Solow model with Cobb-Douglas

We know a production function that satisfies this. The Cobb-Douglas! At the time the original paper was written, the role of technology (the total factor productivity parameter) was not very well understood. So, Solow considered the first version which only deals with K and L. This simpler functional form also comes from the fact that technological progress is considered exogenous, thus the reason why it is categorized as an “Exogenous growth model”.

$F(K, L) = Y = K^{\alpha} \cdot L^{1-\alpha}$, where $0 < \alpha < 1$ is an output elasticity determined by available technology.

As a reminder, output elasticities are the percentage change in output for a percentage change in the input. That is, the responsiveness of output to a change in levels of either labor or capital used in production. Do you remember how to show that this function has CRS? The necessary condition was that the exponents sum to 1, which is guaranteed in this form.

All there is left to do is plug in the production function to our differential equation of capital accumulation and solve it. But there is a small catch here that you must pay attention to. Because this model aims to answer questions in per-capita terms, Solow introduced a new variable $r = \frac{K}{L}$ that allowed him to rewrite the function for capital as $K = r \cdot L_0 e^{nt}$. Our job is now to find the growth rate of the **capital-labor ratio.** We start by differentiating K,

$\frac{dK}{dt} = \frac{dr}{dt} \cdot L_0 e^{nt} + r \cdot L_0 e^{nt} \cdot n$ (via the chain rule)

Now we have two expressions for $\frac{dK}{dt}$ so they must be equal to each other.

$$s \cdot F(K,\ L_0 \cdot e^{nt}) = \frac{dr}{dt} \cdot L_0 \cdot e^{nt} + r \cdot L_0 \cdot e^{nt} \cdot n$$
$$L_0 \cdot e^{nt} \cdot (\frac{dr}{dt} + r \cdot n) = s \cdot F(K, L_0 \cdot e^{nt})$$

Applying the definition of CRS[1], and noting that by definition $L_0 \cdot e^{nt}$ is an exogenous value that is known at any given time t, it follows that

$$\frac{dr}{dt} + r \cdot n = s \cdot F(\frac{K}{L_0 \cdot e^{nt}}, 1)$$

$\frac{K}{L_0 \cdot e^{nt}}$ is the capital-labor ratio, which we defined previously as **r.** Let’s simplify notation

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[1] F(cK,cL) = c * F(K,L). Hence, if L is constant we can divide both arguments by its expression which yields that equation shown in the notes.

$\frac{dr}{dt} + r \cdot n = s \cdot F(r, 1)$

Finally, we isolate the growth rate of capital-labor ratio (the differential)

$\frac{dr}{dt} = s \cdot F(r, 1) - r \cdot n$

This is the **Solow-Swan Growth mode**l! It models the growth of the ratio of capital to labor under the assumptions given earlier. As Solow puts it in his paper this function "states that the rate of change of the capital-labor ratio is the difference of two terms, one representing the increment of capital and one the increment of labor". We can now plug in our Cobb-Douglas to get an explicit expression.

$Y = F(r, 1) = r^{\alpha} \cdot 1^{1-\alpha} = r^{\alpha} \Rightarrow \frac{dr}{dt} = s \cdot r^{\alpha} - r \cdot n$

Write down in paper these steps. Try to understand the logic, reading the original paper as you go through the math helps a lot. Great question for a quiz :)

Numerical Example

Consider a Cobb-Douglas with $\alpha = \frac{1}{3}$

$F(r, 1) = r^{\frac{1}{3}}$

$\frac{dr}{dt} = s \cdot r^{\frac{1}{3}} - r \cdot n$

With differential equations, we are usually interested in examining equilibrium conditions and how, if at all, stable these are. Do you remember how to find the optimal value(s) of a derivative? The fancy term is First-Order Conditions. But all we need to do is set the derivative to 0 and solve. Let's do it.

