ECSE 543 ASSIGNEMNT1

Question 1

(a) Write a program to solve the matrix equation Ax=b by Cholesky decomposition. A is a real, symmetric, positive-definite matrix of order n.

Cholesky decomposition has been implemented using the algorithm from lecture. An input file called matrix.txt has been used to pass matrix A and vector b in the format that each row is separated by a new line and each element in the row is separated by comma. The last line in the input file is taken as the b vector. Five methods have been developed to solve such a matrix problem: *read_matrix(), check_symmetric(A), check_pd(A)*, and *chelosky(A, b).* These four methods are integrated together in *solve_matrix()*. Check Appendix for the implementation and sample input file.

(b) Construct some small matrices (n = 2, 3, 4, or 5) to test the program. Remember that the matrices must be real, symmetric and positive-definite. Explain how you chose the matrices.

(c) Test the program you wrote in (a) with each small matrix you built in (b) in the following way: invent an x, multiply it by A to get b, then give A and b to your program and check that it returns x correctly.

Part (b) and (c) are explained together below.

Five tests were made. The first three tests were designed to validate the algorithm. The last two tests were used to check the error handling ability of the script.

1. Test 1:

   A is SPD, can be used to test result correctness. (Determinant of A is positive)

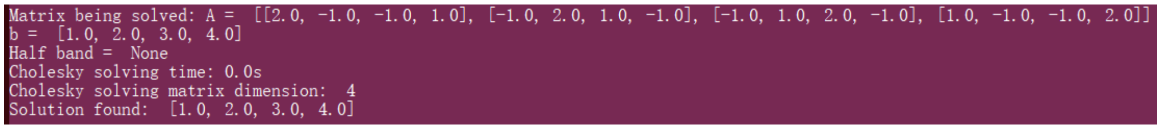
   A = 2, -1, -1, 1
   -1, 2, 1, -1
   -1, 1, 2, -1
   1, -1, -1, 2

   Set x = [1, 2, 3, 4]

   b = A*x = [1, 2, 3, 4]

   Set up matrix.txt file.

   Script output: Solution found: [1.0, 2.0, 3.0, 4.0]



```
Matrix being solved: A =  [[2.0, -1.0, -1.0, 1.0], [-1.0, 2.0, 1.0, -1.0], [-1.0, 1.0, 2.0, -1.0], [1.0, -1.0, -1.0, 2.0]]
b =  [1.0, 2.0, 3.0, 4.0]
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  4
Solution found:  [1.0, 2.0, 3.0, 4.0]
```

2. Test 2:

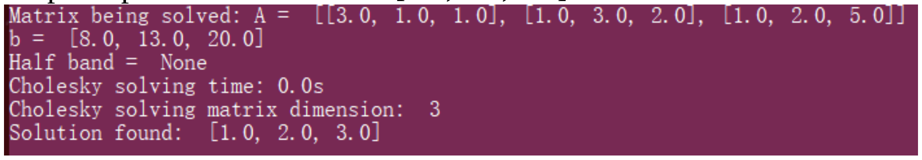
   A is SPD. (Determinant of A is 29.0)

   A = 3,1,1
   1,3,2
   1,2,5

   Set x = [1,2,3]

b = A*x = [8, 13, 20]

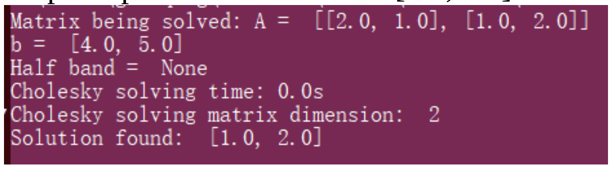
Set up matrix.txt file.
Script output: Solution found: [1.0, 2.0, 3.0]



```
Matrix being solved: A =  [[3.0, 1.0, 1.0], [1.0, 3.0, 2.0], [1.0, 2.0, 5.0]]
b =  [8.0, 13.0, 20.0]
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  3
Solution found:  [1.0, 2.0, 3.0]
```

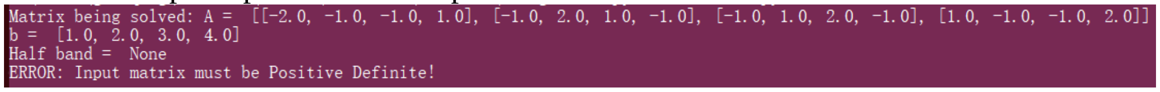
3. Test 3:
   A is PSD. (Determinant of A is 3.0)
   A = 2,1
   1,2
   Set x = [1,2] (transpose)
   b = A*x = [4, 5]

   Set up matrix.txt file.
   Script output: Solution found: [1.0, 2.0]



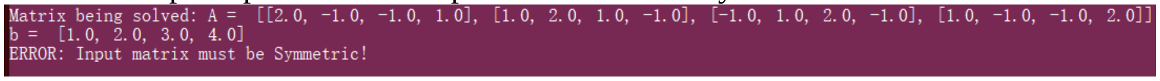
```
Matrix being solved: A =  [[2.0, 1.0], [1.0, 2.0]]
b =  [4.0, 5.0]
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  2
Solution found:  [1.0, 2.0]
```

4. Test 4:
   Error detection. Change the first element of A in Test 1 to -2, which makes A’s determinant negative (-11.0)
   A = -2, -1, -1, 1
   -1, 2, 1, -1
   -1, 1, 2, -1
   1, -1, -1, 2
   b = [1, 2, 3, 4]
   Script output: ERROR: Input matrix must be Positive Definite!



```
Matrix being solved: A =  [[-2.0, -1.0, -1.0, 1.0], [-1.0, 2.0, 1.0, -1.0], [-1.0, 1.0, 2.0, -1.0], [1.0, -1.0, -1.0, 2.0]]
b =  [1.0, 2.0, 3.0, 4.0]
Half band =  None
ERROR: Input matrix must be Positive Definite!
```

5. Test 5:
   Error detection. Change the first element in row 2 of A in Test 1 to 1, which makes A not symmetric.
   A = 2, -1, -1, 1
   1, 2, 1, -1
   -1, 1, 2, -1
   1, -1, -1, 2
   b = [1, 2, 3, 4]
   Script output: ERROR: Input matrix must be Symmetric!



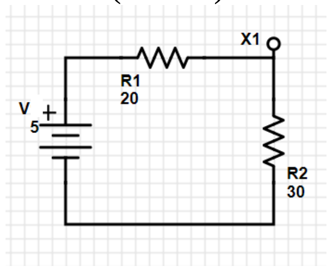
```
Matrix being solved: A =  [[2.0, -1.0, -1.0, 1.0], [1.0, 2.0, 1.0, -1.0], [-1.0, 1.0, 2.0, -1.0], [1.0, -1.0, -1.0, 2.0]]
b =  [1.0, 2.0, 3.0, 4.0]
ERROR: Input matrix must be Symmetric!
```

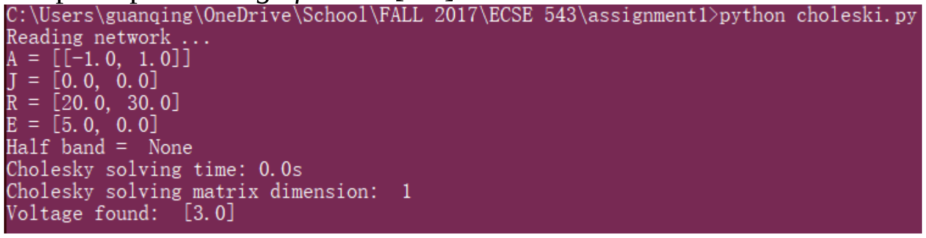
(d) Write a program that reads from a file a list of network branches (Jk, Rk, Ek) and a reduced incidence matrix, and finds the voltages at the nodes of the network. Use the code from part (a) to solve the matrix problem. Explain how the data is organized and read from the file. Test the program with a few small networks that you can check by hand. Compare the results for your test circuits with the analytical results you obtained by hand. Cleary specify each of the test circuits used with a labeled schematic diagram.

Another input file called *networkBranch.txt* is used including four matrices/vectors: circuit relation matrix A, branch model vectors J, R, and E. The last three lines of the input correspond to J, R, E respectively.

Several methods were integrated in method *solve_network()*, including, first, *read_network()* which returns A, J, R, and E matrices. Then based on the formula derived in class: $(A * y * A^T) * v = A * (J - y * E)$, the two inputs required by *cholesky(A, b)* are calculated where $A = (A * y * A^T)$, $b = A * (J - y * E)$. Before feeding A and b into cholesky(), the validation of matrix A has been checked by calling *check_symmetric(A)* and *check_pd(A).*

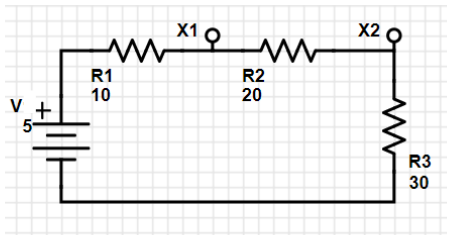
A few test cases can be seen below.

