Quantum Solution To Matrix Riccati Differential Equation: Preliminary Studies

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Abstract

At the time of writing, the most efficient method to solve the Matrix Riccati differential equation is called the Conjugate Gradient (CG) method with complexity $O(N \cdot s \cdot k \cdot log(\frac{1}{\epsilon}))$. In 2009, Harrow-Hassidim-Lloyd from MIT published a quantum algorithm as an alternative to CG with running time complexity of $O(log(N) \cdot s^2 \cdot \frac{k^2}{\epsilon})$ when certain requirements are met. Currently, quantum hardware does not have the properties required to obtain said exponential advantage from these algorithms. Growing emphasis has been recently placed in hybrid architectures, composed of both a classical and a quantum component, better suited for small quantum computers. Here, we propose a mathematical proof and a hybrid algorithm to solve a Matrix Riccati equation using IBM's quantum lab and HHL solver package.

Quantum Linear System Problem (QLSP)

The Quantum Linear System Problem (QLSP) states that given a Hermitian $N \times N$ matrix A, a unit vector $\vec{b}$, and the equation

$$A\vec{x} = \vec{b}$$

prepare a quantum state that approximates

$$|x\rangle = \frac{\sum_{i=1}^{N} x_i |i\rangle}{\sqrt{(\sum_{i=1}^{N} |x_i|^2)}}$$

where $\vec{x} = (x_1, x_2, ..., x_N)^T$. Specifically, for a given precision $\epsilon > 0$, the approximate (mixed or pure) quantum state ρ_x satisfies

$$\frac{1}{2} Tr|\rho_x - |x\rangle\langle x|| \leq \epsilon$$

Re-writing the Matrix Riccati equation

The goal is, given an initial condition $y(0)$, to obtain solutions $y(t)$ for $t > 0$.

Theorem

Let R denote the following Matrix Riccati equation

$$Y' = Y \cdot A(t) \cdot Y + C(t) \cdot Y + Y \cdot B(t) + D(t) \quad (R)$$

where $Y \in \mathbb{R}^{n \times m}$, $D(t) \in \mathbb{R}^{n \times m}$, $C(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{m \times m}$ *and* $A(t) \in \mathbb{R}^{m \times n}$.

Assuming $m = n$, *if* $B(t) = 0$ *and if* $A(t)$ *is invertible, then R can be converted to the following second order matrix differential equation:*

$$u'' - (ACA^{-1} + AA^{-1}) \cdot u' + AD \cdot u = 0$$

by using the change of variables $Y = -A^{-1} \cdot u' \cdot u^{-1}$, *where* u *is invertible.*

Theorem

If $ACA^{-1} + AA^{-1} = S$ *where* S *is a constant and diagonalizable, and* $A \cdot D = -I$, *then (R2) can be converted into the following equation:*

$$v'' - D_1 \cdot v' - v = 0 \quad (R3)$$

where D_1 *is a diagonal matrix.*

A particular case

In the particular case where $m = n = 1$, we get the following Riccati equation:

$$\frac{dy}{dt} = A(t)y^2 + B(t)y + C(t) \quad (1)$$

where $A(t), B(t), C(t)$ are real-valued functions of t.

We convert the Riccati equation to a second order differential equations by applying the change of variables $y = -\frac{v'}{A \cdot v}$, such that we obtain the following equation:

$$v'' + v' \cdot (B - \frac{A'}{A}) + A \cdot C \cdot v = 0$$

Getting the Linear System

Next, let $\alpha = B - \frac{A'}{A}$ and assume $A \cdot C = -1$, which must hold for the resulting matrix to be Hermitian. From this, we obtain a system with vector of unknowns $\vec{u} = [v, z]^T$ and a matrix $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\alpha \end{smallmatrix}\right)$. Therefore, we obtain the following equation $u' = M \cdot u$. We proceed by computing the characteristic polynomial to obtain the following eigenvalues:

$$\lambda_i = \frac{-\alpha \mp \sqrt{\alpha^2 + 4}}{2}, i \in \{1, 2\}$$

and thus the corresponding eigenvectors are:

$$V_1 = [1, \lambda_1]^T; V_2 = [1, \lambda_2]^T$$

Normalized Linear Equation for HHL

Finally, we construct a Passage matrix $P = \left(\begin{smallmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{smallmatrix}\right)$, such that we can rewrite our matrix M as $H = P \cdot D \cdot P^{-1}$ where $D = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ is a diagonal matrix.

It follows that to solve this system we must compute $\vec{x} = e^{tH} \cdot x_0$, where $x_0 = [u(0), z(0)]^T$. Thus, we must rewrite the problem as $e^{-tH} \cdot \vec{x} = x_0$, where $e^{-tH} = P \cdot e^{-tD} \cdot P^{-1}$. Since we now know e^{-tH} is Hermitian, we conclude by normalizing the elements of x_0 such that $u(0)^2 + z(0)^2 = 1$.

Quantum Riccati Solver (QRS)

Algorithm 0.1: QRS($A, A', B, initCondition, T$)

comment: Given initial condition $y(0)$ compute $y(t)$

if $A \cdot C \neq -1$ **and** $(B + \frac{A}{A'})' \neq 0$
then Display error message. Halt program.

