Single Inverted Pendulum System Modeling

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Content

Practical Problem

As the figure below, it is depicted a Single Inverted Pendulum System.What we describe is a model widely used in aerospace field.Inverted Pendulum is installed in a car,moving with the car.

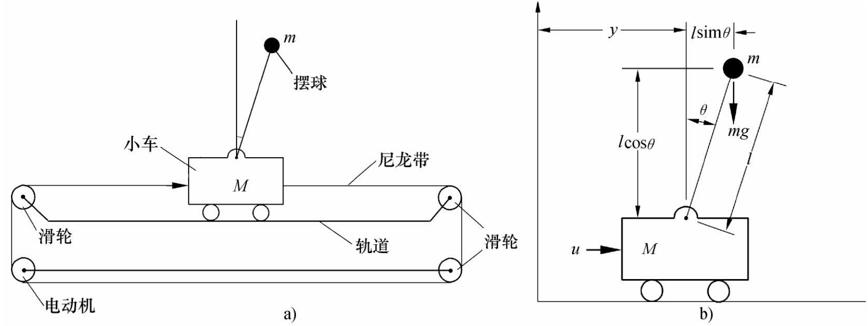


Figure: The Model of Inverted Pendulum

Mathematical Modeling

Then we can derive state-space equation of the Inverted Pendulum.

- In the horizontal direction, we can get

$$M\frac{d^2y}{dt^2} + m\frac{d^2(y + l\sin\theta)}{dt^2} = u \tag{1}$$

- In the direction perpendicular to the pendulum, we can get

$$m\frac{d^2(y + l\sin\theta)}{dt^2}\cos\theta = mg\sin\theta \tag{2}$$

Mathematical Modeling

State-Space Equations

When θ and $\dot{\theta}$ is very small, we have $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and $\theta\dot{\theta}^2 \approx 0$. Thus, from the page before we can get

$$(M+m)\ddot{y} + ml\ddot{\theta} = u \tag{3}$$

$$m\ddot{y} + ml\ddot{\theta} = mg\theta \tag{4}$$

Then we can get:

$$\ddot{y} = -\frac{mg}{M}\theta + \frac{1}{M}u \tag{5}$$

$$\ddot{\theta} = \frac{(M+m)g}{Ml}\theta - \frac{1}{Ml}u \tag{6}$$

Let variables $x_1 = y, x_2 = \dot{y}, x_3 = \theta, x_4 = \dot{\theta}$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix} u \tag{7}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{8}$$

Mathematical Modeling

For the sake of simplicity, assume $m = M = l = 1, g = 10$ by the form of state-space description,

$$\dot{x} = Ax + Bu \tag{9}$$

$$y = Cx \tag{10}$$

Then we can get:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 20 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

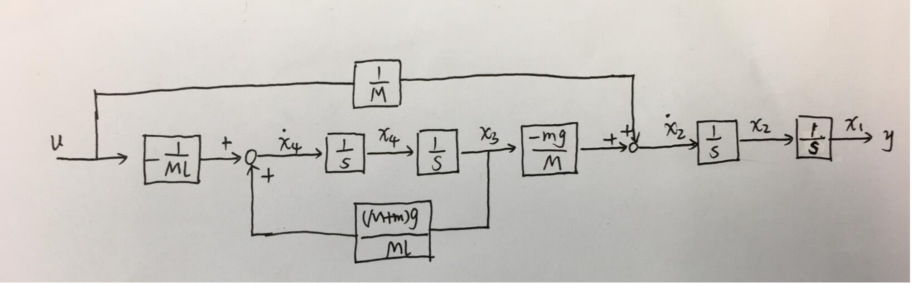


Figure: The State Feedback Diagram of Inverted Pendulum

Controllability

In first step we should judge the controllability to solve the problem. We enter the following program in Matlab,

Program

```
A=[0 1 0 0;0 0 -10 0;0 0 0 1;0 0 20 0];
B=[0;1;0;-1];
C=[1 0 0 0];
rct=rank(ctrb(A,B));
```

The result is that rct=4

When the controllability matrix C=[B AB ... $A^{n-1}B$] has full row rank, it is controllable.

Controllability

Because the system is controllable, we can configure any poles by state feedback. We enter the following program in Matlab,

Program

```
A=[0 1 0 0;0 0 -10 0;0 0 0 1;0 0 20 0];
B=[0;1;0;-1];
P=[-6 -6.6 -7 -7.5];
K=place(A,B,P);
```

The result is K= -207.9000 -123.5700 -502.7000 -150.6700

The state feedback is K= [-207.9 -123.57 -502.7 -150.67]

Simulation Results

As shown in the figure below, we use Simulink to construct Single Inverted Pendulum State Feedback Control System Simulation Model.