1. Test 1
   This circuit has 2 branches and 1 node.
   V1 = V / (R1+R2) * R2 = 3V
   

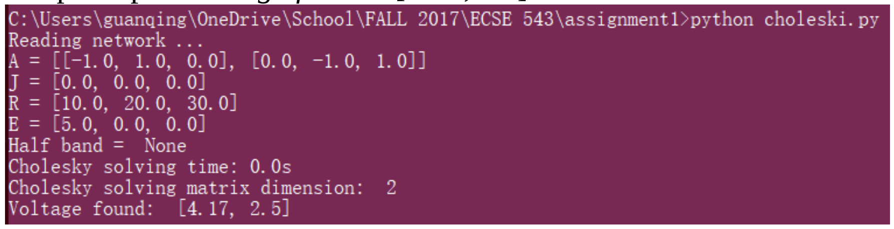
   A = [-1, 1]
   J = [0, 0]
   R = [20, 30]
   E = [5, 0]
   Script output: *Voltage found: [3.0]*
   

```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python choleski.py
Reading network ...
A = [[-1.0, 1.0]]
J = [0.0, 0.0]
R = [20.0, 30.0]
E = [5.0, 0.0]
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  1
Voltage found:  [3.0]
```

2. Test 2
   This circuit has 3 branches and 2 nodes.
   V1 = V / (R1+R2+R3) * (R2+R3) = 4.17V
   V2 = V / (R1+R2+R3) * R3 = 2.5V

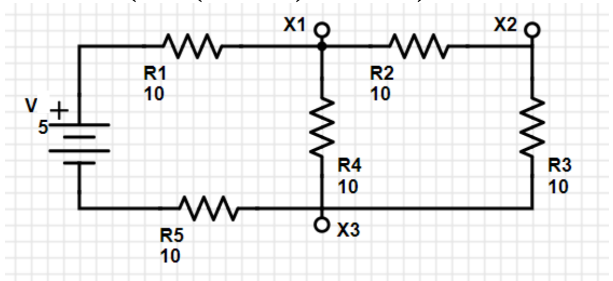


A = [[-1, 1, 0], [0, -1, 1]]
J = [0, 0, 0]
R = [10, 20, 30]
E = [5, 0, 0]
Script output: *Voltage found: [4.17, 2.5]*

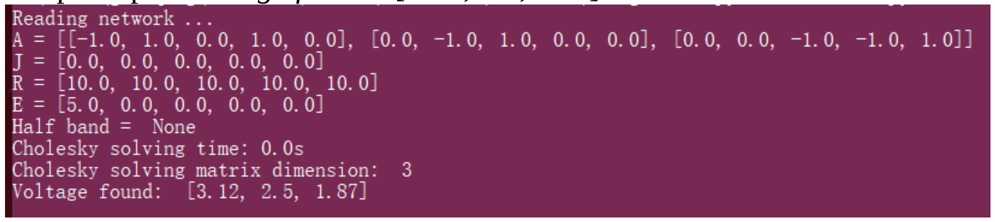


```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python choleski.py
Reading network ...
A = [[-1.0, 1.0, 0.0], [0.0, -1.0, 1.0]]
J = [0.0, 0.0, 0.0]
R = [10.0, 20.0, 30.0]
E = [5.0, 0.0, 0.0]
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  2
Voltage found:  [4.17, 2.5]
```

3. Test 3
This circuit has 5 branches and 3 nodes.
V1 = V / (R1+(R2+R3)//R4+R5) * ((R2+R3)//R4+R5) = 3.12V
V2 = V3 + (V1-V3)/2 = 2.5V
V3 = V / (R1+(R2+R3)//R4+R5) * R5 = 1.87V



A = [[-1, 1, 0, 1, 0], [0, -1, 1, 0, 0], [0, 0, -1, -1, 1]]
J = [0, 0, 0, 0 ,0]
R = [10, 10, 10, 10, 10]
E = [5, 0, 0, 0, 0]
Script output: *Voltage found: [3.12, 2.5, 1.87]*

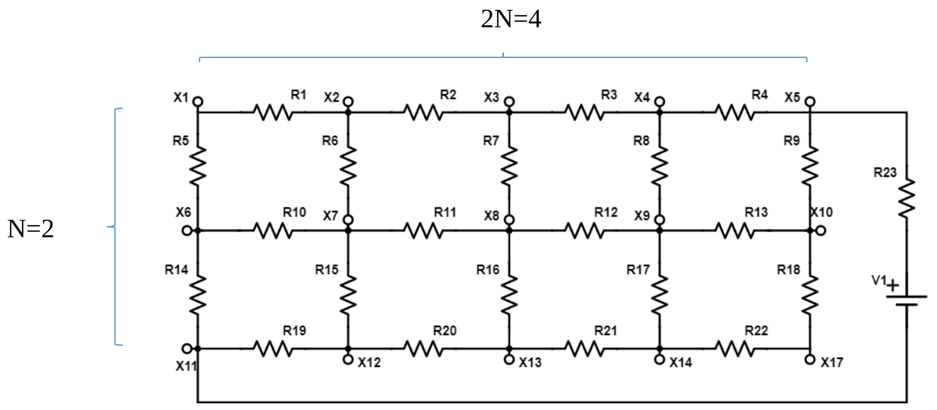


```
Reading network ...
A = [[-1.0, 1.0, 0.0, 1.0, 0.0], [0.0, -1.0, 1.0, 0.0, 0.0], [0.0, 0.0, -1.0, -1.0, 1.0]]
J = [0.0, 0.0, 0.0, 0.0, 0.0]
R = [10.0, 10.0, 10.0, 10.0, 10.0]
E = [5.0, 0.0, 0.0, 0.0, 0.0]
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  3
Voltage found:  [3.12, 2.5, 1.87]
```

Question 2

Take a regular N by 2N finite-difference mesh and replace each horizontal and vertical line by 1 k ohm resistor. This forms a linear, resistive network.

(a) Using the program you developed in question 1, find the resistance, R, between the node at the bottom left corner of the mesh and the node at the top right corner of the mesh, for N = 2, 3, …, 10. (You will probably want to write a small program that generates the input file needed by the network analysis program. Constructing by hand the incidence matrix for a 200-node network is rather tedious).

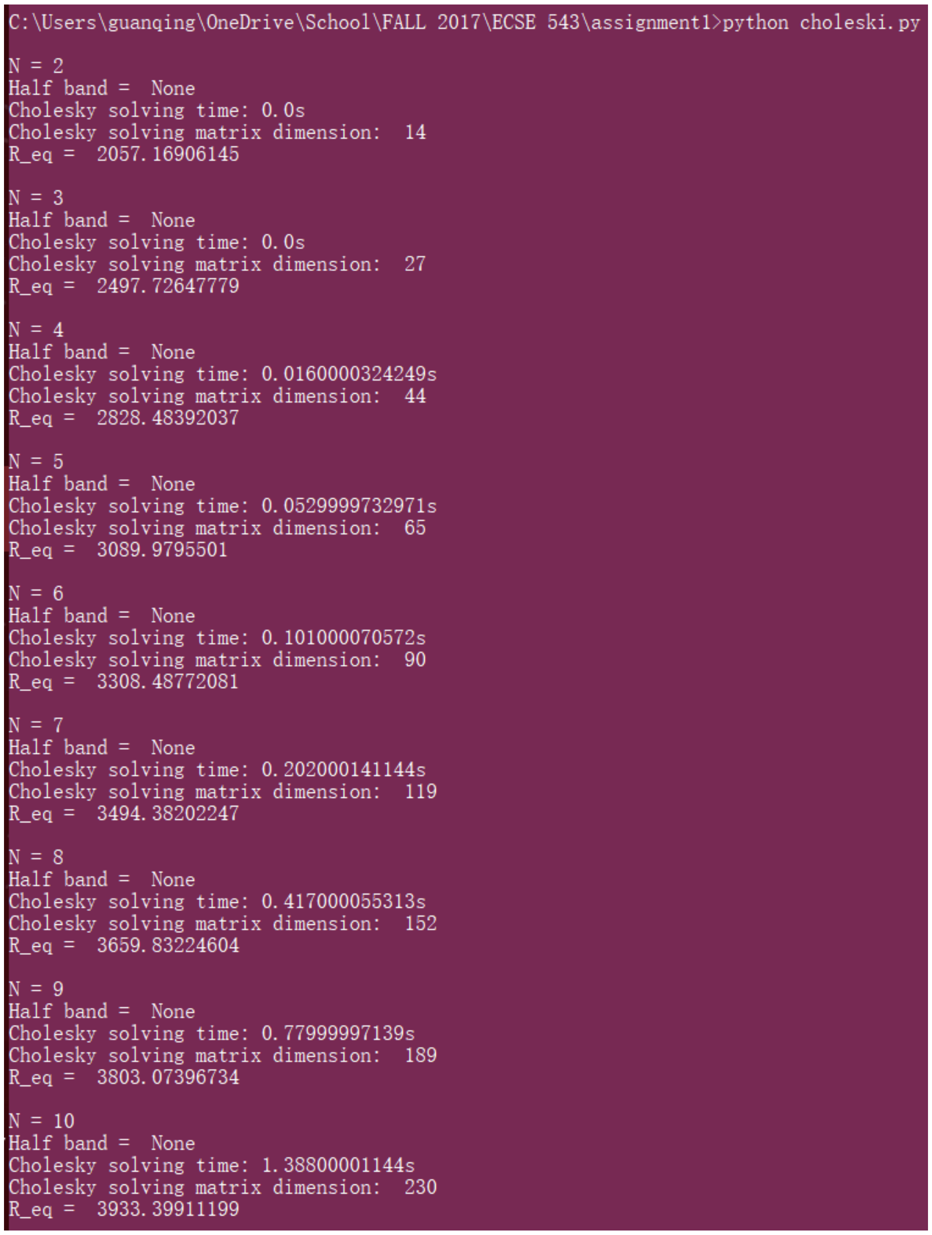


An example of N by 2N finite-difference mesh can be seen above where N=2. N is used as the number of branches, not number of nodes.

The resistance of this mesh network can be found by plugging a voltage source across the two selected nodes. A new method *generate_network(N, R, sJ, sR, sE)* has been developed to generate A, J ,R ,E matrices required by *solve_network()*. The parameter sJ, sR, and sE are the branch descriptions for the source branch. Both voltage and current source can be used to find the equivalent resistance, but using current source is harder due to parallel resistance. Therefore, a voltage source has been used as an example.

The schematic of N = 2 can be seen above. After solving all the node voltages, the current through R23 can be found with $I = ((V_{X11}+5)-V_{X5}/R23)$. Then the mesh equivalent resistance can be found by $R_{eq} = (V_{X1} - V_{X5})/I$. The results for N = 2, 3 … 10 were calculated using the same idea and shown in the table below, followed by a screenshot of the outputs.

