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Dynamic Pricing and Risk Analytics under Competition and Stochastic Reference Price Effects

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***Abstract*—This paper investigates the pricing strategy of firms in the context of uncertain demand. In particular, there are two factors that affect demand dynamics, the influence of reference prices and the price of the competition. In the monopoly case, pricing policy is affected by reference-price effects and in the duopoly case both competitive pricing and reference-price effects are present. In each case, the optimal price paths are derived and simulated. The implications of uncertainty are analysed by comparing the deterministic policy with the stochastic policy. The random variation in price paths are investigated to provide a risk analysis for firms that work in such market conditions. With the advent of the big data era, information about consumers and competitors gives firms a greater control over uncertainty than ever before. Simulations will demonstrate that firms can lower the volatility of their price path if they gather and process this information. Furthermore, the feedback form of the optimal price path are derived in both the absence and presence of both competition and reference-price effects. In general, the impact that demand uncertainty has over the firm's pricing strategy is determined by a combination of the firm's discount rate, demand uncertainty and demand-side/cost-side dynamics.**

***Index Terms*—Demand uncertainty, optimal control, reference price, risk analytics, subgame perfect equilibrium.**

I. Introduction

The pricing of products and services is an important but difficult area in marketing as it must take into consideration the fluctuating and dynamical effects of the market. In particular, the difficulty arises because firms are often uncertain about the demand for their product. This paper will introduce a new pricing model that takes into account two stochastic factors that affect demand uncertainty, competitive pricing and reference-price effects. Competitive pricing provides strategies to set the price of a product based on what the competition is charging. The reference price is what consumers anticipate paying or consider reasonable to pay for a particular good or service. It is an internal price that consumers compare with the observed price. [1] They are formed through the consumers past exposures to the product price, what they paid for similar products and price information such as advertisements. The differences between the reference price and the retail price affect the demand for that product. In particular when the reference price is greater than the retail price, consumers will sense a gain which will lead to increased demand. Conversely when the reference price is less than the retail price, consumers will sense a loss which will lead to decreased demand.

Mathematical models are constructed to determine for firms, the optimum pricing strategy in the context of uncertain demand resulting from the effects of reference prices and competitive pricing. For the monopoly case, only the reference-price effects are considered while in the duopoly case, both reference-price and competitive pricing are present. The random disturbances of reference prices and the competitors' price on retail prices are characterized as stochastic differential equations which turns the optimization problem into a stochastic control problem. Price paths in both the stochastic and feedback form are derived and random variation in the price path which gives firms insight on how to lower the volatility of the prices and thereby reduce risk are investigated.

This study will probably be very useful for firms working under dynamic market conditions in the big data era. In the absence of big data, demand uncertainty was characterized by empirical market conditions with incomplete information on consumers and competitors. Now such information has become readily available for industrial and technological firms. This study will demonstrate that big data allows those firms to control and reduce demand uncertainty that until recently was not possible. It will be shown in the model construction in Section III and with numerical simulations in Section IV that firms that gather and process knowledge about the perception of their product and pricing strategies of their competitors can reduce uncertainty which leads to the reduction of the variance of their projected price path.

Firms that need to constantly introduce new and innovative products into the market would be interested in this study. These markets are characterized by demand and cost dynamics. On the demand side, there is the diffusion effect of increasing market penetration resulting from demand-side learning (eg. word-of-mouth) [2] and the saturation effect with increasing market penetration. On the supply side, there are cost-learning effects. Apple's iPhone 6 is now employed as an illustrative example to show these market characteristics. There may be uncertainty about how well the new product will work. Furthermore, regarding reference prices, consumers will compare the price of iPhone 6 to the price of iPhone 5 and debate whether the expected price increase is reasonable. Moreover, regarding competitive pricing, consumers will also compare the price of the iPhone 6 with its competitor the Samsung Galaxy S5.

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The rest of the paper is organized as follows. Section 2 reviews both the deterministic and stochastic pricing models in literature. Section 3 constructs two models in the monopoly and duopoly cases. Section 4 determines the optimal price path, in particular, the stochastic differential equation that describes the optimal price path. The expected slope of the price path in the stochastic case with the price path in the deterministic case are compared. Furthermore, the random variation in the price path is examined to provide a risk analysis for firms that operate in such dynamic conditions. In particular, numerical simulations will demonstrate that when firms synthesize information from the big data era, they can lower the variance of their projected price path. Section 5 determines the optimal pricing policy in feedback form in both the absence and presence of competition and reference-price effects. Section 6 discusses the assumptions made in the study, the most significant contributions to the state of the art and future research directions that result from the present study. Section 7 summarizes and concludes the paper.

II. Literature Review

[3]–[5] have developed deterministic pricing models which assume that there are no factors of uncertainty that drives the evolution of the price. In particular, it assumed that the market demand for the product is known with certainty over the entire planning horizon. This assumption is unrealistic since demand uncertainty is intrinsic in practice [6] as firms often are not certain about the demand for their product. There are many factors that drive demand uncertainty including changes in customer interests [7] and needs [8], uncertainty about the market's acceptance of the product [9], uncertainty about how well the new product will work [10], uncertainty about the response to the marketing campaign and many others. These factors can all lead to either increases in buying relative to expectations or decline in buying relative to forecasts [11].

To overcome this drawback, [12]–[14] generalize the deterministic pricing model and introduce the monopolist stochastic pricing model for a firm facing uncertain demand. These models now take into account the factor of randomness, in particular a single stochastic factor that affects demand uncertainty. The shortcoming is obvious: in practice, there are often multiple stochastic factors that affect uncertainty in demand and not just one [15]. The most common factor that affect all firms in all industries is price competition since there is always price interaction amongst all competing firms. Firms often base and adjust their prices to the prices of their competition [16]. Many other factors of randomness may also be present and moreover, these random factors are often independent of each other. Therefore, there is a pressing need for a pricing model that allows for multiple and independent factors of randomness affecting uncertain demand.

This paper makes an important contribution to the existing state of the art. Theoretically, the study extends the single factor stochastic pricing model in literature to construct the two factor duopolist stochastic pricing model, taking into account competitive pricing and reference-price effects. It reflects the impact that uncertainty has over pricing decisions more accurately than the present literature because in reality, there are often multiple sources of randomness. And moreover, a new numerical approach for solving the value function in the corresponding Hamilton-Jacobi-Bellman equation is presented. Practically, firms should gain a greater awareness of the role that randomness plays in forming their pricing strategies. They can benefit from the methodology in this study to simulate their own price paths given the specific sources of uncertainty in their industry. Furthermore, the study examines the effect that big data has on demand uncertainty. In particular, without big data, uncertainty was determined strictly by empirical conditions. Now just recently, firms have access to information about their consumers and competitors that would allow them to control and mitigate risk when introducing new products into the market.