$0 = s \cdot r^{\frac{1}{3}} - r \cdot n$

This holds true for $r = 0 \; or \; r = \frac{s}{n}^{\frac{3}{2}}$

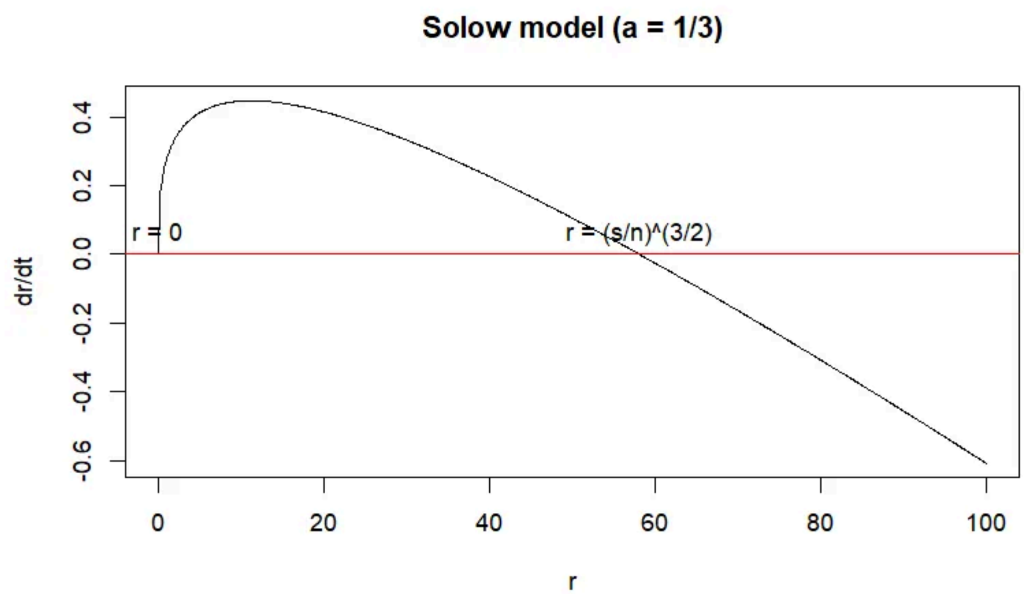
Simulations

The mathematical analysis is pretty abstract, so if you are a bit confused don't worry! This is a perfect situation in which I think simulations can help grasp the intuition. Check out the solow-swan R notebook for replication code.

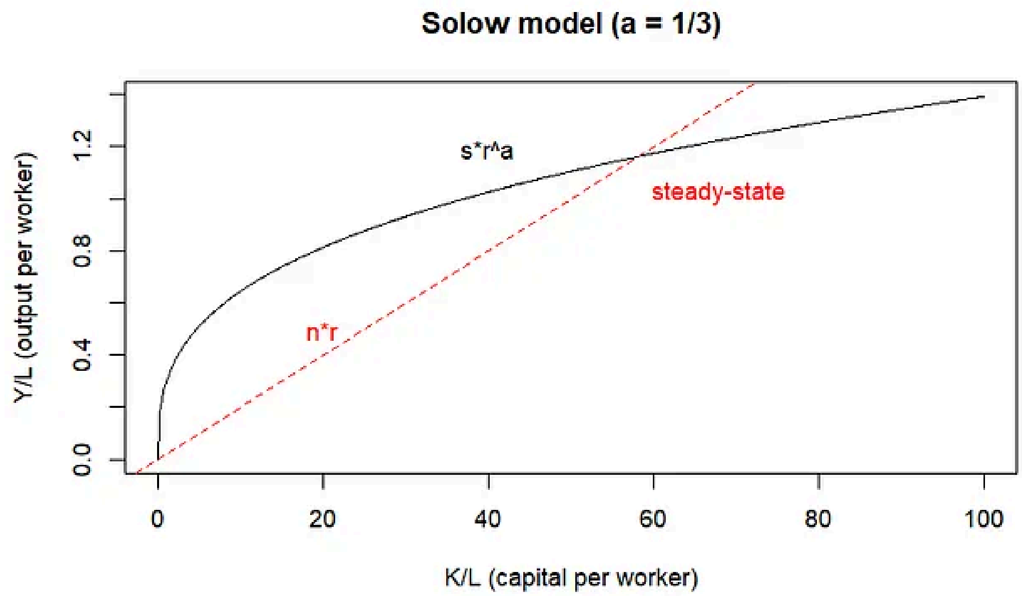
We can simulate the trajectory of this differential equation for our given production function

The first plot shows the rate of change of capital-labor ratio vs the ratio itself. Just like our theoretical prediction, we get these two equilibria 💪

To say a few things about stability… Note the first equilibrium $r = 0$ is unstable because there is an inflection point where dr/dt achieves its maximum value. If the capital-labor ratio goes beyond this point, it will fail to converge back to $r = 0$. In fact, it is guaranteed to converge to the non-zero equilibrium (the second one we found). In other words, $r = \frac{s}{n}^{\frac{3}{2}}$ is asymptotically stable. Any solution that starts near it, beyond the inflection point, will for sure converge to it as time goes to infinity.

