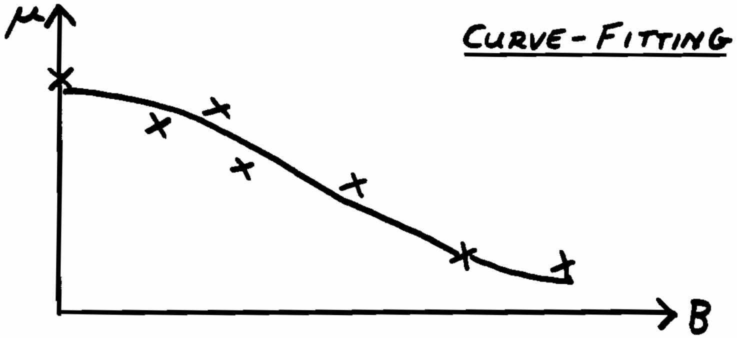
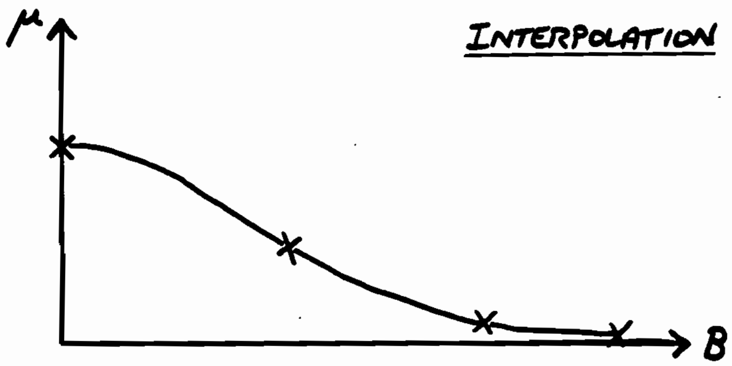
CURVE-FITTING & INTERPOLATION



X = GIVEN DATA

NEED: SMOOTH CURVE CLOSE TO GIVEN POINTS

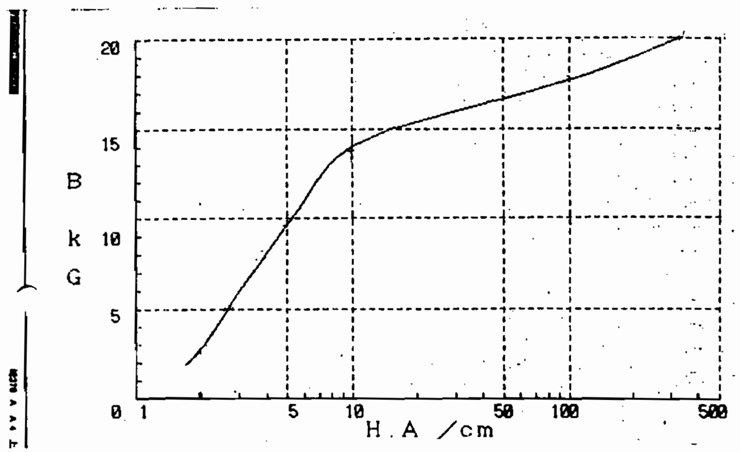


X = GIVEN DATA

NEED: SMOOTH CURVE THROUGH GIVEN POINTS

CV 1

A B-H CURVE



CV 2

LEAST-SQUARES CURVE-FITTING

e.g.,

TRY A SMOOTH CURVE OF THE FORM:

$$y = ax^2 + bx + c$$

CHOOSE THESE 3 PARAMETERS SO THAT THE CURVE LIES AS CLOSE AS POSSIBLE TO THE DATA POINTS:

$$(x_1, y_1), \ldots, (x_N, y_N).$$

i.e., SO THAT

$y(x_k)$ IS CLOSE TO y_k

FOR $k = 1, \ldots, N$.

