this is *true no matter how large* M *is*. This is in sharp contrast to the continuous price sets case where for every $q \in (0, 1)$, the support set of the unique symmetric NE price selection strategy is the set $[\tilde{p}, v]$, which is the set of all the available prices above $\tilde{p}$ (see Section III). Also, it is observed that as q increases from a value in (L^P, U^P) to a value in (L^{P+1}, U^{P+1}), a price (a_{M-P-1}) gets added to the lower end of the support set. This is consistent with the intuition that as q, the probability that a primary has free bandwidth, increases, the price competition becomes more intense, and hence each primary chooses lower prices to get his bandwidth sold.

V. Price Competition for arbitrary n, k and large M

In this section, for the model described in Section II, we show that as $M \to \infty$, the price selection strategies of the primaries under every symmetric NE converge to the unique symmetric NE strategy of the game with continuous price sets.

As in Section IV, let $R(\cdot)$ denote the PMF that each primary adopts over the price set $\{a_1, a_2, ..., a_M, a_{M+1}\}$ in a symmetric NE. Then $R(\cdot)$ satisfies (6a) and (6b). Let $\phi_M(.)$ denote the d.f. corresponding to $R(\cdot)$. Then $\phi_M(a_i) = \sum_{l=1}^{i} R(a_l)\ \forall i \in \{1, ..., M, M+1\}$. Recall from Section III that in the game with continuous price sets, the symmetric NE strategy $\phi(.)$ of a primary is continuous on support set $(\tilde{p}, v]$. However, $\phi_M(\cdot)$ is a discontinuous function with jumps [6] at the prices in its support set (the prices a_i such that $R(a_i) > 0$). Also, $\phi(\cdot)$ is unique. In contrast, as we have seen in Section IV, there may exist multiple symmetric NE in the game with discrete price sets. So $\phi_M(\cdot)$ may not be unique. However, we prove that as $M \to \infty$, *all the possible functions* $\phi_M(\cdot)$ *that constitute a symmetric NE price selection strategy converge to* $\phi(\cdot)$. Since $\phi(\cdot)$ is a continuous d.f., to prove that a discrete d.f. $\phi_M(\cdot)$ converges pointwise to the former, we first show in Lemma 2 that the sizes of the jumps in $\phi_M(\cdot)$ decrease to 0 as $M \to \infty$.

Lemma 2: For every $\epsilon > 0$, $\exists\ M_\epsilon$ such that if $M \geq M_\epsilon$, then in every symmetric NE strategy $\phi_M(\cdot)$, each price $x \in [c+\epsilon, v]$ is played with probability $\leq \epsilon$.

The proof of Lemma 2 is relegated to our technical report [17]. Note that Lemma 2 does not contradict the result stated in Section IV that selection of the price a_M w.p. 1 by each primary that has a free channel constitutes a symmetric NE when $q \in (0, \frac{2}{M}]$. This is because, since each primary has a free channel w.p. q, under this symmetric NE, the effective probability with which a primary selects price a_M is q, which decreases to 0 as $M \to \infty$.

[6] Recall that the function $\phi_M(\cdot)$ has a jump at x, if $\phi_M(x) - \phi_M(x-) > 0$ [18].

Next, we state the main result of this section, which shows that as $M \to \infty$, the price selection d.f. $\phi_M(.)$ under *every* symmetric NE approaches the price selection d.f., $\phi(\cdot)$, in the continuous prices case.

Theorem 1: As $M \to \infty$, the sequence of functions $\phi_M(x)$ converges pointwise to $\phi(x)\ \forall\ x > c + \epsilon'$ when $\epsilon' > \epsilon$, where ϵ is as in Lemma 2.

The proof of Theorem 1 is relegated to our technical report [17] due to space constraints. Theorem 1 shows that as $M \to \infty$, the price selection d.f.s of the primaries under every symmetric NE converge pointwise to the price selection d.f. under the unique symmetric NE of the game with continuous price sets. This is a surprising result since as shown in Section IV, important differences exist between the NE in the games with discrete and continuous price sets for every value of M, no matter how large. Also, Theorem 1 provides a *formal justification for the continuous prices approximation*, which has not been provided in prior work for *any* spectrum pricing game to the best of our knowledge.

