On Unknown Input Observers for LPV Systems

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***Abstract*—This paper is dedicated to the design of unknown input (UI) observers for linear parameter-varying (LPV) systems using algebraic matrix manipulation. First, a discussion is provided concerning the classical design approach, which requires the matching condition. This condition is satisfied for a class of systems having relative degree 1. Second, for the same class of systems, a new generalized approach is proposed for simultaneous state and UI estimation. It is demonstrated that the proposed approach is more general than the classical approach. Finally, an extension of the approach is provided for LPV systems with arbitrary relative degree. Simulation examples are provided to illustrate the main results of the paper.**

***Index Terms*—Algebraic matrix equalities, arbitrary relative degree, linear matrix inequalities (LMIs), linear parameter-varying (LPV) systems, polytopic transformation, sliding-mode differentiators, unknown input observers (UIOs).**

I. Introduction

ON the state and the unknown inputs (UIs) of a dynamical system is a challenging problem. It has been the subject of several studies, and several approaches have been proposed in the last 40 years. An observer can be seen as a software sensor aiming to estimate unmeasured states and UIs (faults, disturbances, etc.). The use of observers is motivated by economic and/or technique reasons. Indeed, it may happen that sensors measuring some internal and external variables of a given system are not available, as it may happen that sensors exist but are expensive or with inadequate size. This problem can then find a solution with the use of observers. For example, in the automotive industry sector, the sensor that measures the lateral speed costs about 15 000; on the other hand, the sensor measuring the road curvature does not exist. In the field of unmanned aerial vehicles (UAVs), the weight of UAVs is an important constraint to take into account. These issues can then be addressed due to observers that overcome the technical and/or economic problems; in addition, they allow providing reliable and robust estimates for control, monitoring, and diagnostics.

From the pioneer result given by Luenberger in [1], several extensions have been considered for linear and nonlinear systems. Among them is the consideration of the case where the system is affected by UIs. The proposed approaches can be classified into two categories: UI decoupling approaches and simultaneous state and UI estimation approaches. The decoupling approaches consist in expressing the state estimation error as a system that is free from any UI. Necessary and sufficient structural conditions for the existence of such observers are proposed in terms of rank conditions (see [2]–[4] and references therein). For the second category, the UIs are considered as a part of the state vector upon assumptions on the variations of the UIs (constant, time polynomial, etc.); the observers are then constructed in such a way to estimate both the state vector and the UI [5], [6]. Necessary and sufficient conditions are also given for the existence of such observers. All these approaches have been first established for linear systems. Afterward, extensions for various types of systems are proposed, e.g., singular systems [7], Lipschitz nonlinear systems [8]–[11], Takagi–Sugeno systems [12], [13], etc.

The previously cited approaches have been also extended for linear parameter-varying (LPV) systems. Generally, the approach used is based on the description of the LPV system in a polytopic form by bounding the parameters and using the sector nonlinearity approach (see, e.g., [14], [15], and references therein). The interest of the polytopic transformation is the ability to express the convergence conditions, which are established generally using the Lyapunov theory, in terms of linear matrix inequalities (LMIs), which provide sufficient conditions for the existence of the observers and a mean for the efficient design of the gains of the observers using dedicated tools (MATLAB LMI Toolbox, YALMIP, etc.). On the other hand, an interesting reduced-order observer for LPV time-delay systems is proposed in [16], where an algebraic approach is presented. Necessary and sufficient conditions are provided for the existence of a reduced-order observer for LPV time-delay systems. It can be seen that the conditions given in [16] depend on parameter-dependent matrices, which take into account the parameter variations. Such conditions cannot be obtained using only the vertices of the polytop after transformation.

In this paper, we are interested by the design of UI observers (UIOs) for LPV systems. First, a discussion about the classical UIOs for LPV systems is given; it will be proven that the classical polytopic approaches are limited to some systems even if the observability and UI decoupling conditions are satisfied. The simple reason of this limitation is related to the transformation of the LPV system into a polytopic form before designing the UIOs; this imposes the structure of the observers. In this paper, a new observer structure is proposed, and it uses algebraic matrix manipulation. This observer was inspired by the one presented for linear systems using the intrinsic approach in [17] and the algebraic approach given in [16] for LPV time-delay systems. It will be illustrated that the

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proposed observer for LPV systems is more general and can estimate the state of an LPV system with UI even if the classical polytopic approaches fail. The second contribution of this paper concerns the generalization of the observer for systems with arbitrary relative degree (i.e., the classical rank conditions are no longer satisfied). All the observers use the time derivatives of the output signals and the varying parameters, which can be obtained by asymptotic differentiators [18], nonasymptotic differentiators [19], or high-order sliding-mode differentiators (HOSMDs) [20]. In this paper, the HOSMDs are adopted due to their efficiency for time-derivative estimations, i.e., the finite time and the exactness properties in noise-free cases.

II. Preliminaries and Notations

In this paper, we are interested by the LPV structure expressed by

$$\begin{cases} \dot{x}(t) = A(\rho(t))\, x(t) + B(\rho(t))\, u(t) + D(\rho(t))\, d(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $d(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$ are the state vector, the control input, the UI, and the output of the system, respectively. The matrices $A(.)$, $B(.)$, and $D(.)$ are parameter varying with appropriate dimensions; the output matrix C is considered constant, but the extension of the approach for a parameter-varying output matrix is straightforward. $\rho^T(t) = (\rho_1, \ldots, \rho_{n_\rho})$ represents the vector of n_ρ time-varying parameters, which are sufficiently smooth and bounded, which means that $\rho(t) \in \Theta$, where Θ is a hyperrectangle defined by

$$\Theta = \left\{ \rho(t) \in \mathbb{R}^{n_\rho} \,\middle|\, \rho_1 \in \left[\rho_1^{\min}, \rho_1^{\max}\right], \ldots, \rho_{n_\rho} \in \left[\rho_{n_\rho}^{\min}, \rho_{n_\rho}^{\max}\right] \right\} \tag{2}$$

where $\rho_i^{\min}$ and $\rho_i^{\max}$, $i = 1, \ldots, n_\rho$ define the upper and lower bounds of parameter $\rho_i(t)$, respectively.

Note that, for the sake of clarity, we deliberately consider the LPV systems with single UI and single output (SUISO). The extension to the case of systems with multiple UIs and multiple outputs (MUIMO) is briefly considered in Remark 2.

Assume that the LPV matrices are defined in polynomial form as follows:

$$X(\rho(t)) = X_0 + \rho_1(t) X_1 + \cdots + \rho_{n_\rho}(t) X_{n_\rho} \tag{3}$$

where $X(\rho(t)) \in \{A(\rho(t)), B(\rho(t)), D(\rho(t))\}$ and X_k, $k = 0, \ldots, n_\rho$ are constant matrices. It follows that the time derivative of $X(\rho(t))$ can be expressed as follows:

$$\frac{dX(\rho(t))}{dt} = \dot{X}(\rho(t)) = \dot{\rho}_1(t) X_1 + \cdots + \dot{\rho}_{n_\rho}(t) X_{n_\rho}. \tag{4}$$

Note that the notation (3) is not restrictive since any parameter-varying matrix can be written under this form, e.g., the parameter-varying matrix

$$X(\rho(t)) = \begin{pmatrix} \rho_1(t) & a + \rho_2(t) \\ b + \rho_3(t) & \rho_4(t) \end{pmatrix} \tag{5}$$

can be expressed as follows:

$$X(\rho(t)) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} + \rho_1(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \rho_2(t) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \rho_3(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \rho_4(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Assume, in addition, that the time derivatives of the parameters belong to the compact sets defined by

$$\Theta_j = \left\{ \rho^{(j)} \in \mathbb{R}^{n_\rho} \,\middle|\, \rho_1^{(j)} \in \left[\rho_{1j}^{\min}, \rho_{1j}^{\max}\right], \ldots, \rho_{n_\rho}^{(j)} \in \left[\rho_{n_\rho j}^{\min}, \rho_{n_\rho j}^{\max}\right] \right\} \tag{7}$$

where j defines the order of the time derivative of the parameters.