III. Models

In this section, models are constructed for both the monopoly and duopoly cases. The assumption is made that the domain is in continuous time and on an infinite horizon. Since the concern is with demand uncertainty in the context of competition and reference-price effect, following [8], in the absence of reference-price effects, the demand function is:

$$Q(p) = a - \delta p \quad a, \delta > 0 \tag{1}$$

In the presence of reference-price effects, the demand following [12] is a function of $p(t)$ and $r(t)$ where p is the set price r is the reference-price of the product.

$$Q(p, r) = a - \delta p(t) - \gamma[p(t) - r(t)], \quad a, \delta, \gamma > 0 \tag{2}$$

In the presence of both competition and reference-price effects, the demand function is then:

$$Q(r_1, p_1, p_2) = a - (\gamma+1)p_1(t) - k(\gamma+1)p_2(t) + \gamma r_1(t) \tag{3}$$

$a, \delta, \gamma > 0$ and $k \in (0, 1)$
where p_1 is the price of Firm 1, r_1 is the reference-price of Firm 1's products, p_2 is the price of Firm 2 and a, δ, γ and k are assumed to be constants.

A. Monopoly

In the scenario of one firm, the pricing strategy is only affected by the presence of reference-price effects. Following [1], the change in reference price is modelled by the demand function which is assumed to be linear and parametric of $p(t)$ and $r(t)$ rather than just t [12]. In particular, reference-price effects are linear in $p - r$ and additive such that demand increases when $p < r$ and decreases when $p > r$. When the demand increases, the price increases which will drive up the reference-price and correspondingly for when the demand decreases. Denote $p(t)$ to be the retail price, c to be the cost, $Q(r, p)$ to be the demand function and $r(t)$ to be the reference price:

$$\frac{dr}{dt} = Q(r(t), p(t)) = a - \delta p(t) - \gamma[p(t) - r(t)], \quad a, \delta, \gamma > 0 \tag{4}$$

Now incorporate the effect of uncertainty in demand by subjecting the reference price to stochastic disturbance. Therefore, the reference price can be characterized by the following stochastic differential equation:

$$dr(t) = Q(r(t), p(t))dt + \sigma(r(t))dW(t) \tag{5}$$

where $dr(t) = r(t+dt) - r(t)$ and $dW(t)$ is the increment of a Wiener process W. $\sigma(r(t))$ is the volatility term which represents the variance from reference-price effects. More specifically, the more information firms have about consumer perception, the lower σ will be. In the past, this term was determined strictly by empirical market conditions since firms had little information about how consumers perceived their product. Now during the big data era where this information is accessible, firms can make a conscious effort to learn about their consumers which allows them to control and reduce σ. As will be seen with simulations in Section IV, this leads to a more stable projected price path.

Together, (5) expresses the idea that the rate of change of reference-price effects is a sum of a deterministic component that is a function of price and reference-price plus a random component whose variance is a function of the reference price. Denote $\mathcal{A}$ to be the set of admissible control of prices and the goal is to optimize over $\mathcal{A}$ the functional

$$J(r,p) = E[\int_0^\infty e^{-\theta t} f(r,p)dt] \tag{6}$$

where f is the reward function and $\theta > 0$ is the discount factor. The value function is defined as

$$V := \max_p J(r,p) \tag{7}$$

The goal is to find the optimal price path $p^* = \{p^*(t) : t \geq 0\} \in \mathcal{A}$ that satisfies the infinite-horizon expected discounted reward defined by

$$\max_{p(t)} E[\int_0^\infty e^{-\theta t}[p(t) - c(r(t))]dr(t)] \tag{8}$$

subject to:

$$dr(t) = Q(r(t), p(t))dt + \sigma(r(t))dW(t), \tag{9}$$

where $c(r(t))$ is the unit cost which is a function of references effects to reflect cost learning effects.

Substituting (9) into (8), the objective value function can be written as

$$\begin{aligned} V(r) = \max_{p(t)} \{ & E[\int_0^\infty e^{-\theta t}(p(t) - c(r(t)))Q(r,p)dt] + \\ & E[\int_0^\infty e^{-\theta t}(p(t) - c(r(t)))\sigma(r(t))dw]\} \end{aligned} \tag{10}$$

which equals

$$\max_{p(t)} \{E[\int_0^\infty e^{-\theta t}(p(t) - c(r(t)))Q(r,p)dt]\} \tag{11}$$

since

$$E[\int_0^\infty e^{-\theta t}(p(t) - c(r(t)))\sigma(r(t))dw] = 0$$

This becomes an infinite horizon stochastic control problem for which the HJB equation is given by:

$$\theta V(r) = \max_{p(t)}\{[p(t) - c]Q(r,p) + V_r Q(r,p) + \frac{1}{2}V_{rr}\sigma^2(r)\} \tag{12}$$

To optimize $p(t)$, solve the first order condition

$$V_r Q_p + Q + (p-c)Q_p = 0 \tag{13}$$

which can be arranged to give

$$p^* = c - \frac{Q}{Q_p} - V_r \tag{14}$$

Following the notation from [12] the price elasticity of demand will be denoted as $\eta = -\frac{Q_p p}{Q}$ and $\lambda = V_r$ which can be interpreted as the shadow price which is the instantaneous change in the objective value function of the optimal solution per unit of the constraint. In this situation, it is the marginal valuation of the reference price $r(t)$. Here, a positive shadow price means lowering the current price and sacrificing present profits for future profits while a negative shadow price is the opposite. Then the price can be written as

$$p^* = \frac{\eta}{\eta+1}(c - \lambda) \tag{15}$$

which has the same form as in the deterministic case.

Since c, λ and η are functions of r, the temporal behaviour of p, more specifically $p(r(t))$ will be stochastic since the evolution of the reference price $r(t)$ is stochastic. However since price is a deterministic function of the reference-price effects, the optimal policy in feedback form $p(r)$ is deterministic given r. To obtain an explicit solution for $p(r)$ substitute p from (14) to (12) to obtain

$$\theta V = -\frac{Q^2}{Q_p} + \frac{\sigma^2(V_{rr})}{2} \tag{16}$$

In general, $\frac{Q^2}{Q_p}$ is a function of r and p. To eliminate p from (16), substitute $p = p(r, V_r)$ from (14) to obtain the second-order differential equation

$$\frac{V_{rr}\sigma^2(r)}{2} - F(r, V_r) - \theta V = 0 \tag{17}$$

where

$$F(r, V_r) := \frac{Q^2}{Q_p} \tag{18}$$

This differential equation is solved for V and V_r. The latter is substituted in (14) to eliminate V_r which will give us $p(r)$. It will be clear that in general the solution of V and V_r will contain σ^2 which means that reference price uncertainty does have an impact on the optimal feedback policy.