VI. Spatial Reuse

In this section, we study a generalization of the model in Section II, in which primaries sell their unused bandwidth at multiple locations. There are n primaries and each primary owns one channel over a large region (e.g., a state) which is divided into several small locations (e.g., towns). There are k secondaries at each location, where $k \in \{1, ..., n-1\}$. In every slot, a primary either uses his channel over the entire region or does not use it anywhere in the region. A typical example of this scenario is when a primary uses its channel to broadcast the signal throughout the region (e.g., TV broadcasting). Like in the single location case, each primary has a free channel w.p. q. A primary that has a free channel in a time slot can lease it out to secondaries at multiple locations. However, simultaneous transmissions on the same channel at two neighbouring locations interfere with each other. So a primary cannot sell his free bandwidth at two locations which are neighbours of each other. The overall region is represented by an undirected graph [19] $G = (V, E)$ where V is the set of nodes (representing locations) and E is the set of edges between the nodes. Two nodes are connected by an edge if transmissions at the corresponding locations interfere with each other. Recall that an *independent set* [19] (I.S.) in a graph is a set of nodes such that there is no edge between any pair of nodes in the set. So a primary can only sell his unused bandwidth at multiple nodes provided they constitute an I.S. Let $\mathscr{I}$ denote the set of all I.S. in G. A primary has to *jointly* select (i) an I.S. from $\mathscr{I}$ at which to offer bandwidth, and (ii) the price at each node in the selected I.S. As in Section II, each price must be from the set $\{a_1, \ldots, a_M\}$. A primary incurs an operational cost c at each node at which he sells bandwidth. So if primary i offers bandwidth at the nodes in I.S. I and selects price $p_{i,z}$ at node $z \in I$, his utility is $\sum_{z \in I}(p_{i,z} - c)$. A primary faces the following tradeoff: if he offers bandwidth at nodes of a large I.S., the number of nodes at which he potentially gets revenue is large; however, he is likely to face intense competition from other primaries at the nodes of the large I.S. We study symmetric NE in the above game.

Consider a symmetric NE in which each primary selects I.S. $I \in \mathscr{I}$ w.p. $\beta(I)$, where $\sum_{I \in \mathscr{I}} \beta(I) = 1$. The probability, say α_z, with which a primary offers bandwidth at a node

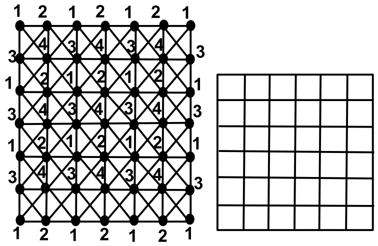

Fig. 4. Part (a) shows a grid graph $\mathcal{H}_{m,m}$ with $m = 7$. It is mean valid with $d = 4$ and the disjoint maximal I.S. $I_1, \ldots, I_4$ (in the notation of the definition of a mean valid graph in Section VI), where the nodes labelled j, $j \in \{1, 2, 3, 4\}$, constitute I.S. I_j. Part (b) shows a tiling of a plane with squares, *e.g.* cells in a cellular network. Transmissions at neighboring cells interfere with each other. The corresponding conflict graph is $\mathcal{H}_{6,6}$.

$z \in V$ equals the sum of the probabilities associated with all the I.S. that contain the node, *i.e.*, $\alpha_z = \sum_{I \in \mathscr{I}: z \in I} \beta(I)$. A set of probabilities, $\{\alpha_z,\ z \in V\}$, is said to be a *valid distribution* [12] if there exists a PMF $\{\beta(I),\ I \in \mathscr{I}\}$ such that $\alpha_z = \sum_{I \in \mathscr{I}: z \in I} \beta(I)\ \forall z \in V$. We analyse the above price competition for a special class of graphs called *mean valid graphs*, which were introduced in [12] and which model the conflict graphs of several topologies that commonly arise in practice, including line graphs, two and three dimensional grid graphs, the conflict graph of a cellular network with hexagonal cells and a clique of size $e \geq 1$. A graph G is mean valid if it satisfies the following two conditions [12]: (i) Its nodes can be divided into d disjoint maximal [7] I.S. $I_1, ..., I_d$. Let $|I_j| = M_j$ and $a_{j,l},\ l = 1, .., M_j$ be the nodes in I.S. I_j. Assume that $M_1 \geq M_2 \geq .... \geq M_d$. (ii) For every valid distribution in which a primary offers bandwidth at a node $a_{j,l}$ w.p. $\alpha_{j,l}$, $j = 1, .., d$, $l = 1, ..., M_j$:

$$\sum_{j=1}^{d} \overline{\alpha}_j \leq 1, \quad \text{where} \quad \overline{\alpha}_j = \frac{\sum_{l=1}^{M_j} \alpha_{j,l}}{M_j}, j \in \{1, ..., d\}.$$

An example of a mean valid graph is the $m \times m$ grid graph, which we denote as $\mathcal{H}_{m,m}$, in part (a) of Fig. 4. In this graph, m^2 nodes (locations) are arranged in a square grid. For example, $\mathcal{H}_{m,m}$ may represent a shopping complex, with the nodes corresponding to the locations of shops with Wi-Fi Access Points (AP) for Internet access. $\mathcal{H}_{m,m}$ is also the conflict graph of a cellular network with square cells as shown in part (b) of Fig. 4.