Definition 1: The system (1) is said to be uniformly strongly algebraically observable, with respect to parameter $\rho(t)$, if there exist positive integers k_y, k_u, and k_ρ, such that $\forall \rho^{(j)}(t) \in \Theta_j$, $j = 0, \ldots, k_\rho$. The state of the system (1) can be expressed as a vector function of the outputs, the inputs, the parameters, and their time derivatives up to finite orders, i.e.,

$$x(t) = F\left(y(t), \ldots, y^{(k_y)}, u(t), \ldots, u^{(k_u)}, \rho(t), \ldots, \rho^{(k_\rho)}\right). \tag{8}$$

Definition 1 is an extension of the algebraic strong observability established for nonlinear and linear systems (see, e.g., [18, Definition 1]). It means that, if a differential function F exists, then the state of the system can be expressed only in terms of the output, the input, the parameters, and their time derivatives up to finite orders, without the UI.

Definition 2: Given the system (1), the number r is called a relative degree of the output $y(t)$, with respect to the UI $d(t)$, if the UI $d(t)$ appears in the equation of the rth time derivative of the output $(y^{(r)}(t))$ [21].

Assumption 1: Throughout this paper, we assume the following:

1) The relative degree is well defined in the compact sets Θ and Θ_j defined in (2) and (7), respectively. It means that the time variations of the parameters and their successive time derivatives do not affect the value of the relative degree r of the output with respect to the UI. We refer to the notion of uniform relative degree r on Θ and Θ_j.
2) The system (1) is assumed to be uniformly strongly algebraically observable (Definition 1).

III. Problem Statement

Generally, LPV systems of the form (1) can be represented, exactly, in a polytopic form, within the compact set Θ, as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{N} \mu_i(\rho(t)) \left(A_i x(t) + B_i u(t) + D_i d(t)\right) \\ y(t) = Cx(t) \end{cases} \tag{9}$$

where $N = 2^{n_\rho}$ is the number of vertices of the polytop (number of linear submodels). The weighting functions $\mu_i(.)$ satisfy the convex sum property, i.e.,

$$\sum_{i=1}^{N} \mu_i\left(\rho(t)\right)=1, \qquad 0\leq \mu_i\left(\rho(t)\right)\leq 1,\ i=1,\ldots,N,\ \forall t. \tag{10}$$

The first objective of this paper is to discuss the existing approaches and to propose a new design approach for a UIO for the system (1). To the best of the author's knowledge, the problem of UIO design for systems of the form (9) is not solved because all the works published until now consider an output equation affected also by the UI $y(t) = Cx(t) + Fd(t)$ with the full row rank matrix F. This is to be able to decouple $d(t)$ from the state estimation error. For example, if we consider the UIO

$$\begin{cases} \dot{z}(t) = \sum\limits_{i=1}^{N} \mu_i\left(\rho(t)\right)\left(N_i z(t) + G_i u(t) + L_i y(t)\right) \\ \hat{x}(t) = z(t) - Hy(t). \end{cases} \tag{11}$$

Assume that the relative degree of the output is $1\ \forall\, \rho(t)\in\Theta$. The state estimation error $e(t)=x(t)-\hat{x}(t)$ can be expressed by

$$e(t) = x(t) - z(t) + HCx(t) + HFd(t) \tag{12}$$
$$= \underbrace{(I + HC)}_{P} x(t) - z(t) + HFd(t). \tag{13}$$

After computations, the dynamics of the state estimation error is

$$\dot{e}(t)=\sum_{i=1}^{N}\mu_i\left(\rho(t)\right)\left(N_i e(t)+(PA_i-L_iC-N_iP)x(t)\right.$$
$$\left.+(PB_i-G_i)u(t)+(PD_i-L_iF)d(t)\right)-HF\dot{d}(t). \tag{14}$$

The decoupling conditions are commonly given by $\forall\, i = 1, \ldots, N$

$$PA_i - L_iC - N_iP = 0 \tag{15}$$
$$PB_i - G_i = 0 \tag{16}$$
$$PD_i - L_iF = 0 \tag{17}$$
$$HF = 0. \tag{18}$$

Under these conditions, the state estimation error dynamics is free from any other signals and given by

$$\dot{e}(t) = \sum_{i=1}^{N} \mu_i\left(\rho(t)\right) N_i e(t). \tag{19}$$

Therefore, in this case, if there exist matrices N_i, L_i, and H, such that the conditions (15)–(18) are satisfied and the system (19) is asymptotically stable, then the state estimation error converges asymptotically toward zero. Now, let us focus on (17). It is clear that, if $F=0$, the equation $PD_i=0$ is difficult to solve, even impossible, unless the matrices D_i are identical $\forall\, i = 1, \ldots, N$. When $F = 0$, even if the conditions $rank(CD_i) = rank(D_i)$ are satisfied $\forall\, i=1,\ldots,N$, obtaining a common matrix H, such that $(I+HC)D_i=0\ \forall\, i=1,\ldots,N$, is very difficult. This problem appears because the LPV system is transformed at the beginning in a polytopic form, which imposes a structure for the gain matrices of the same form.

In this paper, a solution is proposed to overcome this difficulty by considering a new structure of the UIO and design procedure. The arbitrary structure of the LPV model is left unchanged during the observer design, and the polytopic form is used only for establishing LMI conditions at the end of the observer construction, which ease the design of the gains numerically. An extension of the approach for LPV systems with arbitrary relative degree is also proposed.

In the remainder of this paper, let us adopt the notation $Y_\rho = Y(\rho(t))$, where the matrix $Y(\rho(t))$ depends arbitrarily (nonlinearly) on the time-varying parameter vector $\rho(t)$.

IV. Main Results

Without loss of generality, assume that $u(t) = 0$ in the system (1). Note that this is not restrictive since the same approach can be developed for systems with $u(t) \neq 0$. This can be done by generating an auxiliary system in the form

$$\begin{cases} \dot{s}(t) = A\left(\rho(t)\right)s(t) + B\left(\rho(t)\right)u(t) \\ y_s(t) = Cs(t). \end{cases} \tag{20}$$

By defining the error $z(t) = x(t) - s(t)$ and $y_z(t) = y(t) - y_s(t)$, one obtains the dynamics

$$\begin{cases} \dot{z}(t) = A\left(\rho(t)\right)z(t) + D\left(\rho(t)\right)d(t) \\ y_z(t) = Cz(t) \end{cases} \tag{21}$$

which are free from the known input $u(t)$. Thus, after estimating the state $\hat{z}(t)$, the real state is obtained by the equation $\hat{x}(t) = \hat{z}(t) + s(t)$.