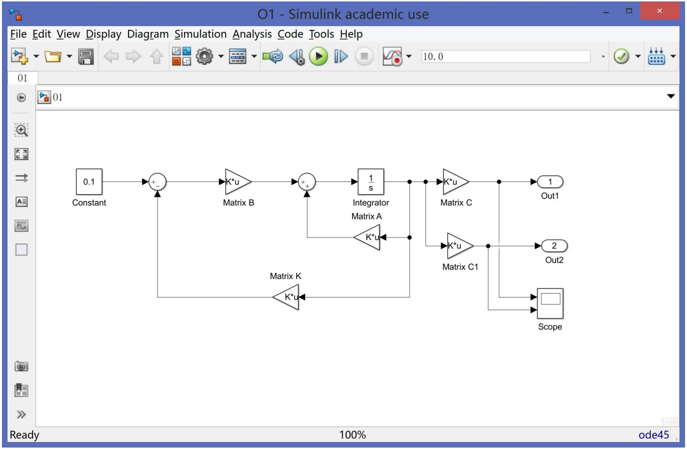


Figure: The State Feedback Control System Simulation Model

Simulation Results

The yellow line in the figure represents the displacement, the blue line represents angle.From the figure, we can get the system will be stable when $\theta = 0$.

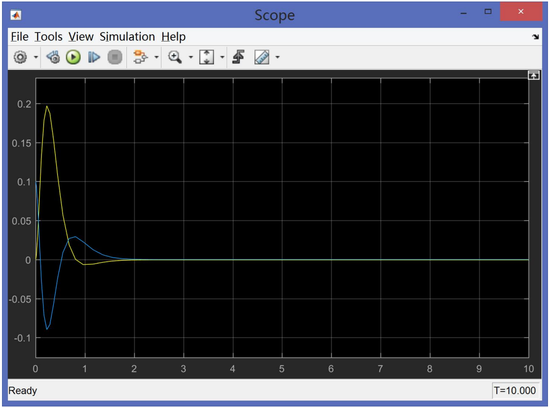


Figure: The State Feedback Control System Simulation Diagram

Observability

We should judge the observability to solve the problem. We enter the following program in Matlab,

Program

```
A=[0 1 0 0;0 0 -10 0;0 0 0 1;0 0 20 0];
B=[0;1;0;-1];
C=[1 0 0 0];
rob=rank(obsv(A,C));
```

The result is that rob=4.

When the observability matrix $O = [C \quad CA \quad \dots \quad CA^{n-1}]^T$ has full column rank, it is observable.
Therefore, we can use the state estimator in state feedback.

Observability

To design the state estimator, we should make the real part of the eigenvalues negative and make absolute value of eigenvalues larger than the absolute value of the eigenvalues of state feedback so that the state estimator has a fast convergence rate.
Take the eigenvalues of the state estimator to be S=[-20 -21 -22 -23]. We can compete the state estimator matrix in Matlab.

Program

```
A=[0 1 0 0;0 0 -10 0;0 0 0 1;0 0 20 0];
B=[0;1;0;-1];
C=[1 0 0 0];
A1=A';
C1=C';
P=[-20 -21 -22 -23];
G1=place(A1,C1,P);
G=G1';
```

The result is G=[86 2791 -4137 -26834]

Then we add state estimator to model the Inverted Pendulum system, the simulation block diagram depicted in the figure below.

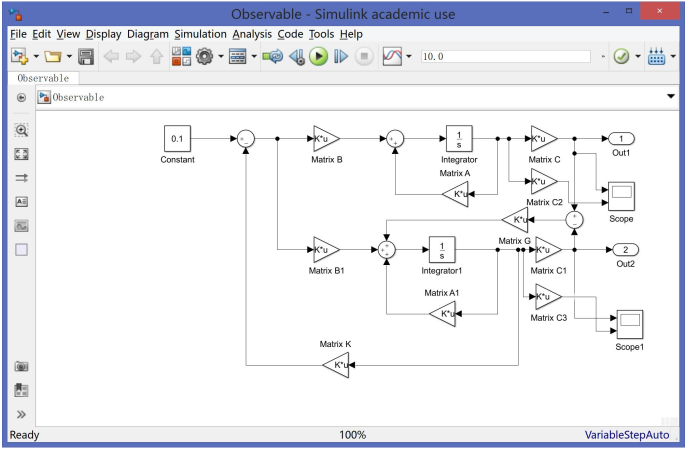
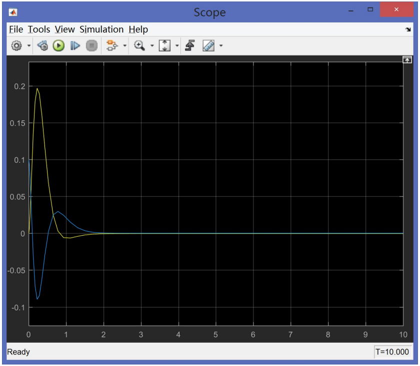


Figure: The System With Adding State Estimator

Then we add state estimator to model the system.From the figure,we can get the system will also be stable when $\theta = 0$. The difference between scope and scope1 is almost equal to 0.