N	2	3	4	5	6	7	8	9	10
R_{eq} (Ω)	2057.17	2497.73	2828.48	3089.98	3308.49	3494.38	3659.83	3803.07	3933.40



```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python choleski.py

N = 2
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  14
R_eq =  2057.16906145

N = 3
Half band =  None
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  27
R_eq =  2497.72647779

N = 4
Half band =  None
Cholesky solving time: 0.0160000324249s
Cholesky solving matrix dimension:  44
R_eq =  2828.48392037

N = 5
Half band =  None
Cholesky solving time: 0.0529999732971s
Cholesky solving matrix dimension:  65
R_eq =  3089.9795501

N = 6
Half band =  None
Cholesky solving time: 0.101000070572s
Cholesky solving matrix dimension:  90
R_eq =  3308.48772081

N = 7
Half band =  None
Cholesky solving time: 0.202000141144s
Cholesky solving matrix dimension:  119
R_eq =  3494.38202247

N = 8
Half band =  None
Cholesky solving time: 0.417000055313s
Cholesky solving matrix dimension:  152
R_eq =  3659.83224604

N = 9
Half band =  None
Cholesky solving time: 0.77999997139s
Cholesky solving matrix dimension:  189
R_eq =  3803.07396734

N = 10
Half band =  None
Cholesky solving time: 1.38800001144s
Cholesky solving matrix dimension:  230
R_eq =  3933.39911199
```

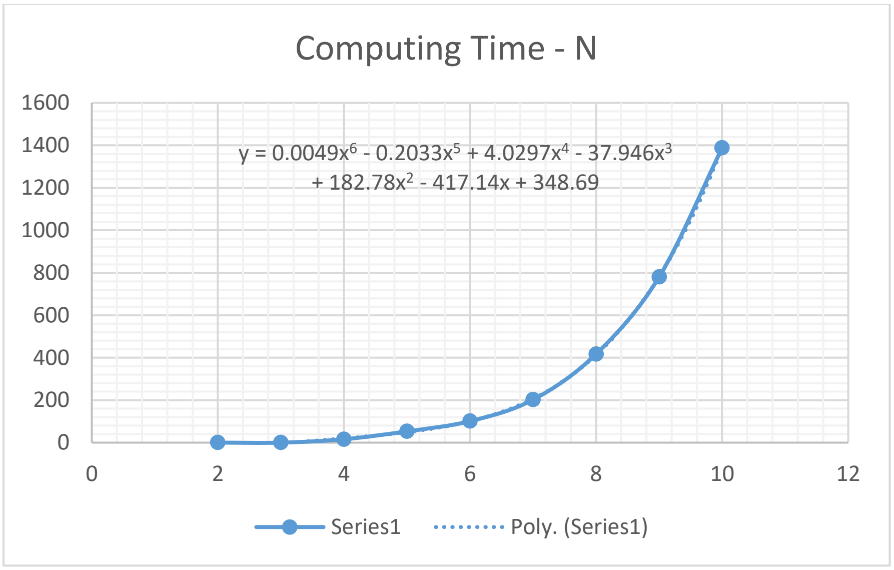
(b) In theory, how does the computer time taken to solve this problem increase with N, for large N? Are the timings you observe for your practical implementation consistent with this? Explain your observations.

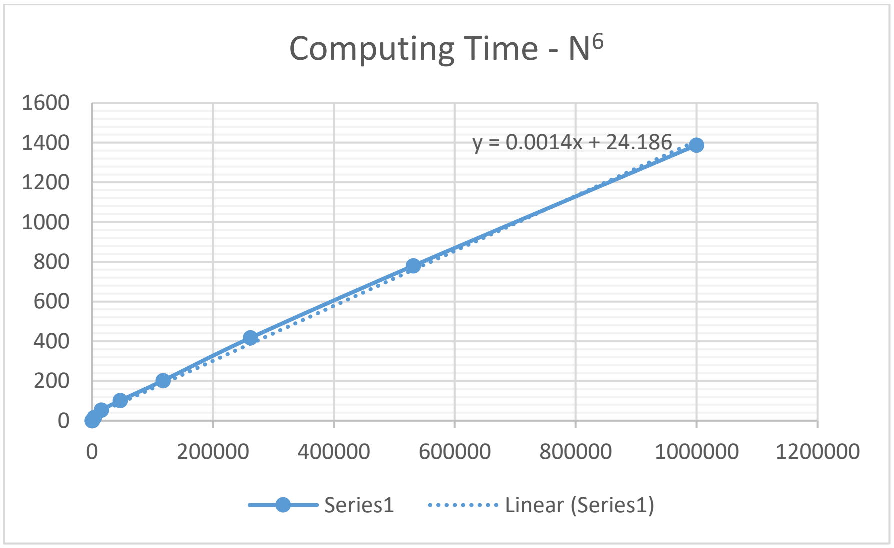
The time complexity to solve Cholesky Decomposition is O(n^3). The parameter n refers to the dimension of the mesh matrix A. In this question, n refers to the number of nodes in the circuit which can be found by $n = (N+1)(2N+1) = 2N^2+3N+1$. Therefore, the time complexity regarding to N is $O((N^2)^3) = O(N^6)$.

The time used to solve Cholesky Decomposition for each value of N is shown below, copied from the screenshot above.

N	2	3	4	5	6	7	8	9	10
Computing Time (ms)	0	0	16	53	101	202	417	780	1388

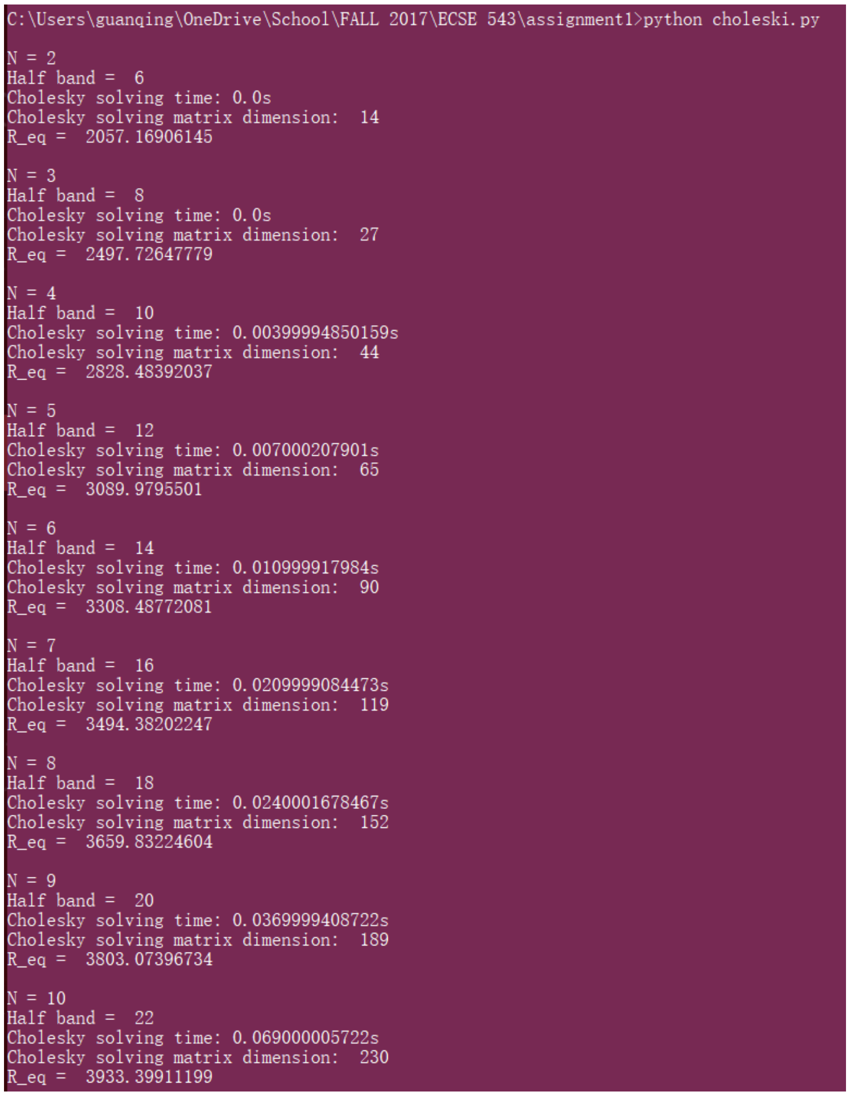
Two line charts were plotted to present how N and N^6 influence the computing time. The relation of computing time with N is exponential, while with N^6 is almost linear. This observation is consistent with the time complexity $O(N^6)$ derived above.





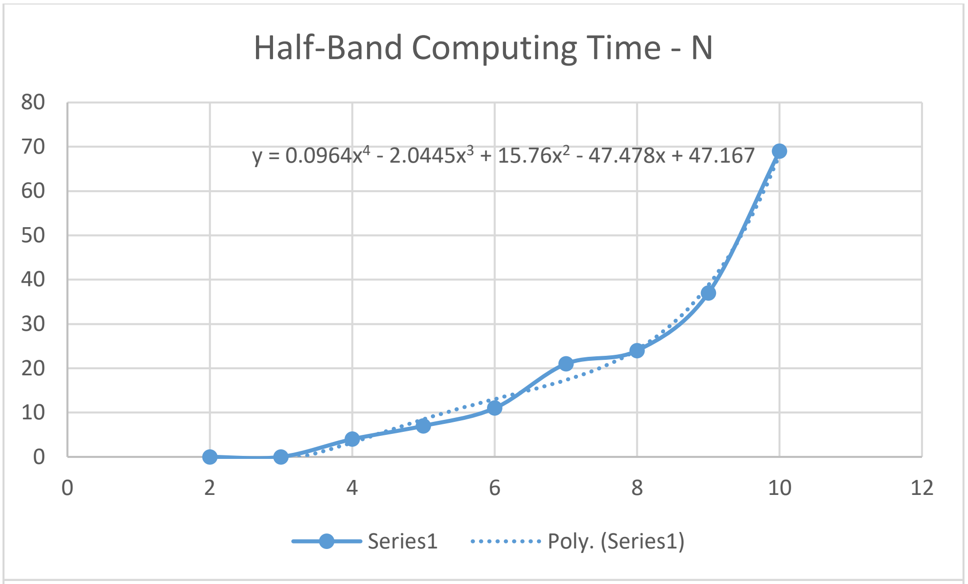
(c) Modify your program to exploit the sparse nature of the matrices to save computation time. What is the half-bandwidth b of your matrices? In theory, how does the computer time taken to solve this problem increase now with N, for large N? Are the timings you for your practical sparse implementation consistent with this? Explain your observations.

