Disjoint Compatible Geometric Matchings*

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ABSTRACT

We prove that for every even set of n pairwise disjoint line segments in the plane in general position, there is another set of n segments such that the $2n$ segments form pairwise disjoint simple polygons in the plane. This settles in the affirmative the *Disjoint Compatible Matching Conjecture* by Aichholzer *et al.* [1]. The key tool in our proof is a novel subdivision of the free space around n disjoint line segments into at most $n+1$ convex cells such that the dual graph of the subdivision contains two edge-disjoint spanning trees.

Categories and Subject Descriptors

G.2.2 [**Discrete Mathematics**]: Graph Theory

General Terms

Theory

Keywords

Geometric graph, matching, convex subdivision, dual graph

1. INTRODUCTION

A *planar straight-line graph* (PSLG, for short) is a graph $G = (V, E)$ where the vertices are distinct points in the plane, the edges are straight-line segments between vertices (not passing through any other vertex) such that any two edges can intersect only at common endpoints. Two PSLGs (V, E_1) and (V, E_2) on the same vertex set are *compatible* if no edge in E_1 crosses any edge in E_2.

Every simple polygon with an even number of edges is the union of two disjoint compatible perfect matchings. (One can think of a straight-line matching as a set of disjoint line segments in the plane.) This simple fact is crucial for the best current upper bound on the number of simple polygons on n points in the plane [6, 9]. Conversely, the union of two compatible disjoint matchings is the union of disjoint simple polygons, each of which has an even number of edges. Aichholzer *et al.* [1] conjectured that for every perfect matching M with an even number of edges, there is a disjoint compatible perfect matching (*Disjoint Compatible Matching Conjecture*). The condition that M has an even number of edges cannot be dropped: there are several constructions for perfect straight-line matchings with an *odd* number of edges that do not admit a disjoint compatible matching [1].

Aichholzer *et al.* [1] also made several stronger conjectures, each of which would immediately imply the Disjoint Compatible Matching Conjecture. For stating some of these conjectures, recall two definitions. For a set M of n disjoint line segments in the plane, a *convex subdivision* of the free space is a set C of interior disjoint cells such that every cell is disjoint from the segments in M, and the union of their closures tile the plane. In a *dual (multi-)graph* G associated with M and C, the vertices correspond to the cells in C, and each endpoint of a segment in M corresponds to an edge between two incident cells. Note that M and C do not always determine the dual graph uniquely, since a segment endpoint may be incident to more than two convex cells, and the corresponding dual edge connects only two of those cells. In any case, the dual multi-graph has exactly $2n$ edges, some of which may be parallel. For brevity, we use the term *dual graph* even if G has parallel edges.

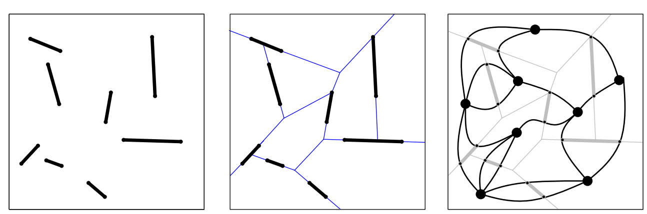

Figure 1: (a) Eight line segments in a bounding box. (b) A convex subdivision with 8 cells. (c) An associated dual graph.

Let M be a set of n disjoint line segments in the plane. According to the *Two Trees Conjecture* [1], there is a convex subdivision with $n+1$ cells such that an associated dual graph G is the edge-disjoint union of two spanning trees, and for every segment $uv \in M$, the edges of G corresponding to u and v are in different spanning trees. The *Extension Conjecture* [1] stipulates that if n is even, then there is a

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convex subdivision into $n+1$ cells such that an associated dual graph G admits an even orientation[1] with the property that whenever a node of G has indegree 2, the two incoming edges do not correspond to the two endpoints of the same segment in M. We no longer believe that these stronger conjectures hold [4], but the ingredients of these conjecture are instrumental for our results.

In this paper, we prove the Disjoint Compatible Matching Conjecture, while bypassing all stronger conjectures made by Aichholzer *et al.* [1]. Specifically, we first prove a weaker form of the Two Trees Conjecture, and show that there exists a convex subdivision with *at most* $n+1$ cells whose dual graph *contains* two edge-disjoint spanning trees.

THEOREM 1. *For every set M of n disjoint line segments in the plane, with no three collinear segment endpoints, there is a convex subdivision of the free space such that an associated dual graph G contains two edge-disjoint spanning trees.*

Combining Theorem 1 with a purely graph theoretic lemma (Lemma 1), which is essentially a weaker form of the Extension Conjecture, we prove the Disjoint Compatible Matching Conjecture.

LEMMA 1. *Let $G=(V,E)$ be a multigraph with $4k$ edges that contains two edge-disjoint spanning trees, and let X be a collection of disjoint pairs of adjacent edges in E, called* conflict pairs. *Then G has an even orientation such that if the indegree of a vertex is 2, then the two incoming edges are not in X.*

THEOREM 2. *For every perfect straight-line matching M with an even number of edges, and no three collinear vertices, there is a disjoint compatible perfect straight-line matching.*

Outline. Recently, Al-Jubeh *et al.* [2] defined a general class of convex subdivisions for a set of disjoint line segments, and proved that there is a convex subdivision in this class whose dual graph is 2-edge-connected. They proved that a bridge (*i.e.*, a cut-edge) in the dual graph is equivalent to the existence of a simple polygon along the boundaries of some convex cells with certain properties (a "forbidden" pattern). Starting from an arbitrary convex subdivision in the class, they applied a sequence of local deformations until all forbidden configurations were removed.

In this paper, we follow a similar strategy with a considerably more sophisticated machinery. We define a class of convex subdivisions (Section 2) that is broader than the one defined in [1, 2]. Specifically, a subdivision for n disjoint segments may have fewer than $n+1$ cells. We identify a class of "critical" closed polygons on the boundaries of convex cells, and show that the dual graph contains two edge-disjoint spanning trees if no critical polygon is present (Section 3). Then we construct a convex subdivision with no critical polygons: we start with a simple initial convex subdivision, and modify it recursively until all critical polygons are removed (Section 4). The number of cells may decrease during the process, but the dual graph of the resulting convex subdivision contains two edge-disjoint spanning trees. Finally we prove Lemma 1 (Section 5), and show that the combination of Theorem 1 and Lemma 1 readily implies the Disjoint Compatible Matching Conjecture.

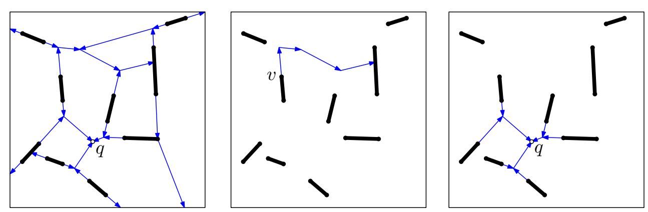
[1] An orientation of a graph is *even* if all in-degrees are even.

2. CONVEX SUBDIVISIONS

We define a general class of convex subdivisions for a set of disjoint line segments in the plane. Let M be a set of n disjoint line segments in the plane, with no three collinear segment endpoints. We can interpret a convex subdivision as a cell complex. A convex subdivision is a set C of pairwise disjoint open convex cells in the plane such that every cell is disjoint from every segment in M and the union of their closures is the entire plane. The *vertices* of the subdivision are the endpoints of the segments in M as well as every point that is incident to three or more cells. We will refer to these two types of vertices as *segment endpoints* and *Steiner points*. The *edges* of the subdivision are line segments between two consecutive vertices along the boundary of a cell, and rays lying on the boundary of a cell incident to a vertex. Note that every segment in M is the union of one or more edges of the subdivision. Every edge of the subdivision is either a *segment edge* which lies along some line segment in M, or an *extension edge* which lies in the free space.