DEFINE AN ERROR E:

$$E = \sum_{k=1}^{N} \left(y(x_k) - y_k \right)^2$$

$$= \sum_{k=1}^{N} \left(a x_k^2 + b x_k + c - y_k \right)^2$$

FIND a, b, & c THAT MINIMIZES THE SQUARED ERROR, E.

i.e., REQUIRE :

$$\left.\begin{aligned}\frac{\partial E}{\partial a} &= 0\\ \frac{\partial E}{\partial b} &= 0\\ \frac{\partial E}{\partial c} &= 0\end{aligned}\right\}$$

3 SIMULTANEOUS EQUATIONS IN 3 UNKNOWNS, a, b, c

OR,

$$\left.\begin{aligned}2\sum_{k=1}^{N} x_k^2\left(a x_k^2 + b x_k + c - y_k\right) &= 0\\ 2\sum_{k=1}^{N} x_k\left(a x_k^2 + b x_k + c - y_k\right) &= 0\\ 2\sum_{k=1}^{N} \left(a x_k^2 + b x_k + c - y_k\right) &= 0\end{aligned}\right]$$

MORE GENERALLY :

$$y = \sum_{j=1}^{n} a_j g_j(x)$$

i.e., $y(x)$ IS A LINEAR COMBINATION OF $g_1(x), \ldots, g_n(x)$.

$$E = \sum_{k=1}^{N}\left(\sum_{j=1}^{n} a_j g_j(x_k) - y_k\right)^2$$

FOR $i = 1, \ldots, n$:

$$\left[\ \frac{\partial E}{\partial a_i} = 0\right.$$

$\Rightarrow$ FOR $i = 1, \ldots, n$:

$$\sum_{k=1}^{N} 2\left(\sum_{j=1}^{n} a_j g_j(x_k) - y_k\right) g_i(x_k) = 0$$

CHANGE ORDER OF THE SUMMATION :

$\Rightarrow$ FOR $i = 1, \ldots, n$:

$$\sum_{j=1}^{n}\left[\sum_{k=1}^{N} g_i(x_k) g_j(x_k)\right] a_j = \sum_{k=1}^{N} g_i(x_k) y_k$$

OR,

$$\underline{\underline{G}}\,\underline{a} = \underline{b}$$

$\Rightarrow$ SOLVE FOR $\underline{a}$.

NOTE :

$$G_{ij} = \sum_{k=1}^{N} g_i(x_k) g_j(x_k)$$

$$b_i = \sum_{k=1}^{N} g_i(x_k) y_k .$$

ALSO, NOTE THAT $\underline{\underline{G}}$ IS SYMMETRIC.

THE CHOICE OF g_j's

(i) WHOLE DOMAIN

i.e., EACH g_j IS NONZERO OVER THE WHOLE RANGE

$$X_{MIN} \longrightarrow X_{MAX}$$

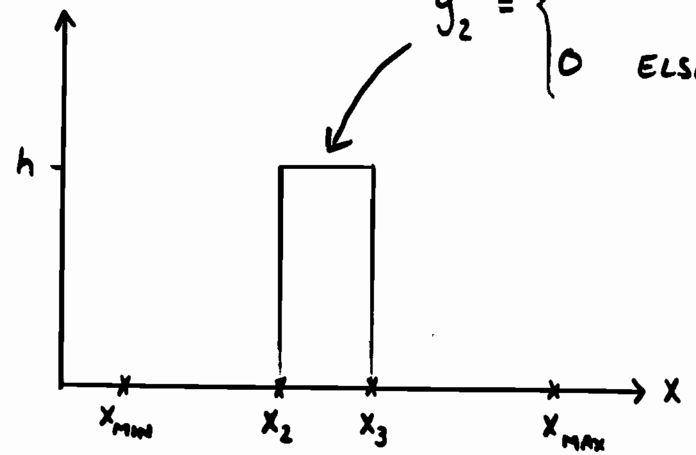
e.g., x^2, x, 1.

