Option Price Intervals Based on Bellman Dynamic Programming Principle

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***Abstract*—The assumption of constant underlying's volatility in Black-Scholes formula cannot be satisfied in financiap market.In this paper ,we get the option price intervals assuming the stock volatility lies within a given interval. First we transform this financial problem to a stochastic optimal control problem,then obtain options' maximum and minimum price models through dynamic programming principle. We solve the nonlinear PDE model and narrow the price interval through optimal static hedging.We conclude this paper by giving its applications in U.S.A option market, get the MCD options intervals ,comparing with Black-scholes,and find a way to identify arbitrage opportunity in option markets.**

Keywords Black-Scholes formula; Assumption of constant volatility; Dynamic Programming Princip; Arbitrage opportunity identifying

I. INTRODUCTION

Option gives the holder a right to buy or sell the underlying at a given future time. In order to obtain this right, the long side needs to pay a fee to the short side, which is called the option premium. How to value the options is a very important issue to both sides.

Black-Scholes formula is the core of the option pricing theory,which has been widely accepted.But the assumptions of Black-Scholes formula cannot be satisfied with the financial data ,such as assuming the underlyings price volatility is a constant until its maturity,But Xu[2] , Ederington find the Volatility Smile and Volatility Skewness implies that the stock volatility is not a constant,which need to be improved. Many practitioners and theorists made certain improvements, the simplest way is to use the underlying's historical volatility or implied volatility instead, other ways are JP Morgan (1994) proposed an exponentially weighted moving average model (EWMA), Engle and other proposed using Garch (1,1) model to estimate volatility, Hull & White (1987) proposed a square root process, Scott considered exponential process.

It seemed that the finance data propose that the stock volatility lies in an interval.In this article,we assumes that the stock volatility is variable, but only assumed changes in a given interval and the other conditions is the same with Black-Scholes's assumptions. As the volatility is uncertain, the stock price trajectory is more difficult to identify, it is hard to find an point estimate of the option price, but it is possible to find the option maximum and minimum price through stochastic optimal control,which results in option price intervals.Such option interval result may have two advantages: first, the confidence of the real price of options fall in an interval is more than equaling a single value; Second, if the length of the price interval can be narrowed to a certain extent, this result will be similar to the Bid-Ask price existing in the trading market.So it is of significance to assume that the volatility lies in a given interval,the maximum and minimum volatility can be estimated from the stock historical price. For the sake of option price interval,we first translate this financial problem into a stochastic optimal control problem, and then use dynamic programming principle to obtain maximum and minimum option price model and discuss the model solution and nature; finally we given model application on the options market taking BaiDu stock options as an example. We use static hedging to narrow interval of options prices, providing the options market arbitrage identification method, and compare the model results with the Black-Scholes formula .

II. OPTION PRICING AND THE STOCHASTIC OPTIMAL CONTROL

In the risk-neutral case,we assume the stock price S (t) is driven by the following stochastic process:

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW(t)$$

Where $r(t), \sigma(t)$ is risk-free rate and the stock volatility, $W(t)$ is a Brownian motion.If the option payoff is $h(X(T))$ at its maturity,then the option discount price is:

$$V(\mathrm{S}) = E\left[e^{-\int_t^T r(s)ds} h(S(T)) \mid F(t)\right]$$

If the risk-free rate is constant then:

$$V(S) = e^{-r(T-t)} E\left[h(S(T)) \mid F(t)\right]$$

For a European call option:

$h(S(T)) = Max(S-E, 0)$ (E is the strike price)

In Black-Scholes formula ,volatlity is assumed to be a constant which does not comply with market,the financail data may support the the stock volatility lies in a given interval .In this paper,we assume y that its volatility lies in a given interval. We will get the option pricing model under the assumption:

$$\sigma^- \le \sigma(t) \le \sigma^+ .$$

A. Translate the option pricing problem to an optimal control problem

As volatility is uncertain, there is an infinite possible stock price trajectory, it is difficult to get a particular option value. But we have the maximum and minimum value of volatility, we can obtain the upper price $V^+(S, t)$ and lower price $V^-(S, t)$, which will be the option price interval . This is an extremal problems which can be solved through stochastic optimal control theory. We can construct the optimal control system in such way: considering the stock price as state variables, the option price as the functionals.then its state equation is :

$$\begin{cases} dS(t) = rS(t)dt + u(t)S(t)dW(t) \\ S(t) = S \end{cases}$$

Since the assumption $\sigma^- \le \sigma(t) \le \sigma^+$

The control set is:

$$u(\cdot) \in \Omega[t,T] = \{u(\cdot) : [t,T] \to [\sigma^-, \sigma^+] \mid u(\cdot) \text{ is measurable}\}$$