A. UIO: Relative Degree 1

For the first main result, let us consider the LPV system

$$\begin{cases} \dot{x}(t) = A_\rho x(t) + D_\rho d(t) \\ y(t) = Cx(t). \end{cases} \tag{22}$$

The observer takes the form

$$\begin{cases} \dot{\hat{x}}(t) = (P_\rho A_\rho - L_\rho C)\hat{x}(t) + Q_\rho \dot{y}(t) + L_\rho y(t) \\ \hat{d}(t) = (CD_\rho)^{-1}\left(\dot{y}(t) - CA_\rho\hat{x}(t)\right). \end{cases} \tag{23}$$

The matrices P_ρ and Q_ρ are determined according to the following theorem 1.

Theorem 1: Under Assumption 1 with relative degree 1, the system (23) is an observer for the system (22) if and only if the following conditions are satisfied:

- The matrices P_ρ and Q_ρ are

$$\begin{cases} Q_\rho = D_\rho\left(CD_\rho\right)^{-1} \\ P_\rho = I_n - Q_\rho C. \end{cases} \tag{24}$$

- The pair $(P_\rho A_\rho, C)$ is detectable $\forall\, \rho \in \Theta$.
- The parameter-varying matrix $(P_\rho A_\rho - L_\rho C)$ is stable $\forall\, \rho \in \Theta$.

Proof: First of all, the time derivative of the output $y(t)$ is

$$\dot{y}(t) = CA_\rho x(t) + CD_\rho d(t). \tag{25}$$

Under Assumption 1, the relative degree of $y(t)$ with respect to $d(t)$ is $1\ \forall\, \rho \in \Theta$, and the term $CD_\rho \neq 0$.

Defining the state estimation error $e(t) = x(t) - \hat{x}(t)$ and the UI estimation error $e_d(t) = d(t) - \hat{d}(t)$ and using (25), it follows that:

$$\begin{aligned}\dot{e}(t) =& A_\rho x(t) + D_\rho d(t) - (P_\rho A_\rho - L_\rho C)\hat{x}(t) \\ & - Q_\rho \dot{y}(t) - L_\rho y(t) \qquad (26)\\ =& A_\rho x(t) + D_\rho d(t) - (P_\rho A_\rho - L_\rho C)\hat{x}(t) \\ & - Q_\rho C A_\rho x(t) - Q_\rho C D_\rho d(t) - L_\rho y(t). \qquad (27)\end{aligned}$$

After computation, one obtains

$$\begin{aligned}\dot{e}(t) = (A_\rho - Q_\rho C A_\rho - L_\rho C)x(t) - (P_\rho A_\rho - L_\rho C)\hat{x}(t) \\ + (D_\rho - Q_\rho C D_\rho)d(t). \qquad (28)\end{aligned}$$

Under the conditions

$$A_\rho - Q_\rho C A_\rho - L_\rho C = P_\rho A_\rho - L_\rho C \tag{29}$$

$$D_\rho - Q_\rho C D_\rho = 0 \tag{30}$$

the state estimation error dynamics become

$$\dot{e}(t) = (P_\rho A_\rho - L_\rho C)e(t). \tag{31}$$

Under the condition that the relative degree of $y(t)$ with respect to the UI is well defined and equal to $1\ \forall\, \rho \in \Theta$, the inverse of CD_ρ exists. Then, from the conditions (29) and (30), the matrices P_ρ and Q_ρ exist and are computed as follows:

$$\begin{cases} Q_\rho = D_\rho (CD_\rho)^{-1} \\ P_\rho = I_n - Q_\rho C. \end{cases} \tag{32}$$

Note that the matrices Q_ρ and P_ρ are imposed by the model matrices D_ρ and C (intrinsic properties). The inverse of CD_ρ exists if the output $y(t)$ has uniform relative degree 1 with respect to the UI $d(t)$, i.e., $rank(CD_\rho) = rank(D_\rho)\ \forall\, \rho(t) \in \Theta$, which is ensured by Assumption 1. Note that this structure can be used in a SUISO system or in square systems with the same number of both UIs and outputs. In the general case $\dim(y(t)) > \dim(d(t))$, the inverse matrix should be replaced by the pseudoinverse under some conditions. This case will be addressed later in Remark 2.

From the condition that the pair $(P_\rho A_\rho, C)$ is detectable $\forall\, \rho \in \Theta$, the gain matrix L_ρ can be computed, in order to ensure the asymptotic stability of the system (31) that generates the state estimation error; consequently, the state of the observer converges asymptotically to the state of the system.

On the other hand, the UI estimation error can be computed as follows:

$$e_d(t) = d(t) - \hat{d}(t) = (CD_\rho)^{-1} e(t). \tag{33}$$

It can be seen that, when $e(t)$ converges asymptotically toward zero, the UI estimation error $e_d(t)$ converges also asymptotically toward zero because (33) is an algebraic equation. This ends the proof. ■

1) Convergence Analysis and LMI Formulation: Let us consider the state and the UI estimation errors $e(t) = x(t) - \hat{x}(t)$ and $e_d(t) = d(t) - \hat{d}(t)$, with the matrices P_ρ and Q_ρ given in (24). Gathering (31) and (33), one obtains

$$\begin{cases} \dot{e}(t) = (P_\rho A_\rho - L_\rho C)e(t) \\ e_d(t) = (CD_\rho)^{-1} e(t). \end{cases} \tag{34}$$

It is possible to design directly the gain L_ρ. However, for more complex systems with great number of parameters, it is more interesting to transform the matrices P_ρ, A_ρ in a polytopic form, which allows fixing the polytopic form of the gain L_ρ. In order to do so, from the structures of the matrices P_ρ, A_ρ and knowing that parameters $\rho(t) \in \Theta$, one obtains exactly the polytopic forms

$$\begin{cases} P_\rho = \sum\limits_{i=1}^{N} \mu_i(\rho(t))\, P_i \\ A_\rho = \sum\limits_{i=1}^{N} \mu_i(\rho(t))\, A_i \end{cases} \quad \forall\, \rho(t) \in \Theta \tag{35}$$

where the functions $\mu_i(\rho(t))$ satisfy the convex sum property (10). From this representation, the gain matrix L_ρ can be defined as

$$L_\rho = \sum_{i=1}^{N} \mu_i(\rho(t))\, L_i. \tag{36}$$

Consequently, the state estimation error dynamics (28) can be expressed by

$$\dot{e}(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} \mu_i(\rho(t))\, \mu_j(\rho(t))\, (P_i A_j - L_j C)\, e(t). \tag{37}$$

Standard LMI conditions can be obtained by using a quadratic Lyapunov function in the form

$$V(e(t)) = e^T(t) X e(t), \qquad X = X^T > 0. \tag{38}$$

After computing the time derivative of the Lyapunov function $V(e(t))$ and by using the state estimation error dynamics (37) and the convex sum property of the weighting functions, sufficient LMI conditions ensuring asymptotic stability are obtained as follows:

$$A_j^T P_i^T X + X P_i A_j - C^T K_j^T - K_j C < 0, \quad i, j = 1, \ldots, N \tag{39}$$

where the gains of the observer are obtained by the equation $L_j = X^{-1} K_j$. This solution ensures the convergence of the state estimation error $e(t)$ toward zero, and then, the UI estimation error e_d converges also toward zero.

