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PDE methods for pricing barrier options☆

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Abstract

This paper presents an implicit method for solving PDE models of contingent claims prices with general algebraic constraints on the solution. Examples of constraints include barriers and early exercise features. In this unified framework, barrier options with or without American-style features can be handled in the same way. Either continuously or discretely monitored barriers can be accommodated, as can time-varying barriers. The underlying asset may pay out either a constant dividend yield or a discrete dollar dividend. The use of the implicit method leads to convergence in fewer time steps compared to explicit schemes. This paper also discusses extending the basic methodology to the valuation of two asset barrier options and the incorporation of automatic time stepping. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Barrier options; Numerical methods; Option pricing; Automatic time stepping

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1. Introduction

The market for barrier options has been expanding rapidly. By one estimate, it has doubled in size every year since 1992 (Hsu, 1997, p. 27). Indeed, as Carr (1995) observes, 'standard barrier options are now so ubiquitous that it is difficult to think of them as exotic' (p. 174). There has also been impressive growth in the variety of barrier options available. An incomplete list of examples would include double barrier options, options with curved barriers, rainbow barriers (also called outside barriers, for these contracts the barrier is defined with respect to a second asset), partial barriers (where monitoring of the barrier begins only after an initial protection period), roll up and roll down options (standard options with two barriers: when the first barrier is crossed the option's strike price is changed and it becomes a knock-out option with respect to the second barrier), and capped options. Barrier-type options also arise in the context of default risk (see for example Merton (1974), Boyle and Lee (1994), Ericsson and Reneby (1998), and Rich (1996) among many others).

The academic literature on the pricing of barrier options dates back at least to Merton (1973), who presented a closed-form solution for the price of a continuously monitored down-and-out European call. More recently, both Rich (1991) and Rubinstein and Reiner (1991) provide pricing formulas for a variety of standard European barrier options (i.e. calls or puts which are either up-and-in, up-and-out, down-and-in, or down-and-out). More exotic variants such as partial barrier options and rainbow barrier options have been explored by Heynen and Kat (1994a,b, 1996) and Carr (1995). Expressions for the values of various types of double barrier options with (possibly) curved barriers are provided by Kunitomo and Ikeda (1992), Geman and Yor (1996), and Kolkiewicz (1997). Broadie and Detemple (1995) examine the pricing of capped options (of both European and American style). Quasi-analytic expressions for American options with a continuously monitored single barrier are presented by Gao et al. (2000).

This is undeniably an impressive array of analytical results, but at the same time it must be emphasized that these results generally have been obtained in a setting which suffers from one or more of the following potential drawbacks. First, it is almost always assumed that the underlying asset price follows geometric Brownian motion, but there is some reason to suspect that this assumption may be undesirable.[1] Second, in most cases barrier monitoring is

[1] Boyle and Tian (1997) examine the pricing of barrier and lookback options using numerical methods when the underlying asset follows the CEV process and report significant pricing deviations from the lognormal model, after controlling for differences in volatility. They conclude that the issue of model specification is much more important in the case of path-dependent options than it is for standard options.

assumed to be continuous, but in practice it is often discrete (e.g. daily or weekly). As noted by Cheuk and Vorst (1996) among others, this can lead to significant pricing errors.[2] Third, any dividend payments made by the underlying asset are usually assumed to be continuous. While this may be reasonable in the case of foreign exchange options, it is less justifiable for individual stocks or even stock indices (see for example Harvey and Whaley, 1992). Fourth, in most cases it is not possible to value American-style securities. Fifth, if barriers change over time, they are assumed to do so as an exponential function of time. Aside from analytical convenience, there would not seem to be any compelling reason to impose this restriction. Finally, it should be noted that the availability of a closed-form solution does not necessarily mean that it is easy to compute. For example, the expression obtained by Heynen and Kat (1996) for the value of a discrete partial barrier option requires high-dimensional numerical integration.

Factors such as these have led several authors to examine numerical methods for pricing barrier options. For the most part, the methods considered have been some form of binomial or trinomial tree.[3] Boyle and Lau (1994) and Reimer and Sandmann (1995) each investigate the application of the standard binomial model to barrier options. The basic conclusion emerging from these studies is that convergence can be very poor unless the number of time steps is chosen in such a way as to ensure that a barrier lies on a horizontal layer of nodes in the tree. This condition can be hard to satisfy in any reasonable number of time steps if the initial stock price is close to the barrier or if the barrier is time varying.

Ritchken (1995) notes that trinomial trees have a distinct advantage over binomial trees in that 'the stock price partition and the time partition are decoupled' (p. 19).[4] This allows increased flexibility in terms of ensuring that tree nodes line up with barriers, permitting valuation of a variety of barrier contracts including some double barrier options, options with curved barriers, and rainbow barrier options. However, Ritchken's method may still require very large numbers of time steps if the initial stock price is close to a barrier.

Cheuk and Vorst (1996) modify Ritchken's approach by incorporating a time-dependent shift in the trinomial tree, thus alleviating the problems arising

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[2] Broadie et al. (1997, 1999) provide an accurate approximation of discretely monitored barrier option values using continuous formulas with an appropriately shifted barrier. This approach works in the case of a single barrier when the underlying asset distribution is lognormal.

[3] One exception is provided by Andersen (1996), who explores the use of Monte Carlo simulation methods.

[4] It should be emphasized that this statement is true only up to a point. Trinomial trees are a form of explicit finite difference method and as such are subject to a well-known stability condition which requires that the size of a time step be sufficiently small relative to the stock price grid spacing.

with nearby barriers. They apply their model to a variety of contracts (e.g. discrete and continuously monitored down-and-outs, rainbow barriers, simple time-varying double barriers). However, even though there is considerable improvement over Ritchken's method for the case of a barrier lying close to the initial stock price, this algorithm can still require a fairly large number of time steps.

Boyle and Tian (1998) consider an explicit finite difference approach. They finesse the issue of aligning grid points with barriers by constructing a grid which lies right on the barrier and, if necessary, interpolating to find the option value corresponding to the initial stock price.

Figlewski and Gao (1999) illustrate the application of an 'adaptive mesh' technique to the case of barrier options. This is another tree in the trinomial forest. The basic idea is to use a fine mesh (i.e. narrower stock grid spacing and, because this is an explicit type method, smaller time step) in regions where it is required (e.g. close to a barrier) and to graft the computed results from this onto a coarser mesh which is used in other regions. This is an interesting approach and would appear to be both quite efficient and flexible, though in their paper Figlewski and Gao only examine the relatively simple case of a down-and-out European call option with a flat, continuously monitored barrier. It also should be pointed out that restrictions are needed to make sure that points on the coarse and fine grids line up. The general rule is that halving the stock price grid spacing entails increasing the number of time steps by a factor of four.

Each of these tree approaches may be viewed as some type of explicit finite difference method for solving a parabolic partial differential equation (PDE). In contrast, we propose to use an implicit method which has superior convergence (when the barrier is close to the region of interest) and stability properties as well as offering additional flexibility in terms of constructing the spatial grid. The method also allows us to place grid points either near or exactly on barriers. In particular, we present an implicit method which can be used for PDE models with general algebraic constraints on the solution. Examples of constraints can include early exercise features as well as barriers. In this unified framework, barrier options with or without American constraints can be handled in the same way. Either continuously or discretely monitored barriers can be accommodated, as can time-varying barriers. The underlying asset may pay out either a constant dividend yield or a discrete dollar dividend. Note also that for an implicit method, the effects of an instantaneous change to boundary conditions (i.e. the application of a barrier) are felt immediately across the entire solution, whereas this is only true for grid points near the barrier for an explicit method. In other words, with an explicit method it will take several time steps for the effects of the constraint to propagate throughout the computational domain. Our proposed implicit method can achieve superior accuracy in fewer time steps.

The outline of the paper is as follows. Section 2 presents a detailed discussion of our methodology, including issues such as discretization and alternative means of imposing constraints. Section 3 provides illustrative results for a variety of cases. Extensions to the methodology are presented in Section 4. Section 5 concludes with a short summary.