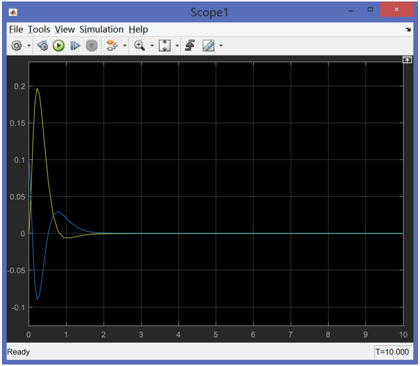


Figure: The System Simulation Diagram With Adding State Estimator

Double Inverted Pendulum Analysis

We expand the Single Inverted Pendulum to Double Inverted Pendulum. As the figure below, it is depicted an Double Inverted Pendulum System.

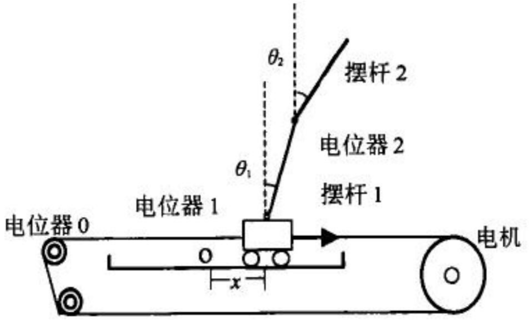


Figure: The Model of Double Inverted Pendulum

The double inverted pendulum system is divided into the following three parts: car, swing rod 1 and swing rod 2 by using the method of isolation in mechanics, the force of these parts is as follows:

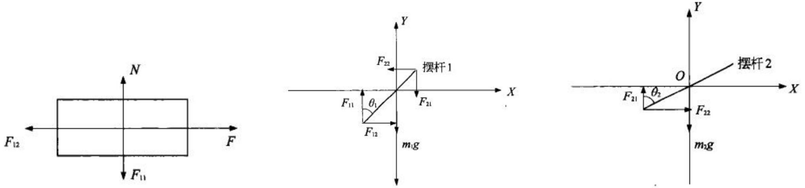


Figure: The Analysis for The Car and Two Swing Rods

Force Analysis

Using Newton's second law and the theorem of moment of momentum, we can get the following equations:

$$F - F_{12} = m_0 \ddot{r} \tag{12}$$

$$F_{12} - F_{22} = m_1 \ddot{r} + m_1 l_1 \ddot{\theta}_1 \cos\theta_1 - m_1 l_1 \ddot{\theta}_1^2 \sin\theta_1 \tag{13}$$

$$m_1 g - F_{11} + F_{21} = m_1 l_1 \ddot{\theta} \sin\theta_1 + m_1 l_1 \ddot{\theta} 1^2 \cos\theta_1 \tag{14}$$

$$d\frac{(J_1\dot{\theta}_1)}{dt} = F_{11} l_1 \sin\theta_1 + F_{21}(L_1 - l_1)\sin\theta_1 - F_{12} l_1 \cos\theta_1 - F_{22}(L_1 - l_1)\cos\theta_1 \tag{15}$$

$$F_{22} = m_2 \ddot{r} + m_2 L_1 \ddot{\theta}_1 \cos\theta_1 + m_2 l_2 \ddot{\theta}_2 \cos\theta_2 - m_2 L_1 \dot{\theta}_1 \sin\theta_1 - m_2 l_2 \dot{\theta} \sin\theta_2 \tag{16}$$

$$m_2 g - F_{21} = m_2 L_1 \ddot{\theta}_1 \sin\theta_1 + m_2 l_2 \ddot{\theta}_2 \sin\theta_2 + m_2 L_1 \dot{\theta}_1^2 \cos\theta_1 + m_2 l_2 \dot{\theta}_2^2 \cos\theta_2 \tag{17}$$

$$d\frac{(J_2\dot{\theta}_2)}{dt} = F_{21} l_2 \sin\theta_2 - F_{22} l_2 \cos\theta_2 \tag{18}$$