Remark 1: Note that relaxed LMI conditions can be obtained by using, for instance, Tuan's lemma [22] or Polya's theorem [23] to derive less conservative LMI conditions, or by considering other types of Lyapunov functions. This is not the scope of this paper.

2) Illustrative Example 1: To illustrate the proposed approach, let us consider the simple LPV system described by

$$\begin{cases} \dot{x}(t) = \underbrace{\begin{pmatrix} 0 & 1+\rho(t) \\ -\rho(t) & -2 \end{pmatrix}}_{A_\rho} x(t) + \underbrace{\begin{pmatrix} \rho(t) \\ 1+\rho(t) \end{pmatrix}}_{D_\rho} d(t) \\ y(t) = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} x(t) \end{cases} \tag{40}$$

with $\rho(t) \in [1,5]$. Note that the system is observable $\forall\ \rho(t) \in [1,5]$ and the output has a relative degree 1 with respect to $d(t)$ and $rank(CD_\rho) = rank(D_\rho),\ \forall\, \rho(t) \in [1,5]$.

a) Classical polytopic approach illustration: From the system (40) and under the assumption that $\rho(t) \in [1,5]$, a polytopic LPV system can be constructed exactly as

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i(\rho(t))\,(A_i x(t) + D_i d(t)) \\ y(t) = Cx(t) \end{cases} \tag{41}$$

where the weighting functions are defined by

$$\mu_1(\rho(t)) = \frac{\rho(t)-1}{4} \tag{42}$$

$$\mu_2(\rho(t)) = \frac{5-\rho(t)}{4} \tag{43}$$

satisfying the convex sum property

$$\begin{cases} \sum_{i=1}^{2} \mu_i(\rho(t)) = 1, \\ 0 \le \mu_i(\rho(t)) \le 1, \forall\, \rho(t) \in \Theta \end{cases} \tag{44}$$

and the matrices are defined by

$$A_1 = \begin{pmatrix} 0 & 6 \\ -5 & -2 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 2 \\ -1 & -2 \end{pmatrix} \tag{45}$$

$$D_1 = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{46}$$

Note that $rank(CD_i) = rank(D_i),\ \forall\, i = 1,2$. Let us go back to the classical observer (11) and the condition (17) given by $PD_i - L_i F = 0$. In the considered example, the matrix $F = 0$; hence, the UI decoupling condition is reduced to $PD_i = 0$, where $P = I + HC$. Thus, decoupling the UI $d(t)$ is obtained if there exists a matrix H, such that

$$HCD_1 = -D_1 \tag{47}$$

$$HCD_2 = -D_2 \tag{48}$$

by considering the matrix H in the form

$$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{49}$$

where h_1 and h_2 are scalars to determine in such a way to satisfy (47) and (48). After replacing the matrices C, D_1, and

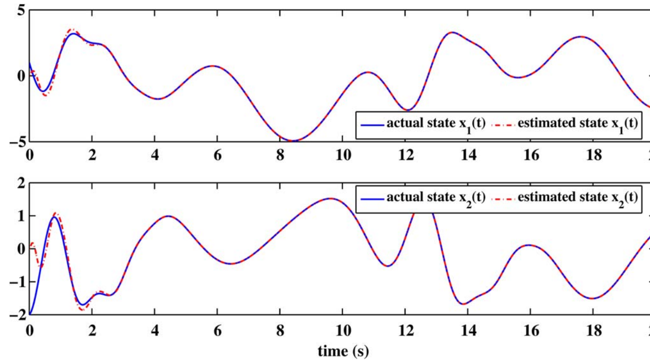

Fig. 1. States and estimates.

D_2 and computation, one obtains $h_1 = -1$ and $h_2 = -6/5$ for satisfying (47), but the solution is $h_1 = -1$ and $h_2 = -2$ for (48). The solutions H for the two equations are different, and there is no common matrix H satisfying both (47) and (48); therefore, a UIO in the form (11) does not exist.

b) Proposed approach illustration: Let us go back to the original system (40) and try to construct the observer (23). The matrices Q_ρ and P_ρ are easily computed algebraically and given by

$$Q_\rho = \begin{pmatrix} 1 \\ \frac{1+\rho(t)}{\rho(t)} \end{pmatrix} \quad P_\rho = \begin{pmatrix} 0 & 0 \\ -\frac{1+\rho(t)}{\rho(t)} & 1 \end{pmatrix}. \tag{50}$$

The state estimation error dynamics is then expressed by

$$\dot{e}(t) = \left(\underbrace{\begin{pmatrix} 0 & 0 \\ -\rho & -\frac{(1+\rho)^2}{\rho} - 2 \end{pmatrix}}_{P_\rho A_\rho} - \underbrace{\begin{pmatrix} l_1(\rho) \\ l_2(\rho) \end{pmatrix}}_{L_\rho} \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} \right) e(t). \tag{51}$$

For this simple example, it is clear that the eigenvalues of the matrix $(P_\rho A_\rho - L_\rho C)$ for all fixed values of parameter ρ in the interval $[1,5]$ are $-((1+\rho(t))^2/\rho(t)) - 2$ and $-l_1(\rho)$, where $l_1(\rho)$ is the first component of the gain vector L_ρ. For all values of $\rho(t) \in [1,5]$, the first eigenvalue is always negative. It is thus enough to fix $l_1(\rho) > 0$ in order to ensure asymptotic stability. The second gain component $l_2(\rho)$ is arbitrary. For example, let us choose the constant scalars $l_1 = 3$ and $l_2 = 0$. With this choice, the state estimation error dynamics is asymptotically stable, with the matrix $(P_\rho A_\rho - L_\rho C)$ having the poles at -3 and $-((1+\rho(t))^2/\rho(t)t) - 2 < 0,\ \forall\, \rho(t) \in \Theta$. One obtains the gain

$$L_\rho = L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}. \tag{52}$$

For this simple example, the gain L_ρ is computed by hand, but it is possible to use the LMI conditions, as presented in (39). The results for state estimation are depicted in Fig. 1, and the UI estimation result is illustrated in Fig. 2. It can be observed that both the states and the UI are estimated asymptotically.

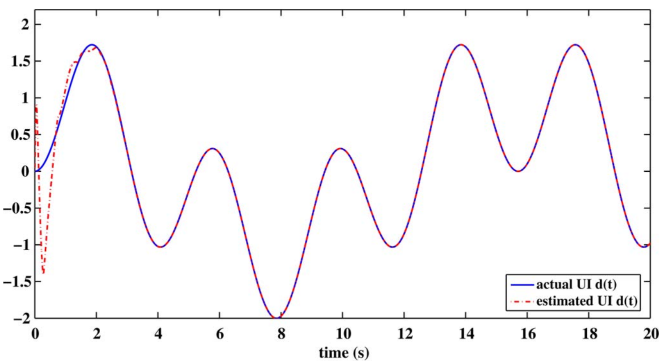

Fig. 2. UI and its estimate.