B. Duopoly

Now the model will be constructed for the case where there are two competing firms where the price of one firm will influence the price of the other firm in dynamic competition. The influence of reference-price effects are still present for both firms. The duopolist pricing model can be interpreted as a form

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of feedback (subgame perfect) Nash equilibrium where firms are allowed to change their strategies in response to the other firm's strategies. This captures the *feedback* reaction of the competition in response to the firm's chosen strategies. This symmetric feedback equilibrium can be found by solving for the value function in the Hamilton-Jacobi-Bellman equation. In particular, the feedback equilibrium strategies satisfy the HJB equations in the infinite horizon setting. The analysis will be carried out from the perspective of Firm 1 however note that both Firm 1 and Firm 2 are simultaneously solving their own revenue optimization problems which are subject to shared constraints [3]. Therefore both firms are playing a symmetric game and playing the same mixed strategy where each firm's strategy depends upon the competitors' strategies. The existence and uniqueness of this feedback equilibrium is shown in [3].

Denote $p_1(t)$ to be the retail price of Firm 1, $p_2(t)$ to be the retail price of Firm 2. Each strategy possess the Markov property of memorylessness which means that each player's mixed strategy can be conditioned only on the state of the game. Together the joint pricing policy $\{p_1, p_2\}$ define the feedback Nash equilibrium and if one firm were to modify its policy away from the equilibrium, then its own pay-off will decrease. Throughout the entire time horizon, there is price interaction between p_1 and p_2. The equilibrium satisfies the backward induction property. Extending (4), the demand function will again be assumed to be linear with the additional term p_2 which takes into account the price of the competition. Denote $r_1(t)$ to be the reference price and $Q_1(r_1, p_1, p_2)$ to be the demand function of Firm 1. For simplification, it will be assumed that $\delta = 1$. Then

$$\begin{aligned}\frac{dr_1}{dt} = Q_1(r_1(t), p_1(t), p_2(t)) = \\ a - (\gamma+1)p_1(t) - k(\gamma+1)p_2(t) + \gamma r_1(t) = \frac{dp_2}{dt}\end{aligned} \tag{19}$$

$k \in (0, 1)$.

The goal is to determine the Markovian equilibrium strategy $p_1(t)$. Incorporate the effect of uncertainty in demand by subjecting the reference-price and the competitor's price to stochastic disturbance. The optimal price path $p_1^*(t) = \{p_1(t) : t \geq 0\}$ will be derived over all admissible price paths that satisfies the infinite-horizon expected discounted reward defined by

$$\max_{p_1(t)} E[\int_0^\infty e^{-\theta t}[p_1(t) - c_1(r_1(t), p_2(t))]dr_1(t)dp_2(t)] \tag{20}$$

subject to:

$$dr_1(t) = Q_1(r(t), p_1(t), p_2(t))dt + \sigma_1(r_1(t))dW(t), \tag{21}$$

$$dp_2(t) = Q_1(r(t), p_1(t), p_2(t))dt + \sigma_2(p_2(t))dW(t), \tag{22}$$

$c(r_1(t), p_2(t))$ is the unit cost which is a function of the reference-price and the competitor's price to reflect cost learning effects. σ_1 and σ_2 are the volatility terms which represent the variance from reference-price effects and price competition respectively. Similarly to the monopoly case, these terms in the past were determined by the specific market conditions and could not be influenced by the firms themselves. This is because information about consumers and competitors were limited. In the big data era, this information has become widely available for the first time. Firms now have the opportunity to control the degree of uncertainty that affect their pricing decisions through learning about consumer perceptions and competing strategies. In particular, firms can limit and reduce the weights of σ_1 and σ_2 which leads to a more stable price path as Section IV will demonstrate.

By the multiplication rules of multivariate stochastic calculus where $dtdt = 0, dtdW = 0$ and $dWdW = dt$:

$$dr_1(t)dp_2(t) = (\sigma_1(r_1(t)) + \sigma_2(p_2(t)))dt \tag{23}$$

Then substituting (23) into (20) the objective value function which coincides with the feedback Nash equilibrium for the two strategies can be written as

$$\begin{aligned}V(r_1, p_2) = \max_{p_1(t)}\{E[\int_0^\infty e^{-\theta t}(p_1(t) - c_1(r_1(t), p_2(t))) \\ (\sigma_1(r_1(t)) + \sigma_2(p_2(t)))dt]\end{aligned} \tag{24}$$

and the HJB equation:

$$\begin{aligned}\theta V(r_1, p_2) = \max_{p_1(t)}\{[p_1(t) - c_1](\sigma_1(r_1) + \sigma_2(p_2)) + \\ V_{rp2}(\sigma_1(r_1) + \sigma_2(p_2))\}\end{aligned} \tag{25}$$

As shown in [3], this set of HJB equations is a sufficient and necessary condition for the feedback equilibrium strategies. It demonstrates the in solving for the value function of the HJB equation, it coincides with the total discounted profit of the firm under the set of equilibrium strategies.

To optimize $p_1(t)$ need to solve the first order condition

$$V_{r1p2}(\sigma_{1p1} + \sigma_{2p1}) + (\sigma_1 + \sigma_2) + (p_1 - c_1)(\sigma_{1p1} + \sigma_{2p1}) = 0 \tag{26}$$

which can be rearranged to give

$$p_1^* = c_1 - \frac{\sigma_1 + \sigma_2}{\sigma_{1p1} + \sigma_{2p1}} - V_{rp2} \tag{27}$$

which following the monopoly case will be rewritten as

$$p^* = \frac{\eta}{\eta+1}(c - \lambda) \tag{28}$$

where $\eta = -\frac{\sigma_{1p1} + \sigma_{2p2}}{\sigma_1 + \sigma_2}$ and $\lambda = V_{rp2}$ which denote price elasticity of demand and shadow price respectively. This will have the same form as in the deterministic competition where $\{p_1, p_2\}$ form an open-loop equilibrium and there are no price interaction. Retailers must commit to their strategy and cannot change it after the planning stage. In dynamic competition, c, λ, η are functions of r_1 and p_2 and the price path $p_1(r_1, p_2)$ will be stochastic since the evolution of r_1 and p_2 are stochastic. Unlike the monopoly price path which can be solved with standard methods in differential equations, the price path in the duopoly case can be found only through approximating the value function of its HJB equation which is detailed in Section 5.

IV. Optimal Price Path Over Time

In this section, the methodology from [12] will be followed to derive the SDE which characterizes the optimal price path. Since the expression itself is very complicated, the equation is broken down into its key components and each component is examined in turn.