DEFINITION 1. *For a set M of disjoint line segments, let $\mathcal{D}(M)$ be the set of all convex subdivisions C that admit an orientation of the extension edges with the following properties.*

- *The out-degree of every segment endpoint is 1;*
- *the out-degree of every Steiner point in the free space is 0 or 1; the Steiner points in the free space whose out-degree is 0 are called* sinks*;*
- *the out-degree of every Steiner point lying in the relative interior of a segment is 0;*
- *every cycle passes through at least one segment endpoint.*



(a) (b) (c)

Figure 2: (a) A convex partition formed by directed extension edges, with a sink q. (b) An extension-path emitted by segment endpoint v. (c) An extension-tree rooted at a sink q.

If no three segment endpoints are collinear, then it is easy to construct a convex subdivision in $\mathcal{D}(C)$ with no sinks as follows. Consider the segment endpoints $v_1, v_2 \dots, v_{2n}$ in an arbitrary order. For each v_i, draw a directed edge (*extension*) along the supporting line of the incident line segment that starts from v_i and ends when it reaches another line segment, a previous extension, or infinity. This simple algorithm does not produce sinks if no three segment endpoints are collinear. (Aichholzer *et al.* [1] considered convex subdivisions obtained by this simple algorithm only.)

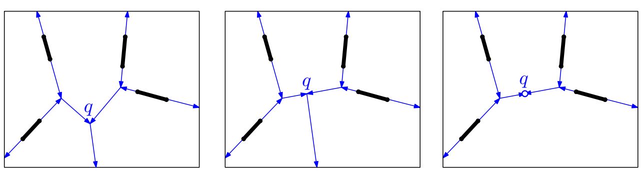
DEFINITION 2. *Let C be a convex subdivision in $\mathcal{D}(M)$.*

- *The* **extension-path** *of a segment endpoint v is a maximal open directed path along extension edges starting from v and ending on a segment, at a sink, or at* *infinity. It does not include the endpoints of the path, and so it is disjoint from all segments.*

- *An* **extension-tree** *is a connected component of the union of all extension paths and sinks. It is acyclic by the definition of $\mathcal{D}(M)$, hence it is a tree rooted at a point in the relative interior of a segment, at a sink, or at infinity.*

Let $S = S(C)$ denote the set of sinks of a convex subdivision $C \in \mathcal{D}(M)$, and let $s = |S(C)|$. It is easy to verify that if there are $|M| = n$ segments, then C has $n-s+1$ cells. Indeed, the complement of all segments and sinks $\mathbb{R}^2 \setminus (M \cup S)$ is a connected region with $n+s$ holes. Insert the $2n$ extension paths one by one. The insertion of each extension path either splits a region into two regions or decreases the number of holes. Eventually, the regions are convex cells, which are simply connected. So there were exactly $n-s$ steps in which a region is split into two.

Al-Jubeh *et al.* [2] considered convex subdivisions with no sinks. By allowing sinks, we have significantly more freedom in constructing and modifying convex subdivisions. Figure 3 illustrates how one can deform adjacent extension edges, create a sink, and reduce the number of cells by one. Note also, that in an extension-tree rooted at a sink, we can change the location of the sink arbitrarily by changing the directions of the extension edges of the tree. We will employ such local deformations in Section 4 for constructing a convex subdivision with no critical polygons.



(a) (b) (c)

Figure 3: (a) A convex subdivision for 4 segments with 5 cells. (b) Extensions edges are deformed so that two incoming edges at q become collinear. (c) By removing the outgoing edge of q, it becomes a sink, and we obtain a convex subdivision with 4 cells.

A segment endpoint v may be incident to more than two cells. For a convex subdivision $C \in \mathcal{D}(M)$, we define the *dual graph* $G = G(C)$ such that the edge of G corresponding to a segment endpoint v connects the two cells adjacent to the unique out-going edge of v. Double edges are possible, corresponding to two endpoints of a line segment lying on the common boundary of two cells. Let $N = N(C)$ denote the 1-skeleton of the convex subdivision C, that is, the planar network formed by all segments and all extension edges.

3. SUFFICIENT CONDITIONS FOR TWO SPANNING TREES

In this section, we establish a sufficient condition for a convex subdivision $C \in \mathcal{D}(M)$ to imply that the dual graph $G(C)$ contains two edge-disjoint spanning trees. Consider a fixed convex subdivision $C \in \mathcal{D}(M)$ for a set M of n disjoint line segments. The dual graph $G = G(C)$ has exactly $2n$ edges and at most $n+1$ nodes, so there are enough edges for two edge-disjoint spanning trees. According to a well known result by Nash-Williams [17] and Tutte [18], a graph $G = (V, E)$ contains two edge-disjoint spanning trees if and only if for any partition $V = \bigcup_{i=1}^r V_i$ of the vertex set into r nonempty classes, there are at least $2(r-1)$ edges between different vertex classes. It is enough to consider vertex sets $V_i \subset V$ that induce a connected subgraph of G, otherwise we could break V_i into two classes without increasing the number of edges between different classes.

For a subset $C' \subseteq C$ of convex cells, let $G(C')$ be the subgraph of G induced by C'. We first deduce upper bounds on the number of edges of each $G(C')$. Then we find a sufficient condition that guarantees that for any partition $C = \bigcup_{i=1}^r C_i$, there are at most $2n - 2(r-1)$ edges in the induced subgraphs $G(C_i)$ and so there are at least $2(r-1)$ edges between different vertex classes $C_1, \ldots, C_r$.

Proposition 1. *If every subset $C' \subseteq C$ induces at most $2(|C'|-1)$ edges of the dual graph G, then G contains two edge-disjoint spanning trees.*

Proof. For every partition $C = \bigcup_{i=1}^r C_i$, the induced subgraphs $G(C_i)$, $i = 1, \ldots, r$, jointly have no more than $\sum_{i=1}^r 2(|C_i| - 1) = 2|C| - 2r \le 2(n+1) - 2r$ edges. The total number of edges is $2n$. Therefore, there are at least $2(r-1)$ edges between distinct vertex classes. □

We can relax this condition noting that the total number of cells in C is only $n - s + 1$. Intuitively, every sink "decreases" the total number of cells by one. We distribute this beneficial effect among disjoint subsets of C. For every subset $C' \subseteq C$, we will assign a set of sinks $S(C')$, of cardinality $s(C') = |S(C')|$, such that disjoint subsets of C are assigned disjoint sets of sinks, that is, for every partition $C = \bigcup_{i=1}^r C_i$ we have $\sum_{i=1}^r s(C_i) \le s$.

Proposition 2. *If every subset $C' \subseteq C$ induces at most $2(|C'|+s(C')-1)$ edges of the dual graph G, then G contains two edge-disjoint spanning trees.*

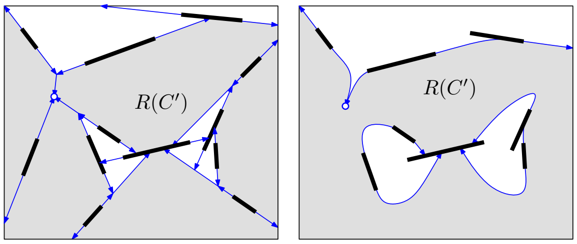
Proof. For every partition $C = \bigcup_{i=1}^r C_i$, the induced subgraphs $G(C_i)$, $i = 1, \ldots, r$, jointly have no more than $\sum_{i=1}^r 2(|C_i| + s(C_i) - 1) \le 2|C| + 2s - 2r = 2(n-s+1) + 2s - 2r = 2(n+1) - 2r$ edges. The total number of edges is $2n$. Therefore, there are at least $2(r-1)$ edges between distinct vertex classes. □

For every subset $C' \subseteq C$, we define the a polygonal domain $R(C')$ as follows (refer to Fig. 4). Let $A(C')$ be the union of the closures of all cells in C', that is, $A(C') = \bigcup_{c \in C'} \mathrm{cl}(c)$. The boundary $\partial A(C')$ may have several components, each component is an (open or closed) simple polygonal curve in $N = N(C)$. The domain $R(C')$ is obtained from $A(C')$ by subtracting

(1) any Jordan arc between different components of $\partial A(C')$ if it is contained in a single line segment or in a single extension-path, and

(2) any extension-path from a component of $\partial A(C')$ to a sink lying in the interior of $A(C')$.