B. Extension to UIO for Arbitrary Relative Degree

In the previous section, the focus is made on the existence and design of an observer for systems with relative degree 1 with respect to the parameter variation. Here, an extension of the previous approach for systems with higher relative degree $1 < r \leq n$ is proposed. Such a problem is studied in several papers for linear systems. One can cite the interesting work in [24] exploiting the time derivatives of the output; however, at this moment, the tools aiming to estimate the time derivatives of a signal are not yet developed. In [25], an auxiliary output is constructed by differentiating the output of the system. The time derivatives of the output are obtained by high-gain differentiators. In [26], an approach based on sliding-mode differentiation of the output and sliding-mode techniques is proposed for linear systems with UIs and arbitrary relative degree. This approach is extended to nonlinear systems in [27] by using the supertwisting algorithm. For LPV discrete-time systems, an interesting approach is also proposed in [28]. It uses a dead-beat functional observer. A linear combination of states is estimated over a finite number of time samples, and the convergence of the observer is proven by using the notions of system inversion, invariant subspaces, and nilpotent semigroups.

Here, the same problem is studied for continuous-time LPV systems with a new observer structure. Let us assume the system (1) with UI $d(t)$ and relative degree r, such that $1 < r \leq n$. By differentiating the output $y(t)$ r times, it follows that

$$y(t) = Cx(t) \tag{53}$$

$$\dot{y}(t) = \underbrace{CA_\rho}_{M_{1\rho}} x(t) \tag{54}$$

$$\ddot{y}(t) = \underbrace{(M_{1\rho}A_\rho + \dot{M}_{1\rho})}_{M_{2\rho}} x(t) \tag{55}$$

$$y^{(3)}(t) = \underbrace{(M_{2\rho}A_\rho + \dot{M}_{2\rho})}_{M_{3\rho}} x(t) \tag{56}$$

$$\vdots \tag{57}$$

$$y^{(r)}(t) = \underbrace{\left(M_{(r-1)\rho}A_\rho + \dot{M}_{(r-1)\rho}\right)}_{M_{r\rho}} x(t) + M_{(r-1)\rho}D_\rho d(t). \tag{58}$$

Note that the matrices $M_{i\rho},\ i = 1, \ldots, r$ may depend on the time derivatives of the parameters; this notation is adopted, in order to ease the mathematical developments.

Consequently, the observer takes the form

$$\begin{cases} \dot{\hat{x}}(t) = (A_\rho - L_\rho C - Q_\rho M_{r\rho})\hat{x}(t) + Q_\rho y^{(r)}(t) + L_\rho y(t) \\ \hat{d}(t) = \left(M_{(r-1)\rho}D_\rho\right)^{-1}\left(y^{(r)}(t) - M_{r\rho}\hat{x}(t)\right) \end{cases} \tag{59}$$

where

$$Q_\rho = D_\rho\left(M_{(r-1)\rho}D_\rho\right)^{-1}. \tag{60}$$

1) Convergence Analysis: Let us define the state and UI estimation errors $e(t) = x(t) - \hat{x}(t)$ and $e_d(t) = d(t) - \hat{d}(t)$. The state estimation error dynamics is given by the following equation:

$$\begin{aligned} \dot{e}(t) =& A_\rho x(t) + D_\rho d(t) \\ & - (A_\rho - L_\rho C - Q_\rho M_{r\rho})\hat{x}(t) \\ & - Q_\rho y^{(r)}(t) - L_\rho y(t) \end{aligned} \tag{61}$$

$$\begin{aligned} =& (A_\rho - L_\rho C - Q_\rho M_{r\rho})x(t) \\ & + \left(D_\rho - Q_\rho M_{(r-1)\rho}D_\rho\right)d(t) \\ & - (A_\rho - L_\rho C - Q_\rho M_{r\rho})\hat{x}(t). \end{aligned} \tag{62}$$

From the term $(D_\rho - Q_\rho M_{(r-1)\rho}D_\rho)$, the UI is completely decoupled in the error dynamics equation if

$$D_\rho - Q_\rho M_{(r-1)\rho}D_\rho = 0 \tag{63}$$

which allows computing the matrix $Q_\rho = D_\rho(M_{(r-1)\rho}D_\rho)^{-1}$. The existence of the matrix Q_ρ is ensured by the uniform relative degree r of the system. This condition can be expressed as

$$rank\left(M_{(r-1)\rho}D_\rho\right) = rank(D_\rho) \tag{64}$$

for all parameter values and their time derivatives. Then, the state estimation error dynamics becomes

$$\dot{e}(t) = (A_\rho - L_\rho C - Q_\rho M_{r\rho})e(t). \tag{65}$$

Finally, if the pair $(A_\rho - Q_\rho M_{r\rho}, C)$ is observable or at least detectable for all values of $\rho(t)$ and its time derivatives, then there exists a gain L_ρ that ensures the asymptotic stability of the system (65), which ensures, then, asymptotic state estimation. In addition, since

$$\hat{d}(t) = \left(M_{(r-1)\rho}D_\rho\right)^{-1}\left(y^{(r)}(t) - M_{r\rho}\hat{x}(t)\right) \tag{66}$$

$$= \left(M_{(r-1)\rho}D_\rho\right)^{-1} M_{r\rho}e(t) + d(t) \tag{67}$$

which leads to

$$e_d(t) = -\left(M_{(r-1)\rho}D_\rho\right)^{-1} M_{r\rho}e(t). \tag{68}$$

When the error $e(t)$ converges toward zero asymptotically, the UI error $e_d(t)$ converges also toward zero.

2) Illustrative Example 2: Let us consider the same example given by (40) with a modified matrix D_ρ as follows:

$$D_\rho = \begin{pmatrix} 0 \\ 1+\rho(t) \end{pmatrix}.$$

It is then clear that, if the rank condition does not hold, i.e., $rank(CD_\rho) = 0 \neq rank(D_\rho)$, then the first observer does not exist. The relative degree of the output $y(t)$ with respect to $d(t)$ is 2. This allows computing the second derivative of the output as follows:

$$\begin{aligned} y(t) &= Cx(t) = (1 \quad 0)\, x(t) \\ \dot{y}(t) &= CA_\rho x(t) = \underbrace{(0 \quad 1+\rho(t))}_{M_{1\rho}}\, x(t) \\ \ddot{y}(t) &= \left(M_{1\rho}A_\rho + \dot{M}_{1\rho}\right) x(t) + M_{1\rho}D_\rho d(t) \\ &= \underbrace{(-\rho(t)(1+\rho(t)) \quad \dot{\rho}(t) - 2\rho(t) - 2)}_{M_{2\rho}}\, x(t) \\ &\quad + (1+\rho)^2 d(t). \end{aligned}$$

The matrix Q_ρ is given by

$$Q_\rho = \begin{pmatrix} 0 \\ \frac{1}{\rho(t)+1} \end{pmatrix}.$$

The state estimation error is then given by

$$\dot{e}(t) = \left(\underbrace{\begin{pmatrix} 0 & \rho(t)+1 \\ 0 & \frac{-\dot{\rho}(t)+2\rho(t)+2}{\rho(t)+1} - 2 \end{pmatrix}}_{A_\rho - Q_\rho M_{2\rho}} - \underbrace{\begin{pmatrix} l_1(\rho) \\ l_2(\rho) \end{pmatrix}}_{L_\rho} \underbrace{(1 \quad 0)}_{C} \right) e(t). \tag{69}$$

The observability matrix of the pair $(A_\rho - Q_\rho M_{r\rho}, C)$ is

$$\mathcal{O} = \begin{pmatrix} C \\ C\left(A_\rho - Q_\rho M_{2\rho}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho(t)+1 \end{pmatrix}. \tag{70}$$

It can be seen that, $\forall\, \rho(t) \in \Theta$, we have $rank(\mathcal{O}) = 2$. Consequently, it is possible to compute the gain matrix L_ρ in such a way to have asymptotic convergence of the state estimation error.

