Claremont Graduate University

MATH 387: Discrete Mathematical Modeling

Implied Volatility Modeling with Markov Switching Autoregression

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1 Introduction

Efforts to understand and predict future movements of uncertainty permeate a diverse set of disciplines. Each with its own perspective of the meaning of uncertainty. Economists attempt to quantify volatility as measures of potential risk of falling into a state of stagnant or decreasing growth. Financial analysts and investors model volatility by capturing the movement of a financial asset as simple standard deviations from expected returns, attempting to maximize profits while minimizing risk. The quest for predicting future states of an underlying system has fueled quantitative and computational innovations, which have helped formalized the field of risk management and develop an appreciation of the complexity of such phenomena.

Two particular measures of asset risk have taken shape over the years. Implied Volatility (IV) captures the uncertainty of the movement of a financial or economic asset (often a stock or index) by reflecting the expected volatility of the options market; thus providing a forward-looking approach by encoding the expected volatility from now until the option's expiration. In contrast, Historical Volatility (HV) offers a backward-looking alternative by leveraging data of known prices over a fixed period of time to reflect the actual volatility realized in the underlying market. There are numerous techniques to measure and model each, we focus on the former.

Successful forecasting of IV from a trader's point of view primarily involves forecasting the direction of IV correctly; a correct magnitude for the change is not as relevant. This is because option positions such as the straddle will generate a profit if the IV moves in the correct direction, ceteris paribus (the size of the profit is affected by the magnitude of change, however). The forecasting accuracy of the various models is first evaluated based on sign: how many times does the sign of the change in the measure of IV correspond to the direction forecasted by the model. However, the point forecasts are also evaluated based on mean squared errors (MSE). Accurate point forecasts can be valuable, for example, in risk management and asset pricing applications [1].

The most widely used measure today is a volatility index constructed by the Chicago Board of Option Exchanges's (CBOE) from S&P 500 index option prices called the VIX, sometimes referred to as the "fear index". Importantly, the VIX is believed to follow a complex autoregressive, multivariate, and heteroskedastic time-series that captures perceptions of risk or market sentiment. We propose a Markov Switching Autoregression (MSAR) framework as an alternative to traditional econometric models that often fall short in this task due to the nuances of the VIX's statistical properties. The paper is organized into 5 sections. After a brief literature review on previous attempts to model implied volatility, particularly the VIX, section 3 describes the multiswitch problem that motivated efforts to merge markov chains with regression models, and introduces the MSAR model. Section 5 presents the dataset used to fit a baseline markov model and our MSAR implementation; we discuss performance and forecasting power of both models. Finally, section 6 offers concluding thoughts and suggests potential improvements.

2 Literature Review

The VIX index, calculated by the Chicago Board Options Exchange (CBOE) from S&P 500 index option prices, represents a critical measure of market volatility. It has garnered significant interest in the literature due to its role as a forward-looking volatility predictor, offering insights into market sentiment and investor expectations. Scholars have emphasized the VIX's utility in financial risk management and market analysis [2].

However, traditional linear models like multivariate regressions or ARIMA face challenges in accurately predicting the VIX [3] due to their inherent linear nature. These models, while effective in many time-series analyses, struggle with the VIX's distinct regimes characterized by shifts in mean and variance. This limitation has prompted the exploration of more advanced methodologies capable of capturing such non-linear and multivariate dynamics of time-series with inherent "turning points". Results from simulations showcasing these limitations are provided in the Appendix.

Markov Switching Regressions (MSRs) [4] emerge as a promising alternative, leveraging the properties of Markov chains to accommodate regime shifts in an autoregressive time-series [5]. MSRs offer a framework where the parameters of interest, such as the mean and variance, can vary across different states. This approach aligns with the observable characteristics of the VIX, which exhibits periods of relative stability interspersed with high volatility. By fitting a system of linear equations for each state and estimating a transition matrix, MSRs enhance the predictive accuracy for such complex time-series data.

In integrating macroeconomic variables, we follow the precedent set by Prasad et al. [6]. Their application of machine learning techniques, such as Light GBM and XG Boost, identified key economic indicators that significantly predict the VIX. These findings guide our selection of exogenous covariates, enhancing the model's robustness and relevance to current economic conditions.