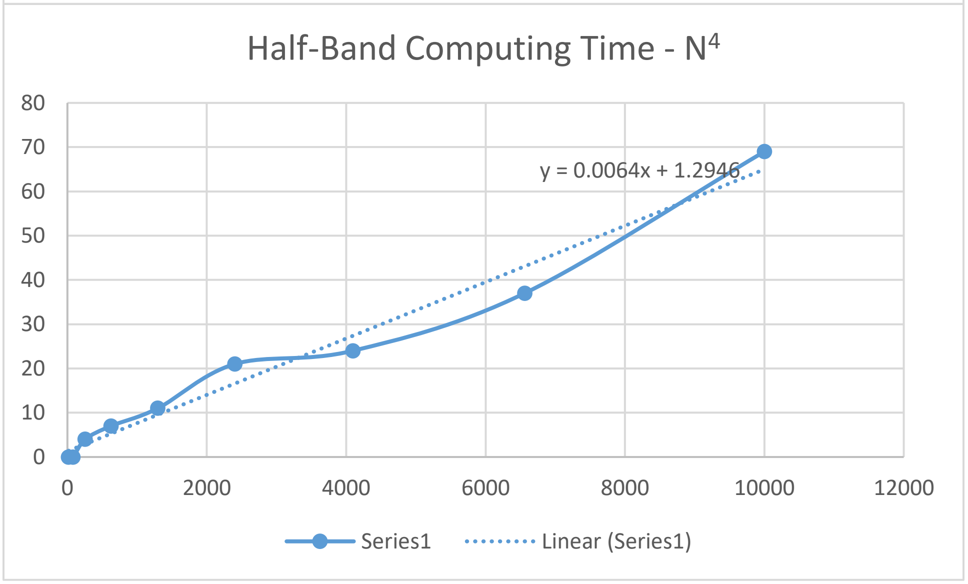
The half band has been found as 2N+2 by simply looking at the Cholesky matrices. The new computing time can be found by O(b^2*n) where both b^2 and n are proportional to N^2. So the computing time should have a linear relation with N^4, instead of N^6. A screenshot of outputs considering half band can be seen below, followed by a summary table, and two plots of computing time vs. N and N^4.



```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python choleski.py

N = 2
Half band =  6
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  14
R_eq =  2057.16906145

N = 3
Half band =  8
Cholesky solving time: 0.0s
Cholesky solving matrix dimension:  27
R_eq =  2497.72647779

N = 4
Half band =  10
Cholesky solving time: 0.00399994850159s
Cholesky solving matrix dimension:  44
R_eq =  2828.48392037

N = 5
Half band =  12
Cholesky solving time: 0.007000207901s
Cholesky solving matrix dimension:  65
R_eq =  3089.9795501

N = 6
Half band =  14
Cholesky solving time: 0.010999917984s
Cholesky solving matrix dimension:  90
R_eq =  3308.48772081

N = 7
Half band =  16
Cholesky solving time: 0.0209999084473s
Cholesky solving matrix dimension:  119
R_eq =  3494.38202247

N = 8
Half band =  18
Cholesky solving time: 0.0240001678467s
Cholesky solving matrix dimension:  152
R_eq =  3659.83224604

N = 9
Half band =  20
Cholesky solving time: 0.0369999408722s
Cholesky solving matrix dimension:  189
R_eq =  3803.07396734

N = 10
Half band =  22
Cholesky solving time: 0.069000005722s
Cholesky solving matrix dimension:  230
R_eq =  3933.39911199
```

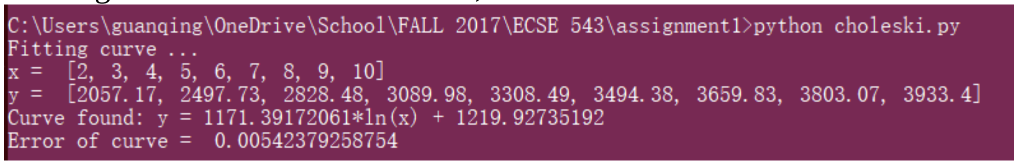
N	2	3	4	5	6	7	8	**9**	**10**
Computing Time (ms)	0	0	4	7	11	21	24	37	69

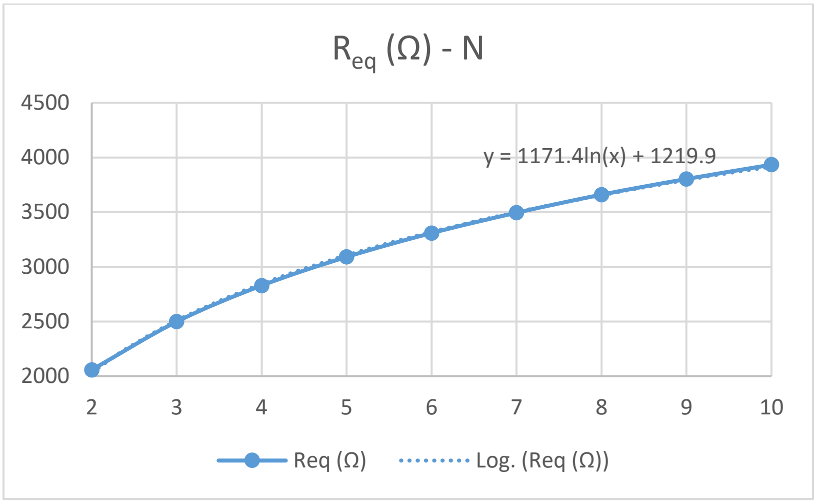




(d) Plot a graph of R versus N. Find a function R(N) that fits the curve reasonably well and is asymptotically correct as N tends to infinity, as far as you can tell.

A method called *fit_curve(x,y)* has been added for this part of question using Python Numpy. The fitting function found is *R = 1171.39*ln(N) + 1219.93* with an error under 0.54% (check screenshot below for more details). This result is consistent with the trending line result found with Excel, which can be seen in the chart below.


```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python choleski.py
Fitting curve ...
x =  [2, 3, 4, 5, 6, 7, 8, 9, 10]
y =  [2057.17, 2497.73, 2828.48, 3089.98, 3308.49, 3494.38, 3659.83, 3803.07, 3933.4]
Curve found: y = 1171.39172061*ln(x) + 1219.92735192
Error of curve =  0.00542379258754
```



Question 3

Figure 1 shows the cross-section of an electrostatic problem with translational symmetry: a coaxial cable with a square outer conductor and a rectangular inner conductor. The inner conductor is held at 15 volts and the outer conductor is grounded.

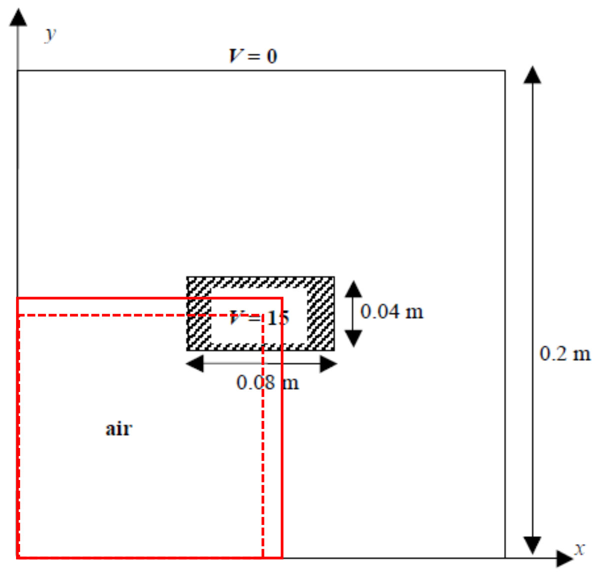
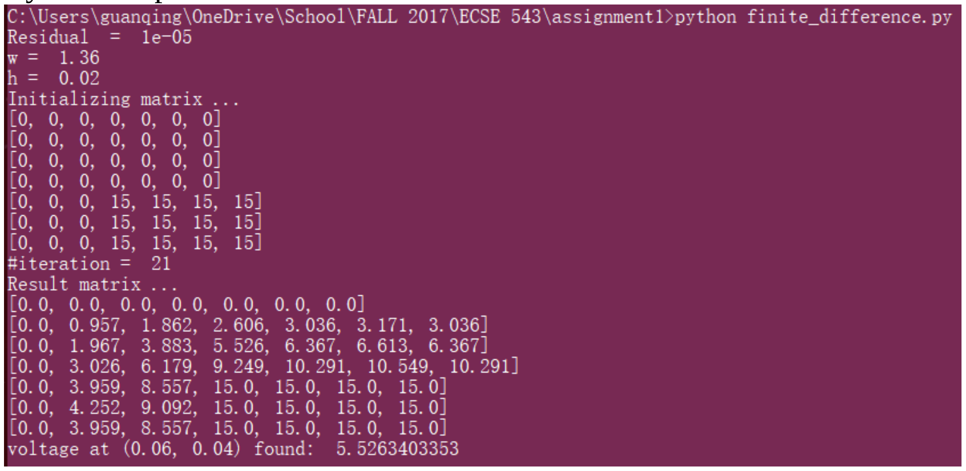


Figure 1.

(a) Write a computer program to find the potential at the nodes of a regular mesh in the air between the conductors by the method of finite differences. Use a five-point difference formula. Exploit at least one of the planes of mirror symmetry that this problem has. Use an equal node-spacing, h, in the x and y directions. Solve the matrix equation by successive over-relaxation (SOR), with SOR parameter w. Terminate the iteration when the magnitude of the residual at each free node is less than 10^{-5}.