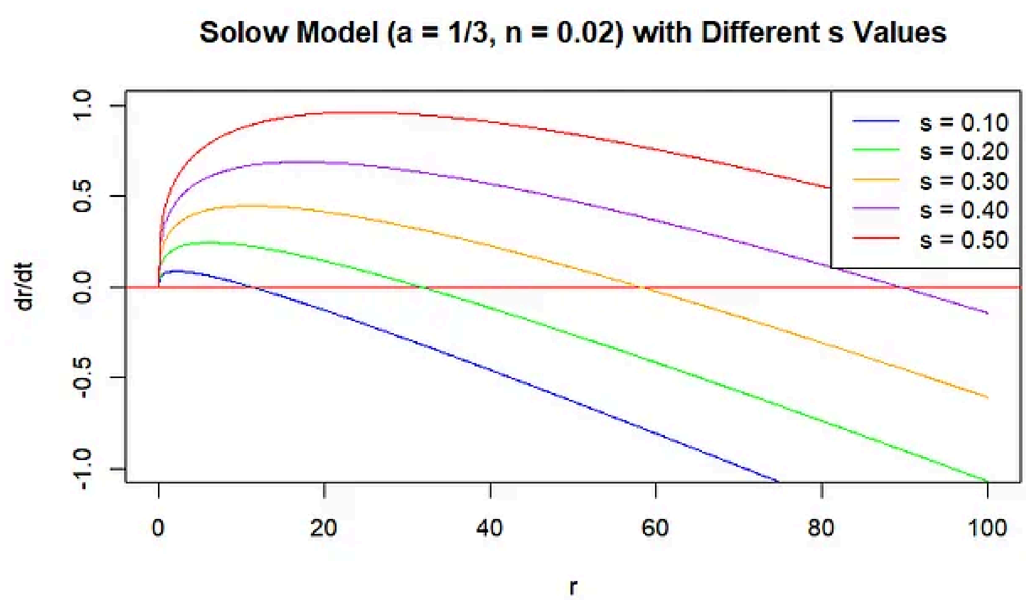


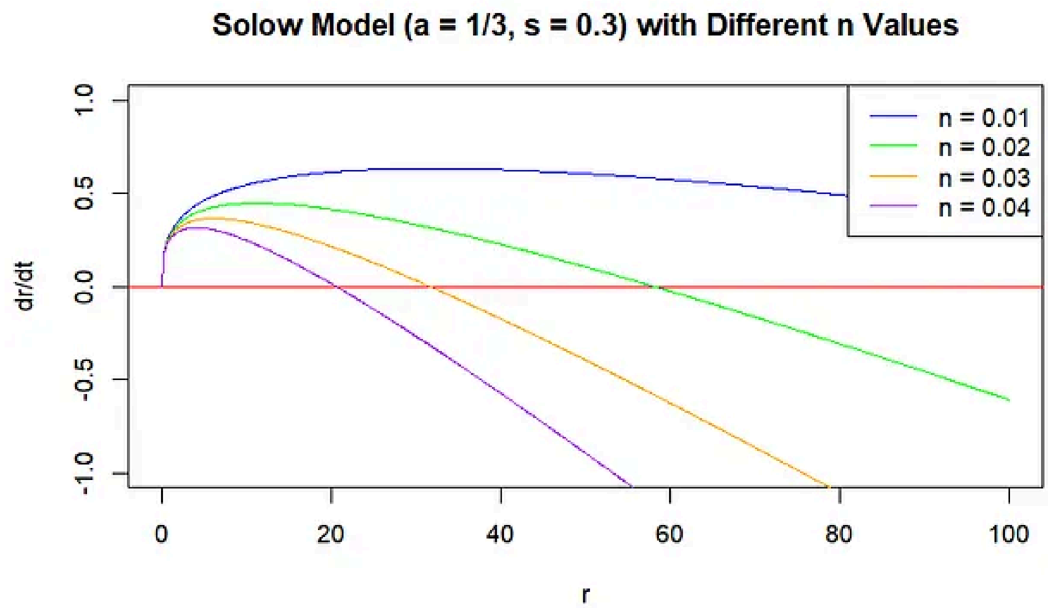
The next plot shows the increment of capital vs the increment of labor. Capital increases with decreasing slope, due to the Cobb-Douglas, and output per worker is constant.