Despite the advantages of MSRs, it's crucial to acknowledge their limitations. The MSR framework, like traditional regressions, is susceptible to endogeneity and relies heavily on assumptions like the Markov property, stationarity, and time-invariant transition probabilities [7]. Moreover, the complexity and stability of the model can be challenged as the number of regimes increases, posing potential issues in model specification and interpretation.

Overall, this review underscores the evolving landscape of volatility prediction. By transitioning from traditional linear models to more sophisticated approaches like MSRs, we address the dynamic nature of financial markets. However, this advancement does not come without new challenges and assumptions, which must be carefully considered in empirical applications.

3 Multiswitch Problem

The concept of multiswitch problems emerges from the need to model systems that exhibit multiple regimes or states, each governed by distinct equations. Consequently, the statistical moments (mean and variance) of the dependent variable of interest are conditionally dependent on the state of the world. Traditional linear models often struggle in these contexts, as they inherently assume a singular, consistent pattern throughout the observed data. Mathematically, this can be conceptualized using a piecewise function, where different linear models apply depending on the current state or regime. A simplified two-state model could be represented as:

$$Y_t = \begin{cases} \alpha_1 + \beta_1 X_t + \epsilon_{1t}, & \text{if State 1} \\ \alpha_2 + \beta_2 X_t + \epsilon_{2t}, & \text{if State 2} \end{cases} \tag{1}$$

where Y_t s the observed dependent variable at time t, X_t is a vector of independent variables, α and β are the parameters unique to each state, and ϵ denotes the state-dependent error terms of each equation.

Early proposals for solving the multiswitch problem relied on deterministic solutions, such as indicator functions. Among these, Quandt's D-method and lambda-method are notable. The D-method involves splitting the sample into different regimes based on an exogenously determined variable, D. Mathematically, it's expressed as [4]:

$$Y_t = (I - D) \cdot X\beta_1 + D \cdot X\beta_2 + W \tag{2}$$

Where $W = (I - D) \cdot \epsilon_1 + D \cdot \epsilon_2$ denotes the latent and heteroskedastic error terms, X is the matrix of observed variables, and D is a diagonal matrix that serves as an indicator function to denote the current state.

The lambda-method, on the other hand, uses an endogenously determined switching variable, λ, to identify the regime. Thus, we estimate the conditional density function of the dependent variable:

$$h(y_i|x_i) = \lambda \cdot f_1(y_i|x_i) + (1 - \lambda) \cdot f_2(y_i|x_i) \tag{3}$$

These methods assume that the transition between states is clearly delineated and can be captured by specific, observable indicators. However, in complex systems such as financial markets, regime switches are often driven by a combination of observable and unobservable factors, making these approaches less effective. The stochastic nature of market dynamics, including unobservable shifts in investor sentiment and market conditions, cannot be adequately captured by these deterministic methods. This limitation paves the way for the introduction of Markov chains as a solution, offering a probabilistic approach to modeling regime switches in time-series, which will be explored in the subsequent sections.

3.1 Markov Switching Models

These models stand out for their ability to incorporate the probabilistic nature of regime changes, drawing on the principles of Markov chains. The key innovation in Markov switching models is the integration of the Markov chain property into a regression framework, allowing for the dynamic modeling of time series data under varying regimes. It extends the $\lambda-$method presented in the previous section by allowing the probabilities of state to be estimated stochastically as transition probabilities via markov chains.

The foundation of a Markov switching model lies in its ability to represent a time series as a mixture of several distinct models, each corresponding to a different regime. The regime at any given time point is determined by a state variable, assumed to follow a non-observable Markov chain. This state variable, typically denoted as s_t, is crucial as it dictates the current regime and, consequently, the parameters of the model.

Consider the structure of a basic Markov Switching Autoregressive (MSAR) model:

$$Y_t = \begin{cases} \alpha_0 + \beta Y_{t-1} + \epsilon_t, & \text{if } s_t = 0, \\ \alpha_1 + \beta Y_{t-1} + \epsilon_t, & \text{if } s_t = 1. \end{cases} \tag{4}$$

In this formulation, the model parameters vary depending on the state variable s_t, which is governed by a Markov process. This state-dependency is the essence of the Markov switching model, enabling the model to adapt its behavior based on the prevailing regime and historical patterns.

