Matrix Factorization with Alternating Least Squares

Introduction

In this lesson, we will look at another matrix factorization technique called Alternating Least Squares (ALS). This method can prove to be much more effective and robust than the SVD we saw earlier. ALS allows you to set regularization measures and minimize a loss function while optimizing the model parameter `k` . We will look at the math behind this approach in this lesson.

Objectives

You will be able to:

- Explain how bias terms can be used to create more accurate embedding matrices
- Describe how ALS is related to matrix decomposition and why it can be parallelized so well

Recap Matrix Factorization

In the past few lessons, we learned matrix factorization functions with the following assumptions:

- Each user can be described by k attributes or features

For example, feature 1 could be a number that says how much each user likes sci-fi movies. 2, how much he/she likes horror movies and so on.

- Each item can be described by an analogous set of k attributes or features

For our `MovieLens` example, feature 1 for a chosen movie might be a number that says how close the movie is to pure sci-fi.

> By multiplying the features of the user by the features of each item and adding these items together, we will approximate what a user would rate a particular item

Introducing a Learning Function

The simplicity in matrix factorization is that we do not have to know what these features are. Nor do we know how many (k) features are relevant. We pick a number for k and learn the relevant values for all the features for all the users and items. This is essentially the same process used for dimensionality reduction with PCA.

> **How do we integrate learning into this problem? By minimizing a loss function, of course**

We can use matrix factorization to represent each user and item by k-dimensional vectors. By letting each item i can be k-dimensional vector q_i and each user u represented by k-dimensional p_u, user u's predicted rating for item i is just the dot product of their two vectors. This means that we can represent the entire utility matrix by multiplying matrices p and q together. So how do we find out what the values for p and q are?

$$R = PQ^T$$

for the entire matrix

or

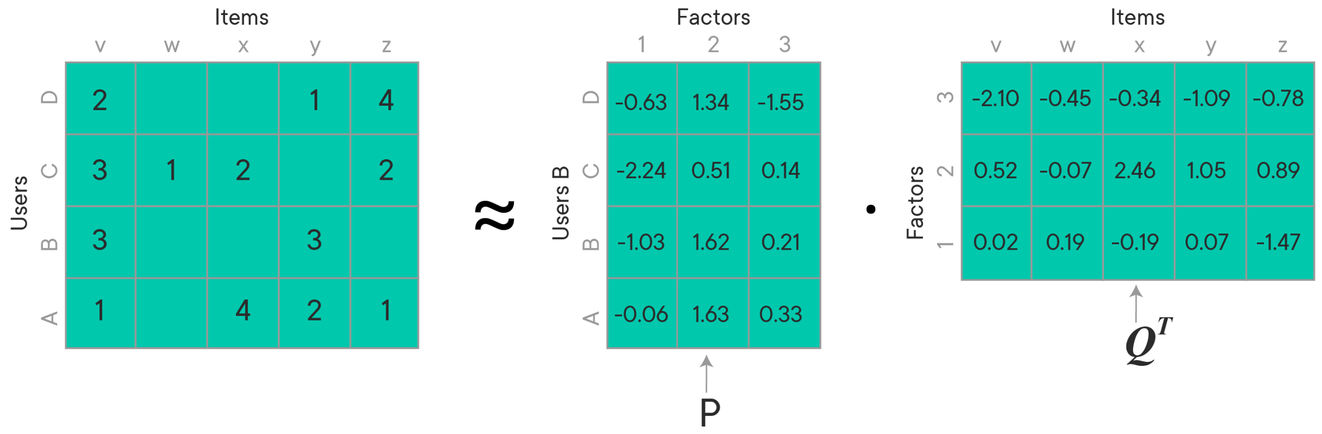
$$\hat{r}_{u,i} = q_i^{\top} p_u$$

for individual ratings

- R is the full user-item rating matrix
- P is a matrix that contains the users and the k factors represented as (user, factor)
- Q^T is a matrix that contains the items and the k factors represented as
- $\hat{r}_{u,i}$ represents our prediction for the true rating r_{ui} . In order to get an individual rating, you must take the dot product of a row of P and a column of Q

These user and item vectors are called **latent vectors**. The k attributes are called **latent factors**.

The image below is a representation of how a matrix is decomposed into two separate matrices:



If we wanted to calculate the rating for user B, item Z, our calculation would be the dot product of `[-1.03 , 1.62, 0.21]` and `[-0.78,0.89,-1.47]` . Let's calculate these values in NumPy:

In [1]:
```
import numpy as np

# users X factors
P = np.array([[-0.63274434,  1.33686735, -1.55128517],
              [-2.23813661,  0.5123861 ,  0.14087293],
              [-1.0289794 ,  1.62052691,  0.21027516],
              [-0.06422255,  1.62892864,  0.33350709]])
```

In [3]:
```
# factors X items
Q = np.array([[-2.09507374,  0.52351075,  0.01826269],
              [-0.45078775, -0.07334991,  0.18731052],
              [-0.34161766,  2.46215058, -0.18942263],
              [-1.0925736 ,  1.04664756,  0.69963111],
              [-0.78152923,  0.89189076, -1.47144019]])
```

In [5]:
```
# the original
R = np.array([[2, np.nan, np.nan, 1, 4],
              [5, 1, 2, np.nan, 2],
              [3, np.nan, np.nan, 3, np.nan],
              [1, np.nan, 4, 2, 1]])
```

In [6]:
```
print(P[2])
```
```
[-1.0289794   1.62052691  0.21027516]
```

In [7]:
```
print(Q.T[:,4])
```
```
[-0.78152923  0.89189076 -1.47144019]
```

In [8]:
```
P[2].dot(Q.T[:,4])
```
Out[8]:
```
1.9401031341455333
```

Now we can do the calculation for the entire ratings matrix. You can see that the values in the predicted matrix are *very* close to the actual ratings for those that are present in the original rating array. The other values are new!