From this, we can see that if $n \cdot r > s \cdot r^{\alpha}$ then the capital-labor ratio **r** will decrease until it reaches the steady-state intersection point $n \cdot r = s \cdot r^{\alpha}$. If $n \cdot r < s \cdot r^{\alpha}$ then it will increase until reaching the equilibrium. Therefore, we call this a **stable equilibrium.**

Of course, changing n or s will change the numerical value of the non-zero equilibrium, but the graph dr/dt vs r will always have the same qualitative shape. We can write a bit more R code to visualize different trajectories for various values of these parameters. The results are shown below





The larger the fraction of income saved, the larger the value of the stable equilibrium in the model. In countries where people save a lot, we would expect the equilibrium level of capital-labor ratio to be much higher as well. In contrast, the faster the population grows in a country the lower the value of the stable equilibrium. Hence, equilibrium is reached much faster, and the country basically saturates its possibility to grow. At this point, it either needs to decrease its population (a bit extreme) or aim for technological breakthroughs that expand the production possibility frontier (thus being able to get more output out of the same capital).

This model predicts that in the long-run, capital will grow exponentially alongside labor. If, for example, capital is too low, it will rapidly increase until it becomes approximately proportional to the labor (catch-up growth), and then it will settle into a long-run behavior where capital stays proportional to labor.

Variations and current applications

The solow-swan model's predictions about growth have been validated by empirical data for the US. According to it, how may we promote growth?

- **Influencing the savings rate**: Recall that higher national saving means higher public saving, higher private saving, or some combination of the two. Much of the debate over policies to increase growth centers on which of these options is likely to be most effective.
    - The government controls public saving (tax revenue minus expenditures), which directly influences national saving. If the government runs budget deficits (i.e.,

spends or borrows more than it raises through taxes), then interest rates will rise and investment will be crowded out. Hence, capital stock will decrease because there are less incentives to invest in capital. High interest rates make it expensive to accumulate capital, which is often financed through loans. This poses a burden on future generations.

  - The government can also influence private saving (how much households and firms save) through taxes. For example, tax rates on capital such as corporate income tax, the federal or state income tax, the estate tax, etc. decrease the incentive to save by lowering potential returns savers may gain. On the other hand, tax-exempt retirement accounts, such as IRAs, are designed to encourage private saving by giving preferential treatment to income saved in these accounts
    - Income tax is quite a hot topic nowadays. In part, it is in stark contrast with what the founding fathers had envisioned for a US led by the working class. It seems ridiculous, to me, that we get taxed so heavily on the income that we have worked to earn. There are many economists supporting this view, and have proposed consumption taxes instead. So, rather than taxing you for working the idea is to tax based on how much you spend. As you might imagine, no consensus has been reached on this regard despite decades of research on the issue.
    - The reasons are closely tied to politics and the workings of the political system. Once lobbying became the primary source of funding for politicians, it is clear that the incentive is for firms to influence policy-making in their favor. Some claim this phenomena has led to Big Tech, Big Pharma, and other oligopolistic industries where only a few firms control the majority of market share. They want people to consume as much as possible, so consumption taxes do not seem desirable for profit-maximizing firms in this regard. I am not an expert in this topic, but I find it super interesting. Could be a great topic for your research project!

- **Allocating investment**: The Solow-Swan model considers a composite good and an abstract representation of capital. While the original analysis interprets it as only physical capital, it can in fact be interpreted as a combination of physical and human capital (e.g., teachers, libraries, education, …). In this general viewpoint, capital is an asset that helps produce more goods and services. Under this view, both physical and human capital can be equally important. In fact, research suggests that both are important factors in determining the growth rates of different countries. Some may have very good infrastructure but poor human capital, or vice versa.
  - Argentina, for example, has poor infrastructure but a surprisingly strong human capital. Not to the government's credit though. Despite all the economic crises and corruption over the last few decades, it is the country in south america with the most unicorn companies, most Nobel laureates, and the highest number of professionals working overseas in complex industries like nuclear energy. The country has found a way to grow because of its culture, rather than specific policies enacted over time.