$\alpha \leftarrow B + \frac{A}{A'}$

$\lambda_1 \leftarrow \frac{(\alpha + \sqrt{\alpha^2+4})}{2}$, $\lambda_2 \leftarrow \frac{(\alpha - \sqrt{\alpha^2+4})}{2}$

Compute components of new matrix **M**
Compute vector **x** of unknowns using initCondition
Feed **M** and **x** to HHL
Extract the right vector components from statevector Ψ

$y_t \leftarrow$-($z(T)/(A(T) \cdot u(T))$)
output (y_t)

Harrow-Hassidim-Lloyd (HHL) algorithm

The graph depicts the circuit for HHL algorithm. After encoding the solution vector $|b\rangle$ as a linear combination of the input matrix A's eigenvectors, the algorithm undergoes an iterative process of inverting A's eigenvalues, by applying Quantum Phase Estimation with $U = e^{iAt}$, as well as A's inverse until the measurement (depicted in between lines 4 and 5) yields 1. Then, it returns the estimate $|\tilde{x}\rangle \approx \sum_{j=1}^{n} \langle u_j|b\rangle / \lambda_j |u_j\rangle$.

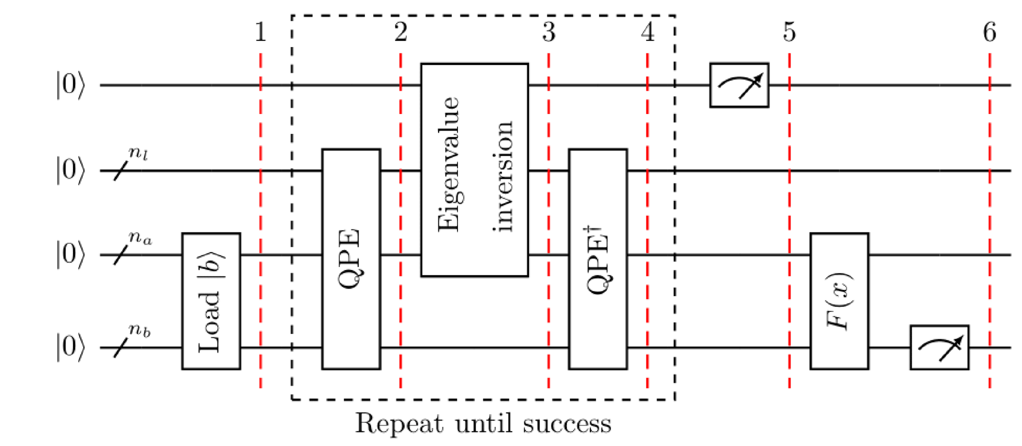

Figure 1. HHL circuit architecture [1]

Results

Consider the following Riccati equation with initial condition $y(0) = 1$

$$\frac{dy}{dt} = (2 - sin(t)) \cdot y^2 + \frac{2 - sin(t) + cos(t)}{2 - sin(t)} \cdot y - \frac{1}{2 - sin(t)}$$

The following pictures depict yt-graphs for both the classical and quantum solver for a time step of 0.001. Note that the equation has a singularity at $t \approx 0.40$ which describes the end of our particular solution.

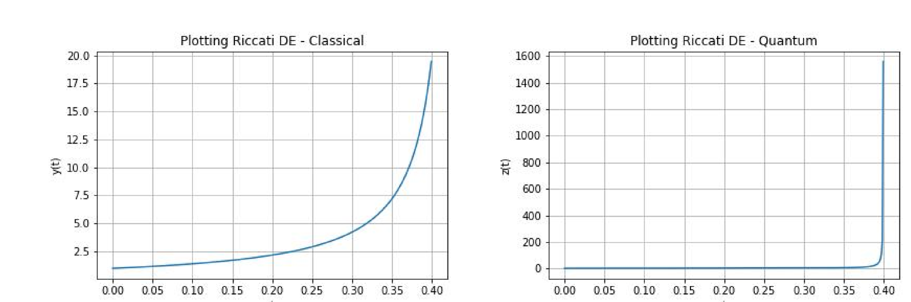

Figure 2. Solutions to the initial value problem

We observe much more accurate approximations from QRS over the local neighbourhood (0.3, 0.375) with a time step of 0.001.

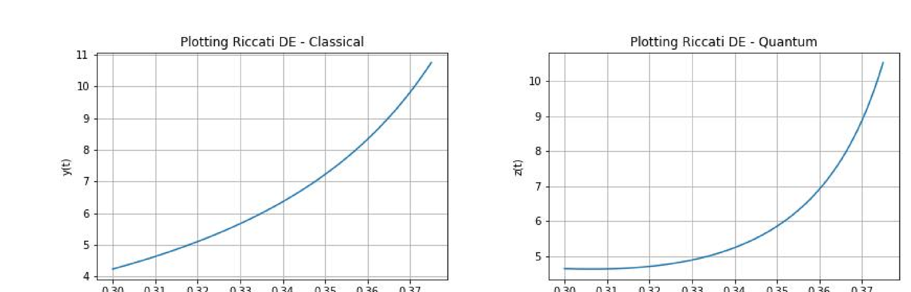

Figure 3. Solutions in local neighbourhood

References

[1] IBM. Solving linear systems of equations using HHL. In *Qiskit Textbook*.

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