UNIVERSITY OF ENERGY AND NATURAL RESOURCES, SUNYANI

PROJECT TOPIC IN CAPS

SURNAME, FIRST NAMES

DEPARTMENT OF MATHEMATICS AND STATISTIC

SCHOOL OF SCIENCE

JUNE 2019

PROJECT TOPIC IN CAPS

by

Name - exams number

B.Sc. (Program) Location

A Thesis submitted to the Department of Mathematics and Statistics, School of Science, University of Energy and Natural Resources, Sunyani in partial fulfillment of the requirements for the degree of Bachelor of Science in Mathematics

JUNE 2019

DECLARATION AND CERTIFICATION

personal declaration here...

Candidate′s Signature: .... Date: ....

Supervisor′s Certification

This study was carried out under the supervisory committee of (Names of all Supervisors) in accordance with the guidelines on supervisions of graduate studies.

Major Supervisor′s Name and Qualifications

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Co-Supervisor′s Certification

Co-Supervisor′s Name and Qualifications

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Co-Supervisor′s Name and Qualifications

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ABSTRACT

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DEDICATION

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ACKNOWLEDGMENTS

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2.1 Introduction

Peralta *et al.*, formulate a susceptible-vaccinated-infected-recovered (SVIR) model by incorporating the vaccination of newborns, vaccine age, and mortality induced by the disease into the SIR epidemic model. It is assumed that the period of immunity induced by vaccines varies depending on the vaccine-age. They performed a nonlinear stability analysis, by means of the Lyapunov function techniques and LaSalle's Invariance Principle for semiflows. They showed that the classical threshold condition for the effective reproductive number, R_v, holds: $R_v > 1$; then the endemic steady E^* is globally asymptotically stable, whereas if $R_v \leq 1$, then the infection-free steady state E_0 is globally asymptotically stable. (**peralta2015global**).

CHAPTER THREE

3.0 METHODOLOGY

3.1 Introduction

In this chapter, the methods used to investigate our model are expounded, clearly stating the mathematical instruments and constructions, theorems, lemmas and their proofs.

3.1.1 Existence & Uniqueness Theorem

Theorem 3.1.1 *Consider the initial value problem* $x' = f(x)$*, with* $x(0) = x_0$*. Suppose that the function* ***f*** *is continuous and that all its partial derivatives* $\frac{\delta f_i}{\delta x_j}$*,* i, j = 1, 2,...,n *are continuous for* x *in some open connected set* $D \subseteq R_n$*, then for* $x_0 \in D$*, the initial value problem has a solution* $x(t)$ *on some interval* (-t, t) *about* $t = 0$*, and the solution is unique. - Nonlinear dynamics and chaos by Steven H. Strogatz (***strogatz2018nonlinear***).*

Theorem 3.1.2 $V(x)$ *is said to be positive (negative) definite in a neighborhood U of the origin if* $V(x) > 0$ $(V(x) < 0)$ *for all* $x \neq 0$ *in* U *, and* $V(0) = 0$*.* $V(x)$ *is positive (negative) semi-definite in a neighborhood U of the origin if* $V(x) \geq 0$ $(V(x) \leq 0)$ *for all* $x \neq 0$ *in* U *, and* $V(0) = 0$*.*

Theorem 3.1.3 *Let* $X^*(t) = 0$*,* $t \geq t_0$ *be the zero solution of the regular system* $X' = X(x)$*, where* $X(0) = 0$*. Then* $X(x(t))$ *is uniformly stable for* $t \geq t_0$ *if there exists V (x) with the following properties in some neighborhood of X = 0:*

i. $V(x)$ *and its partial derivatives are continuous;*

ii. $V(x)$ *is positive definite;*

iii. $\dot{V}(x)$ *is negative semi-definite.*

Theorem 3.1.4 *If we observe all the conditions of the* ***Theorem 6****, except the last condition of* ***iii*** *and instead assume that* ***iii*** $\dot{V}$ *is negative definite. Then the zero solution is asymptotically stable (and such a function* V *is called a strong Lyapunov function for the system).*