For the design of the gain matrix L_ρ, the polytopic approach is used. Let us define parameters

$$\rho_1(t) = \rho(t) \quad \text{and} \quad \rho_2(t) = \frac{-\dot{\rho}(t) + 2\rho(t) + 2}{\rho(t)+1}. \tag{71}$$

The parameter $\rho(t)$ is bounded and sufficiently smooth; then, the time derivative $\dot{\rho}(t)$ is also bounded; consequently, parameters $\rho_1(t)$ and $\rho_2(t)$ are bounded as follows:

$$\rho_1^{\min} \leq \rho_1(t) \leq \rho_1^{\max} \quad \text{and} \quad \rho_2^{\min} \leq \rho_2(t) \leq \rho_2^{\max}. \tag{72}$$

The matrix $(A_\rho - Q_\rho M_{2\rho})$ can be expressed exactly in polytopic form as follows:

$$A_\rho - Q_\rho M_{2\rho} = \sum_{i=1}^{4} \mu_i\left(\rho_1(t), \rho_2(t)\right) \mathcal{A}_i. \tag{73}$$

From this polytopic form, the gain matrix L_ρ takes the form

$$L_\rho = \sum_{i=1}^{4} \mu_i\left(\rho_1(t), \rho_2(t)\right) \mathcal{L}_i. \tag{74}$$

The state estimation error is then expressed in the following form:

$$\dot{e}(t) = \sum_{i=1}^{4} \mu_i\left(\rho_1(t), \rho_2(t)\right) (\mathcal{A}_i - \mathcal{L}_i C) e(t) \tag{75}$$

where the matrices $\mathcal{L}_i, i = 1, \ldots, 4$ are designed using the LMI conditions given in (39). The results are depicted in Figs. 3 and 4. It can be seen that, even if the matching condition is not satisfied, i.e., $rank(CD_\rho) \neq rank(D_\rho), \forall\, \rho \in \Theta$, the state and the UI estimation are realized asymptotically. Note that the time derivatives of the signals $y(t)$ and $\rho(t)$ are computed with HOSMDs, which provide an exact and finite-time estimation of the needed time derivatives of the output and parameters (for more details, the reader can refer to [20]).

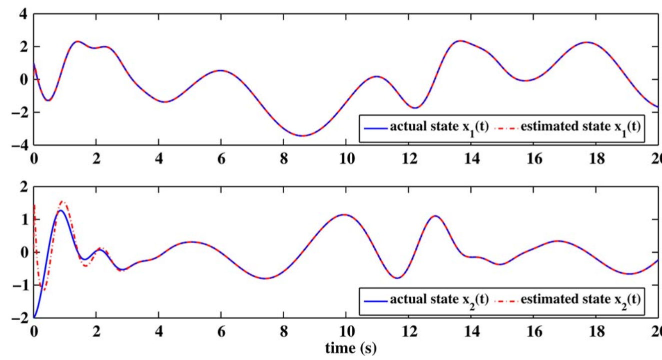

Fig. 3. States and estimates.

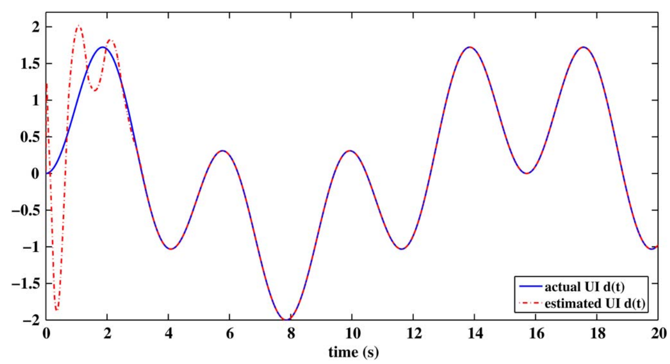

Fig. 4. UI and its estimate.

3) Illustrative Example 3: In this example, the proposed approach is applied for the problem of lateral force estimation in a vehicle. For that purpose, let us consider the bicycle model given by the equations [29], [30]

$$\begin{cases} \dot{v}_y(t) = \frac{1}{m}\left(F_{yf} + F_{yr}\right) - v_x(t)\dot{\psi}(t) \\ \ddot{\psi}(t) = \frac{1}{I_z}\left(a_f F_{yf} - a_r F_{yr}\right) \end{cases} \tag{76}$$

where v_y is the lateral velocity; $\dot{\psi}(t)$ is the yaw rate; $v_x(t)$ is the longitudinal velocity; m is the mass of the vehicle; I_z is the yaw moment of inertia; a_f and a_r are the distances from the front and rear axle to the center of gravity, respectively; F_{yf} and F_{yr} are the front and rear lateral forces, respectively. These forces are nonlinear and commonly represented by nonlinear models as Pacejka's magic formula or Dugoff's model, depending on several parameters, which are related to the characteristics of the tires and the road. It is then difficult to identify these parameters due to their variations. In order to estimate the forces, two observers are designed separately but with the same design approach. In this example, the first observer is provided (the second follows exactly the same procedure). First, since the lateral acceleration is given by $a_y = (1/m)(F_{yf} + F_{yr})$, the model (76) can be rewritten as follows:

$$\begin{cases} \dot{v}_y(t) = a_y(t) - v_x(t)\dot{\psi}(t) \\ \ddot{\psi}(t) = \frac{a_f m}{I_z} a_y(t) - \frac{l}{I_z} F_{yr}(t) \end{cases} \tag{77}$$

where $l = a_f + a_r$. Note that the front force does not appear in this model, which makes the estimation of the rear force insensitive to the parameters of the front force model. Furthermore, in order to cope with eventual parameter uncertainties affecting the lateral rear wheel force F_{yr}, it is interesting to include, in the model (77), the dynamics of F_{yr} by using the relaxation expression $\tau_r \dot{F}_r + F_r = F_r^S$. The inputs of this system is F_r^S, denoting the steady behavior of the lateral rear force, which can be represented by different models as Pacejka's magic formula [31], Dugoff's model, etc. In this paper, Pacejka's model is considered. The parameter τ_r is given by $\tau_r = r/r_D|\Omega|$, where $r_D|\Omega|$ denotes the longitudinal velocity (r_D and Ω represent the dynamic rolling radius and the angular velocity), and r is the relaxation lengths. The whole model is then expressed by

$$\begin{cases} \dot{v}_y(t) = a_y(t) - v_x(t)\dot{\psi}(t) \\ \ddot{\psi}(t) = \frac{a_f m}{I_z} a_y(t) - \frac{l}{I_z} F_{yr}(t) \\ \dot{F}_{yr}(t) = -\frac{v_x(t)}{\tau_r} F_{yr}(t) + \frac{v_x(t)}{\tau_r} F_{yr}^S(t). \end{cases} \tag{78}$$