2. Methodology

For expositional simplicity, we focus on the standard lognormal Black–Scholes setting. We adopt the following notation: t is the current time, T is the time of expiration, V denotes the value of the derivative security under consideration, S is the price of the underlying asset, σ is its volatility, and r is the continuously compounded risk-free interest rate. The Black–Scholes PDE can be written as

$$\frac{\partial V}{\partial t^*} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV, \tag{1}$$

where $t^* = T - t$. We employ a discretization strategy which is commonly used in certain fields of numerical analysis such as computational fluid dynamics, though it appears to be virtually unknown in the finance literature. This is called a point-distributed finite volume scheme. For background details, the reader is referred to Roache (1972). Our reasons for choosing this approach are twofold: (i) it is notationally simple (non-uniformly spaced grids can easily be described); and (ii) it is readily adaptable to more complicated settings. In this setup the discrete version of Eq. (1) has the form

$$\begin{aligned}\frac{V_i^{n+1} - V_i^n}{\Delta t^*} &= \theta F_{i-1/2}^{n+1}(V_{i-1}^{n+1}, V_i^{n+1}) - \theta F_{i+1/2}^{n+1}(V_i^{n+1}, V_{i+1}^{n+1}) + \theta f_i^{n+1}(V_i^{n+1}) \\ &\quad + (1-\theta)F_{i-1/2}^n(V_{i-1}^n, V_i^n) - (1-\theta)F_{i+1/2}^n(V_i^n, V_{i+1}^n) \\ &\quad + (1-\theta)f_i^n(V_i^n),\end{aligned} \tag{2}$$

where V_i^{n+1} is the value at node i at time step $n+1$, Δt^* is the time step size, $F_{i-1/2}$ and $F_{i+1/2}$ are what is known in numerical analysis as flux terms, f_i is called a source/sink term and θ is a temporal weighting factor.

For Eq. (1) the flux and source/sink terms at time level $n+1$ are defined as

$$F_{i-1/2}^{n+1}(V_{i-1}^{n+1}, V_i^{n+1}) = \frac{1}{\Delta S_i}\left[-\left(\frac{1}{2}\sigma^2 S_i^2\right)\frac{(V_i^{n+1} - V_{i-1}^{n+1})}{\Delta S_{i-1/2}} - (rS_i)V_{i-1/2}^{n+1}\right], \tag{3}$$

$$F_{i+1/2}^{n+1}(V_i^{n+1}, V_{i+1}^{n+1}) = \frac{1}{\Delta S_i}\left[-\left(\frac{1}{2}\sigma^2 S_i^2\right)\frac{(V_{i+1}^{n+1} - V_i^{n+1})}{\Delta S_{i+1/2}} - (rS_i)V_{i+1/2}^{n+1}\right], \tag{4}$$

where $\Delta S_i = \frac{1}{2}(S_{i+1} - S_{i-1})$, $\Delta S_{i+1/2} = S_{i+1} - S_i$, and

$$f_i^{n+1}(V_i^{n+1}) = (-r)V_i^{n+1}. \tag{5}$$

Corresponding definitions apply at time level n. Note that flux functions (3) and (4) allow for non-uniform asset price grid spacing. This permits us to construct grids that have a fine spacing near the barriers and a coarse spacing away from the barriers.

To gain some intuition, think of the discrete grid as containing a number of cells. At the center of each cell i lies a particular value of the stock price, S_i. The change in value within cell i over a small time interval arises from three sources: (i) the net flow into cell i from cell $i-1$; (ii) the net flow into cell i from cell $i+1$; and (iii) the change in value over the time interval due to discounting. In Eq. (2), the flux term $F_{i-1/2}$ captures the flow into cell i across the cell interface lying half-way between S_i and S_{i-1}. Similarly, the flux term $F_{i+1/2}$ captures the flow into cell $i+1$ from cell i across the interface midway between S_i and S_{i+1}. The change in value due to discounting is represented by the source/sink term f_i. The temporal weighting factor θ determines the type of scheme being used: fully implicit when $\theta = 1$, Crank–Nicolson when $\theta = \frac{1}{2}$, and fully explicit when $\theta = 0$.

The remaining terms to define in (3) and (4) are $V^{n+1}_{i-1/2}$ and $V^{n+1}_{i+1/2}$. These terms arise from the $(rS\,\partial V/\partial S)$ term in the PDE. Generally, the Black–Scholes PDE can be solved accurately by treating this term using central weighting and we use this approach throughout this study.[5] In this case

$$V^{n+1}_{i+1/2} = \frac{V^{n+1}_{i+1} + V^{n+1}_{i}}{2} \tag{6}$$

in Eq. (4). Furthermore, it is easy to verify that with central weighting the discretization given by Eq. (2) in the special case of a uniformly spaced grid is formally identical to the standard type of discretization described in finance texts such as Hull (1993, Section 14.7).[6]

Some interesting issues arise with respect to choice of the temporal weighting parameter θ. It is well known that explicit methods may be unstable if the time step size is not sufficiently small relative to the stock grid spacing. On the other hand, both fully implicit and Crank–Nicolson methods are unconditionally stable. Both fully explicit and fully implicit methods are first-order accurate in time, whereas a Crank–Nicolson approach is second-order accurate in time.

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[5] In more complex situations, more sophisticated methods may be required. For example, Zvan et al. (1998a) demonstrate the use of one such alternative known as a flux limiter in the context of Asian options. Such methods may also be required if the interest rate is very high relative to the volatility. See Zvan et al. (1998a) and Zvan et al. (1998) for further discussion.

[6] To be completely precise, there is a small difference in that it is traditional in finance to evaluate the rV term in the PDE at time level $n+1$ independent of θ. This permits the interpretation of an explicit method as a trinomial tree where valuation is done recursively using 'risk-neutral probabilities' and discounting at the risk-free rate.

This seems to suggest that a Crank–Nicolson method might be the best choice, but it turns out that this is not correct in the case of barrier options. The reason is that applying a barrier can induce a discontinuity in the solution. A Crank–Nicolson method may then be prone to produce large and spurious numerical oscillations and very poor estimates of both option values and sensitivities (i.e. 'the Greeks').

Zvan et al. (1998a) have shown that in order to prevent the formation of spurious oscillations in the numerical solution, the following two conditions must be satisfied:

$$\Delta S_{i-1/2} < \frac{\sigma^2 S_i}{r} \tag{7}$$

and

$$\frac{1}{(1-\theta)\Delta t^*} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-1/2}\Delta S_i} + \frac{1}{\Delta S_{i+1/2}\Delta S_i}\right) + r. \tag{8}$$

Condition (7) is easily satisfied for most realistic parameter values for σ and r away from $S_i = 0$. However, it may not be possible to satisfy condition (7) for small S_i, but this does not present a problem in practice because the first-order spatial derivative term in the discretized PDE vanishes as S_i tends to zero. See Zvan et al. (1998a) for further discussion.[7] Condition (8) is trivially satisfied when the scheme is fully implicit ($\theta = 1$). For a fully explicit or a Crank–Nicolson scheme, condition (8) restricts the time step size as a function of the stock grid spacing. It is easily verified that the conditions which prevent oscillations in the fully explicit case are exactly the same as the commonly cited sufficient conditions which ensure that it is stable. Furthermore, even though a Crank–Nicolson approach is unconditionally stable, it can permit the development of spurious oscillations unless the time step size is no more than twice that required for a fully explicit method to be stable. Although a fully implicit scheme is only first-order accurate in time, it is our experience that the Black–Scholes PDE can be solved accurately using such a scheme. Hence, we chose to use a fully implicit method. This is advantageous because in order to obtain sufficiently accurate values for barrier options, the grid spacing near the barrier generally needs to be fine. Thus, if a Crank–Nicolson method or a fully explicit scheme were used, the time step size would need to be prohibitively small in order to satisfy condition (8).

The appropriate strategy for imposing an algebraic constraint on the solution depends on the nature of the constraint. If the constraint is of a discrete nature

[7] Restrictions on the time step size similar to condition (8) for the pure heat equation can be found in Smith (1985).