The integration of the Markov chain into the regression model is achieved through the state variable s_t. The Markov chain property posits that the probability of being in a certain state at time t is dependent solely on the state at time $t-1$, encapsulating the memoryless characteristic of Markov processes. This memoryless assumption is supported by the financial literature in Implied Volatility (IV) forecasting that finds the VIX to be accurately represented by a first-order model since "no second lags turned out to be statistically significant" [1].

The transition between states is quantified by a transition probability matrix T, which is central to the estimation of the model:

$$\begin{aligned} P &= \begin{bmatrix} P(s_t = 0 | s_{t-1} = 0) & P(s_t = 1 | s_{t-1} = 0) \\ P(s_t = 0 | s_{t-1} = 1) & P(s_t = 1 | s_{t-1} = 1) \end{bmatrix} \\ &= \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \end{aligned}$$

The elements of this matrix, P_{rs}, indicate the probability of transitioning from state r to state s. These probabilities are estimated from the data and are fundamental in determining the dynamics of the regime switching. It follows that the estimated transition probabilities λ_i are recovered from $\lambda'_i = \lambda'_0 \prod_{j=1}^{i} P_j$ [4].

The superiority of the Markov switching approach in handling financial data stems from its ability to model complex and often non-linear relationships inherent in financial markets. This is crucial for accurately predicting variables like the VIX index, which exhibit distinct regimes with varying means and variances.

3.1.1 Data Requirements and Model Estimation

As in any markov chain framework, stationarity is a crucial prerequisite for the effective application of Markov switching models. In cases where the time series data exhibit non-stationarity, such as a unit root, differencing the series is a necessary step before applying the model. This ensures that the underlying assumptions of the model are satisfied, leading to meaningful and interpretable results.

Markov switching models are estimated from the data by employing the following steps:

1. **Specification of Regimes and Model Structure:** Initially, the model's structure, including the number of regimes and the form of the autoregressive process within each regime, must be specified. This is based on theoretical considerations and the nature of the data.

2. **Estimation of Transition Probabilities:** Central to Markov switching models is the estimation of transition probabilities that govern the likelihood of moving from one regime to another. This is typically done using maximum likelihood estimation methods.

3. **Application of Expectation-Maximization (EM) Algorithm:** The EM algorithm is often employed in the estimation of Markov switching models. The EM algorithm iterates between the following two steps until convergence is reached, providing estimates of both the model parameters and the latent state variables.:

   (a) **E-Step (Expectation):** In this step, the latent state variable (the regime) is estimated using filtering and smoothing algorithms, such as the Kalman smoother.

   (b) **M-Step (Maximization):** Here, the model parameters, including the transition probabilities, are estimated given the current regime estimates. This is typically achieved through maximum likelihood estimation.

In summary, Markov Regime Switching models represent a sophisticated approach to analyzing time series data, especially in contexts where the data exhibit distinct regimes. By integrating the Markov chain property into a regression framework, these models allow for a dynamic and probabilistic understanding of regime changes, thereby enhancing the modeling and prediction of complex time series behaviors.

4 Modeling the VIX

The CBOE VIX Index is a short-term measure of real-time risk in the stock market and is viewed as a fear index. The day-to-day movements in the VIX Index indicate how the market's perceptions fluctuate over time, and it is an important tool for risk management in the capital market. The movements of the VIX Index from day to day are of interest, not only as a good check on the shifting market perceptions of risk, but also for volatility trading, using options strategies, or VIX futures. Some researchers [8, 9] believe the VIX acts as a fear index or a market perception of risk, while others [10, 11] propose risk handling and portfolio diversification.

We start by estimating a baseline Markov model that, at each time step, looks back over a period of time, generates values for a number a variables, matches the current values of the variables to times where the variables have been similar, and takes a weighted average of the next week performance of the VIX. Due to computing limitations and the large time frame that was tested the model was ran once a week, for a total of 520 total datapoints. The variables in question are as follows: Current VIX Price, VIX Previous Week Change, QQQ Previous Week Change, QQQ Onbalance Volume, and the difference between the QQQ 200 Day and 50 Day Moving Averages. For each of the variables above a transition probability matrix is created by counting the amount of times the particular situation occurred in the backtesting data. This is more clear in a simplified example. Take QQQ Previous Week Change, and create a 2x2 matrix: P. In the presented analysis, the transition matrix P is created to denote the relationship between variations in the QQQ Previous Week Change and the levels of the VIX. The rows of the matrix correspond to distinct categories of QQQ Previous Week Change, while the columns signify different VIX values. Specifically, Row 1 denotes instances where QQQ experienced a positive percent change in the preceding week, while Row 2 corresponds to scenarios where QQQ exhibited a negative percent change.