The lateral rear force, which is expressed by Pacejka's magic formula [31], is given by

$$F_{yr}^S = D_r \sin\left(C_r \tan^{-1}\left(B_r\left(1 - E_r\right)\alpha_r + \tan^{-1}\left(B_r \alpha_r\right)\right)\right) \tag{79}$$

where D_r, B_r, E_r, and C_r are parameters that depend on the rear tire characteristics, which are uncertain. α_r is the rear sideslip angle of the rear tire as follows:

$$\alpha_r = -\beta + \tan^{-1}\left(\frac{a_r}{v_x}\dot{\psi}\cos(\beta)\right) \tag{80}$$

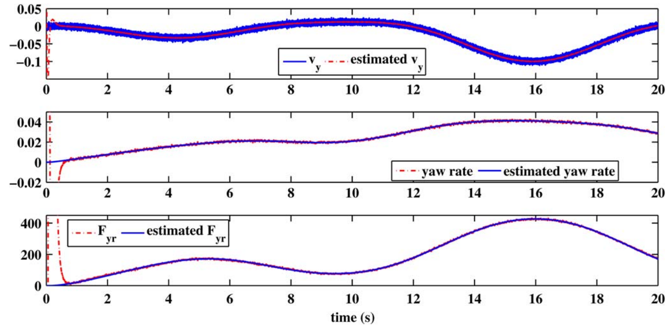

Fig. 5. State estimation.

where β is the body sideslip angle, which is given by $\beta = \tan^{-1}(v_y/v_x)$. For small variations of the sideslip angles, corresponding to the rational driving between the normal and pseudo-sliding regions, i.e., not exceeding 8°, we have $\beta \approx v_y/v_x$; hence, the sideslip angle may be simplified as follows $\alpha_r \approx -(v_y/v_x) + (a_r/v_x)\dot{\psi}$. The LPV system can be written in the following form:

$$\dot{x}(t) = A(v_x(t))x(t) + Bu(t) + D(v_x(t))\xi(x,t) \tag{81}$$

where

$$x(t) = \begin{pmatrix} v_y & \dot{\psi} & F_{yr} \end{pmatrix}^T, \quad u(t) = a_y$$

$$\xi(x,t) = F_{yr}^S - c_r \alpha_r$$

$$A(v_x(t)) = \begin{pmatrix} 0 & -v_x(t) & 0 \\ 0 & 0 & -\frac{l}{I_z} \\ -\frac{c_r}{\tau_r} & \frac{c_r a_r}{\tau_r} & -\frac{v_x}{\tau_r} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ \frac{a_f m}{I_z} \\ 0 \end{pmatrix}$$

$$D(v_x(t)) = \begin{pmatrix} 0 & 0 & \frac{v_x(t)}{\tau_r} \end{pmatrix}^T$$

and $v_x(t)$ is the time-varying parameter. The constant c_r is known. Assume that the lateral velocity is measured or estimated (see, e.g., [32] or [33]); then, the output equation is $y(t) = Cx(t) + \eta(t)$, where $C = (1\,0\,0)$, and η is a white noise affecting the measured output. It is then clear that the relative degree of the output with respect to the uncertain term $\xi(x,t)$ is 3. By following the proposed observer design approach, the state and UI estimations are provided in Figs. 5 and 6, respectively. It can be seen that, even if the output is noisy, the observer provides both acceptable state and UI estimations. Note that the classical UIO does not exist because the condition $rank(CD(v_x(t)) = rank(D(v_x(t)))$ is not satisfied.

In order to design the second observer estimating the front lateral force, the same strategy is applied but contrarily to the first model (77), i.e., the force F_{yr} is substituted in the model, in order to have a new model with F_{yf} as a state component and free from the rear force.

Remark 2: The proposed approach can be extended for LPV MUIMO systems by considering the relative degree of each output with respect to the UIs. Let us consider the LPV system

$$\begin{cases} \dot{x}(t) = A_\rho x(t) + D_\rho d(t) \\ y(t) = Cx(t) \end{cases} \tag{82}$$

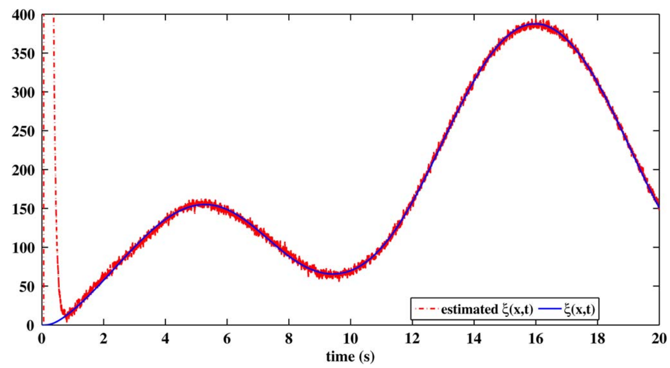

Fig. 6. UI estimation.

where $y(t) \in \mathbb{R}^{n_y}$, and $d(t) \in \mathbb{R}^{n_d}$. Let us assume that $n_y \geq n_d$ and the vector relative order is given by $\{r_1, r_2, \ldots, r_{n_y}\}$. Each value $r_i, i = 1, \ldots, n_y$ describes the relative degree of the ith output with respect to the UIs. The matrices D_ρ and C are expressed as follows:

$$D_\rho = \begin{pmatrix} D_\rho^1 & D_\rho^2 & \cdots & D_\rho^{n_d} \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n_y} \end{pmatrix}. \quad (83)$$

According to the relative degrees of each output and following the same calculus as in (58), the following output time derivatives are obtained in matrix form as follows:

$$\mathcal{Y}(t) = \mathcal{M}_\rho x(t) + \Gamma_\rho d(t) \quad (84)$$

where

$$\mathcal{Y}(t) = \begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_{n_y}^{(r_{n_y})}(t) \end{pmatrix}, \quad \mathcal{M}_\rho = \begin{pmatrix} M_{r_1\rho}^1 \\ M_{r_2\rho}^2 \\ \vdots \\ M_{r_{n_y}\rho}^{n_y} \end{pmatrix}$$

and

$$\Gamma_\rho = \begin{pmatrix} M_{(r_1-1)\rho}^1 D_\rho^1 & M_{(r_1-1)\rho}^1 D_\rho^2 & \cdots & M_{(r_1-1)\rho}^1 D_\rho^{n_d} \\ M_{(r_2-1)\rho}^2 D_\rho^1 & M_{(r_2-1)\rho}^2 D_\rho^2 & \cdots & M_{(r_2-1)\rho}^2 D_\rho^{n_d} \\ \vdots & \vdots & \ddots & \vdots \\ M_{(r_{n_y}-1)\rho}^{n_y} D_\rho^1 & M_{(r_{n_y}-1)\rho}^{n_y} D_\rho^2 & \cdots & M_{(r_{n_y}-1)\rho}^{n_y} D_\rho^{n_d} \end{pmatrix}.$$