A. Stochastic Differential Equation of Price Path in Monopoly

For the monopoly model specified in Section 3, the optimal price path $p^*(t)$ is described by the following SDE:

$$2dp = [K(r) + \sigma^2(T(r) + U(r))]dt + \sigma J(r)dw \qquad (29)$$

where

$$dp = p(t + dt) - p(t), \qquad (30)$$

$$K(r) = -\theta\lambda - 2\frac{QQ_r}{Q_p}, \qquad (31)$$

$$T(r) = \frac{c_{rr}}{2}, \qquad (32)$$

$$U(r) = \frac{\lambda_r \sigma_r}{\sigma}, \qquad (33)$$

$$J(r) = -\frac{Q_r}{Q_p} + c_r - \lambda_r \qquad (34)$$

This result is derived in Appendix A. The optimal price path is a Markov process since $p(t)$ satisfies the stochastic differential equation (29). While the pricing policy is a deterministic function of reference price r, the price path itself represents the uncertain forecast of the evolution of price over time.
The SDE Toolbox from [17] will be used to simulate the price path that is subjected to reference-price effects. The simulation is over a 12 month period and assume that the new product starts with a price of \$10.

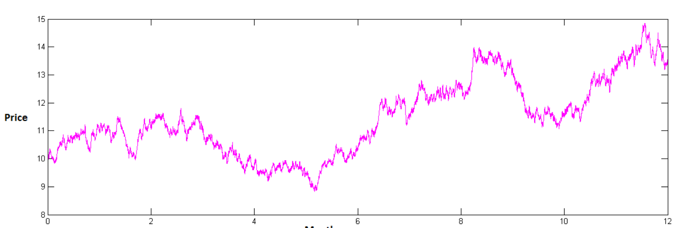

Fig. 1. Simulated price path under reference-price effects

Observe from Figure 1 that the price is initially stable at the beginning. This is because reference-price effects were introduced at month 0 and therefore, there are minimal price learning effects. Afterwards, with more fluctuations in the price path, the impacts of reference-price effects are evident. The amount of market share that the firm possess will determine whether price skimming or price penetration strategies are used. As a result, the trend in the price path will shift between increasing and decreasing.

B. Stochastic Differential Equation of Price Path in Duopoly

For the duopoly model, the optimal price path $p_1^*(t)$ is characterized as:

$$\begin{aligned}(2 - \frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})dp1 = [K(r_1,p_2)+ \\ \sigma_1\sigma_2(S_1(r_1,p_2) + T_1(r_1,p_2) + U_1(r_1,p_2)) \\ \sigma_1^2(S_2(r_1,p_2) + T_2(r_1,p_2) + U_2(r_1,p_2) + \sigma_2^2(S_3(r_1,p_2)+ \\ T_3(r_1,p_2) + U_3(r_1,p_2)))]dt+ \\ [\sigma_1(J_1(r_1,p_2)) + \sigma_2(J_2(r_1,p_2))]dw\end{aligned} \qquad (35)$$

where

$$\begin{aligned}K(r_1,p_2) = c_{r1}Q_1 - \phi_{r1}Q_1 - \lambda_{r1}Q_1 + c_{p2}Q_1 - \phi_{p2}Q_1 - \\ \lambda_{p2}Q_1\end{aligned} \qquad (36)$$

$$\begin{aligned}S_1(r_1,p_2) = -\phi_{r1p2}, S_2(r_1,p_2) = -\frac{1}{2}\phi_{r1r1}, S_3(r_1,p_2) = \\ -\frac{1}{2}\phi_{p2p2}\end{aligned} \qquad (37)$$

$$T_1(r_1,p_2) = c_{r1p2}, T_2(r_1,p_2) = \frac{1}{2}c_{r1r1}, T_3(r_1,p_2) = \frac{1}{2}c_{p2p2} \qquad (38)$$

$$\begin{aligned}U_1(r_1,p_2) = -\lambda_{r1p2}, U_2(r_1,p_2) = -\frac{1}{2}\lambda_{r1r1}, U_3(p_2,p_2) = \\ -\frac{1}{2}\lambda_{p2p2}\end{aligned} \qquad (39)$$

$$J_1(r_1,p_2) = c_{r1} - \phi_{r1} - \lambda_{r1}, J_2(r_1,p_2) = c_{p2} - \phi_{p2} - \lambda_{p2} \qquad (40)$$

This result is derived in Appendix B.
Again, the price paths of both firms subject to competition and reference-price effects over a 12 month period will be simulated. Assume that both firms introduce new products that each start at \$10. Observe from Figure 2 that the price

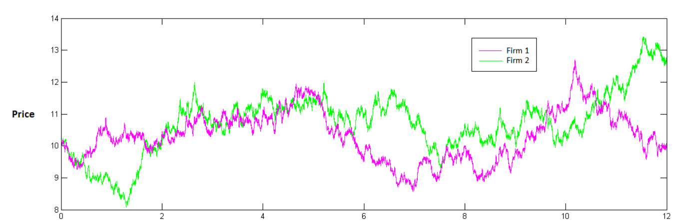

Fig. 2. Simulated price path under competition and reference-price effects

of Firm 1 is stable at the beginning. It can be interpreted from this simulation is that from the beginning Firm 1 is stronger than Firm 2 and in order to compete, Firm 2 must use the penetration pricing strategy of lowering prices to gain market share. Once they gained sufficient market share, Firm 2 used the price skimming strategy and raised their prices above Firm 1. The trend of the price path seems to be determined by the price competition while the fluctuations seems to be determined by the reference price effect.

C. Expected Slope of Price Path in Monopoly

Now the expected slope of the price path will be compared with the slope in the deterministic case by examining the derived SDE (29) and its components. Starting with the monopoly case:
Taking expectations in (29):

$$2E(\frac{dp}{dt}) = K(r) + \sigma^2(T(r) + U(r)) \tag{41}$$

Note that $E[\frac{dp}{dt}]$ has the same sign as $K(r)+\sigma^2(T(r)+U(r))$. In the deterministic case (41) is reduced to

$$2[\frac{dp}{dt}] = K(r)$$

[18]. Therefore the difference between the expected change in price under reference-price effects and the optimal price in the deterministic case is the term:

$$\frac{\sigma^2(T(r) + U(r))}{2}$$

Now examining the terms $K(r), T(r)$ and $U(r)$ in turn:
$K(r)$ is common in both stochastic and deterministic price paths [19]
Looking at the terms $T(r)$ and $U(r)$ which is what is different from the stochastic case to the deterministic one:
$T(r) := \frac{c_{rr}}{2}$ represents the cost-side effect of reference prices on the price slope. Under the influence of learning, it is expected that $c(r)$ is convex [20]. Under reference price effects, cost learning will tend to contribute a positive impact on $E(\frac{dp}{dt})$. This asymmetry in the effect of a random fluctuation in r on the expected cost results in a corresponding asymmetric cost-side effect on the expected price path represented by $\frac{c_{rr}}{2}$.
$U(r) := \frac{\lambda_r \sigma_r}{\sigma}$ represents the effect of reference price dynamics. If σ is constant then $U = 0$ which means $U(r)$ is a factor only if the random error in sales is heteroskedastic. If reference price effects is increasing or decreasing in experience, then $U(r)$ will have the same or different sign as λ_r respectively where λ_r is the marginal impact of reference price effects on the shadow price.