3.1.2 Basic Reproductive Number R_0

In compartmental models for infectious disease transmission, individual are categorized into two: some are called disease compartments if the individuals therein are infected, while others are called non-disease compartments. Suppose that there are $n > 0$ disease compartments and $m > 0$ non-disease compartments. Then a general compartmental disease transmission model can be written as

$$x' = \mathscr{F}(x,y) - \mathscr{V}(x,y), \qquad y' = g(x,y) \tag{3.1}$$

with $g = (g_1, \cdots, g_m)^T$. $x = (x_1, \cdots, x_n)^T \in \Re_+$ and $y = (y_1, \cdots, y_m)^T \in \Re_+$ represent the populations in the disease and non-disease compartments respectively; $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_n)^T$ and $\mathscr{V} = (\mathscr{V}_1, \cdots, \mathscr{V}_n)^T$. Then the assumptions as spelt out in (**shuai2013global**; **van2002reproduction**; **van2008further**) hold and are discussed and interpreted in detail. Following (**van2002reproduction**; **van2008further**), define two $n \times n$ matrices

$$F = \left[\frac{\partial \mathscr{F}_i}{\partial x_j}(0, y_0)\right] \quad \text{and} \quad V = \left[\frac{\partial \mathscr{V}_i}{\partial x_j}(0, y_0)\right]. \tag{3.2}$$

Assume that $F \geq 0$ and $V^{-1} \geq 0$, which are biologically reasonable. Then the next-generation matrix is $K = FV^{-1}$, and the basic reproductive number R_0 can be defined as the spectral radius of K, which is

$$R_0 = \rho(FV^{-1}) \tag{3.3}$$

To this $R_{0]}$ there are several measures considered in several literature one of which is the measure

$$R_0 = \frac{s_{DFE}}{s_{EE}} \tag{3.4}$$

where s_{DFE} is the s value of the DFE and s_{EE} is the s value of the EE.

3.1.3 A Matrix-Theoretic Method

The matrix-theoretic method is used to prove the statement (**shuai2013global**). It is a systematic method, and it is presented to guide the construction of a Lyapunov function. Taking the same path as (**shuai2013global**; **castillo2002computation**; **van2008further**), let us set

$$f(x,y) := (F - V)x - \mathscr{F}(x,y) + \mathscr{V}(x,y) \tag{3.5}$$

Then the equation for the disease compartment can be written as

$$x' = (F - V)x - f(x,y) \tag{3.6}$$

Let $\psi^T \leq 0$ be the left eigenvector of the non-negative matrix $V^{-1}F$ corresponding to the eigenvalue $\rho(V^{-1}F) = \rho(FV^{-1}) = R_0$ (this is proved in the Appendix). The following result provides a general method to construct a Lyapunov function for (**??**). **guo2011global**; **guo2008graph**; **shuai2011global** used this Lyapunov function involving the Perron eigenvectors to study the global dynamics of the several specific disease models while (**shuai2013global**) used it to consider a general case for infectious diseases. In this paper, we are employing this same method to establish the global stability of our system.

Theorem 3.1.5 *Let F, V and $f(x,y)$ be defined as in 3.2 and 3.5 respectively. If $f(x,y) \geq 0$, in the $\Omega \subset \Re_{+}^{n+m}$, $F \geq 0$, $V^{-1} \geq 0$ and $R_0 \leq 1$, then the function $\mathscr{D} = \psi^T V^{-1} x$ is a Lyapunov function for the system 3.6 on Ω.*

Proof. The proof as followed in (**shuai2013global**) gives

$$\mathscr{D}' = \mathscr{D}' \mid_{(\mathbf{??})} = \psi V^{-1} x' = \psi V^{-1}(F - V)x - \psi V^{-1} f(x,y) \tag{3.7}$$

$$= (R_0 - 1)\psi^T x - \psi^T V^{-1} f(x,y) \tag{3.8}$$

Since $\psi^T \geq 0$, $V^{-1} \geq$, and $f(x,y) \geq 0$ in the region Ω, the last term is non-positive. If $R_0 \leq 1$, then $\mathscr{D}^1 \leq 0$ in Ω and thus $\mathscr{D}$ is a Lyapunov function for the system (3.1).

Shuai and Pauline (**shuai2013global**) has proven that the Lyapunov function used to prove the global stability of the DFE in Ω can also be extended to establish a uniform persistence and thus establish the existence of an EE in $\Re_{+}^{n+m}$. Find the *theorem 2.2* in (**shuai2013global**) and the proof thereof.