(ii) SUBDOMAIN

i.e., EACH g_j IS NONZERO ONLY OVER ITS OWN INTERVAL

e.g.,

$$g_2 = \begin{cases} h & x_2 \le x \le x_3 \\ 0 & \text{ELSEWHERE} \end{cases}$$



LAGRANGE'S POLYNOMIALS AS SUBDOMAIN FUNCTIONS

FOR $j = 1, \dots, n$, LET

$$L_j(x) = \frac{F_j(x)}{F_j(x_j)}$$

WHERE

$$F_j(x) = \prod_{\substack{r=1,n \\ r \neq j}} (x - x_r)$$

$x_1, \dots, x_n$ = SET OF GIVEN VALUES

L_k IS AN $(n-1)^{ST}$ ORDER POLYNOMIAL WITH THE PROPERTY:

$$L_k(x_j) = \begin{cases} 1 & \text{IF } j = k \\ 0 & \text{OTHERWISE} \end{cases}$$

e.g., $n = 2$: FIRST-ORDER

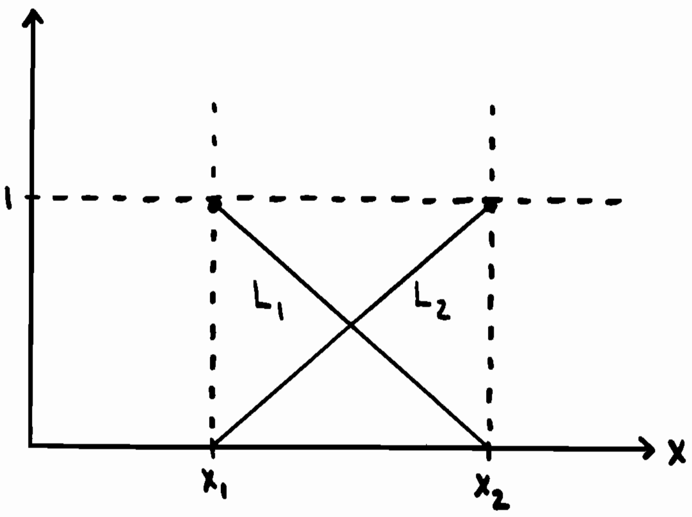
$$F_1(x) = x - x_2$$

$$F_2(x) = x - x_1$$

$$L_1(x) = F_1(x)/F_1(x_1) = \frac{x - x_2}{x_1 - x_2}$$

$$L_2(x) = F_2(x)/F_2(x_2) = \frac{x - x_1}{x_2 - x_1}$$

FIG. 1: FIRST-ORDER 1-D LAGRANGE SUBDOMAIN BASIS FUNCTIONS



$$L_1(x) = \frac{x - x_2}{x_1 - x_2} \quad , \quad x_1 \le x \le x_2$$

$$L_2(x) = \frac{x - x_1}{x_2 - x_1} \quad , \quad x_1 \le x \le x_2$$

AN IMPORTANT FEATURE OF LAGRANGE'S POLYNOMIALS IS:

IF

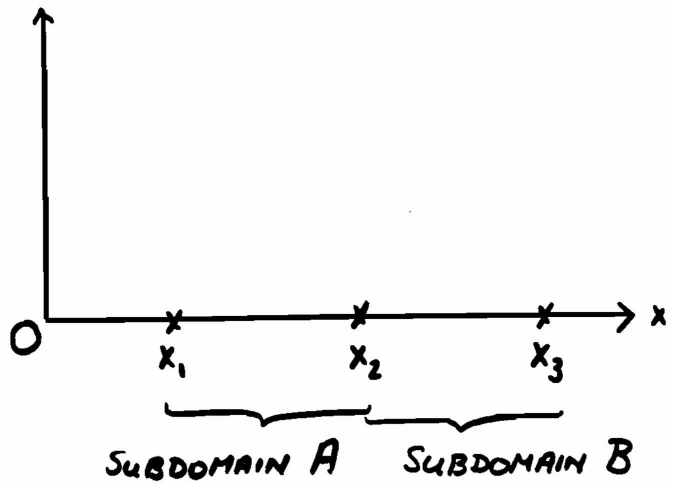
$$y(x) = \sum_{j=1}^{n} a_j L_j(x)$$

THEN

$$a_1 = y(x_1)$$
$$a_2 = y(x_2)$$
$$\vdots$$
$$a_n = y(x_n)$$

i.e., THE PARAMETERS a_j ARE THE VALUES OF y AT THE POINTS $x_1, \ldots, x_j, \ldots, x_n$.

CONSIDER 2 ADJACENT SUBDOMAINS, A AND B :



USE FIRST-ORDER LAGRANGE POLYNOMIALS ON EACH.

$\Rightarrow$ USE x_1, x_2 FOR A

USE x_2, x_3 FOR B

$$y(x) = \begin{cases} a_1^A L_1^A(x) + a_2^A L_2^A(x), & \text{IN } x_1 \le x \le x_2 \\ a_1^B L_1^B(x) + a_2^B L_2^B(x) & \text{IN } x_2 \le x \le x_3 \end{cases}$$

THERE ARE APPARENTLY 4 PARAMETERS:

$$a_1^A, \; a_2^A, \; a_1^B, \; a_2^B$$

BUT, RECALL THAT :

$$a_1^A = y(x_1)$$

$$a_2^A = y(x_2) \quad (\text{VALUE IN } A)$$

$$a_1^B = y(x_2) \quad (\text{VALUE IN } B)$$

$$a_2^B = y(x_3)$$

SO, FOR A CONTINUOUS CURVE :

$$a_2^A = a_1^B$$

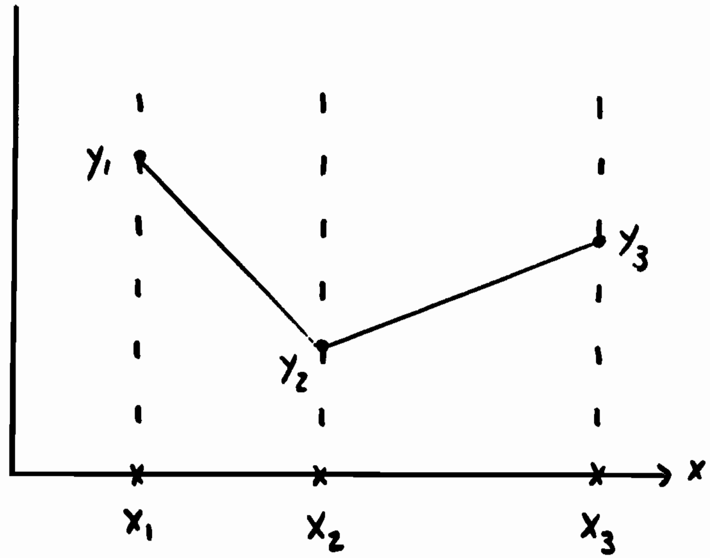
IF WE DENOTE $y(x_j)$ AS y_j :

$$y(x) = \begin{cases} y_1 L_1^A(x) + y_2 L_2^A(x) & \text{IN } x_1 \le x \le x_2 \\ y_2 L_1^B(x) + y_3 L_2^B(x) & \text{IN } x_2 \le x \le x_3 \end{cases}$$

i.e., THERE ARE ONLY 3 PARAMETERS :

y_1, y_2, AND y_3.

A TYPICAL CURVE $y(x)$ OF THIS KIND IS :



SUCH A CURVE IS SAID TO BE PIECEWISE LINEAR.