Finally, we can consider two cases: the first case assumes that $n_y = n_d$ (square system) and the observer is given by the equations

$$\begin{cases} \dot{\hat{x}}(t) = (A_\rho - L_\rho C - Q_\rho \mathcal{M}_\rho)\hat{x}(t) + Q_\rho \mathcal{Y}(t) + L_\rho y(t) \\ \hat{d}(t) = \Gamma_\rho^{-1}(\mathcal{Y}(t) - \mathcal{M}_\rho \hat{x}(t)) \end{cases} \quad (85)$$

where $Q_\rho = D_\rho \Gamma_\rho^{-1}$, and in the more general case where $n_y > n_d$, the second equation of the observer, relating to the UI estimation, takes the form

$$\hat{d}(t) = \Gamma_\rho^\dagger(\mathcal{Y}(t) - \mathcal{M}_\rho \hat{x}(t)) \quad (86)$$

where $\Gamma_\rho^\dagger$ denotes the generalized inverse of Γ_ρ. The matrix Q_ρ is computed by $Q_\rho = D_\rho \Gamma_\rho^\dagger$. In both the two cases, the observers exist if the matrices Γ_ρ^{-1} and $\Gamma_\rho^\dagger$ exist, which are guaranteed by the full column rank of Γ_ρ for all values of parameters ρ and their time derivatives.

Notice that, if the generalized inverse $\Gamma_\rho^\dagger$ does not exist, it does not imply that the UIOs do not exist, as for the SUISO case. Indeed, it may happen that the system is strongly algebraically observable but the generalized inverse of $\Gamma_\rho^\dagger$ does not exist as constructed in (84) [34]. In such a case, in order to overcome this problem, the vector $\mathcal{Y}(t)$, previously defined as a concatenation of the outputs and their time derivatives, is replaced by a new vector consisting of the outputs, their time derivatives, and some parameter-varying linear combinations of them. The generalized inverse of the new matrix $\Gamma_\rho^\dagger$ exists only if the system is strongly algebraically observable. In order to illustrate this fact, consider the following system (inspired by the nonlinear example given in [34]:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\rho(t) & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ \rho(t) & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix} \quad (87)$$

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) \quad (88)$$

with parameter $\rho(t) \in [1, 5]$. It is clear that the relative degrees of $y_1(t)$ and $y_2(t)$ are $r_1 = 2$ and $r_2 = 1$, respectively. In addition, the system is strongly algebraically observable (Definition 1) since the states can be expressed as follows:

$$\begin{cases} x_1(t) = y_1(t) \\ x_2(t) = \dot{y}_1(t) \\ x_3(t) = y_2(t) \\ x_4(t) = \frac{1}{1+\rho(t)}(\rho(t)\dot{y}_2(t) - \rho(t)\dot{y}_1(t) - \ddot{y}_1(t)) \end{cases} \quad (89)$$

which are defined for all $\rho(t) \in [1, 5]$. Following the defined vector $\mathcal{Y}(t) = (\ddot{y}_1(t) \;\; \dot{y}_2(t))^T$ in (84), one obtains

$$\Gamma_\rho = \begin{pmatrix} \rho(t) & 0 \\ 1 & 0 \end{pmatrix} \quad (90)$$

which is singular and does not admit an inverse even if the system is strongly algebraically observable. However, following the discussion previously presented, the vector $\mathcal{Y}(t)$ is replaced by

$$\mathcal{Y}(t) = \begin{pmatrix} \ddot{y}_1(t) \\ \dot{\tilde{y}}_3(t) \end{pmatrix} \quad (91)$$

where $\tilde{y}_3(t) = \ddot{y}_1(t) - \rho(t)\dot{y}(t)$. This leads to the new matrix

$$\Gamma_\rho = \begin{pmatrix} \rho(t) & 0 \\ -\rho^2(t) & -1 \end{pmatrix} \quad (92)$$

which is invertible $\forall\, \rho(t) \in [1, 5]$, and its inverse is given by

$$\Gamma_\rho^{-1} = \begin{pmatrix} \frac{1}{\rho(t)} & 0 \\ -\rho(t) & -1 \end{pmatrix}. \quad (93)$$

The UI decoupling matrix Q_ρ is then given by

$$Q_\rho = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \frac{1}{\rho(t)} & 0 \\ -\rho(t) & -1 \end{pmatrix}. \tag{94}$$

Remark 3: If the system is reduced to a linear model (when the parameter is constant with respect to time)

$$\dot{x}(t) = Ax(t) + Dd(t), \quad y(t) = Cx(t) \tag{95}$$

the same analysis, exploiting the time derivatives of the output $y(t)$, leads to the decoupling condition $rank(CA^{r-1}D) = rank(D)$, where r is the relative degree of the output $y(t)$ with respect to the UI $d(t)$. Note that, if the parameter is constant, the rank condition (64) reduces to $rank(CA^{r-1}D) = rank(D)$. For such systems, the observer is reduced to [17]

$$\begin{cases} \dot{\hat{x}}(t) = (PA - LC)\hat{x}(t) + Qy^{(r)}(t) + Ly(t) \\ \hat{d}(t) = \left(CA^{r-1}D\right)^{-1}\left(y^{(r)}(t) - CA^r\hat{x}(t)\right) \end{cases} \tag{96}$$

where $Q = D\left(CA^{r-1}D\right)^{-1}$, and $P = I_n - QCA^{r-1}$. Note also that the observer structure (96) is very close to

$$\begin{cases} \dot{z}(t) = Nz(t) + Ly(t) \\ \hat{x}(t) = z(t) - Hy^{(r-1)}(t) \end{cases} \tag{97}$$

but has the advantage to reduce the number of output time derivatives from r to $r-1$. For LPV systems, the observer will have the following structure:

$$\begin{cases} \dot{z}(t) = N_\rho z(t) + L_\rho y(t) \\ \hat{x}(t) = z(t) - H_\rho y^{(r-1)}(t). \end{cases} \tag{98}$$

The design of the matrices N_ρ, L_ρ, and H_ρ follows the same procedure as in Section IV-B. These matrices may depend on the parameters and their successive r time derivatives. The structures (97) and (98) are also new and have not been previously proposed for arbitrary relative degree ($r > 1$). However, for relative degree of 1 ($r = 1$), the structure (97) reduced to the classical UIO [2], but in the LPV case, structure (98) is still new since the introduced matrix H_ρ is parameter dependent, which relaxes the decoupling condition, as discussed in Sections III and IV-A2a.

V. Conclusion

This paper has discussed the design of UIOs for LPV systems. It has been shown that the classical approach consisting in the transformation of the system in a polytopic form is limited because it can lead to unsatisfied conditions even if the system is uniformly strongly observable with respect to the parameter domain of variation. Then, a new UIO has been proposed, in order to cope with this problem. The example illustrates clearly that a classical approach for UIO design is not possible, while the proposed observer provides a solution. Finally, an extension of the proposed approach has been provided for systems having output with arbitrary relative degree (greater than one) when the condition $rank(CD_\rho) = rank(D_\rho)$ is not satisfied. In this paper, SUISO systems are considered, in order to develop the proposed strategy. An extension is provided for MUIMO systems, with the notion of uniform relative order, which takes into account the relative degree of each output with respect to each UI.

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