When we subtract a Jordan arc of type (1), we effectively merge the two components of $\partial A(C')$ into one component of $\partial R(C')$. When we subtract an extension-path from the boundary component of $\partial A(C')$ to a sink q, we move q to the boundary of $\partial R(C')$. The boundary $\partial R(C')$ consists of weakly simple polygonal curves.[2] It is clear that if C'



(a) (b)

Figure 4: (a) A subset $C' \subset C$ of 4 cells, where $G(C')$ has 8 edges, and domain $R(C')$ has one hole. (b) A schematic representation, where only $\partial R(C')$ and adjacent line segments are shown.

induces a connected subgraph $G(C')$, then $R(C')$ is a connected polygonal domain.

We can now assign sinks to every subset $C' \subseteq C$ as follows. Let $S(C') \subseteq S$ be the set of sinks that lie in the interior of $R(C')$, on the boundary of a hole of $R(C')$, or in the interior of a hole such that there is an extension-path from $\partial R(C')$ to the sink.

Proposition 3. *If $C_1, C_2 \subset C$ such that $C_1 \cap C_2 = \emptyset$ and $G(C_1)$ and $G(C_2)$ are connected subgraphs, then $S(C_1) \cap S(C_2) = \emptyset$.*

Proof. Suppose to the contrary that there is a sink $q \in S(C_1) \cap S(C_2)$. Since C_1 and C_2 are disjoint, the corresponding domains $R(C_1)$ and $R(C_2)$ are interior disjoint and the boundaries of their holes are disjoint. Therefore q has to lie in the interior of a hole of $R(C_1)$ or $R(C_2)$. Assume w.l.o.g. that q lies in the interior of a hole of $R(C_1)$ such that there is a extension-path from $\partial R(C_1)$ to q. This means that the entire domain $R(C_2)$ lies in a hole of $R(C_1)$. By the definition of $R(C_2)$, the extension-path from $\partial R(C_1)$ to q lies on the outer boundary of $R(C_2)$. Hence, $q \notin S(C_2)$, contradicting our initial assumption. □

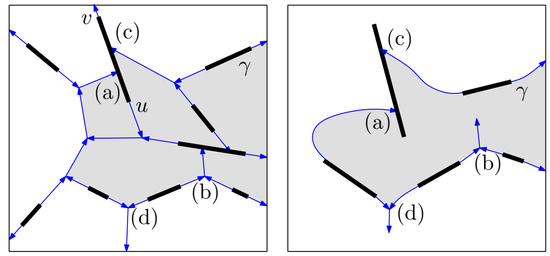
We next give a formula for the number of edges in $G(C')$ for any $C' \subseteq C$. Let Γ denote the set of all (open or closed) oriented[3] weakly simple polygonal curves in $N(C)$. Each curve in Γ decomposes the plane into a *left* and a *right* side (depending the orientation of γ). We define the following *local features* for a curve $\gamma \in \Gamma$. Refer to Fig. 5.

(a) A point $p \in \gamma$ lying on a segment $uv \in M$ such that in any neighborhood of p, part of either the outgoing edge of p or segment uv lies on the *left* side of γ;
(b) a Steiner point $p \in \gamma$ such that in any neighborhood of p, part of the outgoing edge of p lies on the *left* side of γ;
(c) a point $p \in \gamma$ lying on a segment $uv \in M$ such that in any neighborhood of p, part of either the outgoing edge of p or segment uv lies on the *right* side of γ;
(d) a Steiner point $p \in \gamma$ such that in any neighborhood of p, part of the outgoing edge of p lies on the *right* side of γ.

[2]An open (closed) polygonal curve $(p_1, p_2, \ldots, p_n)$ is *weakly simple* if for every $\varepsilon > 0$, there are points p'_i in the ε-neighborhood of each vertex p_i such that $(p'_1, p'_2, \ldots, p'_n)$ is a simple open (closed) polygonal curve.

[3]The orientation of a curve $\gamma \in \Gamma$ is independent from the directions of any extension edges along γ.

Let $a(\gamma)$, $b(\gamma)$, $c(\gamma)$, and $d(\gamma)$ denote the number of features of type (a), (b), (c), and (d), respectively, along γ. Note that an (a)-feature and a (c)-feature may be located at the same point $p \in \gamma$. Furthermore, if $-\gamma$ and γ denote the same curve in Γ taken with opposite orientations, then we have $a(\gamma) = c(-\gamma)$, and $b(\gamma) = d(-\gamma)$.



(a) (b)

Figure 5: (a) Local features along an open polygonal curve γ, the left side of which is gray. (b) A schematic representation, where only the local features and the line segments intersecting γ are shown.

Lemma 2. *Consider a subset $C' \subseteq C$ that induces a connected subgraph $G(C')$. Let h be the number of holes of $R(C')$, and let $s_{\rm in}$ denote the number of sinks in the interior of $R(C')$. Orient all curves along $\partial R(C')$ such that $R(C')$ lies on their* left *side, and let a (resp., b) be the total number of (a)-features (resp., (b)-features) along $\partial R(C')$. Then graph $G(C')$ has $2(|C'|-1) - a - 2b + 2h + 2s_{\rm in}$ edges.*

Proof. Since C' induces a connected subgraph of $G(C')$, the domain $R(C')$ is connected. Let x be the number of line segments lying in the interior of $R(C')$. Now $R(C') \setminus (M \cup S)$ is a connected polygonal domain with $h + s_{\rm in} + x$ holes.

The number of extension-paths that enter the interior of $R(C') \setminus (M \cup S)$ from its boundaries is exactly $a + b + 2x$. Insert these extension-paths one after the other. The insertion of each extension-path either splits a region into two regions or decreases the number of holes. Eventually, we obtain convex cells, which are simply connected. So there were exactly $(a+b+2x) - (h + s_{\rm in} + x) = a + b - h - s_{\rm in} + x$ steps in which a region is split into two regions. Thus the number of cells is $|C'| = 1 + a + b - h - s_{\rm in} + x$.

A segment endpoint v corresponds to an edge between two cells in C' if the outgoing edge of v lies in the interior of $R(C')$. There are exactly $a + 2x$ such segment endpoints. Hence, $G(C')$ has $a + 2x = 2(|C'|-1) - a - 2b + 2h + 2s_{\rm in}$ edges. □

We show next that if $G(C')$ has fewer than $2(|C'| + s(C') - 1)$ edges for some $C' \subseteq C$, then a single component of the boundary $\partial R(C')$ is responsible for this. Let $\Gamma_+ \subset \Gamma$ be the set of all *closed* polygonal curves in Γ oriented *counterclockwise*. For every $\gamma \in \Gamma_+$, let $s(\gamma)$ denote the number of sinks q such that either $q \in \gamma$, or q lies in the interior of the region bounded by γ and some extension-path goes from γ to q.

Lemma 3. *If for every $\gamma \in \Gamma_+$ we have $c(\gamma) + 2d(\gamma) + 2s(\gamma) \geq 2$, then the dual graph $G(C)$ contains two edge-disjoint spanning trees.*

Proof. By Proposition 2, it is enough to show that if $G(C')$ has more than $2(|C'| + s(C') - 1)$ edges for some

$C' \subseteq C$, then there is a curve $\gamma \in \Gamma_+$ such that $c(\gamma) + 2d(\gamma) + 2s(\gamma) < 2$.

Consider a subset $C' \subseteq C$ that induces more than $2(|C'|+s(C')-1)$ edges. By Lemma 2, we have $2(|C'|+s(C')-1) < 2(|C'|-1) - a - 2b + 2h + 2s_{\text{in}}$, that is, $a + 2b + 2(s(C') - s_{\text{in}}(C')) < 2h$. Let $\gamma_1, \ldots, \gamma_h \in \Gamma_+$ denote the boundaries of the h holes of $R(C')$. It is clear that we have

$$a \geq \sum_{i=1}^{h} c(\gamma_i), \quad b \geq \sum_{i=1}^{h} d(\gamma_i), \quad s(C') - s_{\text{in}}(C') = \sum_{i=1}^{h} s(\gamma_i).$$

Thus, $\sum_{i=1}^{r} (c(\gamma_i) + 2d(\gamma_i) + 2s(\gamma_1)) < 2h$, and for at least one of the holes γ_i, we have $c(\gamma_i) + 2d(\gamma_i) + 2s(\gamma_i) < 2$. $\square$

A closed counterclockwise polygonal curve $\gamma \in \Gamma_+$ is called *critical* if $c(\gamma) + 2d(\gamma) + 2s(\gamma) \leq 1$. It is *0-critical* if $c(\gamma) + 2d(\gamma) + 2s(\gamma) = 0$, and *1-critical* if $c(\gamma) + 2d(\gamma) + 2s(\gamma) = 1$. Observe that if $\gamma \in \Gamma_+$ is critical, then it has no (d)-features, it does not pass through any sink, and there is no extended-path from γ to a sink in the interior of γ.