Because of symmetry, only quarter of the cable needs to be modeled. An example is shown below with h = 0.02m, resulting in a regular 6-by-6 grid plus boundaries (7x7 matrix). There are totally 24 unknowns including 5 boundary nodes and 19 interior nodes. There are 25 fixed nodes with voltage potential equals to 0V or 15V.

There are several major methods in the program developed to solve this problem. First is *generate_mesh()* method which analyzes width, height, fixed-node ranges, and fixed voltage values of the quarter cable and initializes a matrix expression of all the nodes in the mesh based on the value of node-spacing (h). The matrix below is an example of the output matrix with h = 0.02m. The nodes highlighted in blues are the unknowns and the nodes highlighted in yellow are the boundary nodes. Nodes circled in red are the ones locate on the symmetric plane.



```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python finite_difference.py
Residual  =  1e-05
w =  1.36
h =  0.02
Initializing matrix ...
[0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 15, 15, 15, 15]
[0, 0, 0, 15, 15, 15, 15]
[0, 0, 0, 15, 15, 15, 15]
#iteration =  21
Result matrix ...
[0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]
[0.0, 0.957, 1.862, 2.606, 3.036, 3.171, 3.036]
[0.0, 1.967, 3.883, 5.526, 6.367, 6.613, 6.367]
[0.0, 3.026, 6.179, 9.249, 10.291, 10.549, 10.291]
[0.0, 3.959, 8.557, 15.0, 15.0, 15.0, 15.0]
[0.0, 4.252, 9.092, 15.0, 15.0, 15.0, 15.0]
[0.0, 3.959, 8.557, 15.0, 15.0, 15.0, 15.0]
voltage at (0.06, 0.04) found:  5.5263403353
```

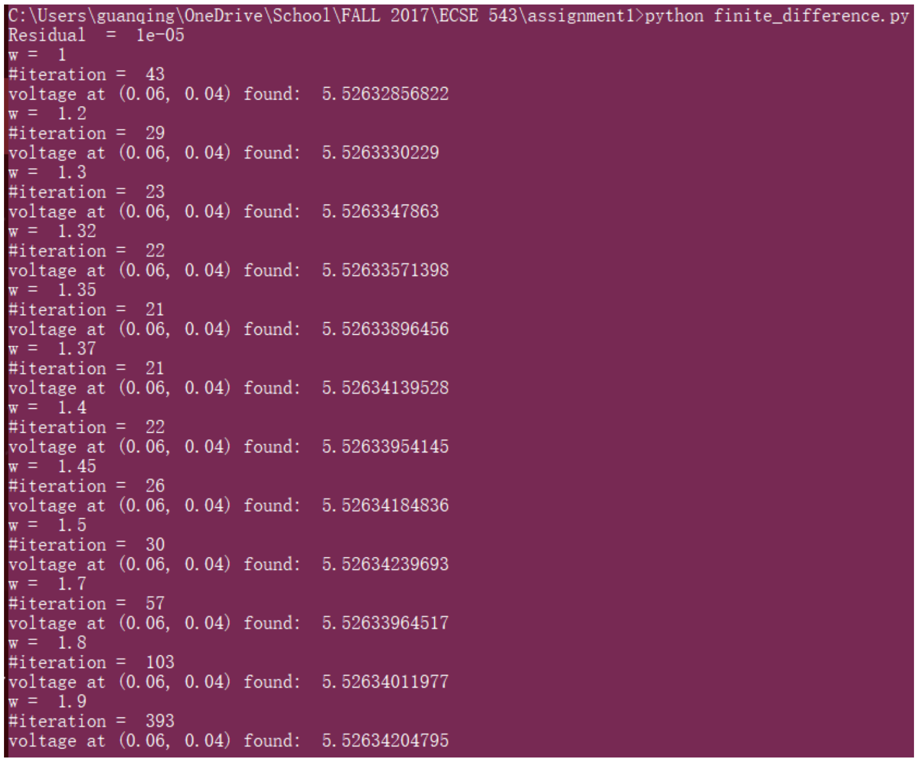
```
[0, 0, 0,  0,  0,  0,  0],
[0, 0, 0,  0,  0,  0,  0],
[0, 0, 0,  0,  0,  0,  0],
[0, 0, 0,  0,  0,  0,  0],
[0, 0, 0, 15, 15, 15, 15],
[0, 0, 0, 15, 15, 15, 15],
[0, 0, 0, 15, 15, 15, 15]
```

After the grid matrix has been initialized, a method called *update_matrix(n, unknown_range, boundary_range)* is called where the matrix gets updated node by node using five-point difference formula and SOR-Gauss Method.

(b) With h = 0.02, explore the effect of varying w. For 10 values of w between 1.0 and 2.0, tabulate the number of iterations taken to achieve convergence, and the corresponding value of potential at the point (x , y) = (0.06, 0.04). Plot a graph of number of iterations versus w.

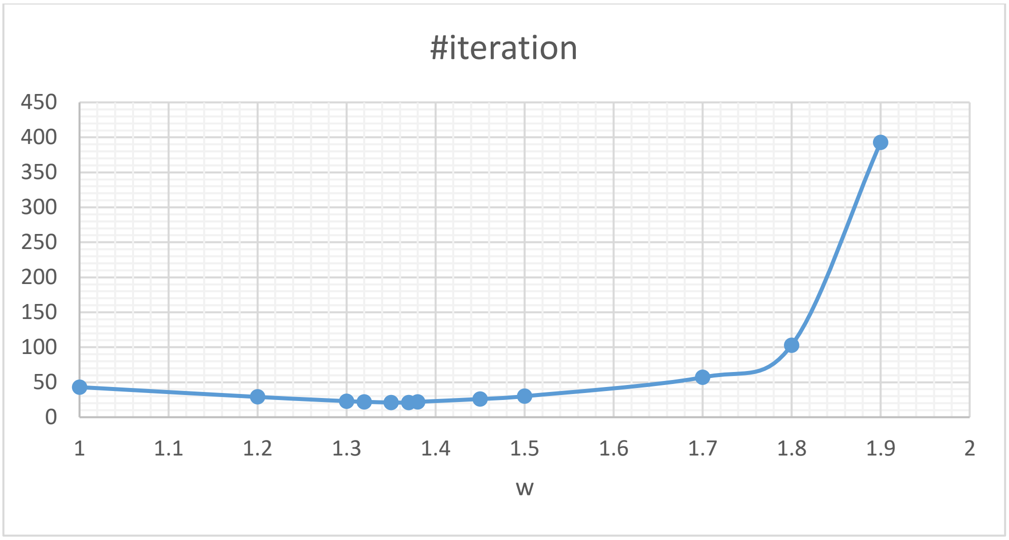
Some values of w between 1 and 2 have been tested. Points between 1.32 and 1.4 give the least number of iterations needed to converge. The average was taken as the estimated best choice for w, which is 1.36.

An interesting find is that the sequence of updating the unknowns influences number of iterations required to converge. The data above got from updating points from the top-left most unknown in the matric. Another way is starting the update from the bottom-right most unknown, which will take less iterations to converge because the algorithm is able to skip a lot of zero updates.



```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python finite_difference.py
Residual  =  1e-05
w =  1
#iteration =  43
voltage at (0.06, 0.04) found:  5.52632856822
w =  1.2
#iteration =  29
voltage at (0.06, 0.04) found:  5.5263330229
w =  1.3
#iteration =  23
voltage at (0.06, 0.04) found:  5.5263347863
w =  1.32
#iteration =  22
voltage at (0.06, 0.04) found:  5.52633571398
w =  1.35
#iteration =  21
voltage at (0.06, 0.04) found:  5.52633896456
w =  1.37
#iteration =  21
voltage at (0.06, 0.04) found:  5.52634139528
w =  1.4
#iteration =  22
voltage at (0.06, 0.04) found:  5.52633954145
w =  1.45
#iteration =  26
voltage at (0.06, 0.04) found:  5.52634184836
w =  1.5
#iteration =  30
voltage at (0.06, 0.04) found:  5.52634239693
w =  1.7
#iteration =  57
voltage at (0.06, 0.04) found:  5.52633964517
w =  1.8
#iteration =  103
voltage at (0.06, 0.04) found:  5.52634011977
w =  1.9
#iteration =  393
voltage at (0.06, 0.04) found:  5.52634204795
```

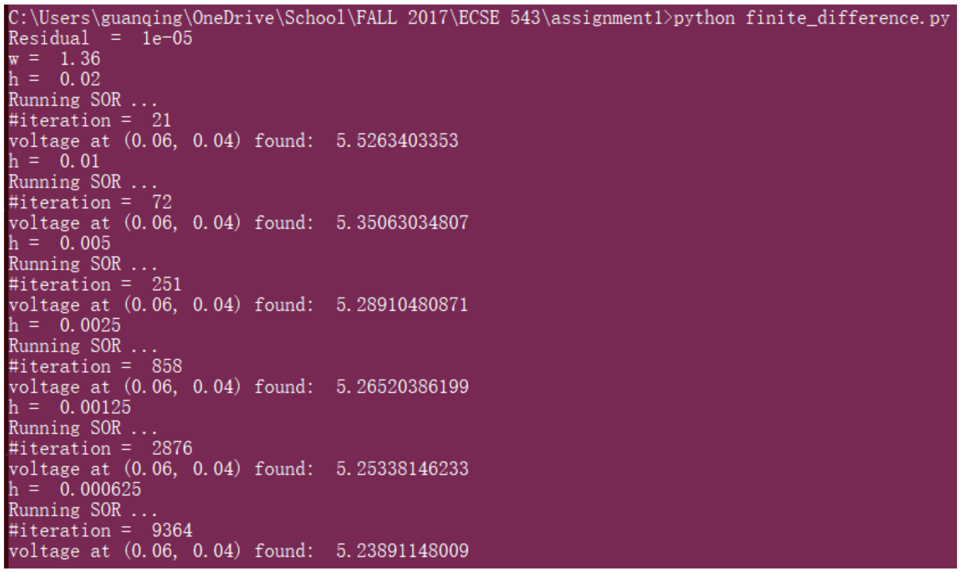
w	**1**	**1.2**	**1.3**	**1.32**	**1.35**	**1.37**	**1.4**	**1.45**	**1.5**	**1.7**	**1.8**	**1.9**
#iteration	43	29	23	22	21	21	22	26	30	57	103	393
V(0.06, 0.04)	5.526	5.526	5.526	5.526	5.526	5.526	5.526	5.526	5.526	5.526	5.526	5.526