D. Expected Slope of Price Path in Duopoly

The components of the SDE in the duopoly case will be investigated here.
Following the same methodology, taking expectations:

$$(2 - \frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})E(\frac{dp1}{dt}) = [K + \sigma_1\sigma_2(S_1 + T_1 + U_1) + \frac{\sigma_1^2}{2}(S_2 + T_2 + U_2) + \frac{\sigma_2^2}{2}(S_3 + T_3 + U_3)]dt \tag{42}$$

In the deterministic case (42) is reduced to

$$(2 - \frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})(\frac{dp1}{dt}) = K \tag{43}$$

Therefore in the duopoly case, the difference between the expected change in price under reference-price and competition effects and the optimal price in the deterministic case is the term:

$$\frac{\sigma_1\sigma_2(S_1+T_1+U_1)}{(2 - \frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})} + \frac{\sigma_1^2(S_2+T_2+U_2)}{2(2 - \frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})} + \frac{\sigma_2^2(S_3+T_3+U_3)}{2(2 - \frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})} \tag{44}$$

Now the terms $K, S_i's, T_i's$ and $U_i's$ will be analysed.
Similar to the monopoly scenario, K is common in both the deterministic and stochastic cases. More importantly for the analysis are the $S_i's, T_i's$ and $U_i's$ which are the terms which separate the price slope in the deterministic case and the expected price slope under demand uncertainty.
The $S_i's$ represent the demand-side uncertainty on the price slope from the reference-price effects and from the competitor's prices since they are composed of the term $\phi = \frac{\sigma_1+\sigma_2}{(\sigma_1+\sigma_2)_p}$
The $T_i's$ which are composed of $c_{r1p2}, c_{r1r1}, c_{p2p2}$ represent the cost-side effect of demand uncertainty on the price slope.
The $U_i's$ represents the dynamics of effects of reference price and the competitor's price. λ_{r1p2} is the even marginal impact of both reference price and the competitor's price on the shadow price. And the term λ_{r1r1} represent the weight where the influence of reference-price effects is greater than the influence of the competitor's price.

E. Random Variation of Price Path in Monopoly

In the monopoly case, the instantaneous variance of the continuous Markov process $p(t)$ is

$$\frac{\sigma^2 J(r)^2}{4} \tag{45}$$

and also, it is known that $p_r = \frac{J(r)}{2}$. Therefore, it can be seen that the degree of the random variation in the price path about the mean increases with both σ and p_r. The dependence on σ is because uncertainty in demand interacts with the stochastic nature of reference-price effects which would result in the variability of the price path. Furthermore, the amount of variation in the price path that results from random fluctuations from reference-price effects depends upon the marginal impact of reference-price effects on the price. Therefore there is a positive interactive effect of σ and p_r on the random variation in price.
From the perspective of risk analysis, even though the derived price path tells the firm the optimal price to set given the market conditions, a price path that is less volatile is more desired than a price that is highly volatile. Higher volatility carry higher risk and this is undesirable for a firm that just introduced a new product into the market. Also, it is detrimental for marginal firms that cannot afford such risk. In the big data era, if firms takes advantage of the available and accessible market data to study and influence the perception of the product in the consumers' minds *prior* to setting a price then they could reduce σ which leads to a less volatile price path.
The price path of a firm that makes a conscious effort to take advantage of market information from big data to

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study reference-price effects versus one that have not will be simulated. The previous assumptions that assumed a firm will introduce a new product starting at \$10 will be followed. Observe from Figure 3 that if a firm makes a conscious

Fig. 3. Simulated price paths for firms that influence reference-price effects vs. firms that does not influence reference-price effects

choice to study and influence the reference-price effects in their favour prior to setting the price, then there will be less volatility in the price path. This is because σ has decreased which decreases the instantaneous variance of the optimal price path (46).

F. Random Variation of Price Path in Duopoly

In the duopoly case, the instantaneous variance of the multivariate continuous Markov process $p_1(t)$ is

$$[\frac{\sigma_1 J_1(r_1,p_2)+\sigma_2 J_2(r_1,p_2)}{(2-\frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})}]^2 \tag{46}$$

Know that

$$p_{r1} = [\frac{J_1(r_1,p_2)}{(2-\frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})}] \tag{47}$$

and

$$p_{p2} = [\frac{J_2(r_1,p_2)}{(2-\frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}})}] \tag{48}$$

Therefore in the duopoly case, the degree of the random variation in the price path increases with $\sigma_1, \sigma_2, p_{r1}$ and p_{p2}. Clearly, σ_1 and σ_2 will positively affect the variation in the price path because demand uncertainty interacts with the random nature of competition and reference-price effects. Firms that study their competition in addition studying the reference price of their products can reduce σ_1 and σ_2 which will lead to a less volatile price path. These actions were not possible before because firms had limited access to market information. Now with big data, firms could analyse consumer and competitor trends from economic indicators and reduce demand uncertainty and variation from their projected price path.

Also the marginal impact of reference-price and the marginal impact of the competitor's price will determine the extent of variability of the firm's price. Similar to the monopolist case, firms that are either start-ups or introducing new products into the market would want the optimal price path to have low volatility. In general firms want to stabilize and establish their presence in the market market before they can afford to or are willing to take greater risks with high volatility.

In addition to making a conscious effort through marketing campaigns to influence reference-prices like in the monopolist case, in the duopolist case where there are price competition between firms, firms would need to study the competition carefully. For competing firms, depending on the relative strength of each firm, it may be either advantageous or disadvantageous to introduce new products in the same time interval as other firms. Larger firms can afford to take higher risks with higher volatility in prices than smaller firms. The price paths for firms that make the necessary preparations and firms that do not will be simulated in Figure 4.

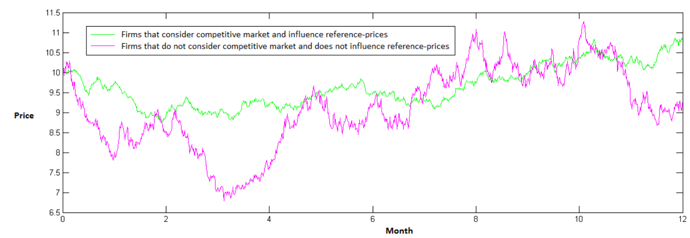

Fig. 4. Simulated price paths for firms that make prior preparation vs. firms that do not

V. Optimal Policy in Feedback Form

In this section the optimal pricing policy in feedback form will be examined. Consider three cases, the first is in the absence of reference price effects, the second is in the context of reference price effects and the third is in the context of both competition and reference price effects. In the case involving the presence of reference-price effects, the procedure for obtaining an explicit solution $p(r)$ involves solving the second order differential equation (17) and then substituting V_r obtained from the solution in (14) to obtain $p(r)$ and $p(r_1,p_2)$ respectively. In the case involving both competition and reference-price effects, the HJB-PDE will be solved using numerical methods. Specific assumptions will be made about the demand function $Q(r,p), c(r)$ and $\sigma(r)$ because non-linearity of the equation makes the possibility of a closed-form solution difficult.