In [9]:
```
P.dot(Q.T)
```
Out[9]:
```
array([[ 1.99717984, -0.10339773,  3.80157388,  1.00522135,  3.96947118],
       [ 4.95987359,  0.99772807,  1.9994742 ,  3.08017572,  1.99887552],
       [ 3.00799117,  0.38437256,  4.30166793,  2.96747131,  1.94010313],
       [ 0.99340337, -0.02806164,  3.96943336,  2.00841398,  1.01228247]])
```

This should remind you of how things were calculated for the SVD array, so let's see what is different. We want our predictions to be as close to the truth as possible. In order to calculate these matrices, we establish a loss function to minimize. To avoid overfitting, the loss function also includes a regularization parameter λ. We will choose a λ to minimize the square of the difference between all ratings in our dataset R and our predictions.

The loss function L can be calculated as:

$$L = \sum_{u,i \in \kappa} (r_{u,i} - q_i^T p_u)^2 + \lambda(||q_i||^2 + |p_u||^2)$$

Where κ is the set of (u, i) pairs for which $r_{u,i}$ is known.

In the equation, there are two L2 regularization terms to prevent overfitting of the user and item vectors. As always, our goal is to minimize this loss function. This could be done with something like gradient descent of course, but due to the massive size of sparse matrices so frequently associated with recommendation system datasets, this is not always feasible. That's where Alternating Least Squares comes into play.

Alternating Least Squares

For ALS minimization, we hold one set of latent vectors constant. Essentially ALS alternates between holding the q_i's constant and the p_u's constant. While all q_i's are held constant, each p_u is computed by solving the least squared problem, independently. After that process has taken place, all the p_u's are held constant while the q_i's are altered to solve the least squares problem, again, each independently. This process repeats many times until you've reached convergence (ideally).

If we assume either the user-factors or item-factors was fixed, this should be just like a regularised least squares problem. Let's look at these least squares equations written out mathematically.

First, let's assume the item vectors are fixed, we first solve for the user vectors:

$$p_u = \Big(\sum_{r_{u,i} \in r_{u*}} q_i q_i^T + \lambda I_k\Big)^{-1} \sum_{r_{u,i} \in r_{u*}} r_{ui} q_i$$

Then we hold the user vectors constant and solve for the item vectors:

$$q_i = \Big(\sum_{r_{u,i} \in r_{i*}} p_u p_u^T + \lambda I_k\Big)^{-1} \sum_{r_{u,i} \in r_{u*}} r_{ui} p_u$$

This process repeats until convergence.

Above two steps are iterated until convergence or some stopping criterion is reached. [Literature on ALS suggests 10 iterations for optimal results (https://endymecy.gitbooks.io/spark-ml-source-analysis/content/%E6%8E%A8%E8%8D%90/papers/Large-scale%20Parallel%20Collaborative%20Filtering%20the%20Netflix%20Prize.pdf)](https://endymecy.gitbooks.io/spark-ml-source-analysis/content/%E6%8E%A8%E8%8D%90/papers/Large-scale%20Parallel%20Collaborative%20Filtering%20the%20Netflix%20Prize.pdf). Here is [another good source (https://datajobs.com/data-science-repo/Collaborative-Filtering-%5BKoren-and-Bell%5D.pdf)](https://datajobs.com/data-science-repo/Collaborative-Filtering-%5BKoren-and-Bell%5D.pdf).

Modification to Include Bias

Although this can produce great results on its own, it doesn't take into account trends related to different characteristics of certain items and certain users as well as the data as a whole. To account for this, more advanced implementations of ALS include a **bias** term to capture some of these aspects of the overall utility matrix. A common implementation of this is captured with three bias terms:

- μ : a global average - the overall average rating of all items
- b_i : item bias - the deviations of item i from the average
- b_u : user bias - the deviations of user u from the average

Let's look at a basic example of how this would work. Imagine we're trying to calculate the rating for *The Shawshank Redemption* for user Matt. Assume the overall average rating is 3.5 and that *The Shawshank Redemption* tends to be rated 0.4 points higher than average. Matt is a generous rater and tends to rate 0.2 higher than the average. This would make his rating for the Shawshank Redemption 3.5 + 0.4 + 0.2 = 4.1

Putting these biases into an equation becomes:

$$\hat{r}_{ui} = \mu + b_i + b_u + q_i^T p_u$$

and the overall loss function becomes:

$$L = \sum_{u,i \in \kappa} (r_{u,i} - \mu - b_u - b_i - p_u^T q_i)^2 + \lambda(||q_i||^2 + |p_u||^2 + b_u^2 + b_i^2)$$

ALS vs SVD

ALS is generally less computationally efficient than directly computing the SVD solution, but it shines when you are dealing with giant, sparse matrices. SVD requires that all entries of the matrix be observed, and this is not a requirement with ALS. Because of ALS's "alternating" nature, it lends itself to performing computations in parallel. This can make it extremely beneficial to use distributed computing when using ALS.

Additional Resources

- A detailed explanation of ALS (https://datasciencemadesimpler.wordpress.com/tag/alternating-least-squares/)
- The math behind ALS (http://stanford.edu/~rezab/classes/cme323/S15/notes/lec14.pdf)

Summary

In this lesson, we looked at another matrix factorization technique, called alternating least squares and learned how we can train such a model to minimize a loss function, based on least squares.