The functional is：

$$J(u(\cdot)) = E\left(h(S(T))\right)$$

We can conclude

$$V(t,S) = e^{-r(T-t)} J(u(\cdot))$$

Then getting the minimum option price is to solve the following stochastic optimal control problem:

Problem S_{tS}^- :Finding a optimal control function $\underline{u}(\cdot)$,such that:

$$J(t,S;\underline{u}(\cdot)) = \inf_{u(\cdot)\in\Omega[t,T]} J(u(\cdot))$$

Likewise, getting the maximum option price is to solve the following stochastic optimal control problem:

Problem (S_{tS}^+) :Finding a optimal control function $\overline{u}(\cdot)$,such that:

$$J(t,S;\overline{u}(\cdot)) = \sup_{u(\cdot)\in\Omega[t,T]} J(u(\cdot))$$

This is a type of Bolza functionals optimal control problem,which can be solved through Pontryagin maximum principle and Bellman dynamic programming principle,we first use Bellman dynamic programming principle to get the corresponding Hamilton-Jacobi-Bellman equation. It should be noted that, in the equation of state will be set equal to the risk-free rate control function, by continuously adjusting the control function, can make the performance indicators to achieve the most functional value. It can be proved on the risk-free rate of the general settings are also able to get back to the same conclusion.

B. Bellman dynamic programming principle

We now consider the problem S_{tS}^-

The value function is

$$\begin{cases} U(t,S) = \inf_{u(\cdot)\in\Omega} J(t,S;u(\cdot)) \\ U(T,S) = h(S) \end{cases}$$

It should satisfying the following Hamilton-Jacobi-Bellman equation which introduced in [4]:

$$\begin{cases} -U_t^- + \sup_{u\in\Omega} G(t,S,u,-U_S^-,-U_{SS}^-) = 0 \\ U^- |_{t=T} = h(S) \end{cases}$$

Where G is the generalized Hamilton function:

$$G(t,S,u,p,P) = \frac{1}{2}u^2S^2P + rSp$$

Then the Hamilton—Jacobi—Bellman equation is

$$\begin{cases} \dfrac{\partial U^-}{\partial t} + \dfrac{1}{2}(\rho(\dfrac{\partial^2 U^-}{\partial S^2}))^2 S^2 \dfrac{\partial^2 U^-}{\partial S^2} + rS\dfrac{\partial U^-}{\partial S} = 0 \\ U^-(S)|_{t=T} = e^{r(T-t)}h(S(T)) \end{cases}$$

$\rho(x) = \sigma^+ I_{\{x<0\}} + \sigma^- I_{\{x>0\}}$, I is the inductor function.

Since the relation between the option price and the functional:

$$U(t,S) = e^{r(T-t)}V(t,S)$$

We have:

$$\begin{cases} U_t = -re^{r(T-t)}V + e^{r(T-t)}V_t \\ U_S = e^{r(T-t)}V_S \\ U_{SS} = e^{r(T-t)}V_{SS} \end{cases}$$

Then the option lower price should satisfy the following terminal problem(1)

$$\begin{cases} \dfrac{\partial V^-}{\partial t} + \dfrac{1}{2}(\rho(\dfrac{\partial^2 V^-}{\partial S^2}))^2 S^2 \dfrac{\partial^2 U^-}{\partial S^2} + rS\dfrac{\partial V^-}{\partial S} - rV^- = 0 \\ V^-(S)|_{t=T} = h(S(T)) \end{cases}$$

Likewise,the option upper price should satisfy the following terminal problem(2):

$$\begin{cases} \dfrac{\partial V^+}{\partial t} + \dfrac{1}{2}(\rho(-\dfrac{\partial^2 V^+}{\partial S^2}))^2 S^2 \dfrac{\partial^2 V^+}{\partial S^2} + rS\dfrac{\partial V^+}{\partial S} - rV^+ = 0 \\ V^+(S)|_{t=T} = h(S(T)) \end{cases}$$

Both problems contain non linear partial differential equations,which is a kind of Black-Scholes-Barrenblatt equations.

C. The Regularity of Problem (1) and (2)

Since the model (1) - (2) contain nonlinear partial differential equations, it is difficult to discuss their solutions regularity, We will discuss their regularity by means of optimal control theory ,we get the following theorem.