Remark 1. There is a simple intuitive interpretation of critical polygons. Recall that $N = N(C)$ denotes the planar network formed by all segments and extensions of a convex subdivision $C \in \mathcal{D}(M)$. The line segments in M are undirected, and the extensions are directed. We define a *path* in N such that it follows given directions along extension edges (and has arbitrary direction along segment edges). From a point p on a 0-critical polygon γ, no path can enter the exterior of γ. From a point p on a 1-critical polygon γ, all paths to the exterior of γ must pass through the (c)-feature of γ, (and then follow the line segment containing the (c)-feature to its endpoint in the exterior of γ).

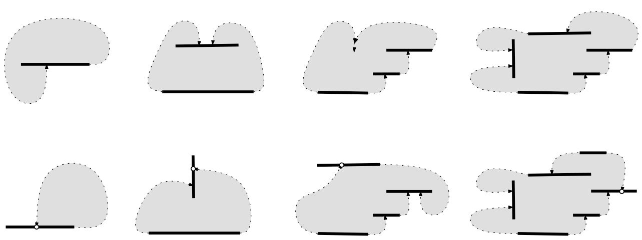


Figure 6: Schematic drawing of some critical curves. Upper row: 0-critical curves. Lower row: 1-critical curves.

Maximal Critical Polygons.

We define maximal critical polygons and show that they have disjoint interiors. In the next section, we will construct a convex subdivision recursively, by modifying the boundary of a maximal critical polygon. The disjointness of maximal critical polygons is essential for this approach.

For every closed polygon $\gamma \in \Gamma_+$, let $C(\gamma)$ denote the set of cells in the interior of γ; and let $R(\gamma) = R(C(\gamma))$. We define the *size* of γ to be the cardinality of $C(\gamma)$. We say that a critical polygon γ is *maximal* if there is no other critical polygon γ' such that $C(\gamma) \subset C(\gamma')$. Similarly, we define maximal 0-critical and maximal 1-critical polygons.

PROPOSITION 4.

(i) If $\gamma_1, \gamma_2 \in \Gamma_+$ are maximal 0-critical, then $R(\gamma_1)$ and $R(\gamma_2)$ are disjoint.

(ii) If there is no 0-critical polygon in Γ_+ and $\gamma_1, \gamma_2 \in \Gamma_+$ are 1-critical, then $R(\gamma_1)$ and $R(\gamma_2)$ are disjoint.

(Some proofs are omitted from this extended abstract. For complete proofs, refer to the full version of this paper.)

4. BUILDING A CONVEX SUBDIVISION

Let M be a set of n disjoint line segments, with no three collinear endpoints. In this section we construct a convex subdivision $C \in \mathcal{D}(M)$ with no critical polygon. We start with an initial convex subdivision $C_0 \in \mathcal{D}(M)$, which has no sinks, and then apply a sequence of *modifications* in order to eliminate critical polygons. For a convex subdivision $C \in \mathcal{D}(M)$, let $s(C)$ denote the total number of sinks, and let M_C be the set of segments that intersect or lie in the interior of some critical polygons. A modification step may have two possible outcomes: (1) We create a new sink (refer to Fig. 3) and $s(C)$ strictly increases. In this case, we redraw the convex subdivision by a simple "left-right" algorithm described below that preserves all existing sinks. (2) There are no new sinks but we destroy a maximal critical polygon and M_C strictly decreases. In this case we consider the next maximal critical polygon and recurse. In each modification step, the function $f(C) = n \cdot s(C) - |M_C|$ strictly increases. Since the value of $f(C)$ is always between $-n$ and n^2, after fewer than $n^2 + n$ modification steps, we obtain a convex subdivision $C \in \mathcal{D}(M)$ with $M_C = \emptyset$, that is, with no critical polygons.

Left-right subdivisions. Our initial convex subdivision $C_0 \in \mathcal{D}(M)$ is a "left-right" subdivision defined below. If a modification step creates a new sink, we redraw a left-right subdivision that preserves all extension-trees rooted at sinks, and draws new extensions for all other segment endpoints.

Let M be a set of n disjoint line segments in the plane such that no three endpoints are collinear, and no two segment endpoints have the same x-coordinate. Assume that we are given a set T of $s \geq 0$ pairwise disjoint extension trees, which terminate at s distinct sinks (initially $s = 0$), and have convex angles at their vertices. Recall that one can move a sink to anywhere within its own extension-tree. Move each sink to the leftmost point of its extension-tree (the leftmost point is the right endpoint of some segment, and so the outgoing edge of this segment endpoint degenerates to a point). We construct extensions for all remaining segment endpoints in two phases.

Sort the segments in M according to the x-coordinates of their left endpoints in increasing order. Let u_i and v_i be the left and right endpoint, respectively, of the ith segment. In the first phase, consider the left endpoints $u_1, \ldots u_n$, successively. If u_i is not incident to an extension-tree in T, then draw a directed segment (*left extension*) along $\overrightarrow{v_i u_i}$ from u_i until it reaches another segment, a previous extension, or infinity. In the second phase, consider the right endpoints $v_1, \ldots, v_n$. If v_i is not incident to an extension tree in T, then draw a directed segment (*right extension*) along $\overrightarrow{u_i v_i}$ from v_i until it reaches another segment, a previous extension, or infinity. The algorithm extends all segments beyond both of their endpoints, all angles are convex, and so it produces a convex subdivision in $\mathcal{D}(M)$.

Recall that $N = N(C)$ is the union of all segments and extensions, and a path in N has to follow the given directions along extensions. Recall also that we assume that every sink is located at the leftmost vertex (*i.e.*, a right segment endpoint) of its extension tree. With this convention, we observe a key property of left-right subdivisions.

Proposition 5. *If $C \in \mathcal{D}(M)$ is a left-right subdivision, then N contains a path from any point $p \in N$ to infinity.*

It follows, by Remark 1, that left-right subdivisions have no 0-critical polygons.

Subdivisions with no 0-critical polygons. Our algorithm performs a sequence of local modifications on the left-right subdivision that will either eliminate all critical polygons from the resulting subdivisions or else introduce a new sink. It maintains the invariant that the subdivision always remains in the class $\mathcal{D}^*(M) \subseteq \mathcal{D}(M)$ defined here.

Definition 3. *A subdivision $C \in \mathcal{D}(M)$ is in the class $\mathcal{D}^*(M)$ if it has the following properties.*

1. *N contains a path from every point $p \in N$ to infinity;*
2. *For every vertex $v \in N$ lying on or in the interior of a critical polygon, any outgoing edge of v is collinear with an incoming edge or with a line segment incident to v.*

By Remark 1, if $C \in \mathcal{D}^*(M)$, then it has no 0-critical polygons in $\Gamma_+(C)$. We can now focus on 1-critical polygons, and derive some immediate consequences of the definition.

Proposition 6. *Let $C \in \mathcal{D}^*(M)$. If $\gamma \in \Gamma_+(C)$ is a 1-critical polygon, then for every $p \in N(C)$ in the closed polygonal domain bounded by γ, there is a path $\delta(p) \subset N(C)$ from p to the (c)-feature of γ.*

Proof. By Definition 3, N contains a path $\delta(p)$ from p to infinity. If p lies in the closed polygonal domain bounded by γ, then $\delta(p)$ has to pass through the (c)-feature of γ. $\square$

When the number of sinks is unchanged, any reduction in $|M_C|$, the number of segments that intersect or lie in the interior of some critical polygon, represents progress towards decreasing the number of critical polygons. If $C \in \mathcal{D}^*(M)$, there is a simple equivalent condition for $m \in M_C$. Two paths in N are *endpoint-disjoint* if no segment endpoint lies in the relative interior of both paths.