A. Absence of Reference-Price Effects

This is the classical supply-demand problem for a firm which is to find the optimal price $p(t)$ that maximizes its profits $\Pi(p)$ in the absence of reference-price effects. In this situation, the demand function is $Q(p) = a - \delta p \quad a, \delta > 0$

$$\Pi(p) = (p-c)Q \tag{49}$$

With $\Pi(p)' = 0$ gives that the optimal price policy without reference-price effects which is

$$p(t) = \frac{a+\delta c(t)}{2\delta} \tag{50}$$

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B. Presence of Reference-Price Effects

Now consider the stochastic case with reference-price effects. To simplify, assume that $\delta = 1$ Therefore the demand function is:

$$Q(p, r) = a - (\gamma + 1)p(t) + \gamma r(t), \quad a, \gamma > 0 \tag{51}$$

and write this as

$$Q(p, r) = \gamma(N - Gp - r) \tag{52}$$

where $N = \frac{a}{\gamma}$ and $G = \frac{\gamma+1}{\gamma}$. This is the Price-Timing model as specified in [12], [21]. Here γ is market potential at zero price, N is the diffusion coefficient and G is the price sensitivity parameter. Cost learning as well as demand uncertainty that may decrease with experience will be assumed. Thus

$$c(r) = c_0 - c_1 r \tag{53}$$

and

$$\sigma^2(r) = \sigma_0^2(N - r) \tag{54}$$

With these assumptions, the differential equation (17) becomes

$$\frac{\sigma^2(r)V_{rr}}{2} + \frac{\gamma}{4}[\frac{N-r}{G} - c_0 - c_1 r - V_r]^2 - \theta V = 0 \tag{55}$$

From (14):

$$p(r) = \frac{1}{2}[c_0 - c_1 r + \frac{N-r}{G} - V_r] \tag{56}$$

Next, determine the exact solution of the differential equation (55) with the method of undetermined coefficients. [22] To solve using the method of undetermined coefficients, try a solution

$$V(r) = k_0 + k_1(r) + k_2(r^2) \tag{57}$$

and the coefficients k_0, k_1, k_2 will be determined so that (55) is satisfied. Substitute (57) into (55) and then rearrange in terms of the powers of r. Then the coefficients of each power must equal zero which will give us a system of equations that can be solved for k_0, k_1, k_2 and thus giving us $V(r)$. This will give us the desired analytical solution and it can be seen that the optimal policy in feedback form is

$$p(r) = A + Br - C\sigma_0^2 \tag{58}$$

where A, B, C are positive terms that is composed of the various parameters γ, G, N, c_0, c_1. When $A - C\sigma_0^2 < 0$ and $B < 1$ this means that given the market conditions, firms would need to increase their market penetration. In order to attract consumers firms should decrease their prices such that it becomes lower than the current reference price. Conversely when $A - C\sigma_0^2 > 0$ and $B > 1$ it demonstrates that the firm has a firm control over the market and therefore consumers' have a greater tolerance for a higher price than the reference price. In this case firms can raise their prices.
Overall in the context of reference-price effects and under the assumptions that have been made, the optimal price path is increasing and linear in r. If uncertainty is linearly decreasing in r, the effect is that it will increase the optimal price without affecting the slope of the price path.

C. Presence of Competition and Reference-Price Effects

Now consider the stochastic case with both competition and reference price effects. Here, to solve for the feedback form of the price, numerical methods to solve for the value function of the HJB-PDE will be used. Again cost learning and also demand uncertainty that may decrease with experience will be assumed.

$$c(r) = c_0 - c_1 r_1 - c_2 p_2 \tag{59}$$

and

$$\sigma_1(r_1) = \sigma_0(N - r_1) \tag{60}$$

$$\sigma_2(p_2) = \sigma_0(N - p_2) \tag{61}$$

where N is the diffusion coefficient. Under these assumptions, substitute (59), (60) and (61) into (25) and (27) to get the HJB-PDE in the duopoly case:

$$\begin{aligned}\theta V = [p_1(t) - c_0 + c_1(r_1) + c_2(p_2)](\sigma_0(2N - r_1 - p_2)) \\ + V_{rp2}(\sigma_0(2N - r_1 - p_2))\end{aligned} \tag{62}$$

with

$$p_1 = c_0 - c_1(r_1) - c_2(p_2) - \frac{\sigma_0}{\sigma_{0p1}} - V_{r1p2} \tag{63}$$

Analytical solutions of the HJB equation are only known for special cases with simple state equations and cost functional. In this particular case, there is no known analytical solution and therefore a numerical approach to solve (62) will be needed. The penalty method from [23] will be used to approximate the value function V in (62). Since the domain is a continuous infinite horizon, V is a unique viscosity solution. Penalty methods replaces the constrained optimization problem with a series of unconstrained problems whose solutions then converge to the solution problem. For this problem specifically, the algorithm discretises the HJB equation (62) using standard finite differences and then approximates the resulting nonlinear discrete problem with a penalisation technique. The procedure is given by the following flowchart:

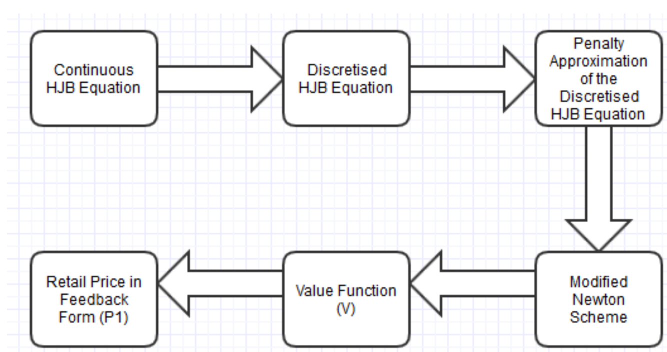

Fig. 5. Penalty method for HJB

Continuous HJB:
Let $\mathbb{S}, |\mathbb{S}| < \infty$ be the finite set of admissible controls and $\Omega \subset \mathbb{R}^n$ be an open set. Then let $L_p, p \in S$ be the

differential operator on Ω. The task is to find the value function $V : \Omega \to \mathbb{R}$ such that

$$\max\{L_p V : p \in S\} = 0 \tag{64}$$

where

$$L_p V := [p - c_0 + c_1 + c_2](\sigma_0(2N - r_1 - p_2)) + V_{rp2}(\sigma_0(2N - r_1 - p_2)) - \theta V \tag{65}$$

The next step is to reduce the problem to obtain a nonlinear discrete approximation to (64)