(i.e. it holds at a point in time, not over an interval of time), such as a discretely monitored barrier, then it can be applied directly in an explicit manner. In other words, we compute the solution for a particular time level, apply the constraint if necessary, and move on to the next time level.[8] Consider the example of a discretely monitored down-and-out option with no rebate. If time level $n+1$ corresponds to a monitoring date, we first compute V^{n+1} and then apply the constraint

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1})H, \\ V_i^{n+1} & \text{otherwise}, \end{cases} \tag{9}$$

where H is the initial level of the barrier, h is a positive function which allows the barrier to move over time, and α^{n+1} is an arbitrary parameter. Note that for constant barriers h is always equal to one. Similarly, for a discretely monitored double knock-out option we compute V^{n+1} and if necessary apply the constraint

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1})H_{\text{lower}} \text{ or } S_i \geq h(t^{n+1}, \beta^{n+1})H_{\text{upper}}, \\ V_i^{n+1} & \text{otherwise}, \end{cases} \tag{10}$$

where H_{lower} (H_{upper}) is the initial level of the lower (upper) barrier, and β^{n+1} is an arbitrary parameter.

If the constraint under consideration is not of a discrete nature, then it may be better to use an alternative strategy which imposes the constraint in an implicit fully coupled manner. This ensures that the constraint holds over a time interval (from time level n to $n+1$), not only at one point in time. Suppose for example that we want to value some kind of barrier call option with continuous early exercise opportunities. The constraint is $V_i^{n+1} \geq \max(S_i - K, 0)$, where K is the exercise price of the option. Zvan et al. (1998a) demonstrate how to impose this in an implicit fully coupled manner. Instead of solving the discrete system given by (2) we solve (for a call) the following differential algebraic equation (DAE) (Brenan et al., 1996):

$$\frac{\Phi_i^{n+1} - V_i^n}{\Delta t^*} = F_{i-1/2}^{n+1}(V_{i-1}^{n+1}, V_i^{n+1}) - F_{i+1/2}^{n+1}(V_i^{n+1}, V_{i+1}^{n+1}) + f_i^{n+1}(V_i^{n+1}),$$

$$V_i^{n+1} = \max(\Phi_i^{n+1}, S_i - K, 0). \tag{11}$$

This non-linear system is solved by using Newton iteration. In Eq. (11), Φ_i^{n+1} can be interpreted as the unconstrained value. American options can also be

[8] Note that this is exactly the way that the early exercise feature for American options has been traditionally handled in finance applications.

formulated as linear complementarity (Jaillet et al., 1990; Wilmott et al., 1993) or linear programming problems (Dempster and Hutton, 1997). Dempster and Hutton (1997) use projected successive over-relaxation (PSOR) and dual simplex methods to solve American option pricing problems formulated as linear complementarity and linear programming problems, respectively.

Similar to American options, options with continuously applied barriers can be valued using (11) where for down-and-out barrier options (with Φ_i^{n+1} given by (11))

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1})H, \\ \Phi_i^{n+1} & \text{otherwise.} \end{cases} \tag{12}$$

American barrier options where the barriers are applied continuously can be valued by incorporating the early-exercise feature into constraint (12) as follows (Φ_i^{n+1} as in (11)):

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1})H, \\ \max(\Phi_i^{n+1}, S_i - K, 0) & \text{otherwise.} \end{cases} \tag{13}$$

Of course, continuously applied constant barriers can be modelled easily by using the appropriate Dirichlet conditions. We include such cases in our framework for the purpose of generality. The importance of evaluating a constraint implicitly or explicitly appears to depend on the constraint itself. Zvan et al. (1998a) report very little difference either way in computed values for standard American put options. However, as will be shown below, there can be a significant advantage to using the implicit fully coupled approach in the case of barrier constraints.

Situations where the underlying asset features a known dividend yield can be dealt with in the usual way. The case of a known discrete dollar dividend is slightly more complex. We assume that the holder of the option contract is not protected against dividend payouts. Continuity of the option value across the ex-dividend date leads to a jump condition, as in Ingersoll (1987, p. 366). In particular, for a European option we impose $V(S_i, t^*_+) = V(S_i - D, t^*_-)$ with linear interpolation, where D is the discrete dividend, and t^*_+ and t^*_- are times just before and after the ex-dividend date, respectively. For an American call option, the possibility of early exercise to capture the dividend leads to the condition $V(S_i, t^*_+) = \max(V(S_i - D, t^*_-), S_i - K)$. An analogous condition applies in the case of an American put (see Geske and Shastri, 1985, p. 210).

3. Results

This section presents a set of illustrative results. We focus on knock-out options with zero rebate in order to maximize comparability with existing

Table 1
Down-and-out call values when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $K = 100$ and $S = 100$. C & V denotes results for barriers applied daily and weekly obtained by Cheuk and Vorst (1996). Dividend denotes European option values where the underlying asset pays a discrete dividend of \$2 at $T - t = 0.25$. American denotes values for options that are continuously early-exercisable where the underlying asset does not pay dividends. Execution times (in seconds) are in parentheses

Barrier application	European		Dividend	American	American with dividend
	PDE	Analytic/ C & V			
Continuously	0.164 (0.05)	0.165	0.141 (0.05)	0.164 (0.05)	0.144 (0.05)
Daily	1.506 (13.96)	1.512	1.309 (13.96)	1.506 (15.34)	1.316 (15.39)
Weekly	2.997 (2.80)	2.963	2.599 (2.81)	2.997 (3.08)	2.599 (3.08)

published results. European knock-in option values may be calculated either directly or by using the fact that the sum of a knock-out option and the corresponding knock-in option generates a standard European option, at least when the rebate is zero. If the rebate is non-zero, or if the knock-in option is American style, the methods described by Reimer and Sandmann (1995) in the binomial context may be applied.

For the examples in this section, a year was considered to consist of 250 days and a week consisted of five days. Thus, discrete daily and weekly barrier applications occurred in time increments of 0.004 and 0.02, respectively. The results were obtained using a fully implicit scheme ($\theta = 1$ in Eq. (2)) unless noted otherwise. All runs were carried out on a 233 MHz Pentium-II PC. A detailed description of the grids used in the computations is provided in the appendix.

Results for European down-and-out call options where the barrier is applied continuously and discretely are contained in Table 1. The results are for cases where the barrier is close to the point of interest ($H = 99.9$ and $S = 100$). Although the continuous application of the constant barrier effectively establishes a boundary condition at the same point throughout the life of the option, discretization (11) and constraint (12) were used to obtain the numerical solution for this case in order to maintain generality.

The results in Table 1 were obtained using non-uniform grids. A grid spacing of $\Delta S = 0.1$ near the barrier and $\Delta t^* = 0.05$ were used when the barrier was continuously applied. For the barrier applied daily $\Delta t^* = 0.0005$ and $\Delta S = 0.01$ near the barrier. A grid spacing of $\Delta S = 0.01$ near the barrier was used for the barrier applied weekly with $\Delta t^* = 0.0025$. The PDE results in Table 1 can be considered accurate to within \$0.01, since reduction of ΔS and Δt^* changed the solution by less than \$0.005. Table 1 indicates that the PDE method converges to a slightly higher value than obtained by Cheuk and Vorst (1996) for options where the barrier is applied weekly. This may be due to a difference in the times