Proposition 7. *Let $C \in \mathcal{D}^*(M)$ and $m \in M$. We have $m \notin M_C$ if and only if $N(C)$ contains two endpoint-disjoint paths from any point $p \in m$ to infinity.*

A convex subdivision $C \in \mathcal{D}^*(M)$ may have collinear extension edges. Let a *straight-extension* be a maximal path along collinear extension edges in N. A straight-extension is composed of one or more extension edges. If a straight-extension ends at a Steiner point in the free space, lying on or in the interior of a critical polygon, then its endpoint must be in the relative interior of another straight-extension by Definition 3.

One step of the recursion. It remains to show that if a convex subdivision in $\mathcal{D}^*(M)$ has a critical polygon, then we can perform a local modification step that strictly increases $f(C) = n \cdot s(C) - |M_C|$.

Lemma 4. *If $C \in \mathcal{D}^*(M)$ and C has a 1-critical polygon, then there is a convex subdivision $C' \in \mathcal{D}^*(M)$ such that $n \cdot s(C) - |M_C| < n \cdot s(C') - |M_{C'}|$.*

Proof (sketch). Let $C \in \mathcal{D}^*(M)$ and let $\gamma \in \Gamma_+$ be a maximal 1-critical polygon. Let p_0 be the unique (c)-feature of γ, and let q_0 be the endpoint of the segment containing p_0 that lies in the exterior of γ. Let H_γ denote the geodesic hull of $\gamma \cup \{q_0\}$ with respect to the line segments of M lying in the exterior of γ (refer to Fig. 7). The geodesic hull H_γ is a closed polygonal domain, with boundary ∂H_γ. Note that every convex vertex of H_γ is either q_0 or a convex vertex of γ; and every reflex vertex of H_γ is a segment endpoint in the exterior of γ. Let $(q_0, q_1, \ldots, q_t)$ denote the convex vertices of H_γ in counterclockwise order.

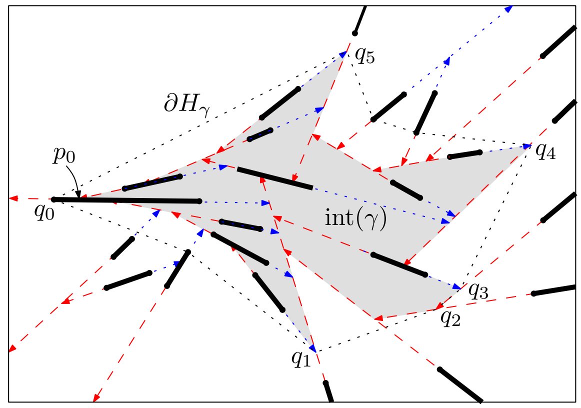


Figure 7: A left-right subdivision, the interior of a 1-critical polygon γ is shaded. The boundary of the geodesic hull H_γ is dotted. The convex vertices of H_γ are $(q_0, \ldots, q_5)$, where q_1 is counterclockwise and $q_2, \ldots, q_5$ are clockwise.

Each convex vertex q_i, $i = 1, \ldots, t$, is incident to two edges of γ: one incoming and one outgoing edge (since q_i cannot be an (a)-, (b)-, or (c)-feature of γ). We define an *orientation* for these vertices as follows. Vertex q_i is *counterclockwise* (resp., *clockwise*) if the edge of γ directed into q_i precedes (follows) q_i in a counterclockwise walk along γ.

A very brief overview of the remainder of the proof of Lemma 4 follows. Choose a convex vertex q_i, $1 \leq i \leq t$, modify the outgoing edge of q_i, and make some additional adjustments to obtain a convex subdivision $C' \in \mathcal{D}^*(M)$. One of the following two strategies will suffice. (1) Replace the outgoing edge of q_i with $q_i q_{i+1}$, and introduce a new sink at q_{i+1}. In this case, recompute a left-right subdivision (including the new sink) C', and the number of sinks increases by one. (2) Replace the outgoing edge of q_i with an edge of ∂H_γ, and recompute some of the extension edges lying in the geodesic hull H_γ. In this case, the number of sinks remains the same but M_C strictly decreases. Distinguish two cases.

Case 1: There are two consecutive convex vertices, q_i and q_{i+1}, $1 \leq i \leq t-2$, such that q_i is oriented counterclockwise and q_{i+1} is oriented clockwise.
Case 2: Vertex q_1 is oriented clockwise or q_t is oriented counterclockwise.

Case 1 is split into two subcases.

Subcase 1a: $q_i q_{i+1}$ is an edge of H_γ (*c.f.* Fig. 7 with $i = 1$). Replace the outgoing edge of q_i by the directed edge $q_i q_{i+1}$, and delete the outgoing edge of q_{i+1}, creating a new sink at q_{i+1}. Since both q_i and q_{i+1} are vertices of the geodesic hull H_γ, the angles around both q_i and q_{i+1} remain con-

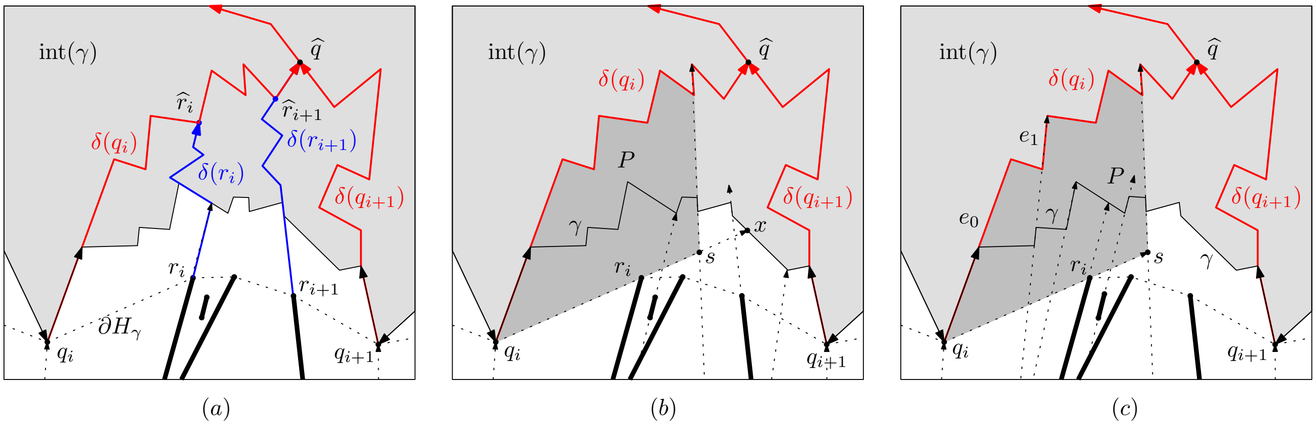


Figure 8: (a) The portions of δ and ∂H_γ between a counterclockwise vertex q_i and a clockwise vertex q_{i+1}. The paths $\delta(q_i)$, $\delta(r_i)$, $\delta(r_{i+1})$, and $\delta(q_{i+1})$ go to the (c)-feature of γ (which is outside of the figure). The interior of γ is shaded light gray. (b) There straight-extensions cross segment r_ix, one of them contains an edge of $\delta(q_i)$. Polygon P is shaded dark gray. (c) A straight-extension crosses q_ir_i, and it contains an edge of $\delta(q_i)$.

vex. Then recompute a left-right subdivision C' preserving these $s(C)+1$ sinks (the $s(C)$ sinks of C and a new sink at q_{i+1}). To see that C' is a convex subdivision in $\mathcal{D}^*(M)$, it is enough to show that all angles of the $s(C)+1$ extension trees terminating at sinks are convex. As noted above, the angles around q_i and q_{i+1} are convex. Other extension trees terminating at sinks of C are not effected because γ is 1-critical and so no extension tree that terminates at a sink intersects γ. So in case 1a, we have $C' \in \mathcal{D}^*(M)$ and C' has one more sinks than C.