HERMITE'S POLYNOMIALS AS SUBDOMAIN FUNCTIONS

CONSIDER THE POLYNOMIALS

FOR $j = 1, \ldots, n$:

$$U_j(x) = \left[1 - 2L_j'(x_j)(x-x_j)\right] L_j^2(x)$$

$$V_j(x) = (x-x_j)\, L_j^2(x)$$

WHERE $x_1, \ldots, x_n$ = GIVEN x VALUES

$L_j(x)$ = LAGRANGE POLYNOMIAL, DEGREE $(n-1)$

$$L_j'(x) = \frac{dL_j}{dx}$$

IT IS POSSIBLE TO PROVE THE FOLLOWING :

$$U_j(x_i) = \delta_{ij}$$

$$U_j'(x_i) = 0$$

$$V_j(x_i) = 0$$

$$V_j'(x_i) = \delta_{ij}$$

(δ_{ij} IS THE KRONECKER DELTA)

u_j AND v_j ARE POLYNOMIALS OF DEGREE $(2n-1)$.

e.g., $n = 2 \Rightarrow$ DEGREE 3 (CUBIC)

$$L_1(x) = \frac{x - x_2}{x_1 - x_2} \; ; \quad L_1' = \frac{1}{x_1 - x_2}$$

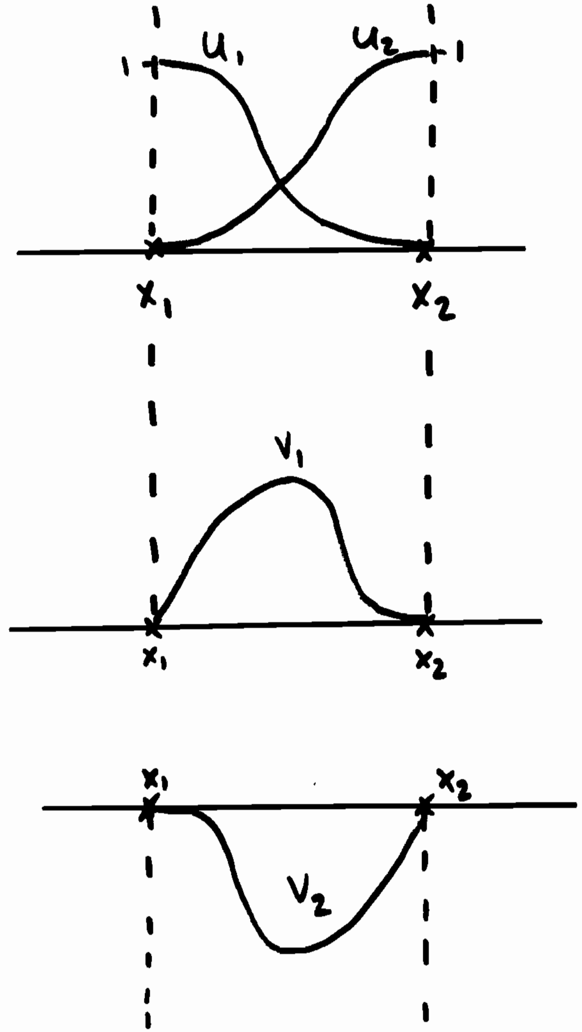
$$L_2(x) = \frac{x - x_1}{x_2 - x_1} \; ; \quad L_2' = \frac{1}{x_2 - x_1}$$

$$u_1(x) = \left[1 - \frac{2}{x_1 - x_2}(x - x_1)\right]\left(\frac{x - x_2}{x_1 - x_2}\right)^2$$

$$u_2(x) = \left[1 - \frac{2}{x_2 - x_1}(x - x_2)\right]\left(\frac{x - x_1}{x_2 - x_1}\right)^2$$

$$v_1(x) = (x - x_1)\left(\frac{x - x_2}{x_1 - x_2}\right)^2$$

$$v_2(x) = (x - x_2)\left(\frac{x - x_1}{x_2 - x_1}\right)^2$$



NOW IF :

$$y(x) = \sum_{j=1}^{n} a_j U_j(x) + b_j V_j(x)$$

THEN :