- Should the government promote a level playing field for capital to get allocated through market mechanisms, like competition? Or should it promote specific types of capital that may be more relevant in the current economic landscape? In other words, what kinds of capital yield the highest marginal products?
    - These incentives are influenced through specific taxes. If there is no preferential tax structure, then markets are left to deal with the allocation issue. Those industries with the highest marginal products of capital will naturally be most willing to borrow at market interest rates to finance new investment.
    - In contrast, some specific taxes or tax cuts may be enacted for promoting development of specific technology. In this scenario, instead of relying on the market mechanism to yield the highest marginal products, the economy might hope for certain by-products to emerge. For example, encouraging production of microchips has had positive externalities on many technology markets. With cheaper chips firms can produce cheaper computers, cheaper software, more patents, and eventually also cheaper robots. This phenomenon is called technological or knowledge diffusion.
    - A third alternative is through direct investment in infrastructure. Like building more bridges, public spaces, and other types of public infrastructure. This is a bit tricky, because it is hard to measure the precise effect public investment has. We know of issues with free riding and public common goods. It also opens a door for corruption, because politicians can leverage their political power to gather large investments for infrastructure that may not be justified or are simply inefficiently produced.
    - As Mankiw puts it “Once the government gets into the business of rewarding specific industries with subsidies and tax breaks, the rewards are as likely to be based on political clout as on the magnitude of externalities”

An important takeaway from these examples, particularly the case of Argentina, is that culture plays a primary role in driving or limiting economic growth. A nation’s culture arises from various historical, anthropological, and sociological forces and is not easily controlled by policymakers. But culture evolves over time, and policy can play a supporting role. In economics, we study culture through the quality of institutions. The particular branch that focuses on this is called Institutional Economics. Thomas Veblen “Theory of the Leisure Class” from 1890 is a great read if you are interested in this perspective.

An institution can be defined as “habits of thought”. Cultural traditions create patterns of behavior that differ in every country. These habits are further influenced by laws or norms specific to those countries. So when we talk about institutions we think about the legal or judicial system, the constitution, social norms and traditions. The quality and effectiveness of these institutions directly affect the level of trust individuals have in their institutions and by consequence in their government. And since a good working economy depends heavily on trust, it is a fundamental determinant of economic growth among different countries as well. In terms of capital, we would denote these factors of growth as “social capital”.

For example, if people perceive that fraud or corruption is not prosecuted as it should. Then what incentives do you think will emerge? Well, probably people will grow resentful and these behaviors will become part of the culture. Countries need good laws, effective judicial systems, and trust just as much as they need good investment in capital and technology in order to foster growth. Arguably, without trust or well established rules of the game, any investment or new technology will have a much lower impact than it may otherwise have.

To illustrate this, a very practical example is currency crises through hyperinflation. As we saw last week, hyperinflation erodes people’s trust in their currency because it increases the uncertainty of its true value. Saving in local currency is thus discouraged and people start seeking alternatives. In Argentina, for example, there is so little trust on the peso that everyone tries to save in us dollars. The unstable demand for dollar denominated currency, coupled with government policies that limit the avenues via which it can be legally acquired, has led to the emergence of parallel or shadow markets for USD. Today there have been at least 6 different exchange rates for a USD. There is the official exchange rate, the dollar blue (illegal and bought on the streets. Often the only way individuals can get USD), the dollar “tourist” charged on credit card expenses in USD, the savings dollar (official + 60%), and we even had a specific dollar for the world cup called “Qatar” dollar.

Anyways, this is why economics is a social science. We study people, not money. Keep this in mind as you advance with your studies. Build up macroeconomic insights and models from microeconomic foundations.

Additional Reading

1. Solow.pdf (nyu.edu)
2. The Empirics of the Solow Growth Model: Long-Term Evidence (tandfonline.com)
3. Parameter estimation of the Solow-Swan fundamental differential equation (sciencedirectassets.com)
4. Economic Growth and Development in the Undergraduate Curriculum: The Journal of Economic Education: Vol 44, No 2 (tandfonline.com)
5. Contribution to the Empirics of Economic Growth* | The Quarterly Journal of Economics | Oxford Academic (oup.com)
6. All of Argentina’s dollar exchange rates, explained - Buenos Aires Herald

Endogenous Growth Theory

Endogenous growth models are a set of theories that relax the assumption of exogenous technological progress. In other words, these models aim to explain how knowledge makes labor effective (an assumption in solow-swan) and how technology improves the usage of capital (again assumed previously).