Subcase 1b: q_iq_{i+1} is not an edge of H_γ. Replace the outgoing edge of q_i or q_{i+1} with the incident edge of ∂H_γ. This provides a new edge from γ to the exterior of γ. This new edge guarantees that some segment in M_C has two endpoint-disjoint paths to infinity (*c.f.* Proposition 7). After modifying the outgoing edge of q_i or q_{i+1}, a careful rebuilding nearby extensions yields a subdivision $C' \in \mathcal{D}^*(M)$ with $M_{C'} \subset M_C$. To control the extent of the modifications, we define a polygonal domain $P \subset H_\gamma$: all extension edges in the exterior of P remain unchanged, and the extension edges inside P are redrawn. Polygon P is defined in the next two paragraphs.

By Proposition 6, $N(C)$ contains paths $\delta(q_i)$ and $\delta(q_{i+1})$ from q_i and q_{i+1}, respectively, to the (c)-feature of γ. The paths $\delta(p_i)$ and $\delta(q_{i+1})$ meet at some point $\widehat{q}$, and we denote by $\delta(q_i, q_{i+1})$ the union of the initial portions of $\delta(q_i)$ and δ_{i+1} up to $\widehat{q}$ (see Fig. 8a). Let q_ir_i and $r_{i+1}q_{i+1}$ be the edges of ∂H_γ incident to q_i and q_{i+1}, respectively (possibly $r_i = r_{i+1}$). Since r_i and r_{i+1} are reflex vertices of the geodesic hull H_γ, they are endpoints of some line segments lying in the exterior of γ. The extension paths emitted by r_i and r_{i+1} reach γ, so $N(C)$ contains paths $\delta(r_i)$ and $\delta(r_{i+1})$ from r_i and r_{i+1}, respectively, to the (c)-feature of γ. Both $\delta(r_i)$ and $\delta(r_{i+1})$ reach $\delta(q_i, q_{i+1})$ at some points $\widehat{r}_i$ and $\widehat{r}_{i+1}$, respectively (possibly $\widehat{r}_i = \widehat{r}_{i+1}$). Since $\delta(r_i)$ and $\delta(r_{i+1})$ do not cross, the points $(q_i, \widehat{r}_i, \widehat{r}_{i+1}, q_{i+1})$ appear in this order along path $\delta(q_i, q_{i+1})$. Suppose (by applying a reflection in the x-axis if necessary) that $\widehat{q}$ does not lie between q_i and $\widehat{r}_i$ along $\delta(q_i, q_{i+1})$.

The ray $\overrightarrow{q_ir_i}$ hits γ at some point x (see Fig. 8b). Note that any extension edge that crosses q_ix must cross it from the right side because no extension leaves γ between q_i and q_{i+1}. Let $\alpha(r_i, q_{i+1})$ be the directed path[4] from r_i to q_{i+1} that starts with the directed segment segment r_ix and then follows γ counterclockwise from x to q_{i+1}. Let s be the first point on $\alpha(r_i, q_{i+1})$ that lies on a straight-extension that contains an edge of $\delta(q_i, q_{i+1})$. Point $s \in \alpha(r_i, q_{i+1})$ exists, since the outgoing edge of q_{i+1} is part of both $\alpha(r_i, q_{i+1})$ and $\delta(q_i, q_{i+1})$. Now let P be the polygonal domain enclosed by segment q_ir_i, the initial portion of path $\alpha(r_i, q_{i+1})$ from r_i to s, and paths $\delta(q_i)$ and $\delta(s)$.

Let e_0 be the straight-extension emitted by q_i. We will replace e_0 with q_ir_i. Any extension edge that crosses q_ir_i in $N(C)$ will be truncated. Let E be the set of all edges of polygon P that are contained in some straight-extensions of $N(C)$ that intersect q_ir_i. Clearly, we have $e_0 \in E$. Let every edge $e \in E$ inherit its direction from the straight-extension it is contained in.

We are now ready to construct a new convex subdivision C', by modifying some extensions of C. All extensions in the exterior of P remain the same (but some new extensions starting from P may reach into the exterior of P). Replace the straight-extension e_0 by the directed segments q_ir_i and r_ix (these become the outgoing edges of q_i and r_i, respectively). Redraw the extensions in P as follows. The edges of $N(C)$ that cross q_is (from the exterior of P into the interior of P) are truncated such that they now end at q_is. In particular, all edges in E are erased. The edges of $N(C)$ that hit some edge $e \in E$ from the exterior of P are extended successively into the interior of P until they hit a line segment, the boundary of ∂P, or another extension. For each segment in the interior of P, Draw new extensions in two phases: first extend them successively beyond their endpoints closest to the supporting line of q_is; and then extend them beyond their other endpoints (these extensions may exit from P through the erased edges of E). Denote by $C' \in \mathcal{D}(M)$ the resulting convex subdivision of M.

One can show, using Proposition 7, that $M_{C'} \subsetneq M_C$, hence $|M_{C'}| < |M_C|$. The conditions in Definition 3 are maintained, hence $C' \in \mathcal{D}^*(M)$, as required.

(Further details and the discussion of Case 2 are omitted from this extended abstract.) $\square$

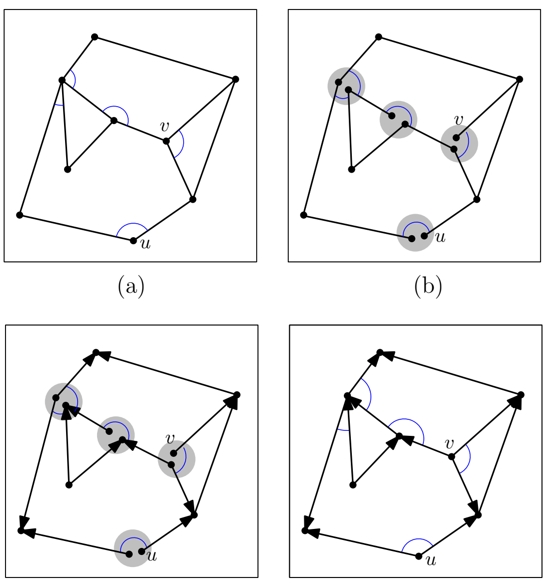
[4]Path $\alpha(r_i, q_{i+1})$ is not part of $N(C)$, and does not have to follow the directions of extension edges along γ.

5. FROM TREES TO MATCHINGS

This section begins with a proof of Lemma 1, that uses purely graph theoretic arguments and then demonstrates that Theorem 1 and Lemma 1 readily imply the existence of a compatible disjoint matching for any even matching M.

A related result is due to Farber *et. al.* [10] and Cordovil and Moreira [8]: If $G = (V, E)$ is the union of two disjoint spanning trees on n vertices, then one can exchange all edges of the two trees in $n-1$ steps such that each step exchanges one pair of edges and maintains a partition of E into two spanning trees. These exchange steps, however, do not seem to be sufficient to reach a partition of E into two *conflict-free* spanning trees.

The proof of Lemma 1 is based on the concept of "conflict-free tree representation," defined below. Recall that *vertex splitting* in a (multi-)graph $G = (V, E)$ means that a vertex $v \in V$ is replaced by two vertices v_1 and v_2, and each edge $vu \in E$ incident to v is replaced by a new edge incident to either v_1 or v_2 (see Fig. 9(a-b)). The replacement edge is identified with the original edge, and in particular it preserves the same conflicts.



(a) (b)

(c) (d)

Figure 9: (a) A graph G with an even number of edges. Conflicting pairs of edges are joined by small circular arcs. (b) A conflict-free tree representation with leaves at u and v after some vertex splitting steps. (c) An even orientation on the tree. (d) The induced even orientation of the original graph G.