Discrete HJB:
Now solve (64) numerically backwards in time. Then for every $j \in R$, the goal is to find $V^{j-1} \in \mathbb{R}^n$ such that $V^j \in \mathbb{R}^n$ such that

$$\max\{A_p V^{j-1} - V^j : p \in \mathbb{S}\} = 0 \tag{66}$$

where $V^j \in \mathbb{R}^n$ is the vector known from the previous time step, $A_p \in \mathbb{R}^{n\times n}, p \in \mathbb{S}$ is a matrix with $(b_i)_{i=1}^N$ on the diagonal, $(c_i)_{i=1}^{N-1}$ on the upper diagonal and $(a_i)_{i=2}^N$ on the lower diagonal where the entries are given by

$$a_i = -\frac{1}{2}i^2\sigma^2 + \frac{1}{2}i\theta$$
$$b_i = 1 + i^2\sigma^2 + r$$
$$c_i = -\frac{1}{2}i^2\sigma^2 - \frac{1}{2}i\theta$$

[24]

Now it remains to penalise the discrete problem

Penalise Discrete HJB: The nonlinear discrete equation (66) can be approximated by another nonlinear problem using a penalisation technique and then the penalised formulation can be solved very efficiently with numerical methods.
The penalty system is this: Define

$$P := \rho(A - I) \tag{67}$$

where ρ is the penalty parameter and $I \in \mathbb{R}^n$ is the identity matrix. Then the penalised reformulation is: Find $(V^j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ such that

$$V^{j-1} - V^j + \max\{PV^{j-1}\} = 0 \tag{68}$$

Now numerical methods will be employed to solve the non-linear discrete equation (68)

Iterative Method for Solving Penalised Discrete HJB: An iterative method similar to Newton's method will be used which has finite termination and converges to the solution of (68). Let $x^0 \in \mathbb{R}^n$ be some starting value. Then for some known $x^n, n \geq 0$ find x^{n+1} such that

$$(I + P_n)x^{n+1} = V^j \tag{69}$$

Now the finite-difference schemes from MathPDE package will be used to approximate the solution for (69).
Having estimated the value function for (62), the feedback equilibrium strategies is of the form:

$$p(r_1, p_2) = A_1 + B_1 r_1 + C_1 p_2 - D\sigma_0 \tag{70}$$

where A_1, B_1, C_1, D are positive terms which depend on demand function parameters $N, c_0, c_1, c_2, k, \gamma$. Therefore, they can be estimated empirically. The equilibrium strategies are symmetric because they are the same for all firms. Based on this equation, it is known that both the reference price and the competitor's price has an influence on the firm's price. The values of B_1 and C_1 will determine whether the reference-price or the competitor's price has a greater effect. Likewise in the case of only reference-price effects, if uncertainty is decreasing in r_1 and decreasing in p_2 this will increase the optimal price.

VI. Assumptions/Contributions/Future Research

In this section, the assumptions that were made in constructing the stochastic models will be discussed, followed by a recapitulation of the theoretical and practical contributions of the study and finally, future research directions will be suggested.
In constructing the duopoly stochastic model, the assumption was made that there is an equal impact of randomness from reference price effects and from competitive pricing on the firm's pricing strategy. In practice, this is usually not the case as depending on the firm and the industry that it is in, one factor will usually have a greater influence than the other one. Practitioners should apply the theory by adjusting the stochastic weights based upon each specific situation.
This study has made important contributions both in theory and in practice. In theory, this study has extended the single variable monopoly stochastic model consisting of one factor of randomness to the multivariable duopoly stochastic model comprised of two factors of randomness. Furthermore, this study introduced and applied a novel approach to numerically solve the multivariate stochastic control problem resulting from industrial problems.
In practice, this study has provided businesses a clearer picture about the impacts of uncertainty on a firm's pricing decisions. Firms would benefit from this study by simulating their own optimal price paths, taking into consideration the uncertainties in their specific industries. Over multiple simulations, each business would have a projection of the future forecast of their price path and therefore, make the necessary preparations to mitigate risk. More importantly, because of big data, there are more information about consumer perception and past price changes of the competitors. This has allowed firms to reduce uncertainty and the variance in their price path.
The two-factor stochastic model developed here is an important stepping stone for many areas of future investigation. In general, especially in high-paced industries, there are many other sources of randomness. Future research could follow the methodology of this study to extend the two factor model to multi-factor model. Moreover, for specific markets and industries, the weights of each factor of randomness could be determined beforehand. Furthermore, while this study has provided novel insights into the nature of uncertainty on the demand side, it would be a natural extension to consider factors of supply-side uncertainty.

VII. Conclusion

In this paper, the optimal pricing strategy under demand uncertainty, specified by the influence of competition and reference price effects have been considered. The dynamics of both competition and reference-price effects have been characterized as a stochastic differential equation and the problem of determining the optimal pricing policy becomes stochastic control problem. The expected price path in the stochastic case have been compared with the one in the deterministic case. In the past, the degree of a firm's demand uncertainty was determined by the empirical conditions in that particular market and was beyond the influence for most firms. With big data, information about consumer perception and past price changes of the competitors have become readily available. This has given firms a greater control of uncertainty than ever before. This can be validated by examining the random variations in the price paths. The optimal policy in feedback form have been derived in three cases, the absence of reference-price effects, the presence of reference-price effects and the presence of both competition and reference-price effects. With technology playing an ever more prominent role in many industries, markets that were traditionally static have become more and more uncertain. Far-sighted firms that work in such dynamic market conditions subject to competition and reference-price effects would be prudent to consider and take advantage of information from the big data era to reduce uncertainty for their pricing decisions.

Appendix A
Derivation of Stochastic Differential Equation of Optimal Price in Monopoly Case

The demand function that is considered is $Q(r,p) = a - \delta p - \gamma[p-r], \quad a,\delta,\gamma > 0$
From (14)

$$p = c - \phi - \lambda \tag{71}$$

where $\phi(r,p) = \frac{Q}{Q_p}$ and $\lambda(r) = V_r$. Taking total derivatives:

$$dp = dc(r) - d\phi(r,p) - d\lambda(r) \tag{72}$$

Note that $\phi_p = 1, \phi_r = \frac{Q_r}{Q_p}$ and $\phi_{rr} = 0$
Now expand each term on the right with Ito's Lemma:

$$\begin{aligned} dc(r) = c_r dr + \frac{1}{2}c_{rr}(dr)^2 = c_r(Qdt + \sigma dw) + \\ \frac{1}{2}c_{rr}(Qdt+\sigma dw)^2 = (c_r Q + \frac{\sigma^2 c_{rr}}{2})dt + \sigma c_r dw \end{aligned} \tag{73}$$

$$\begin{aligned} d\phi(r,p) = \phi_p dp + \phi_r dr + \frac{1}{2}(\phi_{rr})(dr)^2 = \\ \phi_p dp + (\phi_r Q + \frac{\sigma^2 \phi_{rr}}{2})dt + \sigma\phi_r dw \end{aligned} \tag{74}$$

$$\begin{aligned} d\lambda(r) = \lambda_r dr + \frac{1}{2}\lambda_{rr}(dr)^2 = (\lambda_r Q + \sigma^2\frac{\lambda_{rr}}{2})dt + \\ \sigma\lambda_r dw \end{aligned} \tag{75}$$