Linearization

In this system, m_0, m_1, m_2 is the quantity of the car, swing rod 1 and swing rod 2.L_1 is the length of swing rod, r is the displacement of the car relative to the central position,l_1, l_2 is the distance from the center of swing rod 1 and 2 to the point.Then by linearation we can get the following equations:

$$(m_0 + m_1 + m_2)\ddot{r} + (m_1 l_1 + m_2 L_1)\ddot{\theta_1} + m_2 l_2 \ddot{\theta_2} = F \tag{19}$$

$$(m_1 l_1 + m_2 L_1)\ddot{r} + (J_1 + m_1 l_1^2 + m_2 L_1^2)\ddot{\theta_1} + m_2 L_1 l_2 \ddot{\theta_2} = (m_1 l_1 + m_2 L_1) g \theta_1 \tag{20}$$

$$m_2 l_2 \ddot{r} + m_2 L_1 \ddot{\theta_1} + (J_2 + m_2 l_2^2)\theta_2 = m_2 g l_2 \theta_2 \tag{21}$$

Similar as the single inverted pendulum simulation, then we use Matlab Simulink tools to simulate the Double Inverted Pendulum system, the simulation block diagram is depicted in the figure below.

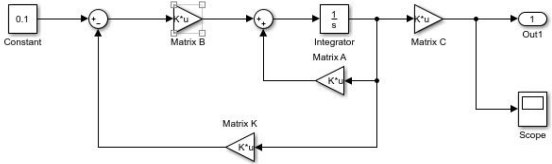


Figure: The Double Inverted Pendulum System With Feedback

Then we can get the simulation results by the scope, which is depicted in the figure below.

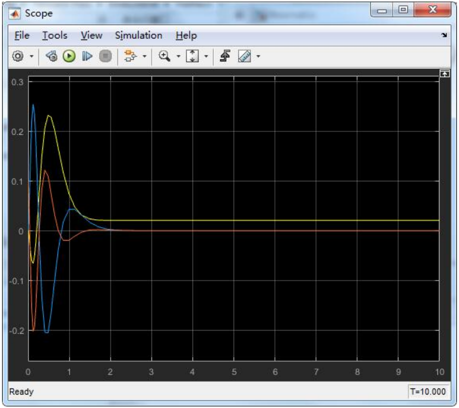


Figure: Simulation Result

Simulation Results

When we add state estimator to model the Double Inverted Pendulum system, the simulation block diagram is depicted in the figure below.

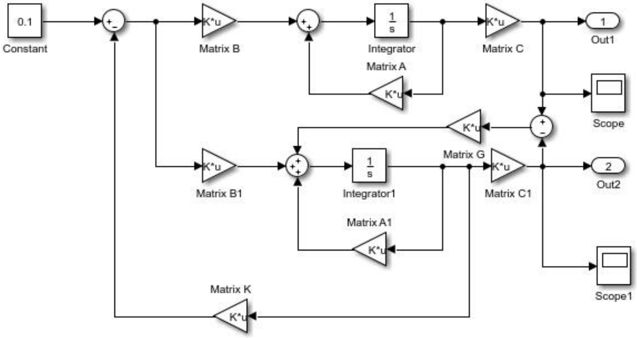


Figure: The System With Adding State Estimator

Simulation Results

Then we can also get the simulation results by the scope and scope1, which is depicted in the figure below.

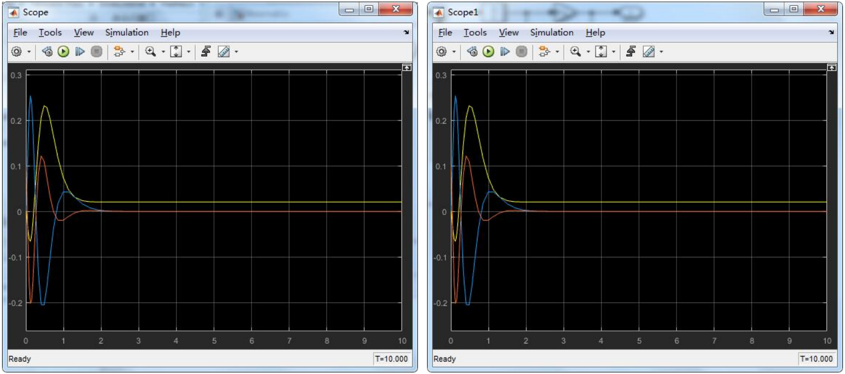


Figure: The System Simulation Result With Adding State Estimator

Conclusion

By comparison, whether we add state estimator or not,we can make good control of the Inverted Pendulum.The transfer function is invariant whether the state estimator is in.More important, adding state estimator can let us to know whether the state is close to input and make better performance than not adding state estimator.

Thank you!