3.1.4 Lokta-Volterra Criterion for Lyapunov Functions

A general form of Lyapunov functions coined from the first integral of the Lokta-Volterra system which is often used in the literature of mathematical biology is used to prove the global stability of the EE. This function takes the form

$$\mathscr{L} = \Sigma_{i=1}^{n} c_i \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*} \right) \tag{3.9}$$

where x are the variables and c_i are carefully selected constants. This criterion has been used many times in establishing the stability or otherwise of many disease models and also present in **shuai2013global**.

CHAPTER FOUR

4.0 ANALYSIS, RESULTS & DISCUSSION

4.0.1 Variable & Parameter Definition

Find in the tables (**4.2**) and (**4.3**) for the definition of variables and parameters respectively as used in this study.

4.0.2 Model Equations

The following is the model equation gleaned from the schematic diagram above - Figure (**4.1**) for the formulation of a SEIRS model;

$$\begin{aligned}
S^{'} &= \alpha N + \epsilon R - (\mu + \beta I/N + \lambda)\, S \\
E^{'} &= \beta SI/N - (\mu_E + \sigma + \delta_E)\, E \\
I^{'} &= \sigma E - (\tau + \gamma) I \\
R^{'} &= \gamma I - (\mu + \epsilon) R + \lambda S + \delta_E E
\end{aligned} \tag{4.1}$$

Table 4.1: Variable Definition.

Variable	Definition
S	Susceptible individuals
E	Exposed individuals
I	Infected individuals
R	Recovered and Immunized individuals

Table 4.2: Parameter Definition

Parameters	Definitions
α	New births and Immigration rates
μ	Mortality rate
μ_E	$(\mu + k_E)$, where k_E is the rate at which exposed individuals die
λ	Proportion of children vaccinated successfully at birth
β	Average number of adequate contacts of a person per unit time
σ	The rate at which exposed individuals get infectious
γ	The rate at which infected individuals get recovered
δ_E	Number of natural immunes per thousands
τ	$(\mu + k_I)$, where k_I is the rate at which infected individuals die

And we find the dynamics 4.1 of the total population by adding the equations above to get;

$$\begin{aligned} S^{'} + E^{'} + I^{'} + R^{'} = \alpha N + \epsilon R - (\mu + \beta I/N + \lambda)\, S + \beta S I/N \\ - (\mu_E + \sigma + \delta_E)\, E + \sigma E - (\tau + \gamma) I \\ + \gamma I - (\mu + \epsilon) R + \lambda S + \delta_E E \end{aligned} \tag{4.2}$$

$$\begin{aligned} F(\lambda) = & -\frac{\beta h_1 h_3 h_4 k_I \sigma}{h_3 h_4 k_I - \beta h_3 h_4} - \frac{\lambda \beta h_1 h_4 k_I \sigma}{h_3 h_4 k_I - \beta h_3 h_4} - \frac{\lambda \beta h_1 h_3 k_I \sigma}{h_3 h_4 k_I - \beta h_3 h_4} - \frac{\lambda^2 \beta h_1 k_I \sigma}{h_3 h_4 k_I - \beta h_3 h_4} \\ & + \frac{\beta^2 h_1 h_3 h_4 \sigma}{h_3 h_4 k_I - \beta h_3 h_4} + \frac{\lambda \beta^2 h_1 h_4 \sigma}{h_3 h_4 k_I - \beta h_3 h_4} + \frac{\lambda \beta^2 h_1 h_3 \sigma}{h_3 h_4 k_I - \beta h_3 h_4} + \frac{\lambda^2 \beta^2 h_1 \sigma}{h_3 h_4 k_I - \beta h_3 h_4} \\ & + \frac{h_2 {h_3}^2 {h_4}^2 k_I}{h_3 h_4 k_I - \beta h_3 h_4} + \frac{\lambda h_2 h_3 {h_4}^2 k_I}{h_3 h_4 k_I - \beta h_3 h_4} + \frac{\lambda h_2 {h_3}^2 h_4 k_I}{h_3 h_4 k_I - \beta h_3 h_4} + \frac{\lambda^2 h_2 h_3 h_4 k_I}{h_3 h_4 k_I - \beta h_3 h_4} \\ & - \frac{\beta h_2 {h_3}^2 {h_4}^2}{h_3 h_4 k_I - \beta h_3 h_4} - \frac{\lambda \beta h_2 h_3 {h_4}^2}{h_3 h_4 k_I - \beta h_3 h_4} - \frac{\lambda \beta h_2 {h_3}^2 h_4}{h_3 h_4 k_I - \beta h_3 h_4} - \frac{\lambda^2 \beta h_2 h_3 h_4}{h_3 h_4 k_I - \beta h_3 h_4} \\ & - \lambda h_2 h_4 - \lambda^2 h_4 - \lambda h_2 h_3 - \lambda^2 h_3 - \lambda^2 h_2 - \lambda^3 \end{aligned} \tag{4.3}$$