Definition 4. *Let $G = (V, E)$ be a multigraph in which some pairs of edges are in conflict (conflict pairs).*

- *A subset of edges $R \subset E$ has* a conflict-free tree representation *if, after some appropriate vertex splitting operations, the edges in E form a tree in which adjacent edges are not in conflict.*
- *Let $U \subseteq V$ be a subset of vertices. A subset of edges $R \subset E$ has* a conflict-free tree representation with leaves at U *if, after some appropriate vertex splitting operations, the edges in E form a tree in which adjacent edges are not in conflict and for every $u \in U$ some vertex corresponding to u is a leaf.*

Proposition 8. *Let $G = (V, E)$ be a multigraph and let P be a collection of disjoint edge pairs (conflict pairs). If $R \subseteq E$ has a conflict-free tree representation and $|R|$ is even, then (V, R) has an even orientation such that if the indegree of a vertex is 2, then the two inbound edges are not in conflict.*

Proof. It is well known that every connected graph with an even number of edges has an even orientation [16]. After some vertex splitting operations, (V, R) becomes a spanning tree (V', R') with no two adjacent edges in conflict. (Refer to Fig. 9.) Fix an arbitrary even orientation for the tree (V', R'). Since all indegrees are even, for every directed edge $\overrightarrow{e} \in R'$, there is another edge $\overrightarrow{f} \in R'$ directed into the same vertex such that e and f are not in conflict. This orientation is an even orientation of (V, R) with the same property. Hence, if a vertex has in-degree 2, then the two inbound edges are not in conflict. $\square$

In the proof of Lemma 1, we construct an edge-partition of $G(V, E)$ into even subgraphs, each of which has a conflict-free tree representation. For the initialization of this algorithm, we will use the following simple observation.

Proposition 9. *Let $G = (V, E)$ be a multigraph with a collection of disjoint pairs of adjacent edges, called* conflict pairs. *If G contains two edge-disjoint spanning trees, then it also contains two edge-disjoint spanning trees, (V, B) and (V, R), with the property that if two parallel edges are in conflict then either both of them are contained in $B \cup R$ or neither of them are in $B \cup R$.*

Proof of Lemma 1. Partition E into a family $\mathcal{P}$ of even subsets, each having a conflict-free tree representations. Apply Proposition 8 for each edge set in $\mathcal{P}$ independently, and obtain a desired orientation of E. In the remainder of the proof, we construct the edge partition $\mathcal{P}$. Initially $\mathcal{P}$ is empty. In each step, our algorithm increments $\mathcal{P}$ with even edge sets that have conflict-free tree representations.

Maintain an auxiliary multi-graph $H_i = (V_i, B_i \cup R_i)$ for $i = 0, 1, \ldots, n$, such that $V = V_0 \supset V_1 \supset \ldots \supset V_n = \emptyset$, and each H_i is the edge-disjoint union of two spanning trees, denoted (V_i, B_i) and (V_i, R_i). Each edge $uv \in B_i \cup R_i$ corresponds to a nonempty set $E(uv) \subset E$ of edges with a conflict-free tree representation and leaves at $\{u, v\}$. This means that in some conflict-free tree representation of $E(uv)$, there is a unique edge incident to $u \in V_i$ and a unique edge (possibly the same edge) incident to $v \in V$. Fix such designated edges in $E(u, v)$ incident to u and v. Two edges of H_i, say uv and uw, are *in conflict at vertex* u if the designated edges of $E(uv)$ and $E(uw)$ incident to u are in conflict.

For each vertex $v \in V_i$, maintain two sets of edges: a blue set $b_i(v) \subset E$ and a red set $r_i(v) \subset E$. Each of them contains at most one edge, which is incident to v. They have the property that an edge in $b_i(v)$ is not in conflict with any edge of B_i at vertex v; and an edge in $r_i(v)$ is not in conflict with any edge in R_i at v.

Throughout our algorithm, the following partition of E is maintained. The edges of E are partitioned into sets in $\mathcal{P}$ (these are even sets and have conflict-free tree representations); all remaining edges are partitioned into sets $E(uv)$

corresponding to edges of H_i, and sets $b_i(v)$, $r_i(v)$ corresponding to the vertices of H_i. Throughout the algorithm, for all $i = 0, 1, \ldots, n$, both

$$B_i \cup \bigcup_{v \in V_i} b_i(v) \qquad \text{and} \qquad R_i \cup \bigcup_{v \in V_i} r_i(v)$$

are even, that is, the total number of blue and red edges remain even.

Initialization. By Proposition 8, G has a partition $E = B_0 \cup R_0 \cup Y_0$ such that (V, B_0) and (V, R_0) are edge-disjoint spanning trees, and for any pair of conflicting parallel edges, either both are in $B_0 \cup R_0$ or both are in Y_0. Since E is even, Y_0 is also even. Color half of the edges in Y_0 red and the other half blue arbitrarily. Recall that every edge is in conflict with at most one other edge. Assign every blue edge $uv \in Y_0$ to either u or v such that the possible edge in conflict with uv has not been assigned to the same vertex. Let $b_0(v)$ (resp., $r_i(v)$) be the set of blue (resp.,) edges assigned to v. Note that the edges in $b_0(v)$ and $r_0(v)$ are conflict free, hence $b_0(v)$ and $r_0(v)$ have conflict-free tree representation. If $b_0(v)$ (or $(r_0(v))$ contains two or more edges, then move $2\lfloor |b_0(v)|/2 \rfloor$ edges from $b_0(v)$ to $\mathcal{P}$, as an even conflict-free tree representation. Now we have $|r_i(v)| \leq 1$ and $|b_i(v)| \leq 1$.

Recursion step. H_i is the union of two edge-disjoint spanning trees. It has $n_i = |V_i| = n - i$ vertices and $2(n_i - 1)$ edges. Every vertex in V_i is incident to at least two edges in H_i. Ideally, H_i has a vertex of degree 2; but if H_i has no vertex of degree 2, then there are at least 4 vertices of degree 3. In each step, prune one vertex of V_i of degree 2 or 3 and recurse until $n_i = 1$. *Pruning* a vertex $u \in V_i$ means removing vertex u; reassigning the incident edges, and the sets $b_i(u)$ and $r_i(u)$. We describe a single prune step below.

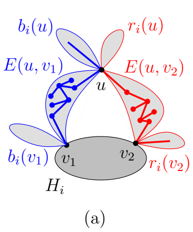


Figure 10: (a) Graph H_i with a vertex u of degree 2. (b) Sets $E(v_1)$ and $E(V_2)$. (c) Sets $E'(v_1)$ and $E'(v_2)$.

Pruning a vertex $u \in V_i$ of degree 2. Refer to Fig. 10. Since H_i is the union of two spanning trees, u is incident to a red and a blue edge. Assume that $uv_1 \in B_i$ and $uv_2 \in R_i$ (possibly $v_1 = v_2$). By construction, $b_i(u)$ is not in conflict with uv_1, and $r_i(u)$ is not in conflict with uv_2. Let $E(v_1) := E(uv_1) \cup b_i(u)$ and $E(v_2) = E(uv_2) \cup r_i(u)$. By construction, $E(v_1)$ and $E(v_2)$ have conflict-free tree representations, with leaves at v_1 and v_2, respectively. Let $e_1 \in E(V_1)$ and $e_2 \in E(v_2)$ be their designated edges incident to v_1 and v_2, respectively. Note that $E(v_1)$ or $E(v_1) \setminus \{e_1\}$ is even and can be placed into $\mathcal{P}$. Let $E'(v_1)$ be the remainder of $E(v_1)$ (with $E'(v_1) = \emptyset$ if $|E(v_1)|$ is even and $E'(v_1) = \{e_1\}$ if $|E(v_1)|$ is odd). Similarly, let $E'(v_2)$ be $\{e_2\}$ or the empty set after moving the remainder of $E(v_1)$ into $\mathcal{P}$. If $b_i(v_1) \cup E'(v_1)$ is odd, then let $b_{i+1}(v_1) = b_i(v_1) \cup E'(v_1)$; otherwise we can put the even set $b_i(v_1) \cup E'(v_1)$ into $\mathcal{P}$ and set $b_{i+1}(v_1) = \emptyset$. Similarly, if $r_i(v_2) \cup E'(v_2)$ is odd, then let $r_{i+1}(v_2) = r_i(v_2) \cup E'(v_2)$; otherwise we can put the even set $r_i(v_2) \cup E'(v_2)$ into $\mathcal{P}$ and let $r_{i+1}(v_2) = \emptyset$. It is clear that the total number of red and blue edges remains even, and H_{i+1} is the edge-disjoint union of two spanning trees.