$$a_1 = y(x_1) \quad ; \quad b_1 = y'(x_1)$$

$$a_2 = y(x_2) \quad ; \quad b_2 = y'(x_2)$$

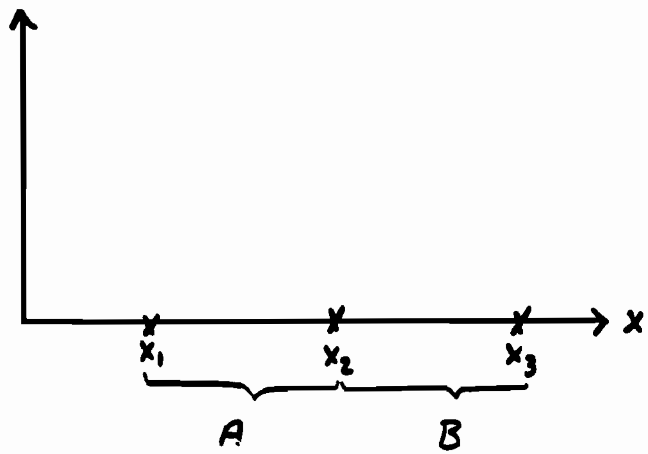
$$\vdots \qquad\qquad\qquad \vdots$$

$$a_n = y(x_n) \quad ; \quad b_n = y'(x_n)$$

i.e., THE PARAMETERS a_j ARE VALUES OF y AT THE POINTS $x_1, \ldots, x_j, \ldots, x_n$.

THE PARAMETERS b_j ARE VALUES OF dy/dx AT THE POINTS $x_1, \ldots, x_j, \ldots, x_n$.

CONSIDER 2 ADJACENT SUBDOMAINS, A AND B :



USE CUBIC HERMITE POLYNOMIALS ON EACH.

$\Rightarrow$ USE x_1, x_2 FOR A

USE x_2, x_3 FOR B

$$
y(x) = \begin{cases} a_1^A U_1^A(x) + a_2^A U_2^A(x) \\ + b_1^A V_1^A(x) + b_2^A V_2^A(x) \\ \qquad \text{IN SUBDOMAIN } A \\ \\ a_1^B U_1^B(x) + a_2^B U_2^B(x) \\ + b_1^B V_1^B(x) + b_2^B V_2^B(x) \\ \qquad \text{IN SUBDOMAIN } B \end{cases}
$$

THERE ARE APPARENTLY 8 PARAMETERS

$$a_1^A,\ a_2^A,\ b_1^A,\ b_2^A,$$

$$a_1^B,\ a_2^B,\ b_1^B,\ b_2^B.$$

BUT, RECALL THAT :

$$a_1^A = y(x_1)$$

$$b_1^A = y'(x_1)$$

$$\left.\begin{aligned} a_2^A &= y(x_2) \\ b_2^A &= y'(x_2) \end{aligned}\right\} \text{VALUES IN } A$$

$$\left.\begin{aligned} a_1^B &= y(x_2) \\ b_1^B &= y'(x_2) \end{aligned}\right\} \text{VALUES IN } B$$

$$a_2^B = y(x_3)$$

$$b_2^B = y'(x_3)$$

So, FOR A CONTINUOUS (C_0) CURVE :

$$a_2^A = a_1^B$$

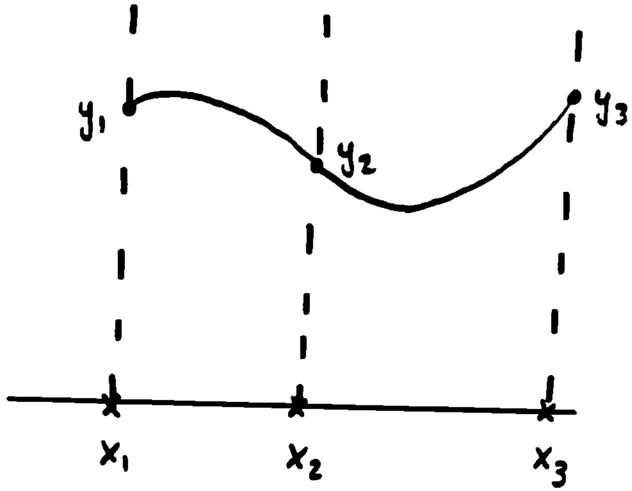
FOR A CURVE THAT ALSO HAS CONTINUOUS SLOPE (C_1) :

$$b_2^A = b_1^B$$

i.e., THERE ARE ONLY 6 PARAMETERS:

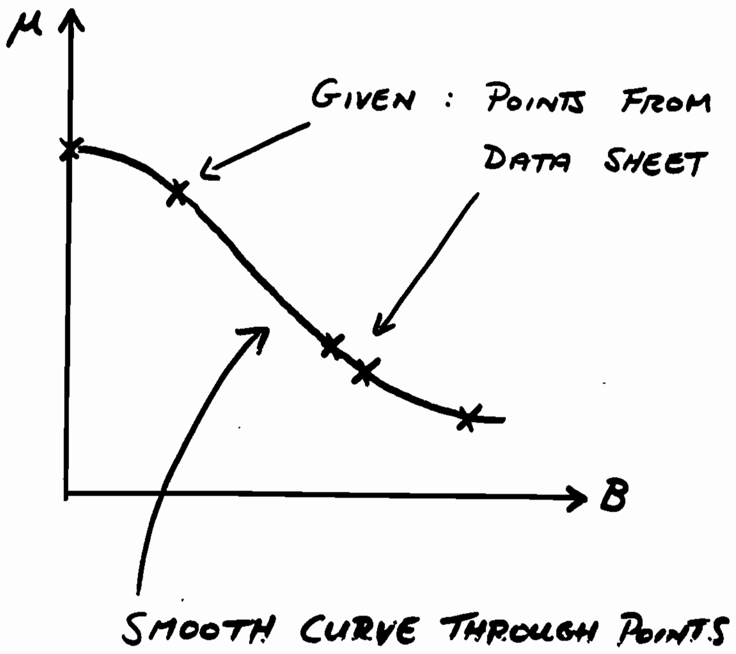
$$y_1 , y_1' , y_2 , y_2' , y_3 , y_3'$$

A TYPICAL CURVE $y(x)$ IS :



INTERPOLATION

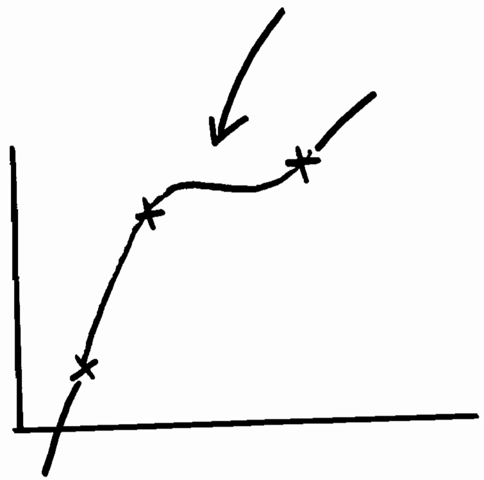
INTERPOLATION MEANS FINDING A CURVE THAT PASSES EXACTLY THROUGH A GIVEN SET OF POINTS.