Theorem: The model (1) - (2)has the solution is of existence, uniqueness and continuity.

Proof: First we discuss the option lower price casel(1).

Since the model (1) is a Hamilton-Jacobi-Bellman equation, according to[9], Hamilton-Jacobi-Bellman equation in the sense of having a viscosity solution existence, uniqueness and continuity, and because the transformation $U(t,x)=e^{r(T-t)}V(t,x)$ is a one-one transformation, then we can get the model (2) has the solution existence, uniqueness and continuity. Likewise the modelThe solution also has the existence, uniqueness and continuity, thus the theorem is proved.

D. Solving the terminal problem through Hamilton System

According to [4],we can solve the terminal problem(1) following the three steps below:

Firstly we can get its Hamilton through Pontryagin maximum principle:

$$\begin{cases} dp(t)=-\left(rp(t)+u(t)q(t)\right)dt+q(t)dW(t) \\ p(t)=-h_S(S(T)) \end{cases}$$

$$\begin{cases} dP(t)=-\left(r+rP(t)+P(t)+2u(t)Q(t)\right)dt+Q(t)dW(t) \\ P(t)=-h_{xx}(x(T)) \end{cases}$$

We will get the optimal control *u(t)* and *P(t).p(t)*

Secondly,we get the functional $U(t,S)\backslash$

Thirdly,we obtain the option lower or upper price through

$$V(t,S)=e^{-r(T-t)}U(t,S)$$

E. Solving the terminal problem through numerical methods

Since there is a nonlinear partial differential equation in the terminal problem, the option price is obtained mainly by numerical solution method ,we now discuss its finite difference grids.

Because the terminal problem is bachward ,we approximate their derivatives as following:

$$V_i^k=V(i\delta x,T-k\delta t) \quad , \quad \frac{\partial V}{\partial t}\approx\frac{V_i^k-V_i^{k+1}}{\delta t} \quad ,$$

$$\frac{\partial V}{\partial S}\approx\frac{V_{i+1}^k-V_{i-1}^k}{2\delta S}\ , \quad \frac{\partial^2 V}{\partial S^2}\approx\frac{V_{i+1}^k-2V_i^k+V_{i-1}^k}{\delta S^2}\ ,$$

Thus the finite difference is :

$$V_i^{k+1}=A_i^kV_{i-1}^k+(1+B_i^k)V_i^k+C_i^kV_{i+1}^k$$

where：

$$A_i^k=\frac{1}{2}(\sigma(\cdot)^2i^2-ri)\delta t \quad , \qquad B_i^k=-(\sigma^2(\cdot)i^2+r)\delta t \quad ,$$

$$C_i^k=\frac{1}{2}(\sigma^2(\cdot)i^2+ri)\delta t 。$$

where：

$$\sigma(\cdot)=\begin{cases}\sigma^+ & \dfrac{\partial^2V^-}{\partial S^2}<0 \\ \sigma^- & \dfrac{\partial^2V^-}{\partial S^2}>0\end{cases} \quad \text{or} \quad \sigma(\cdot)=\begin{cases}\sigma^+ & \dfrac{\partial^2V^+}{\partial S^2}>0 \\ \sigma^- & \dfrac{\partial^2V^+}{\partial S^2}<0\end{cases}$$

The terminal conditions is：

$$V_N=2V_{N-1}-V_{N-2}$$

III. MODEL EMPIRCAL AND APPLICATION

A. Narrowing the Option Price interval through optimal static hedging

Since the terminal problem(1) - (2) contain nonlinear partial differential equations, if the option Payoff function is not smooth enough, then the option price interval will be too long, which will undermine the significance of the model. We will propose optimal static hedging methods to narrow price interval of options appeared in Avellaneda et al (1995).Since the nonlinearity of nonlinear partial differential equations in terminal problem(1) - (2),as for a portfolio, the Black-Scholes formula is a linear partial differential equations, option price result of the Black-Scholes formula calculated are equal to the combined portfolio of options respectively we can't narrow price interval through constructing a portfolio. But terminal problem(1) - (2) is nonlinear, the entire investment portfolio optimal path is not necessarily the optimal option price of each path, so the whole portfolio price interval does not mean that the price interval of the various options, and can be reduced through a certain way. To narrow the price interval of options,we select some existing on the market as static hedging instruments, if this portfolio payoff function smooth enough, portfolio price interval will be smaller through terminal problem(1) - (2). Then the length of the smaller portfolio minus the price interval of the upper and lower hedging costs is the target price interval of options, constantly adjusting the number of hedging instruments, the price interval of options will be narrowed.