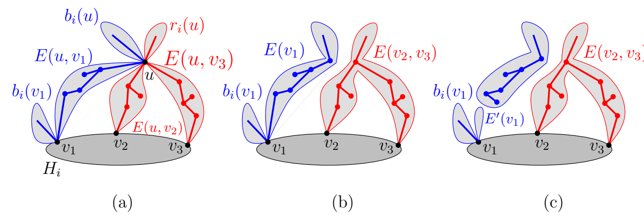


Figure 11: (a) Graph H_i with a vertex u of degree 3. (b) Set $E(v_1)$ and $E(v_2, v_3)$. (c) Set $E'(v_1)$.

Pruning a vertex of $u \in V_i$ of degree 3. Refer to Fig. 11. Since H_i is the union of two spanning trees, u has degree 1 in one of B_i and R_i, and degree 2 in the other. Assume w.l.o.g. that $uv_1 \in B_i$ and $uv_2, uv_3 \in R_i$. We distinguish two cases:

Case 1: no color exchange is necessary. First assume that uv_2 and uv_3 are not in conflict at u. It follows that $E(v_2, v_3) := E(uv_2) \cup E(uv_3) \cup r_i(u)$ has a conflict-free tree representation with leaves at v_2 and v_3; and $E(v_1) := E(uv_1) \cup b_i(u)$ has a conflict-free tree representation with a leaf at v_1. Let e_1 be the designated edge of $E(v_1)$ incident to v_1. Note that $E(v_1)$ or $E(v_1) \setminus \{e_1\}$ is even and can be placed into $\mathcal{P}$. Let $E'(v_1)$ be the remainder of $E(v_1)$ (with $E'(v_1) = \emptyset$ if $|E(v_1)|$ is even and $E'(v_1) = \{e_1\}$ if $|E(v_1)|$ is odd). If $b_i(v_1) \cup E'(v_1)$ is odd, then let $b_{i+1}(v_1) = b_i(v_1) \cup E'(v_1)$; otherwise we can put the even set $b_i(v_1) \cup E'(v_1)$ into $\mathcal{P}$ and set $b_{i+1}(v_1) = \emptyset$.

Case 2: color exchange. Assume now that uv_2 and uv_3 are in conflict at u. The deletion of the red edges uv_2 and uv_3 would break the red spanning tree R_i into three components, which is denoted by $U_1 = \{u\}$, U_2, and U_3 such that $v_2 \in U_2$ and $v_3 \in U_3$. Assume w.l.o.g. that $v_1 \in U_2$. Observe that $B'_i = B_i - uv_1 + uv_2$ and $R'_i = R_i - uv_2 + uv_1$ partition H_i into two edge-disjoint spanning trees, where the two edges of R'_i incident to u are not in conflict anymore. Exchange the colors of uv_1 and uv_2. Since the color of uv_1 changes from blue to red, we need to adjust $r_i(u)$ and $r_i(v_1)$: if $r_i(u)$ or $r_i(v_1)$ is in conflict with uv_1, then recolor their edges blue (merging them into $b_i(u)$ and $b_i(v_1)$, respectively). Similarly, the color of uv_2 changes from red to blue and we need to adjust $b_i(u)$ and $b_i(v_2)$: if $b_i(u)$ or $b_i(v_2)$ is in conflict with uv_2, then recolor their edges red. These color adjustments might change the parity of the total number of red and blue edges. Let $\tilde{E}(uv_1)$ be the union of $E(uv_1)$ and any edge in $r_i(u)$ and $r_i(v_1)$ in conflict with uv_1; and let $\tilde{E}(uv_2)$ be the union of $E(uv_2)$ and any edge in $b_i(u)$ and $b_i(v_2)$ in conflict with uv_2. Exactly the edges in $\tilde{E}(uv_1)$ and $\tilde{E}(uv_2)$ change colors, so the color exchange preserve the parity of the color classes if and only if $|\tilde{E}(uv_1)| + |\tilde{E}(uv_2)|$ is even. If the parity is preserved, one can prune u after the color exchange as described above.

Color exchange at two vertices of degree 3. Assume that H_i has no vertices of degree 2, and no vertex of degree 3 can be pruned (because the color exchange would change the parity of red and blue edges). That is, at every vertex of degree 3, the two same-color edges are in conflict and a color exchange would change the parity of the total number of red and blue edges. Assume first that there are two nonadjacent vertices $u_1, u_2 \in V_i$, each of which has degree 3. Exchange colors at both u_1 and u_2, and prune u_1 and then u_2. Observe that the set of edges that change colors due to the color exchanges at u_1 and at u_2 are disjoint. The number of red and blue edges change parity twice, so the two steps together result in an even number of blue and red edges.

Assume next that any two vertices of degree 3 are adjacent. Since H_i has at least four vertices of degree 3, which must be pairwise adjacent, H_i is isomorphic to K_4. Denote the vertices by u_1, u_2, u_3, and u_4. Each of the two spanning trees, B_i and R_i, is a path of 3 edges. Without loss of generality, we may assume that $B_i = \{u_1u_2, u_2u_3, u_3u_4\}$ and $R_i = \{u_2u_4, u_4u_1, u_1u_3\}$. Each vertex in V_i is incident to two edges of the same color, and one edge of the other color. At each vertex, the two edges of the same color are in conflict (otherwise one could prune a vertex of degree 3). We show that there are two consecutive color exchanges that preserve the parity of red and blue edges.

Since a single color exchange at any vertex would change the parity of red and blue edges, the sets $\tilde{E}(u_1u_2)$ and $\tilde{E}(u_1u_4)$ have *different* parities. Suppose, to the contrary, that two consecutive color exchanges would not preserve the parity of red and blue edges. That is, a first color exchange changes the parity, and any second color exchange would preserve the parity of blue and red edges. After a color exchange at u_1, the pairs $(\tilde{E}(u_2u_1), \tilde{E}(u_2u_3))$, $(\tilde{E}(u_3u_1), \tilde{E}(u_3u_4))$, and $(\tilde{E}(u_4u_2), \tilde{E}(u_4u_3))$ have the *same* parity. After color exchange at u_2, pairs $(\tilde{E}(u_1u_2), \tilde{E}(u_1u_3))$, $(\tilde{E}(u_3u_1), \tilde{E}(u_3u_4))$, and $(\tilde{E}(u_4u_1), \tilde{E}(u_4u_2))$ have the *same* parity. It follows that $\tilde{E}(u_1u_2)$ and $\tilde{E}(u_1u_4))$ have the same parity, contradicting our earlier observation. This proves that there are two consecutive color exchanges that together preserve the parity of red and blue edges. □

Disjoint Compatible Matchings. We can now prove Theorem 2 and settle the Disjoint Compatible Matching Conjecture.

PROOF OF THEOREM 2. Let M be a set of disjoint line segments in the plane, with no three collinear segment endpoints. By Theorem 1, there is a convex subdivision $C \in \mathcal{D}(M)$ with no critical polygons. In the dual graph, the nodes correspond to cells in C, and the edges correspond to segment endpoints. Two edges of the dual graph are *in conflict* if they are adjacent and correspond to the two endpoints of the same segment. By Lemma 1, the dual graph has an even orientation such that if a vertex has indegree 2, then the two incoming edges are not in conflict. Fix such an orientation for the remainder of the proof.

For each cell $c \in C$, assign every incident segment endpoint to c if the corresponding dual edge is oriented into c. In every cell $c \in C$, independently, we construct a perfect matching on the segment endpoints assigned to c. Since the orientation is even, every cell is assigned to an even number of segment endpoints. Furthermore, if a cell is assigned to exactly two segment endpoints, then these endpoints are not connected by a segment in M. By a result of Aichholzer *et al.* [1][Lemma 2], there is a perfect matching on the segment endpoints assigned to each c. The union of these perfect matchings is a perfect matching on all segment endpoints. It is disjoint from M and compatible with M, as required. □

6. REFERENCES

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