Global Stability Analysis of the Endemic Equilibrium

A general form of Lyapunov functions coined from the first integral of the Lokta-Volterra system which is often used in the literature of mathematical biology is used to prove the global stability of the EE. This function takes the form

$$\mathscr{L} = \Sigma_{i=1}^{n} c_i \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*} \right)$$

where x are the variables and c_i are carefully selected constants. This criterion has been used many times in establishing the stability or otherwise of many disease models and also present in (**shuai2013global**).

Theorem 4.0.1 *The EE is globally stable.*

Proof. Let

$$\begin{aligned}
\mathscr{L}_1 &= s - s^* - s^* \ln \frac{s}{s^*} \\
\mathscr{L}_1' &= -\left(\frac{s^* - s}{s}\right) s' \\
&\leq -\left(\frac{s^* - s}{s}\right) (h_1 - h_2 s - (\beta - k_I) s i), \\
&\quad \text{and the equilibrium relation for } h_1 = h_2 s^* + (\beta - k_I) s^* i^* \\
&\leq -\left(\frac{s^* - s}{s}\right) (h_2 (s^* - s) + (\beta - k_I)(s^* i^* - s i)) \\
\mathscr{L}_1' &\leq 0, \text{ irrespective of the values assumed by } s^*, s\ i^* \text{ and } i \text{ in the region } \Re_+
\end{aligned} \tag{4.4}$$

Again, let

$$\mathscr{L}_2 = e - e^* - e^* \ln \frac{e}{e^*}$$

$$\mathscr{L}_2' = -\left(\frac{e^* - e}{e}\right) e'$$

$$\begin{aligned}
\mathscr{L}_2' &\leq -\left(\frac{e^* - e}{e}\right)(\beta s i - h_3 e) \\
&\leq -(e^* - e)\left(\frac{\beta s i}{e} - h_3\right) \\
&\leq -(e^* - e)\left(\frac{\beta s^* i^*}{e^*} - h_3\right),
\end{aligned} \tag{4.5}$$

then from the system under study, (**??**), $\dfrac{i^*}{e^*} = \dfrac{\sigma}{h_4}$ and $s^* = \dfrac{h_3 h_4}{\beta\sigma}$

$$\leq -(e^* - e)\left(\frac{\beta h_3 h_4 \sigma}{\beta h_4 \sigma} - h_3\right) = -(e^* - e) \times 0 = 0$$

$$\mathscr{L}_2' \leq 0$$

Lastly, let us set

$$\mathscr{L}_3 = i - i^* - i^* \ln \frac{i}{i^*}$$

$$\mathscr{L}_3' = -\left(\frac{i^* - i}{i}\right) i'$$

$$\begin{aligned}
\mathscr{L}_3' &\leq -\left(\frac{i^* - i}{i}\right)(\sigma e - h_4 i) \\
&\leq -(i^* - i)\left(\sigma \frac{e}{i} - h_4\right) \\
&\leq -(i^* - i)\left(\sigma \frac{e^*}{i^*} - h_4\right), \ \frac{e^*}{i^*} = \frac{h_4}{\sigma} \\
&\leq -(i^* - i)\left(\sigma \frac{h_4}{\sigma} - h_4\right) = 0
\end{aligned} \tag{4.6}$$

$$\mathscr{L}_3' \leq 0$$

Therefore $\mathscr{L}$ defined as $\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3$ is a Lyapunov function for the sys-

Figure 4.1: UENR Logo

tem 10. Arbitrary constants c_i can be chosen from $\Re_+$ and any linear combination of $\mathscr{L}$ would be a Lyapunov function for the system. Hence the proof.

CHAPTER FIVE

5.0 CONCLUSION & RECOMMENDATIONS

5.1 Introduction

This chapter contains the summary of our findings and the recommendations from our findings. These recommendations are necessary information for the Health Directorate in Sunyani and also for Mathematicians in the study of non-linear systems.

5.2 Conclusion

Appendix Chapter

Appendix B Chapter