We now show how to narrow the option price interval through optimal static hedging in the options in US market. On Dec 15, 2014, McDonald's Crop (NYSE ,Code: MCD) stock closed at $90.96.Its put option prices are $ 0.46, $ 2.91 and $ 4.51 ,whose strike price are 85,90 and the $ 95 respectively, all will matury on Feb 63, 2015 (Data from Finance.yahoo.com). According to the U.S. Federal Reserve interest rate one-year risk-free rate,we set *r*=0.25%.We estimates McDonald's Crop stock volatility bound from 10% to 40% using GARCH (1,1) model.Maturity T = 53/365 = 0.143 years.We want to obtain the price interval of a put options striking at $90 (denoted as: 90Put). Because 90Put' s payoff function is unbounded, the price interval is[2.63,3.37] ,the price length is relatively large, then we select (-0.5) 85Put and (-0.5) $ 95Put as hedging instrument("-" implies the short side), to form a option portfolio, whose payoff function is

$$V(S,T) = Max(90-S,0) - \frac{1}{2}[Max(85-S,0) + Max(95-S,0)]$$

The payoff funcion is bounded, the price interval we obtain is [2.79,3.19], its length is 0.4,which is shoter than Black-Scholes model result（See Table1）.If we adjust the number of hedging instruments, the best price interval is [2.87,3.02], whose interval length is $ 0.15, is one-tenth of the results of the Black-Scholes formula. (See Table 1).

Four kinds of price interval of an exercise price of $90 McDonald's Crop put are listed in table 1. The first one is the result of the Black-Scholes formula; second is the result of the terminal problem (1) - (2); the third price interval is obtained through the model (1) - (2) after static hedging,the hedging instructment is (-0.5) 85Put and (-0.5) 95Put; the fourth result is based on the third case, adjusting the number of hedging instruments, narrowing the option price target interval length.

Table 1 Four price intervals of an McDonald's Crop 90 put Option

	Lower price	Upper price
Result of Black-Scholes Formula	2.11	3.79
Result of Terminal Problen (1)~(2)	2.58	3.37
Result of Static hedging	2.79	3.19
Result of optimal Static hedging	2.87	3.02

The Bid-Ask option price is 2.88 and 3.03,the error between with the upper and lower price is very small,thus the model is very practical.

The price interval obtained by optimal static hedge have two applications: the first one is to recognize the existence of some kind of option prices on arbitrage opportunities; second one is to absoult risk control, we can give an option price maximum error.

B. *Identifying Arbitrage opportunity on option price*

Different maturity options can be issued based on the same stock, whether these options market price is relatively reasonable and if there are arbitrage opportunities on some options, we can identify these following three steps below:

Step One: Choose an target option and other options as hedging instruments , forming a new option portfolio, whose payoff is relatively smooth;

Step two: Calculating the target option price interval and narrow the option interval through static hedging;

The third step: Identifying the arbitrage opportunity. If the target option market price falls in the price interval in Step two , then we conclude that there is no arbitrage opportunities on the price of the target option; otherwise we consider there are arbitrage opportunities.

In Table 1, the market price of $ 90Put is $2.91,it falls in all of four price interval,so there are no arbitrage opportunities. Although consistent with the conclusion of four results obtained, but in the latter two cases, the price interval length is relatively smaller.Increasing the option hedging instrument amount, the interval length will be further reduced.

C. *Absolute risk Control*

VaR method can give us a possible loss in a certain probability. But the option price interval will give options and option portfolio maximum loss, which is the price interval in length. The example above, you can get the maximum loss of trading a $90Put McDonald's Crop is 3.02-2.87 = $ 0.15.

IV. CONCLUSIONS

This paper focus on finding option price intervals through stochastic optimal control,which can be seen as an amendment to Black-scholes formula which assume that the stock volatility is constant :

1) We translate the option price intervals problem into a stochastic optimal control problem, and get the nonlinear partial differential equation to obtain the option price intervals, providing a stochastic optimal control framework for discussing nonlinear partial differential equations;

2) Narrowing the option price interval through optimal static hedging,which is more effective than the result of Black-Scholes formula ,and there is a small error with the option Bid-Ask quoted;

3) Providing a method of arbitrage opportunities for the options trading.

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