We now state a separation lemma due to which once the PMF, $\{\beta(I),\ I \in \mathscr{I}\}$, that each primary uses in selecting I.S. under a symmetric NE is known, his price selection strategy at each node follows.

Lemma 3: Suppose a primary selects each node $z \in V$ w.p. α_z under a symmetric NE. Then the price selection d.f. that each primary uses at node z in the symmetric NE will be $\phi_M(\cdot)$, which is as in Section V but with q replaced with $q\alpha_z$ throughout.

The proof of Lemma 3 is similar to that of Lemma 2 in [12] and is omitted. Next, let $W(\alpha) = (v-c)(1-w(q\alpha, n))$, where $w(q, n)$ is as in (1). For the game with continuous price sets, *i.e.*, the above game with the change that the price of a primary at a node can be any *real number* from the interval $(c, v]$, it was shown in [12] that there exists a unique symmetric NE. The following result from [12] characterizes that NE.

Theorem 2: In a mean valid graph, for every $q \in (0, 1)$, there is a unique symmetric NE in the game with continuous price sets. In this NE, each primary offers bandwidth at every node in I_j, $j \in \{1, ..., d\}$ w.p. t_j, i.e., $\alpha_{j,l} = t_j$, $l = 1, ..., M_j$, where $(t_1, ..., t_d)$ is the unique distribution satisfying the following conditions:

1. There exists $d' \in \{1, ..., d\}$ such that $t_j = 0$ if $j > d'$.
2. $M_1 W(t_1) = ... = M_{d'} W(t_{d'}) > M_{d'+1}(v - c)$.

It was shown in [12] that when primaries $2, \ldots, n$ play their symmetric NE strategies, primary 1 gets an expected payoff of $M_j W(t_j)$ if it selects I_j. Thus, condition 2 in Theorem 2 states that a primary gets equal expected payoffs by choosing I.S. in $I_1, I_2, \ldots, I_{d'}$ and this payoff exceeds the maximum payoff it could have got by selecting an I.S. in $I_{d'+1}, \ldots I_d$; hence, it never opts for the latter choice, *i.e.*, condition 1 holds.

Next, we state a theorem that characterizes the symmetric NE in the game in which there are M available prices at each location, for the case where M is large.

Theorem 3: Let $\{\alpha_z^M : z \in V\}$ be node selection probabilities that constitute a symmetric NE in the game with M available prices at each location. Let $\{\alpha_z : z \in V\}$ be the node selection probabilities that constitute the unique symmetric NE in the game with continuous prices described in Theorem 2, i.e., $\alpha_z = t_j$ if $z \in I_j$. Given $\epsilon > 0$, there exists M_ϵ such that for all $M \geq M_\epsilon$, $|\alpha_z^M - \alpha_z| < \epsilon$ for all $z \in V$.

Theorem 3 says that *as $M \to \infty$, the strategies of the primaries under all symmetric NE in the game with discrete prices converge to those in the unique symmetric NE in the game with continuous prices*. Thus, Theorem 3 provides a *formal justification for the continuous prices approximation* for the above price competition game with spatial reuse. Theorem 3 can be proved using Theorem 1 and techniques similar to the proof of Theorem 2 in [12]. We omit the proof due to space constraints.

VII. Simulations

So far, we have studied NE in the price competition game in a single time slot. However, in practice, primaries in a region would repeatedly interact with each other in different time slots. To model this situation, in this section, we consider a scenario in which there are an infinite number of time slots, and in each slot, n primaries sell bandwidth to k secondaries as in the model in Section II. Also, in practice, the players (primaries) would not know all the parameters of the game (e.g., n, k, q) and hence would use *learning algorithms* to adapt their price selection strategies based on the prices they selected and the payoffs they got in previous slots. We assume that each primary independently adapts his price selection strategy using the *Softmax learning algorithm*, which was proposed to solve the multi-armed bandit problem in [20], and investigate under what conditions the strategies of the primaries converge to the NE of the one-shot game. The algorithm is initiated by each player i by playing all the available prices $\{a_1, ..., a_M\}$ randomly atleast once. Then, the utilities obtained by primary i in time slots $1, ..., t-1$ are used to compute the PMF that is used by primary i to select the price in time slot t. Specifically, in slot t, primary i selects price a_j, $j \in \{1, ..., M\}$ with the following probability:

$$R_{i,t}(a_j) = \frac{exp^{\frac{u_{i,t-1}(a_j)}{\tau N_{i,t-1}(a_j)}}}{\sum_{l=1}^{M} exp^{\frac{u_{i,t-1}(a_l)}{\tau N_{i,t-1}(a_l)}}}, \tag{13}$$