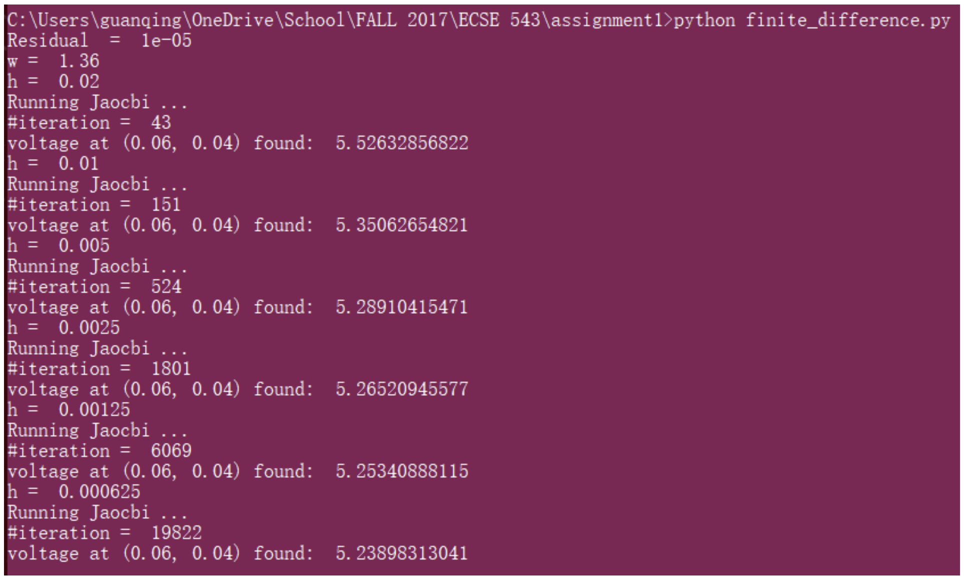
(c) With an appropriate value of w, chosen from the above experiment, explore the effect of decreasing h on the potential. Use values of h = 0.02, 0.01, 0.005, etc., and both tabulate and plot the corresponding values of potential at (x, y) = (0.06, 0.04) versus 1/h. What do you think is the potential at (0.06, 0.04), to three significant figures? Also, tabulate and plot the number of iterations versus 1/h. Comment on the properties of both plots.

(d) Use the Jacobi method to solve this problem for the same values of h used in part (c). Tabulate and plot the values of the potential at (x, y) = (0.06, 0.04) versus 1/h and the number of iterations versus 1/h. Comment on the properties of both plots and compare to those of SOR.

Since (c) and (d) have very similar logic. The results of them are explained together. Screenshots of outputs using SOR and Jacobi can be seen first, followed by a summary table and plots.

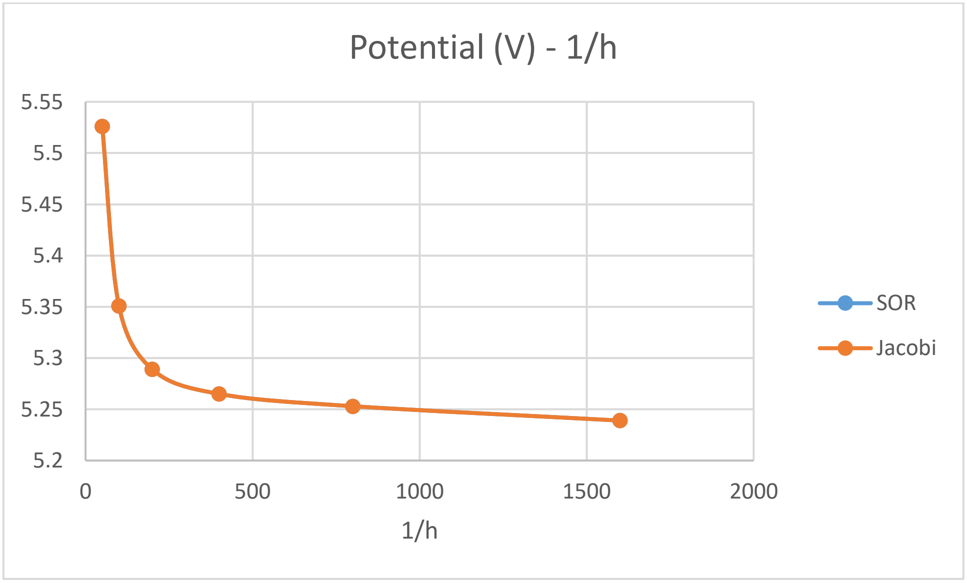


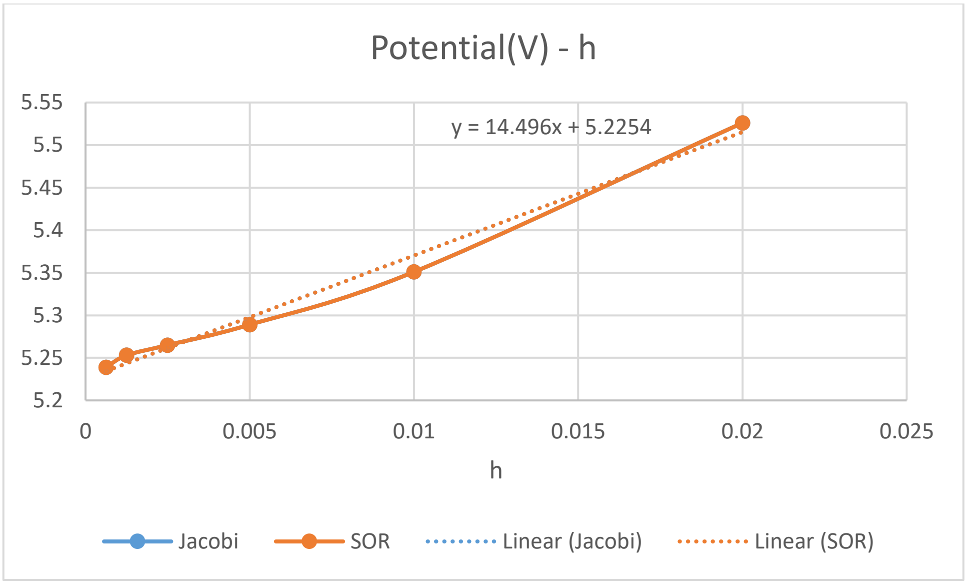
```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python finite_difference.py
Residual  =  1e-05
w =  1.36
h =  0.02
Running SOR ...
#iteration =  21
voltage at (0.06, 0.04) found:  5.5263403353
h =  0.01
Running SOR ...
#iteration =  72
voltage at (0.06, 0.04) found:  5.35063034807
h =  0.005
Running SOR ...
#iteration =  251
voltage at (0.06, 0.04) found:  5.28910480871
h =  0.0025
Running SOR ...
#iteration =  858
voltage at (0.06, 0.04) found:  5.26520386199
h =  0.00125
Running SOR ...
#iteration =  2876
voltage at (0.06, 0.04) found:  5.25338146233
h =  0.000625
Running SOR ...
#iteration =  9364
voltage at (0.06, 0.04) found:  5.23891148009
```

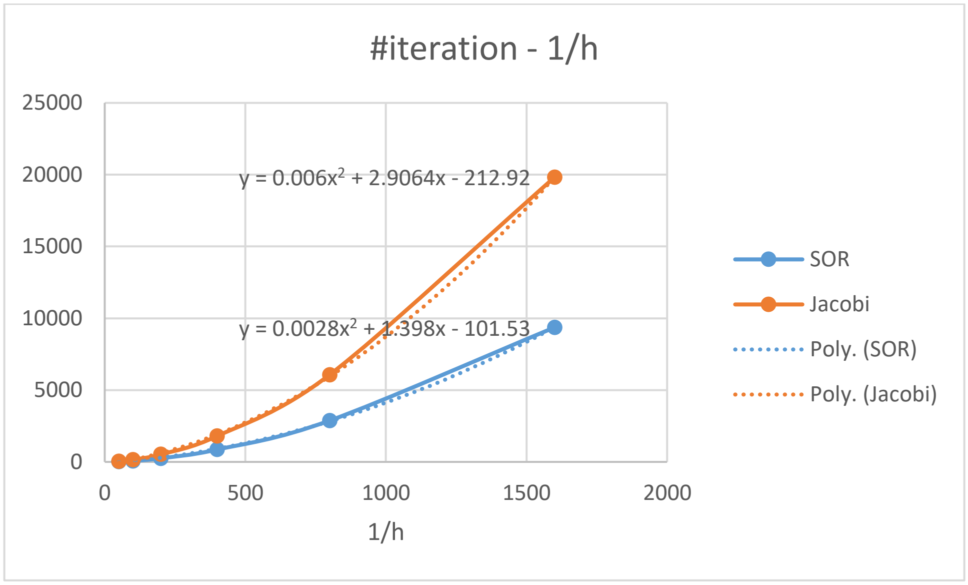


```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python finite_difference.py
Residual  =  1e-05
w =  1.36
h =  0.02
Running Jaocbi ...
#iteration =  43
voltage at (0.06, 0.04) found:  5.52632856822
h =  0.01
Running Jaocbi ...
#iteration =  151
voltage at (0.06, 0.04) found:  5.35062654821
h =  0.005
Running Jaocbi ...
#iteration =  524
voltage at (0.06, 0.04) found:  5.28910415471
h =  0.0025
Running Jaocbi ...
#iteration =  1801
voltage at (0.06, 0.04) found:  5.26520945577
h =  0.00125
Running Jaocbi ...
#iteration =  6069
voltage at (0.06, 0.04) found:  5.25340888115
h =  0.000625
Running Jaocbi ...
#iteration =  19822
voltage at (0.06, 0.04) found:  5.23898313041
```

h		**0.02**	**0.01**	**0.005**	**0.0025**	**0.00125**	**0.000625**
1/h		50	100	200	400	800	1600
SOR	#iteration	21	72	251	858	2876	9364
	Potential (V)	5.526	5.351	5.289	5.265	5.253	5.239
Jacobi	#iteration	43	151	524	1801	6069	19822
	Potential (V)	5.526	5.351	5.289	5.265	5.253	5.239







From the graph of potential at point (0.06, 0.04) vs. 1/h, it can be seen that, for both SOR and Jacobi method, the value is approaching 5.24V dramatically at the beginning, but smoother after 1/h = 400. The precision using either method is the same. The potential values corresponding to 1/h = 1600 is the most accurate one, giving 5.239V.