Brief recap. The rate of economic growth in the long-run, as measured by the growth rate of output per person, depends on the growth rate of total factor productivity (TFP), which is determined in turn by the rate of technological progress. ***Remember our discussions of classical theory and the full version of the Cobb-Douglas?*** The neoclassical growth theory of Solow (1956) and Swan (1956) assumes the rate of technological progress to be determined by a scientific process that is separate from, and independent of, economic forces. Neoclassical theory thus implies that economists can take the long-run growth rate as given exogenously from outside the economic system.

Endogenous growth theory challenges this neoclassical view by proposing channels through which the rate of technological progress, and hence the long-run rate of economic growth, can be influenced by economic factors. It starts from the observation that technological progress takes place through innovations, in the form of new products, processes and markets, many of which are the result of economic activities. In effect, we are trying to model how new ideas are formed and what effect this endeavor can have in innovation. Thus, we can consider a new sector we may call Research & Development (R&D).

This set of models involve a more intricate usage of mathematics and microeconomic theory. We would need a few weeks to fully work out the inner workings of one of the established models of endogenous growth. I hate presenting topics incompletely, or abstracting many details. But it is very important you are aware of these lines of work, which remain actively researched. So, in the hope of introducing the main ideas of knowledge creation, we will follow Makiw's note on the simplest version of endogenous growth: The AK model.

For further reading, please refer to David Romer's Advanced Macroeconomics Chapter 3 or Peter Howitt's notes from his class at Brown.

The main property of the AK model is the absence of diminishing returns to capital. The mechanism justifying this assumption is technology spillovers (or knowledge diffusion as we defined it on the previous lecture) from capital investment. That is, investing in knowledge or capital tend to have positive externalities which maintain the same returns for an additional unit of capital.

Mathematically, we introduce a new variable called Technology (or Total Factor Productivity) denoted **A.**

$Y = AK^{\alpha}L^{1-\alpha}$, $0 < \alpha < 1$ is our output elasticity.

For the special case $\alpha = 1$ (i.e. output only responds to changes in capital, and the productivity of that capital depends on current technology), we have

$Y = AK$, thus the AK model where K encompasses both physical and human capital.

Growth is always about per-capita output (productivity) or per-capita income (purchasing power) Dividing by available labor yields per capita measures

$Y/L = A \cdot K/L \Rightarrow y = A \cdot k$

Here, k is the capital-labor ratio we studied in the Solow-Swan model. This functional form elucidates the **absence** of diminishing marginal returns to capital. One extra unit of capital produces A extra units of output, regardless of how much capital there is.

The savings rate remains an exogenous variable, so a constant fraction of total income is saved and the remainder is invested back into the economy. We can follow similar steps as in our mathematical derivation of the solow growth equation to find that the rate of capital accumulation is given by

$\Delta K = sY - \delta K$, where δ is the depreciation rate. Applying the production function yields

$\Delta K = s \cdot A \cdot K - \delta \cdot K \Rightarrow \Delta K/K = sA - \delta \Rightarrow \dot{k} = sA - \delta$

In the Solow model, saving temporarily leads to growth, but diminishing returns to capital eventually force the economy to approach a steady state in which growth depends only on exogenous technological progress. By contrast, in this endogenous growth model, saving and investment can lead to persistent growth. Note that if $sA > \delta$ then we have growth forever.

This is all I will say about endogenous growth models. Please review the book's chapter for more detail and the role of the R&D sector. Your main takeaway should be the distinctive constant or increasing marginal returns of capital, justified by knowledge spillovers, and the idea of a second sector which focuses on creating new knowledge.

To further these models, we need a strong microeconomic foundation. After all, what ends up happening with the knowledge created or new technology is largely a factor of economic decisions by households and firms. Aggregating these microeconomic facts into purely macro models is very hard, if not futile. Remember when I discussed the ideas of Friedrich Hayek on the use of knowledge in society? And the new paradigms of economic modeling that complexity economics promotes? A lot of these criticisms stem from modeling problems such as this one. At its core, I see the complexity approach as the modern effort to model the ideas of Austrian economists such as Hayek or Schumpeter.

Further Reading

1. AK model - Wikipedia
2. Mankiw's Macroeconomics Chapter 9
3. David Romer's Advanced Macroeconomics Chapter 3
4. Endogenous Growth (mit.edu)