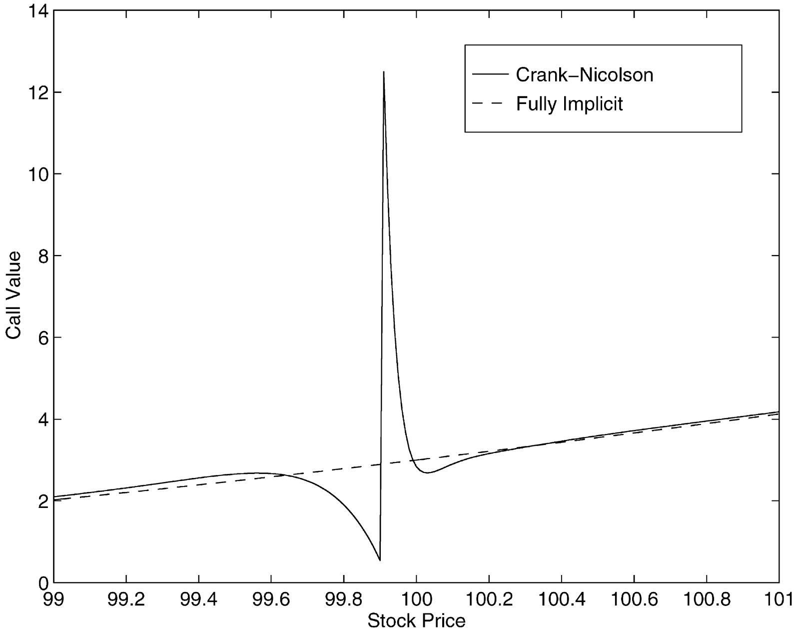

Fig. 1. European down-and-out call option with a constant barrier applied weekly calculated using Crank–Nicolson and fully implicit schemes when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H = 99.9$ and $K = 100$. A non-uniform spatial grid with $\Delta S = 0.01$ near the barrier was used and $\Delta t^* = 0.0025$.

at which the barrier is applied, because of a discrepancy in the definition of a weekly time interval.

As noted by Cheuk and Vorst (1996), Table 1 illustrates that there is a considerable difference between continuous monitoring and discrete monitoring, even with daily monitoring. It is clearly inappropriate in some circumstances to use continuous models in the case of discrete barriers.

Fig. 1 demonstrates the oscillatory solution obtained using the Crank–Nicolson ($\theta = \frac{1}{2}$ in Eq. (2)) method to value a European down-and-out call where the barrier is applied weekly. The grid spacing and time step size are identical to that used to obtain accurate solutions with a fully implicit scheme. The oscillations result because condition (8) was not satisfied. In order to satisfy condition (8) in the region of the barrier when a Crank–Nicolson scheme is used, the time step size must be less than 5.00×10^{-7}. This time step size is several orders of magnitude smaller than the time step size of $\Delta t^* = 2.50 \times 10^{-3}$ needed to obtain accurate results using a fully implicit scheme. Note that if a fully explicit scheme were used the stable time step size is less than 2.50×10^{-7}.

We also point out that oscillations are a potential problem with the Cheuk and Vorst (1996) algorithm, at least in some circumstances. As noted by Cheuk

Table 2
Double knock-out call values with continuously and discretely applied constant barriers when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{\text{lower}} = 95$, $H_{\text{upper}} = 125$, $K = 100$ and $S = 100$. C & V denotes results obtained by Cheuk and Vorst (1996). Dividend denotes European option values where the underlying asset pays a discrete dividend of \$2 at $T - t = 0.25$. American denotes values for options that are continuously early-exercisable where the underlying asset does not pay dividends. Execution times (in seconds) are in parentheses

Barrier application	European		Dividend	American	American with dividend
	PDE	C & V			
Continuously	2.037 (0.48)	2.033	1.915 (0.46)	5.462 (0.50)	4.794 (0.49)
Daily	2.485 (37.93)	2.482	2.325 (38.31)	5.949 (42.80)	5.201 (42.83)
Weekly	3.012 (9.47)	2.989	2.795 (9.48)	6.444 (10.70)	5.610 (10.71)

and Vorst, if the time step size is too large, then their tree probabilities can be negative. In such cases, their algorithm is not guaranteed to prevent oscillations.

We next consider double knock-out call options. Table 2 contains results for cases where the barriers are applied continuously and discretely. In Table 2, the results for the continuously applied barriers were obtained using a uniform spacing of $\Delta S = 0.5$ with $\Delta t^* = 0.0025$. The results for the discretely applied barriers were obtained using a non-uniform grid spacing of $\Delta S = 0.01$ near the barriers. The time step size was $\Delta t^* = 0.00025$ and $\Delta t^* = 0.001$ for barriers applied daily and weekly, respectively. Reduction of ΔS and Δt^* changed the European PDE results in Table 2 by less than \$0.005.

In Table 2 we also include results for cases where the underlying asset pays a discrete dividend of \$2 at $T - t = 0.25$ and where the option is American. The table shows that the early exercise premia for the American cases are very large. Note that (at least in the continuously monitored case) this is due to the presence of the upper barrier – by Proposition 5(c) of Reimer and Sandmann (1995), a continuously monitored American down-and-out call on a non-dividend-paying stock will not be optimally exercised early if the barrier is lower than the strike price.

Table 3 demonstrates convergence for down-and-out and double knock-out options when the barriers are applied weekly. The grid refinements were achieved by successively halving the grid spacing over the entire domain. For double knock-out options, Table 3 includes results for both non-uniform and uniform grids. The results demonstrate that having a fine grid spacing far from the barriers does not improve the accuracy of the solutions. Using a uniform grid increased the execution time by almost a factor of five.

Fig. 2 is a plot of the oscillations that result when the Crank–Nicolson method is used for a double knock-out barrier with the same grid spacing and time step size as was used to produce sufficiently accurate results with a fully

Table 3
Successive grid refinements demonstrating convergence for down-and-out and double knock-out call options with barriers applied weekly when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $K = 100$, and $S = 100$. For the down-and-out case, $H = 99.9$. For the double knock-out case, $H_{\text{lower}} = 95$ and $H_{\text{upper}} = 125$. ΔS denotes the grid spacing near the barrier. The grids used were non-uniform unless noted otherwise. Execution times (in seconds) are in parentheses

	$\Delta S = 0.1$ $\Delta t^* = 0.01$	$\Delta S = 0.05$ $\Delta t^* = 0.005$	$\Delta S = 0.025$ $\Delta t^* = 0.0025$	$\Delta S = 0.0125$ $\Delta t^* = 0.00125$	$\Delta S = 0.00625$ $\Delta t^* = 0.000625$
Down-and-out	2.934 (0.15)	2.972 (0.59)	2.991 (2.31)	3.000 (9.25)	3.004 (37.47)
Double knock-out Uniform	3.071 (1.09)	3.040 (3.81)	3.024 (15.74)	3.015 (65.42)	3.011 (267.36)
Double knock-out	3.070 (0.21)	3.040 (0.84)	3.023 (3.33)	3.015 (13.42)	3.011 (55.64)
American double knock-out	6.486 (0.28)	6.464 (1.06)	6.453 (4.14)	6.446 (16.67)	6.443 (69.93)

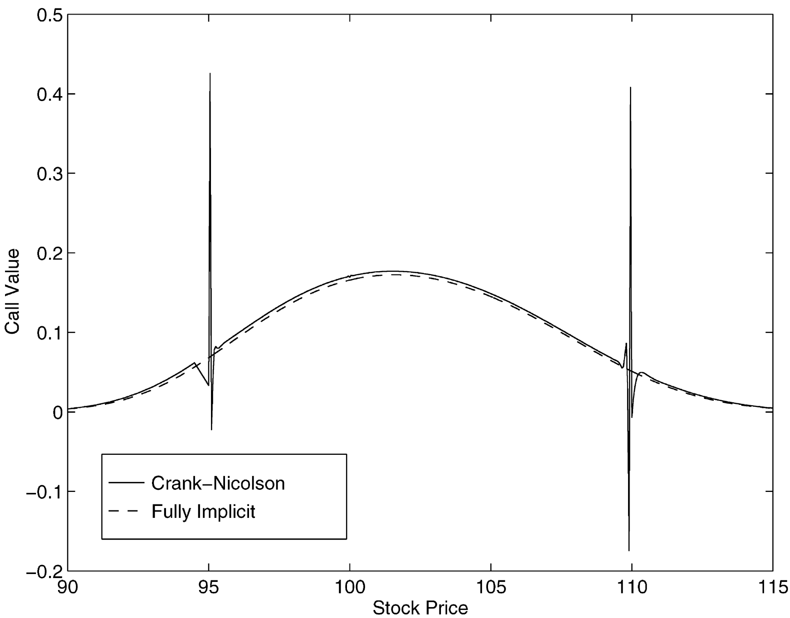

Fig. 2. European double knock-out call option with a constant barrier applied weekly calculated using Crank–Nicolson and fully implicit schemes when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{\text{lower}} = 95$, $H_{\text{upper}} = 110$ and $K = 100$. A non-uniform spatial grid with $\Delta S = 0.05$ near the barrier was used and $\Delta t^* = 0.002$.