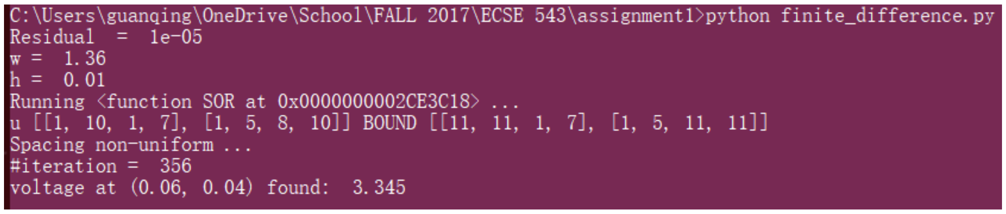
Number of iterations increases quadratic with 1/h in both cases. But obviously, SOR method needs much less number of iterations than Jacobi does to reach the same precision. In the data points that have been tested, the number of iterations required by SOR is about half of that required by Jacobi and the difference is going to be even more with increasing 1/h.

As a conclusion, SOR is a more efficient method than Jacobi and its advantage in computing efficiency becomes larger when the size of the problem increases.

(e) Modify the program you wrote in part (a) to use the five-point difference formula derived in class for non-uniform node spacing. An alternative to using equal node spacing, h, is to use smaller node spacing in more "difficult" parts of the problem domain. Experiment with a scheme of this kind and see how accurately you can compute the value of the potential at (x, y) = (0.06, 0.04) using only as many nodes as for the uniform case h = 0.01 in part (c).

When non-uniformed spacing is used, the five-point difference formula needs to be generalized. A flag called *even_spacing* was added to the SOR method. When flag values is False, SOR enters non-uniform spacing mode, where two extra arrays were used to specify horizontal and vertical node positions.

With non-uniform spacing and the same amount of nodes, slightly more accurate result can be found with more iterations as tradeoff.



```
C:\Users\guanqing\OneDrive\School\FALL 2017\ECSE 543\assignment1>python finite_difference.py
Residual  =  1e-05
w =  1.36
h =  0.01
Running <function SOR at 0x0000000002CE3C18> ...
u [[1, 10, 1, 7], [1, 5, 8, 10]] BOUND [[11, 11, 1, 7], [1, 5, 11, 11]]
Spacing non-uniform ...
#iteration =  356
voltage at (0.06, 0.04) found:  3.345
```

Appendix

Q1 & Q2

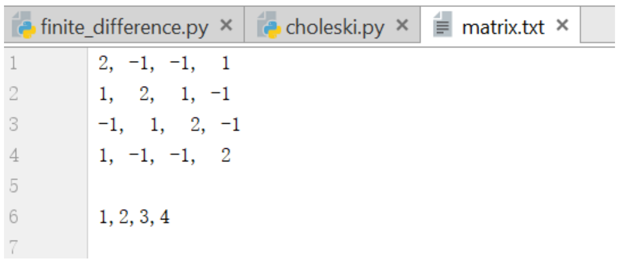
Cholesky.py

```
"""
This script solves Ax=b matrix problem using Cholesky Decomposition, where A is represented by L*transpose(L)
Author: Guanqing Hu, Oct 3rd, 2017
"""
import math
import numpy as np
import time

# check S.P.D
def check_symmetric(A):
    n = len(A)
    for i in range(n):
        for j in range(i + 1, n):
            if A[i][j] != A[j][i]:
                exit('ERROR: Input matrix must be Symmetric!')


# positive definite if determinant of A is positive
# determinant equals to the product of all the eigenvalues of A
def check_pd(A):
    det = np.linalg.det(A)
    print det
    if det <= 0:
        exit('ERROR: Input matrix must be Positive Definite!')

def read_matrix():
    # read matrix, last line is b
    with open('matrix.txt', 'r') as f:
        A = []
        for line in f:
            if line.strip() == '':
                continue
            A.append([float(x) for x in line.split(',')])
            # check if data complete
            # todo
        b = A[-1]
        A = A[:-1]
        print 'Matrix being solved: A = ', A, '\nb = ', b
        return A,b


def cholesky(A, b, hb=None):
    # Cholesky implementation
    n = len(A)
    print 'Half band = ', hb
    start_time = time.time()
    for j in range(n - 1):
        if A[j][j] <= 0:
            exit('ERROR: Input matrix must be Positive Definite!')
        A[j][j] = math.sqrt(A[j][j])
        b[j] = b[j] / A[j][j]

        i_range = range(j + 1, n)
        if (hb!=None):
            if ((j+hb+1)<n):
                i_range = range(j+1, (j+hb+1))

        for i in i_range: #range(j+1, n):
            #print 'updating element A{}{}'.format(i,j)
            A[i][j] = A[i][j] / A[j][j]
            b[i] = b[i] - A[i][j] * b[j]
            for k in range(j + 1, i + 1):
                A[i][k] = A[i][k] - A[i][j] * A[k][j]
    # Back Substitution: T(L)x = y
    x = [0.0 for i in range(n)]
    for i in range(n - 1, -1, -1):
        x[i] = round((b[i] - sum([A[j][i] * x[j] for j in range(i + 1, n)])) / A[i][i], 2)
    solving_time = time.time()-start_time
    print 'Cholesky solving time: {}s'.format(solving_time)
    print 'Cholesky solving matrix dimension: ', len(A)

    return x

def solve_matrix():
    A, b = read_matrix()
    check_symmetric(A)
    #check_pd(A)
    results = cholesky(A, b, None)
    print 'Solution found: ', results

def read_network():
    with open('networkBranch.txt', 'r') as f:
        A = []
        for line in f:
            if line.strip() == '':
                continue
            A.append([float(x) for x in line.split(',')])
        J, R, E = A[-3], A[-2], A[-1]
        A = A[:-3]
        return A, J, R, E


def m_product(A, B):
    # return A*B
    # check dimension
    if len(A[0]) != len(B):
        exit('ERROR: Matrix production failed due to wrong dimensions!')
    # create correct dimension output matrix
    #print 'product', A, B
    C = [[0 for col in B[0]] for row in A]
    for i in range(len(C)):
        for j in range(len(C[0])):
            C[i][j] = sum([A[i][k] * B[k][j] for k in range(len(B))])
    return C


def transpose(A):
    A2 = [[0 for i in A] for j in A[0]]
    for i in range(len(A)):
        for j in range(len(A[0])):
            A2[j][i] = A[i][j]
    return A2

def m_substract(A, B):
    # check dimension
    if (len(A)!=len(B))|(len(A[0])!=len(B[0])):
        exit('ERROR: Matrix dimension error in substraction!')
    C = []
    for i in range(len(A)):
        C.append([A[i][k]-B[i][k] for k in range(len(A[0]))])
    return C


# this method generates network for a regular N by 2N finite-difference mesh and replace each horizontal and vertical
# line by a 1 kW resistor
def generate_network(N, R, sJ, sR, sE):
    # calculate number of branches
    nbranch = (N+1)*2*N+N*(2*N+1)
    J = [0]*nbranch
    R = [R]*nbranch
    E = [0]*nbranch
    # add teasting branch: voltage source branch
    J.append(sJ)
    R.append(sR)
    E.append(sE)
    # calculate nodes
    nnode = (N+1)*(2*N+1)
    A = [[0]*(nbranch+1) for i in range(nnode)]
    #print len(A), len(A[0])
    for i in range(1, nnode+1):
        level = i/(2*N+1)
        offset = i%(2*N+1)
        if offset==0:
            level -= 1
            offset = 2*N+1
        branch_per_level = (2*N+(2*N+1))
        # calculate surrounding branch indices
        right = branch_per_level * level + offset
        left = right - 1
        top = 2*N*level + (2*N+1)*(level-1)+ offset
        bottom = top + branch_per_level
        #print i, 'r',right, 'l', left, 't',top, 'b', bottom

        i -= 1
        # node on top border
        if (level==0):
            # top left
            if (offset==1):
                A[i][right - 1], A[i][bottom - 1] = -1, 1
            # top right
            elif (offset==2*N+1):
                A[i][left - 1], A[i][bottom - 1] = 1, 1
            else:
                A[i][right - 1], A[i][left - 1], A[i][bottom - 1] = -1, 1, 1
        # node on bottom border
        elif (level == N):
            # bottom left
            if (offset==1):
                A[i][right - 1], A[i][top - 1] = -1, -1
            # bottom right
            elif (offset==2*N+1):
                A[i][top - 1], A[i][left - 1] = -1, 1
            else:
                A[i][right - 1], A[i][top - 1], A[i][left - 1] = -1, -1, 1
        # node on left border
        elif (offset == 1):
            A[i][right - 1], A[i][top - 1], A[i][bottom - 1] = -1, -1, 1
        # node on right border
        elif (offset == 2*N+1):
            A[i][top - 1], A[i][left - 1], A[i][bottom - 1] = -1, 1, 1
        # nodes in middle
        else:
            A[i][right - 1], A[i][top - 1], A[i][left - 1], A[i][bottom - 1] = -1, -1, 1, 1

        # setup the voltage source
        # right top corner
        if (level==0)&(offset==2*N+1):
            A[i][-1] = -1
        elif (level==N)&(offset==1):
            A[i][-1] = 1
    # set up ground node by removing that node from A
    # ground left-bottom corner node
    del A[-(2*N+1)]

    return A, J, R, E


def solve_network(A, J, R, E, hb=None):
    # (A * y * T(A)) * v = A * (J - y * E)

    y = [[0 for r in R] for r in R]
    for i in range(len(y)):
        y[i][i] = 1.0/R[i]

    A2 = m_product(m_product(A, y), transpose(A))
    check_symmetric(A2)
    #check_pd(A2)

    '''print '### start of matrix'
    for i in A2:
        print i
    '''
    b = m_product(A, m_substract([[j] for j in J], m_product(y, [[e] for e in E])))
    # example of b: [[0.5], [0.0]], flat it
    b = [i[0] for i in b]
    #print 'b', b, 'A2', A2
    v = cholesky(A2, b, hb)
    return v

def fit_curve(x,y):
    print 'Fitting curve ... '
    print 'x = ',x
    print 'y = ',y
    x = np.array(x)
    y = np.array(y)
    k, b = np.polyfit(np.log(x), y, 1)
    print 'Curve found: y = {}*ln(x) + {}'.format(k,b)
    r = []
    for n in x:
        r.append(k * math.log(n, math.e) + b)
    error = max([(a0-a)/a for (a0,a) in zip(r, y)])
    print 'Error of curve = ', error


if __name__ == '__main__':
    ### q1
    solve_matrix()

    ### q1
    A, J, R, E = read_network()
    print 'Reading network ...\nA = {}\nJ = {}\nR = {}\nE = {}'.format(A,J,R,E)
    v = solve_network(A, J, R, E)
    print 'Voltage found: ',v
```