HIGH - ORDER FULL-DOMAIN POLYNOMIALS

e.g. LAGRANGE POLYNOMIALS OF ORDER (N-1), THROUG N POINTS

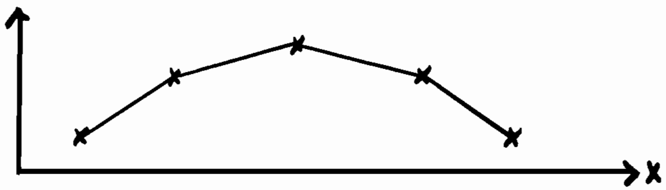
- TEND TO GET "WIGGLES":



PIECEWISE POLYNOMIALS:

a) FIRST-ORDER LAGRANGE

- SIMPLE, BUT
- NOT ALWAYS SMOOTH ENOUGH.



b) CUBIC HERMITE

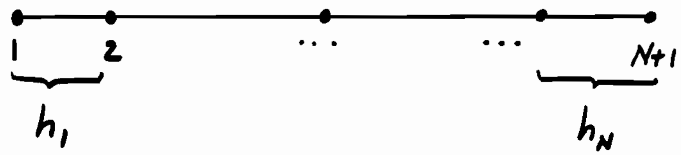
i.e. INTERPOLATE y AND y'

- C_1 CONTINUITY, BUT
- DERIVATIVES AT DATA POINTS ARE NOT ALWAYS AVAILABLE.

c) CUBIC SPLINES

START WITH CUBICS ON N INTERVALS:	$4N$ DEGREES OF FREEDOM (DOF)
INTERPOLATE y AT $N+1$ DATA POINTS:	$2N$ CONSTRAINTS
MATCH y' AND y'' AT $N-1$ DATA POINTS:	$2(N-1)$ CONSTRAINTS
FIX y' AT 2 ENDS:	2 CONSTRAINTS
$\Rightarrow$ REMAINING DOFs:	0

"DRAFTING" SPLINE: BENDS SO SLOPE AND CURVATURE ARE CONTINUOUS (C_2).



ON SEGMENT i :

$$S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

LET y_i = VALUE OF y AT x_i

y_i'' = VALUE OF $\dfrac{d^2y}{dx^2}$ AT x_i

IT CAN BE SHOWN THAT:

$$a_i = \frac{1}{6h_i}(y''_{i+1} - y_i'')$$

$$b_i = y_i''/2$$

$$c_i = \frac{1}{h_i}(y_{i+1} - y_i) - \frac{h_i}{6}(y''_{i+1} + 2y_i'')$$

$$d_i = y_i$$

$\Rightarrow$ GIVEN y_i AND y_i'' AT ALL THE NODES, WE COULD USE THE EQUATIONS ABOVE TO GET A CUBIC IN EACH SEGMENT.

BUT WE DON'T USUALLY KNOW y_i''.

INSTEAD : IMPOSE SLOPE CONTINUITY:

FOR $i = 1, \ldots, N-1$:

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$$

OR :

$$y_i'' \frac{h_i}{6} + y_{i+1}'' \left(\frac{h_i}{3} + \frac{h_{i+1}}{3} \right) + y_{i+2}'' \frac{h_{i+1}}{6} = \frac{y_{i+2} - y_{i+1}}{h_{i+1}} - \frac{y_{i+1} - y_i}{h_i}$$

CHOOSE $y_1'' = y_{N+1}'' = 0$ (FOR SIMPLICITY).

THEN :

$$\begin{pmatrix} \frac{h_1+h_2}{3} & \frac{h_2}{6} & & \\ \frac{h_2}{6} & \frac{h_2+h_3}{3} & \frac{h_3}{6} & \\ & & \ddots & \end{pmatrix} \begin{pmatrix} y_2'' \\ y_3'' \\ \vdots \\ y_N'' \end{pmatrix} = \begin{pmatrix} \frac{y_3-y_2}{h_2} - \frac{y_2-y_1}{h_1} \\ \frac{y_4-y_3}{h_3} - \frac{y_3-y_2}{h_2} \\ \vdots \\ \frac{y_{N+1}-y_N}{h_N} - \frac{y_N-y_{N-1}}{h_{N-1}} \end{pmatrix}$$

↑ SYMMETRIC TRIDIAGONAL MATRIX

SOLVE FOR $y_2'', y_3'', \ldots, y_N''$

THEN GET a_i, b_i, c_i, d_i

FOR $i = 1, 2, \ldots, N$.