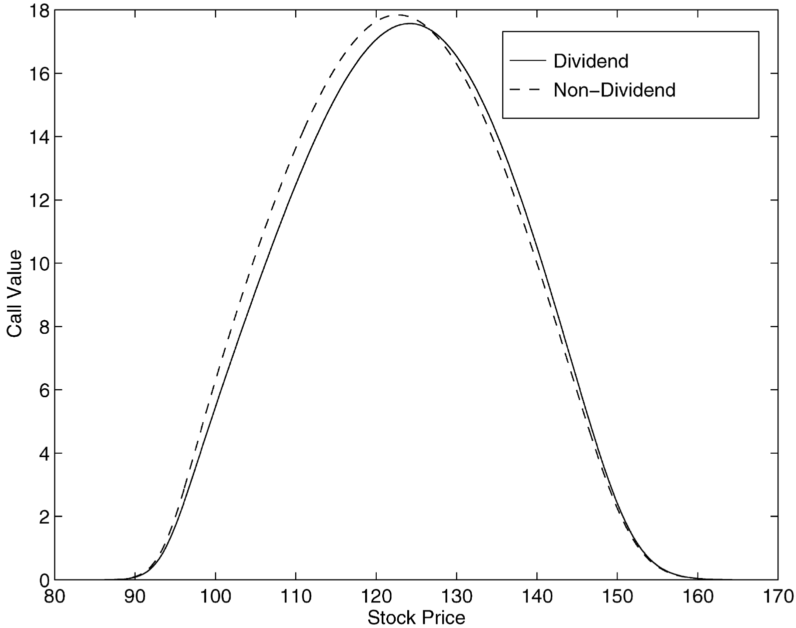

Fig. 3. European double knock-out call options with a constant barrier applied weekly where the underlying asset does not pay a dividend and where the underlying asset pays a discrete dividend (no dividend protection) of \$2 at $T-t=0.25$, when $r=0.10$, $\sigma=0.2$, $T-t=0.5$, $H_{\text{lower}}=95$, $H_{\text{upper}}=150$ and $K=100$. A non-uniform spatial grid with $\Delta S=0.05$ near the barrier was used and $\Delta t^*=0.002$.

implicit scheme. Again, condition (8) was violated, which resulted in severe oscillations near the barriers for the Crank–Nicolson method. Fig. 3 is a plot of a European double knock-out option where the underlying asset pays a discrete dividend of \$2 at $T-t=0.25$, and the barriers are applied weekly. Note that the dividend case produces lower values than the non-dividend case, unless the stock price is relatively close to the upper barrier. This reflects the reduced probability of crossing the upper barrier due to the dividend. A plot of an American double knock-out option where the barriers are applied daily is contained in Fig. 4. Clearly, discrete monitoring has a large impact. With continuous monitoring, the option would be worthless for all stock price values less than \$95 or above \$110. The positive value in the region below \$95 is due to the probability of the stock climbing back above the boundary before the next day.

It is interesting to note that to obtain accurate solutions for the double knock-out options with continuously applied barriers considered here, only a relatively large grid spacing of $\Delta S=0.5$ was needed. This is due to the fact that the continuous application of the constant barriers effectively establishes

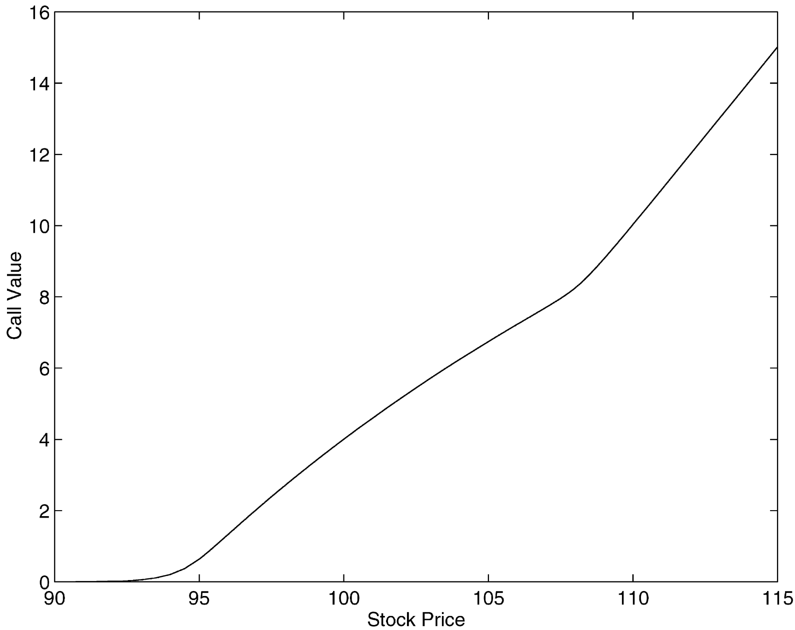

Fig. 4. American (continuously early-exercisable) double knock-out call option with a constant barrier applied daily when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{\text{lower}} = 95$, $H_{\text{upper}} = 110$, and $K = 100$. A non-uniform spatial grid with $\Delta S = 0.05$ near the barrier was used and $\Delta t^* = 0.002$.

boundary conditions at the same points throughout the life of the option. Whereas a fine grid spacing is needed near the barriers when they are applied discretely in order to resolve the discontinuities formed by the discrete application. An analogous situation exists for down-and-out options with continuous barriers considered here. However, a finer grid spacing near the barrier was used for such options because the barrier was close to the region of interest.

Although the grids for the examples with constant barriers considered here were constructed such that a node fell directly on the barrier, we found that it was not actually necessary to do so if the grid spacing was fine. However, if a large grid spacing was being used, then it was necessary to place a node right on the barrier or substantial pricing errors could result.

Table 4 contains results for European double knock-out options with time-varying continuous and weekly barriers where $h(t^{n+1}, \alpha^{n+1}) = \mathrm{e}^{\alpha^{n+1} t^{n+1}}$ and $h(t^{n+1}, \beta^{n+1}) = \mathrm{e}^{\beta^{n+1} t^{n+1}}$. For inward moving barriers $\alpha^{n+1} = 0.1$ and $\beta^{n+1} = -0.1$. For outward moving barriers $\alpha^{n+1} = -0.1$ and $\beta^{n+1} = 0.1$. Note that discretely applied time-varying barriers can be viewed as step barriers.

A grid spacing with $\Delta S = 0.5$ near the barriers and $\Delta t^* = 0.001$ was chosen in order to obtain option values that differed by no more than 0.01% of the

Table 4
European double knock-out call values for continuously and discretely applied time-varying barriers when $r = 0.05$, $T - t = 0.25$, $H_{\text{lower}} = 800$, $H_{\text{upper}} = 1200$, $K = 1000$ and $S = 1000$. A non-uniform spatial grid with $\Delta S = 0.5$ near the barriers was used and $\Delta t^* = 0.001$. K & I denotes results obtained by Kunitomo and Ikeda (1992). The execution time is in seconds

σ	Continuous				Weekly	
	Outward		Inward		Outward	Inward
	PDE	K & I	PDE	K & I		
0.20	35.146	35.13	24.717	24.67	38.486	29.836
0.30	24.986	24.94	14.111	14.02	32.982	21.646
0.40	14.878	14.81	7.241	7.17	24.383	14.799
Exec. time	3.51	N/A	3.51	N/A	2.51	2.51