ECSE 543 Assignment 1
Fall 2017 Guanqing Hu

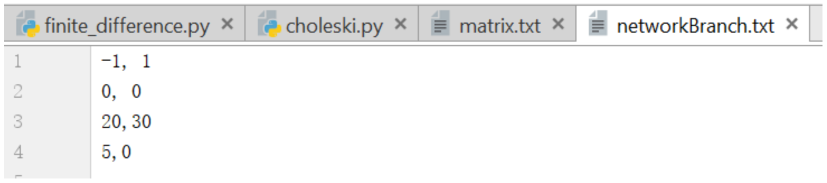
```
### q2
# J, R ,E of the source branch
sJ = 0
sR = 1000.0
sE = 100.0
for N in range(2,2):
    print '\nN = {}'.format(N)
    A, J, R, E = generate_network(N, 1000, sJ, sR, sE) #N, R, sJ, sR, sE
    # half band, not sure ...
    #hb = 2*N+2
    v = solve_network(A, J, R, E)
    # get the voltage at top right and bottom left
    v1 = v[2*N]
    v2 = 0#v[-1-2*N]
    # R_total = sR+R_mesh
    I  = ((v2+sE)-v1)/sR
    R_mesh = (v1-v2)/I
    #print 'Voltage of {} nodes found. '.format(len(v))#, v
    print 'R_eq = ', R_mesh

# fit curve
N = [2, 3, 4, 5, 6, 7, 8, 9, 10]
R = [2057.17, 2497.73, 2828.48, 3089.98, 3308.49, 3494.38, 3659.83, 3803.07, 3933.4]
fit_curve(N, R)
```

Matrix.txt



```
2,  -1,  -1,   1
1,   2,   1,  -1
-1,   1,   2,  -1
1,  -1,  -1,   2

1, 2, 3, 4
```

networkBranch.txt



```
-1,  1
0,   0
20, 30
5, 0
```

Q3

```
import math

def initial_matrix(r, n, value):
    i_min, i_max, j_min, j_max = r
    for j in range(j_min, j_max + 1):
        for i in range(i_min, i_max + 1):
            n[j][i] = value
    return n

def generate_meshnodes():
    # find matrix dimension, +h : add boundary
    j = height/h
    i = width/h
    #check if h qualifies
    if (j!=int(j))|(i!=int(i)):
        print 'ERROR: h should be selected so that width/d and hight/h are both integers.'
        exit()
    # build matrix:
    n = [[None for x in range(int(i+1))] for y in range(int(j+1))]
    # decide boundary, unknown, and fixed nodes
    ground_range = []
    for area in ground:
        # [i_min i_max, j_min, j_max]
        ground_range.append([int(area[0][0]/h), int(area[1][0]/h), int(area[0][1]/h), int(area[1][1]/h)])
    for r in ground_range:
        n = initial_matrix(r, n, 0)
    v1_range = []
    for area in v1:
        v1_range.append([int(area[0][0]/h), int(area[1][0]/h), int(area[0][1]/h), int(area[1][1]/h)])
    for r in v1_range:
        n = initial_matrix(r, n, v1_value)

    #todo make it more dynamic
    # [i_min i_max, j_min, j_max]
    unknowns = [[1, len(n)-2, 1, v1_range[0][2]-1], [1,v1_range[0][0]-1,v1_range[0][2],v1_range[0][3]-1]]
    boundary = [[len(n[0])-1, len(n[0])-1, 1, v1_range[0][2]-1],[1,v1_range[0][0]-1, len(n)-1, len(n)-1]]
    for r in unknowns+boundary:
        n = initial_matrix(r, n, 0)
    #print ground_range, v1_range, unknowns, boundary
    return n, unknowns, boundary

def error_small_enough(n,unknowns, thr):
    error = []
    for r in unknowns:
        for j in range(r[2],r[3]+1):
            for i in range(r[0],r[1]+1):
                error.append(n[j][i-1]+n[j][i+1]+n[j-1][i]+n[j+1][i]-4*n[j][i])
    if max(error)>thr:
        print max(error)
        return False
    return True

def SOR(unknown, boundary, n, even_spacing=True, horizontal_split=[], vertical_split=[]):
    if even_spacing:
        for j in range(unknown[2],unknown[3]+1):
            for i in range(unknown[0],unknown[1]+1):
                n[j][i] = (1-w)*n[j][i] + w*0.25*(n[j][i-1]+n[j][i+1]+n[j-1][i]+n[j+1][i])
                # check orientation of boundary
                if boundary[2]==boundary[3]:
                    n[boundary[2]][i] = n[boundary[2]-2][i]
            # check orientation of boundary
            if boundary[0]==boundary[1]:
                n[j][boundary[0]] = n[j][boundary[0]-2]
    else:
        for j in range(unknown[2],unknown[3]+1):
            for i in range(unknown[0],unknown[1]+1):
                a1 = horizontal_split[i] - horizontal_split[i-1]
                a2 = horizontal_split[i+1] - horizontal_split[i]
                b1 = vertical_split[j] - vertical_split[j-1]
                b2 = vertical_split[j+1] - vertical_split[j]
                e = (math.pow(a1,2)+math.pow(a2,2))
                f = (math.pow(b1,2)+math.pow(b2,2))
                c1 = f/2/(e+f)
                c2 = e/2/(e+f)
                #print 'c1,c2 = ',c1,c2
                n[j][i] = (1-w)*n[j][i] + w*(c1*(n[j][i-1]+n[j][i+1]) + c2*(n[j-1][i]+n[j+1][i]))
                # check orientation of boundary
            if boundary[2]==boundary[3]:
                n[boundary[2]][i] = n[boundary[2]-2][i]
        # check orientation of boundary
        if boundary[0]==boundary[1]:
            n[j][boundary[0]] = n[j][boundary[0]-2]
        #print '###',i,j,a,b,c,d,coeff1,coeff2
    return n

def Jacobi(r,b,n_old):
    n_new = [i for i in n_old]
    for j in range(r[2],r[3]+1):
        for i in range(r[0],r[1]+1):
            n_new[j][i] = 0.25*(n_old[j][i-1]+n_old[j][i+1]+n_old[j-1][i]+n_old[j+1][i])
            # check orientation of boundary: if horizontal
            if b[2]==b[3]:
                n_new[b[2]][i] = n_old[b[2]-2][i]
        # check orientation of boundary: if vertical
        if b[0]==b[1]:
            n_new[j][b[0]] = n_old[j][b[0]-2]
    return n_new

if __name__ == '__main__':
    thr = math.pow(10, -5)
    print 'Residual  = ',thr

    # width or hight / h should both be integer
    for w in [1]:#[1, 1.2, 1.3, 1.32, 1.35, 1.37, 1.4, 1.45, 1.5, 1.7, 1.8, 1.9]:
        print 'w = ', w
        for h in [0.01]:#[0.02, 0.01, 0.005, 0.0025, 0.00125, 0.000625]:
            print 'h = ', h
            # specify cable dimension in meter
            width = 0.1 + h
            height = 0.1 + h
            # specify fixed voltage area
            ground = [[(0, 0), (0, 0.1 + h)], [(0, 0), (0.1 + h, 0)]]
            v1 = [[(0.06, 0.08), (0.1 + h, 0.1 + h)]]
            v1_value = 15

            target = [0.06, 0.04]
            target_pos = [int(x / h) for x in target]

            methodToRun = SOR
            print 'Running {} ... '.format(methodToRun)
            #run(method, h, even_spacing)

            n, unknowns, boundarys = generate_meshnodes()
            print 'u', unknowns, 'BOUND', boundarys

            even_spacing = True
            horizontal_split =
[0,0.001,0.002,0.003,0.004,0.005,0.006,0.007,0.008,0.009,0.01,0.011]
            vertical_split =
[0,0.001,0.002,0.003,0.004,0.005,0.006,0.007,0.008,0.009,0.01,0.011]
            i = 0
            while not error_small_enough(n, unknowns, thr):
                i += 1
                #n = update_matrix(n, unknowns, boundary, methodToRun)
                for unknown, boundary in zip(unknowns, boundarys):
                    n = methodToRun(unknown, boundary, n, even_spacing=even_spacing,
horizontal_split=horizontal_split, vertical_split=vertical_split)
```

```
        #print n[target_pos[1]][target_pos[0]]
        #for row in n:
        #    print row

    print '#iteration = ', i
    # print 'Result matrix ... '
    # for i in n:
    #    print [round(x,3) for x in i]
    print 'voltage at (0.06, 0.04) found: ', n[target_pos[1]][target_pos[0]]
```