[7] An I.S. I is maximal if for every $z \in V \setminus I$, $I \cup z$ is not an I.S [19].

where $N_{i,t-1}(a_j)$ is the number of time slots in which primary i played the price a_j so far, $u_{i,t-1}(a_j)$ is the total utility that primary i got in the time slots in which he played the price a_j so far, and τ is the temperature constant [20]. Note that the algorithm assigns a probability to each price which is an increasing function of the payoffs that player i got by playing that price in the time slots elapsed so far.

We simulated a scenario in which primaries adapt their price selection strategies using the Softmax algorithm and the *steady state probability distributions* to which the price selection PMFs, $R_{i,t}(\cdot)$, of the primaries converge after the simulation has run for a large number of time slots were obtained for different values of n, k, M, q and τ. Throughout, we observed that whenever a *pure strategy* symmetric NE exists in the one-shot game for given values of n, k, M and q (*i.e.*, the NE price selection strategy support set contains a single price), the price selection PMFs of the primaries under the Softmax algorithm converge to their price selection PMFs under at least one of the pure strategy NE for some values of τ. However, when only mixed strategy NE exist in the one-shot game, the price selection PMFs of the primaries under the Softmax algorithm do not converge to the NE price selection PMFs *for any value of* τ. For example, with $M = 7$ throughout, (i) with $n = 8$, $k = 4$ and $q = 0.2$, one pure strategy NE exists and has support set $\{a_7\}$; the top-left plot in Fig. 5 shows that the Softmax algorithm converges to this NE at $\tau = 0.4$, (ii) with $n = 8$, $k = 4$ and $q = 0.8$, there exist two pure strategy NE with support sets $\{a_2\}$ and $\{a_3\}$, and the top-right plot shows that the Softmax algorithm converges to $\{a_2\}$ at $\tau = 0.1$, (iii) with $n = 2$, $k = 1$ and $q = 0.5$, only a mixed strategy NE exists, and the bottom-left plot shows the steady state probability distributions under Softmax. It is easy to check that the probabilities in the plot do not equal the price selection strategy probabilities under the mixed NE for any value of τ. The design of learning algorithms that converge to the NE even when only mixed strategy NE exist is a direction for future work.

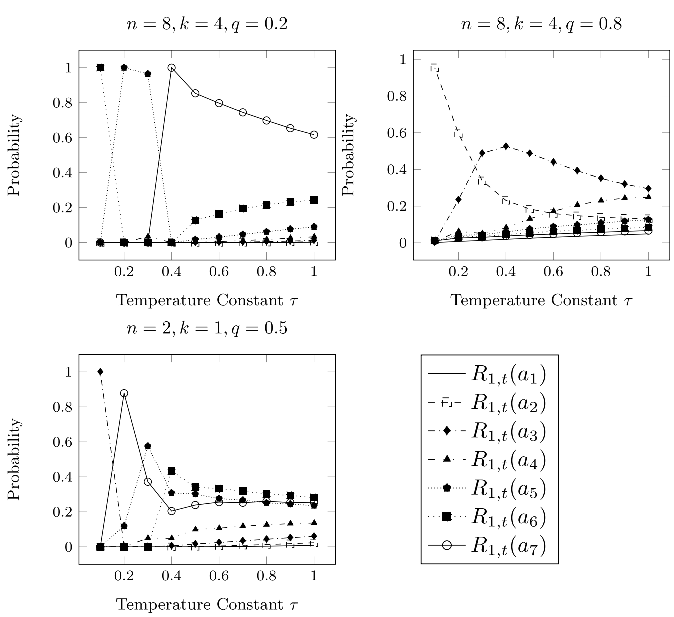

Fig. 5. The figure shows the steady state probability distributions versus the temperature constant τ for different values of n, k and q.

VIII. Conclusions and Future Work

In this paper, we investigated the fundamental question of how the behavior of the players involved in price competition in a DSA market changes when the widely used continuous prices approximation is removed. Our analysis reveals several important differences between the games with continuous and discrete price sets. However, we show, for the games at a single location as well as at multiple locations, that as the number of available prices becomes large in the discrete prices game, the strategies of the primaries under every symmetric NE converge to the unique NE strategy of the game with continuous price sets. For simplicity, we assumed in this paper that q, the probability with which a primary has unused bandwidth, is the same for each primary. A direction for future work is to generalize our results to the case where these probabilities are different for different primaries.

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