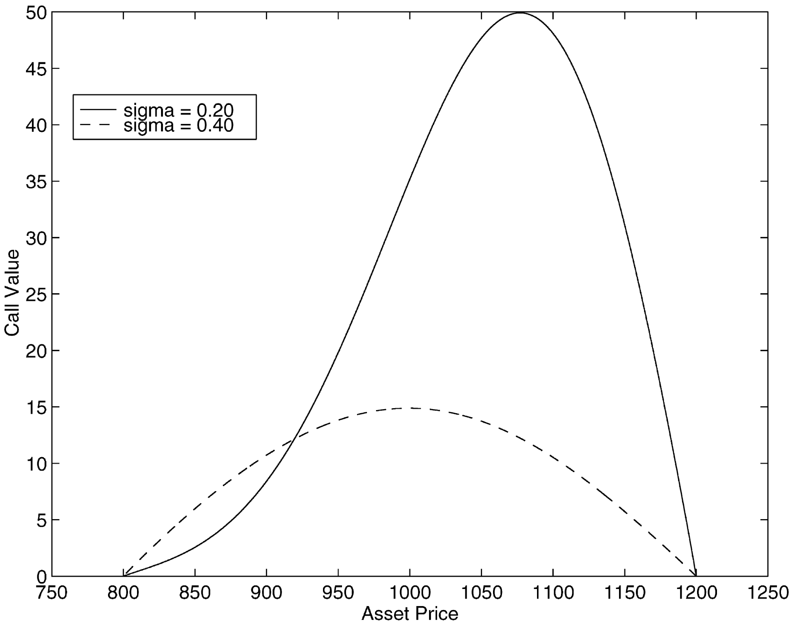

Fig. 5. European double knock-out call options when $\sigma = 0.20$ and $\sigma = 0.40$, $r = 0.05$, $T - t = 0.25$, $H_{\text{lower}} = 800$, $H_{\text{upper}} = 1200$ and $K = 1000$. The barriers are outward moving and continuously applied. A uniform spatial grid with $\Delta S = 0.5$ was used and $\Delta t^* = 0.001$.

exercise price from the results obtained by Kunitomo and Ikeda (1992) for the case of continuously applied barriers. The large impact of discrete monitoring is once again readily apparent, particularly for higher values of σ.

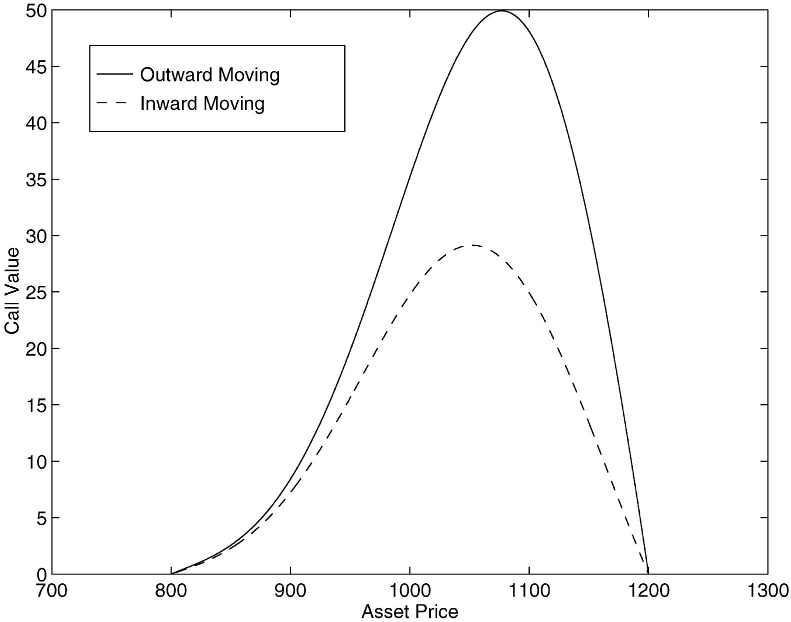

Fig. 6. European double knock-out call options with outward and inward moving continuously applied barriers when $r = 0.05$, $\sigma = 0.20$, $T - t = 0.25$, $H_{\text{lower}} = 800$, $H_{\text{upper}} = 1200$ and $K = 1000$. A uniform spatial grid with $\Delta S = 0.5$ was used and $\Delta t^* = 0.001$.

Fig. 5 is a plot of European double knock-out options with differing volatilities where the barriers are outward moving and continuously applied. Note that the option value may or may not be increasing with volatility. The intuition for this is that higher volatility implies an increased probability of a relatively high payoff but also a greater chance of crossing a barrier. A plot of European double knock-out options with inward and outward moving barriers is contained in Fig. 6. As we would expect, shrinking the distance between the barriers causes a large drop in the initial option value, especially for stock price values midway between the barriers.

The convergence of the method for pricing time-varying barrier options is demonstrated in Table 5. Table 6 contains option values where the barriers are applied in an implicit fully coupled manner or explicitly. The table demonstrates that the implicit fully coupled application of the constraint for barrier options leads to more rapid convergence. The accuracy of the solution obtained using an explicit application of the barriers is very poor. Table 7 demonstrates that in order for the explicit method to achieve comparable accuracy to the implicit method, substantially more CPU time is required due to the need for smaller time steps.

Table 5
Successive grid refinements demonstrating convergence for European double knock-out calls with continuously applied time-varying barriers when $r = 0.05$, $\sigma = 0.4$, $T - t = 0.25$, $H_{\text{lower}} = 800$, $H_{\text{upper}} = 1200$, $K = 1000$ and $S = 1000$. K & I denotes results obtained by Kunitomo and Ikeda (1992). The execution time is in seconds

Barrier movement	$\Delta S = 0.5$ $\Delta t^* = 0.001$	$\Delta S = 0.25$ $\Delta t^* = 0.0005$	$\Delta S = 0.125$ $\Delta t^* = 0.00025$	$\Delta S = 0.0625$ $\Delta t^* = 0.000125$	$\Delta S = 0.03125$ $\Delta t^* = 0000625$	K & I
Outward	14.878	14.844	14.825	14.816	14.812	14.81
Inward	7.241	7.206	7.188	7.177	7.172	7.17
Exec. time	3.51	13.55	54.36	220.08	928.90	

Table 6
Explicit and implicit application of continuously applied time-varying barriers for European double knock-out calls when $r = 0.05$, $T - t = 0.25$, $H_{\text{lower}} = 800$, $H_{\text{upper}} = 1200$, $K = 1000$ and $S = 1000$. A non-uniform spatial grid with $\Delta S = 0.5$ near the barriers was used and $\Delta t^* = 0.001$. K & I denotes results obtained by Kunitomo and Ikeda (1992). The execution time is in seconds

σ	Outward			Inward		
	Explicit	Implicit	K & I	Explicit	Implicit	K & I
0.20	34.233	35.146	35.13	25.955	24.717	24.67
0.30	25.100	24.986	24.94	15.816	14.111	14.02
0.40	15.738	14.878	14.81	8.833	7.241	7.17
Exec. time	2.56	3.51	N/A	2.56	3.51	N/A

Table 7
Explicit application of continuously applied outward moving barriers for European double knock-out calls when $r = 0.05$, $\sigma = 0.40$, $T - t = 0.25$, $H_{\text{lower}} = 800$, $H_{\text{upper}} = 1200$, $K = 1000$ and $S = 1000$. A non-uniform spatial grid with $\Delta S = 0.5$ near the barriers was used

Δt^*	Result	Exec. time (s)
0.00000500	14.961	510.63
0.00000250	14.920	1021.70
0.00000125	14.892	2048.43

4. Extensions

4.1. Automatic time stepping

Although the results in Section 3 were obtained using constant time stepping, automatic time stepping procedures can be employed. Automatic time stepping

Table 8
Double knock-out call values with continuously applied constant barriers computed using automatic time stepping when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{\text{lower}} = 95$, $H_{\text{upper}} = 125$, $K = 100$ and $S = 100$. ε denotes the specified global time truncation error. The initial time step used was 0.001. Execution times (in seconds) are in parentheses

ε				Converged solution
0.10	0.04	0.02	0.01	
2.070 (0.21)	2.048 (0.48)	2.040 (0.94)	2.035 (1.89)	2.037 (0.48)

routines will select time steps so that the global time truncation error will be less than or equal to a specified target error. Thus, the trial and error involved in selecting a constant time step that achieves a desired level of accuracy is removed.