Putting this together:

$$\begin{aligned} 2dp = (c_r Q - \phi_r Q - \lambda_r Q + \frac{\sigma^2 c_{rr}}{2} - \frac{\sigma^2\lambda_{rr}}{2})dt + \\ (c_r - \phi_r - \lambda_r)dw \end{aligned} \tag{76}$$

Then taking the partial derivative of (12) with respect to r will be zero. Therefore:

$$\theta\lambda = (p - c + \lambda)Q_r - c_r Q + \lambda_r Q + \frac{\sigma^2\lambda_{rr}}{2} + \sigma\sigma_r\lambda_r \tag{77}$$

Know that from (14)

$$p - c + \lambda = -\frac{Q}{Q_p} \tag{78}$$

Substituting (78) into (77):

$$\frac{\sigma^2\lambda_{rr}}{2} = \theta\lambda + \frac{QQ_r}{Q_p} + c_r f - \lambda_r Q - \sigma\sigma_r\lambda_r \tag{79}$$

Then substituting (79) into (76):

$$\begin{aligned} 2dp = (-\phi_r Q + \frac{\sigma^2 c_{rr}}{2} - \theta\lambda - \frac{QQ_r}{Q_p} + \sigma\sigma_r\lambda_r)dt - \\ (c_r - \phi_r - \lambda_r)dw \end{aligned} \tag{80}$$

and this is the desired stochastic differential equation for the optimal price in the monopoly case.

Appendix B
Derivation of Stochastic Differential Equation of Optimal Price in Duopoly Case

From (27):

$$p_1 = c - \phi - \lambda \tag{81}$$

where $\phi(r_1,p_1,p_2) = \frac{\sigma_1+\sigma_2}{(\sigma_1+\sigma_2)_{p1}} \implies \phi_{p1} = 1 - \frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_2)_{p1p1}}{(\sigma_1+\sigma_2)^2_{p1}}$ and $\lambda(r,p_2) = V_{rp2}$. Taking total derivatives:

$$dp_1 = dc(r_1,p_2) - d\phi(r_1,p_1,p_2) - d\lambda(r_1,p_2) \tag{82}$$

Now expand each term on the right with the two-dimensional Ito's Lemma:

$$\begin{aligned} dc(r_1,p_2) &= c_{r1}dr1 + c_{p2}dp_2 + c_{r1p2}dr1dp2 \\ &+ \frac{1}{2}c_{r1r1}(dr_1)^2 + \frac{1}{2}c_{p2p2}(dp2)^2 \\ &= c_{r1}(Q_1 dt + \sigma_1 dw)p_2(Q_1 dt + \sigma_2 dw) \\ &+ c_{r1p2}(Q_1 dt + \sigma_1 dw)(Q_1 dt + \sigma_2 dw) \\ &+ \frac{1}{2}c_{r1r1}(Q_1 dt + \sigma_1 dw)^2 + \frac{1}{2}c_{p2p2}(Q_1 dt + \sigma_2 dw) \\ &= (c_{r1}Q_1 + c_{p2}Q_1 + c_{r1p2}\sigma_1\sigma_2 + \frac{1}{2}c_{r1r1}\sigma_1^2 + \frac{1}{2}c_{p2p2}\sigma_2^2)dt \\ &+ (c_{r1}\sigma_1 + c_{p2}\sigma_2)dw \end{aligned} \tag{83}$$

$$\begin{aligned} d\phi(r_1,p_1,p_2) &= \phi_{p1}dp1 + \phi_{r1}dr1 + \phi_{p2}dp2 \\ &+ \phi_{r1p2}dr1dp2 + \frac{1}{2}\phi_{r1r1}(dr1)^2 + \frac{1}{2}(\phi_{p2p2})(dp2)^2 \\ &= \phi_{p1}dp1 + (\phi_{r1}Q_1 + \phi_{p2}Q_1 + \phi_{r1p2}\sigma_1\sigma_2 + \frac{1}{2}\phi_{r1r1}\sigma_1^2 \\ &+ \frac{1}{2}c_{p2p2}\sigma_2^2)dt + (\phi_{r1}\sigma_1 + \phi_{p2}\sigma_2)dw \end{aligned} \tag{84}$$

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$$d\lambda(r_1,p_2) = \lambda_{r1}dr1 + \lambda_{p2} + \lambda_{r1p2}dr1dp2 + \frac{1}{2}\lambda_{r1r1}(dr1)^2 + \frac{1}{2}\lambda_{p2p2}(dp2)^2 = (\lambda_r Q + \sigma^2\frac{\lambda_{rr}}{2})dt + \sigma\lambda_r dw = (\lambda_{r1}Q_1 + \lambda_{p2}Q_1 + \lambda_{r1p2}\sigma_1\sigma_2 + \frac{1}{2}\lambda_{r1r1}\sigma_1^2 + \frac{1}{2}c_{p2p2}\sigma_2^2)dt + (\lambda_{r1}\sigma_1 + \lambda_{p2}\sigma_2)dw \tag{85}$$

Putting this together:

$$(2-\phi_{p1})dp1 = (c_{r1}Q_1 - \phi_{r1}Q_1 - \lambda_{r1}Q_1 + c_{p2}Q_1 - \phi_{p2}Q_1 - \lambda_{p2}Q_1 + c_{r1p2}\sigma_1\sigma_2 - \phi_{r1p2}\sigma_1\sigma_2 - \lambda_{r1p2}\sigma_1\sigma_2 + \frac{1}{2}c_{r1r1}\sigma_1^2 - \frac{1}{2}\phi_{r1r1}\sigma_1^2 - \frac{1}{2}\lambda_{r1r1}\sigma_1^2 + \frac{1}{2}c_{p2p2}\sigma_2^2 - \frac{1}{2}\phi_{p2p2}\sigma_2^2 - \frac{1}{2}\lambda_{p2p2}\sigma_2^2)dt + (c_{r1}\sigma_1 - \phi_{r1}\sigma_1 - \lambda_{r1}\sigma_1 + c_{p2}\sigma_2 - \phi_{p2}\sigma_2 - \lambda_{p2}\sigma_2^2)dw \tag{86}$$

and this is the desired stochastic differential equation for the optimal price in the duopoly case.

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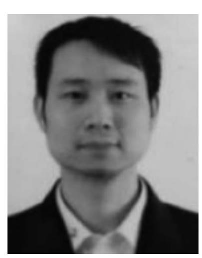

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