One such automatic time stepping procedure for fully implicit schemes solving linear equations is

$$\Delta t^{n+1} = 2\varepsilon \Big/ \sqrt{\left\| \frac{\partial^2 \boldsymbol{V}^n}{\partial (t^n)^2} \right\|_\infty}, \tag{14}$$

where ε is the target global time truncation error and $\boldsymbol{V}^n$ is the solution vector at time step n (Lindberg, 1977; Sammon and Rubin, 1986). In Eq. (14), $\partial^2 \boldsymbol{V}^n / \partial (t^n)^2$ is approximated by

$$\frac{\partial^2 \boldsymbol{V}^n}{\partial (t^n)^2} = \frac{1}{\Delta t^n} \left[\frac{\partial \boldsymbol{V}^n}{\partial (t^n)} - \frac{\partial \boldsymbol{V}^{n-1}}{\partial (t^{n-1})} \right],$$

where

$$\frac{\partial \boldsymbol{V}^n}{\partial (t^n)} = -\boldsymbol{J}^{-1} \frac{\partial \gamma^n}{\partial (t^n)},$$

$\boldsymbol{J}$ is the Jacobian of the discrete system (2) and $\partial \gamma^n / \partial (t^n) = -(\boldsymbol{V}^n - \boldsymbol{V}^{n-1})/(\Delta t^n)^2$ (see Mehra et al., 1982). Note that if the fluxes are dependent on t^n, as is the case when the barriers are time varying, then the flux functions and source terms should be included in the calculation of $\partial \gamma^n / \partial (t^n)$.

When using (14), a time step size must be specified for the initial two time steps and the two time steps immediately following the application of a barrier (since $\partial^2 \boldsymbol{V}^n / \partial (t^n)^2$ is meaningless at such points in time). In practice a small time step size is specified for the first two steps. The time step selector will then increase the time step significantly if appropriate.

Table 8 contains results obtained using (14) for a double knock-out call with continuously applied constant barriers. In Table 8 *Converged solution* refers to

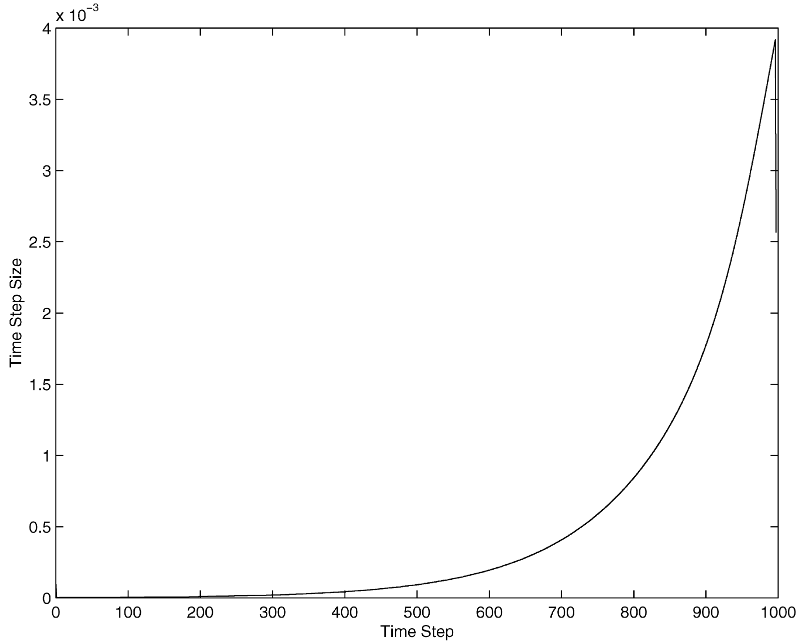

Fig. 7. Time steps using automatic time stepping for a double knock-out call option with constant barriers applied continuously when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{\text{lower}} = 95$, $H_{\text{upper}} = 125$ and $K = 100$. The initial time step size was 0.0001 and $\varepsilon = 0.01$. Note that the final time step was cut in order not to exceed $T - t = 0.5$.

the converged option value obtained using a constant time step size (see Table 2) where the solution is accurate to within \$0.01. The spatial grid used for the results obtained with automatic time stepping was the same as that used for obtaining the converged option value. Hence, errors in the results are solely due to time truncation. Table 8 demonstrates that the actual errors are generally less than the specified global time truncation errors (ε). Thus, the method produces time step sizes that are conservative, which is consistent with Lindberg (1977) since ε in (14) is an upper bound for the error. Fig. 7 is a plot of time steps chosen by the automatic time stepping procedure.

Although the computation of $\partial V^n/\partial(t^n)$ requires an additional matrix solve, this does not introduce a great amount of additional overhead since the Jacobian has already been constructed. In fact, computational savings can be gained when the time step size can grow to be sufficiently large. In Table 8 the time required to achieve a level of accuracy of $\varepsilon = 0.01$ is almost a factor of four greater than the time required for a constant time step of $\Delta t^* = 0.0025$ when $T - t = 0.5$. Note that the constant time step was chosen empirically using many trial runs. Table 9 demonstrates how automatic time

Table 9
Double knock-out call option values computed using constant and automatic time stepping when $r = 0.10$, $\sigma = 0.2$, $K = 100$ and $S = 100$. The barriers are continuously applied with $H_{\text{lower}} = 95$ and $H_{\text{upper}} = 125$. The initial time step used for automatic time stepping was 0.001. Execution times (in seconds) are in parentheses

Maturity $(T - t)$	Constant $\Delta t^* = 0.0025$	Automatic $\varepsilon = 0.01$
1.0	0.520 (0.94)	0.522 (2.08)
1.5	0.129 (1.41)	0.132 (2.19)
2.0	0.032 (1.88)	0.034 (2.25)
2.5	0.008 (2.33)	0.009 (2.26)

stepping can produce lower execution times for options with longer maturities since the time step grows to become quite large. In particular, Table 9 shows that increasing the maturity of the option by a factor of 2.5 only results in less than a 10% increase in execution time. Thus, automatic time stepping will be useful in reducing execution times for options with long maturities.

Automatic time stepping procedures can also be used to obtain the desired level of time truncation error when options with discretely applied barriers are being priced. However, there will be a greater increase, compared to continuously applied barriers, in execution time over that of the required constant time step when pricing options with discretely applied barriers. This is due to the fact that if the barriers are applied frequently, the time step size cannot become very large. It will also be more difficult to obtain computational savings for such problems. Consequently, automatic time step selectors will be of benefit with respect to efficiency for options with long maturities and continuously applied barriers. For discretely applied barriers a tuned (optimal) constant time step will generally be more efficient than using a time step selector. However, the optimal time step must be determined by trial and error.

4.2. Two asset barrier options

The purpose of this section is to demonstrate that the above methods can be successfully adapted to pricing barrier options written on two assets. The only difference is that it is advantageous to use a different spatial discretization method in this circumstance. In particular, we exploit the superior grid flexibility and computational efficiency offered by a finite element approach. As it is not our main focus here, to save space we will not describe the finite element discretization techniques used in detail, referring the reader instead to the two factor discretization described in Zvan et al. (1998b). If the two assets, S_1

and S_2, evolve according to

$$dS_1 = \mu_1 S_1 \, dt + \sigma_1 S_1 \, dz_1,$$

$$dS_2 = \mu_2 S_2 \, dt + \sigma_2 S_2 \, dz_2$$

where z_1 and z_2 are Wiener processes with correlation parameter ρ, then the option value, $V = V(S_1, S_2, t^*)$, is given by

$$\frac{\partial V}{\partial t^*} = \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV. \tag{15}$$

We will consider pricing call options where the barriers are defined as

$$V(S_1, S_2, t^*_{\text{app}}) = \begin{cases} V(S_1, S_2, t^*_{\text{app}}) & \text{if } 90 \le S_1,\ S_2 \le 120, \\ 0 & \text{otherwise,} \end{cases}$$

where t^*_{app} is the application date of the barriers.[9] Note that the barriers do not need to be rectangular. Irregular barriers can be implemented by using an unstructured grid finite element method. The payoff for the sample problems is based on the *worst of* the two assets:

$$V(S_1, S_2, 0) = \max(\min(S_1, S_2) - K, 0).$$

For the examples considered here $T - t = 0.25$, $r = 0.05$ and $K = 100$. The barriers are applied weekly (in time increments of 0.02) and at maturity. As mentioned earlier, the use of Crank–Nicolson time weighting would result in large oscillations, so a fully implicit method was used. Table 10 contains results computed using coarse and fine grids when $\rho = -0.50$, $\sigma_1 = 0.40$ and $\sigma_2 = 0.20$. Table 11 contains option values when $\rho = 0.50$, $\sigma_1 = 0.25$ and $\sigma_2 = 0.25$. Figs. 8 and 9 show the contours of constant value for the options. A comparison of the figures reveals how the option value is affected by the volatility and correlation parameters.

5. Conclusions

We have described an implicit PDE approach to the pricing of barrier options and illustrated its application to a variety of different types of these contracts.

[9] Note that our methodology allows us to apply barriers to either asset or both assets. Previous research by Ritchken (1995) and Cheuk and Vorst (1996) examines two-dimensional problems but where barriers are only applied to one of the assets.

Table 10
Two asset barrier call option values when $\rho = -0.50$, $\sigma_1 = 0.40$ and $\sigma_2 = 0.20$. The execution time is in seconds

S_1	S_2	Grid	
		89×89	179×179
90	90	0.002	0.003
90	100	0.062	0.062
90	110	0.163	0.163
90	120	0.087	0.086
100	90	0.019	0.020
100	100	0.261	0.258
100	110	0.507	0.495
100	120	0.146	0.143
110	90	0.037	0.038
110	100	0.332	0.327
110	110	0.480	0.464
110	120	0.086	0.083
120	90	0.029	0.030
120	100	0.161	0.160
120	110	0.171	0.167
120	120	0.016	0.015
Exec. time		45.80	338.20

Table 11
Two asset barrier call option values when $\rho = 0.50$, $\sigma_1 = 0.25$ and $\sigma_2 = 0.25$. The execution time is in seconds

S_1	S_2	Grid	
		89×89	179×179
90	90	0.099	0.102
90	100	0.215	0.219
90	110	0.115	0.117
90	120	0.012	0.012
100	90	0.215	0.219
100	100	0.946	0.942
100	110	0.821	0.811
100	120	0.159	0.159
110	90	0.115	0.117
110	100	0.821	0.811
110	110	1.063	1.037
110	120	0.308	0.305
120	90	0.012	0.012
120	100	0.159	0.159
120	110	0.308	0.305
120	120	0.165	0.162
Exec. time		46.67	336.40

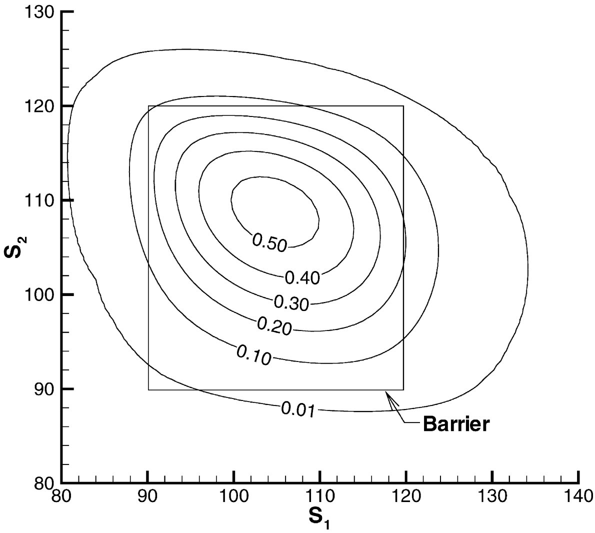

Fig. 8. Two asset barrier call option on the worst of two assets where $\sigma_1 = 0.40$, $\sigma_2 = 0.20$, $\rho = -0.50$, $r = 0.05$, $K = 100$, and $T - t = 90$ days. The barriers are applied weekly.

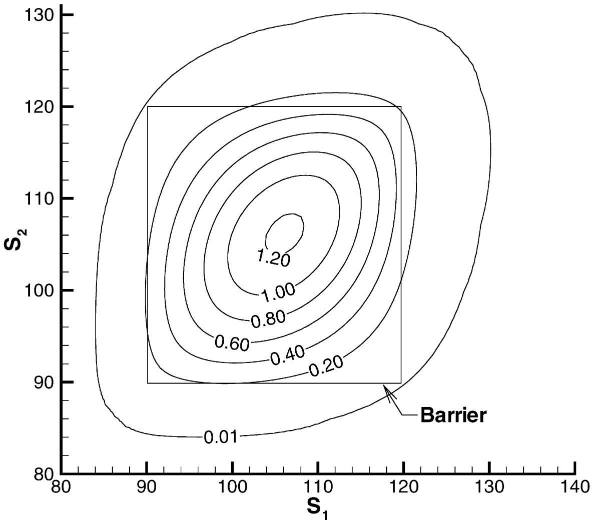

Fig. 9. Two asset barrier call option on the worst of two assets when $\sigma_1 = \sigma_2 = 0.25$, $\rho = 0.50$, $r = 0.05$, $K = 100$, and $T - t = 90$ days. The barriers are applied weekly.

We have also shown that a Crank–Nicolson approach, though stable, can produce very poor solutions if conditions which prevent the formation of spurious oscillations are not met. Furthermore, due to the very small grid

Table 12
Grid used for European and American down-and-out pricing problems for both continuously and discretely applied barriers. * denotes 0.1 for continuously applied barriers and 0.01 for discretely applied barriers

S	70–90	90–97	97–103	103–110	110–140	140–200	200–300	300–400	400–1000
ΔS	2	0.5	*	0.5	2	5	10	20	200

Table 13
Grid used for European and American double knock-out problems where the barriers were continuously applied and constant

S	95–125
ΔS	0.5

Table 14
Grid used for European double knock-out problems where the barriers were discretely applied and constant

S	70–80	80–93	93–97	97–123	123–127	127–140	140–150	150–160
ΔS	1	0.5	0.01	0.5	0.01	0.5	1	2

spacing required near the barrier (in order to obtain accurate solutions), the time step size restrictions for explicit and partially explicit methods are very severe. Examples in this paper show that an accurate explicit method would require time steps four orders of magnitude smaller than a fully implicit scheme (which admittedly has more computational overhead per time step). We have demonstrated that applying barrier constraints in an implicit fully coupled manner for barriers that are applied continuously leads to more rapid convergence than if the constraints are applied explicitly. Automatic time stepping routines can be incorporated into the basic methodology in order to eliminate the trial and error involved in finding the appropriate constant time step size. However, automatic time stepping will reduce execution times primarily for options with long maturities and continuous barriers. The same techniques used to solve one-dimensional barrier option models can also be applied successfully to solve two-dimensional (two asset) problems.

Appendix

Tables 12–17 contain the grids used for the sample problems considered in this paper. The extremes of the grids used for discrete double knock-out pricing

Table 15
Grid used for American double knock-out problems where the barriers were discretely applied and constant

S	70–80	80–93	93–97	97–123	123–127	127–140	140–150	150–160	160–200
ΔS	1	0.5	0.01	0.5	0.01	0.5	1	2	5
S	200–300	300–400	400–1000						
ΔS	10	20	200						

Table 16
Grid used for double knock-out problems where the barriers were continuously applied and time varying

S	775–830	830–1165	1165–1235
ΔS	0.5	5	0.5

Table 17
Grid used for double knock-out problems where the barriers were discretely applied and time varying

S	0–500	500–775	775–830	830–1165	1165–1235	1235–1800	1800–2000
ΔS	100	5	0.5	5	0.5	5	100
S	2000–3000	3000–10000					
ΔS	500	1000					

problems were determined empirically. That is, extending the grids further made no discernable difference to the solution. The grid used for the discrete time-varying problems (see Table 17) was large because a barrier application did not fall on the maturity date.

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