Furthermore, Column 1 denotes instances where the VIX is above 25, while Column 2 signifies occurrences with the VIX below 25. The analytical procedure involves conducting tests for each week over the past decade.

Should the QQQ demonstrate a positive percent change in the preceding week and the VIX register a value above 25, 1 is added to the corresponding cell in the matrix, in this case the first row and the first column of P.

Following the accumulation of data points over 520 weeks, the transition matrix (P) is computed. This matrix is derived by dividing the frequency of occurrences of a particular event by the total number of data points. An illustrative example of P is provided below:

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

Then, after P is fully created, the program calculates the current state of the QQQ Previous Week Change variable. If the QQQ declined the previous week, than the VIX would have a 40 percent change of below 25 and a 60 percent change of above 25 the following week. This process is repeated for each of the five variables. Note, the example above was a simplified matrix to demonstrate how the transition matrices are calculated, the true matrices in the program are 8x8 matrices with more specific cutoffs to generate a more accurate prediction. After the transition matrices have been created for each of the five variables, a weighted average is taken to create the final VIX prediction.

The R^2 value of this method is 0.315. The graph can be seen below. The model follows the direction of the changes, and is able to follow the spikes in the testing data, however, it fails to account for sharp spikes, most notably, the COVID spike. This is due to the fact that the model did not have reliable past data to compare the values of the indicators in beginning of the COVID pandemic because nothing like that had happened in the previous 10 years. Figure 1 depicts the results of the baseline model, the blue line is the actual VIX data, and the orange line is the predicted VIX.

We followed by fitting a multivariate MSAR model. The dataset used consists of data for the VIX ($\hat{\text{VIX}}$), and the following macroeconomic variables that the literature found to be relevant predictors: gold prices (GC=F), Crude Oil Prices (CL=F), USD Index (DX-Y.NYB), and the financial stress index (STLFSI4). The weekly sample size spans 13 years of data, from 01/17/2010 to 01/01/2023. Data for the financial stress index was fetched from FRED, and the remaining variables were obtained from yahoo finance. Finally, to correct for stationarity and/or large outliers, we applied a logarithmic transformation to the first three covariates and a first-differences to

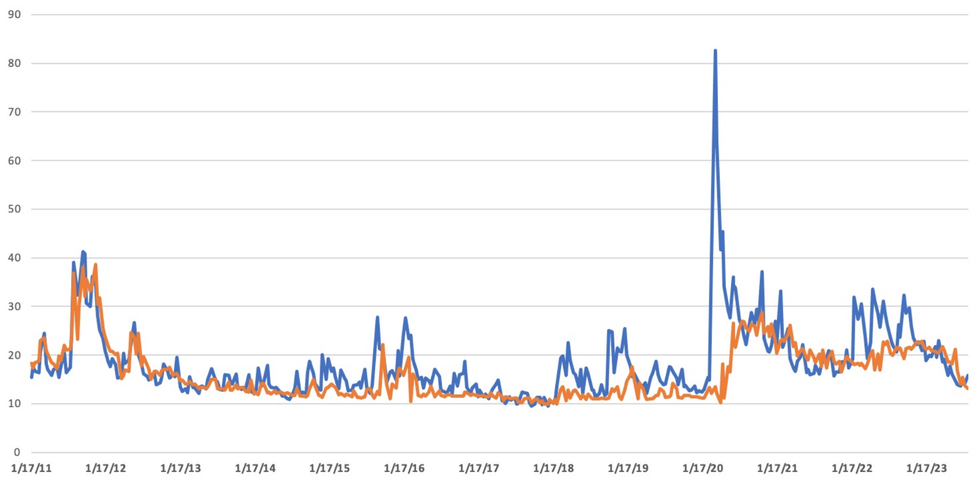

Figure 1: Baseline model predictions

the VIX data; leaving FSI data unchanged. Figure 2 shows the resulting series.

Interestingly, transforming the dependent variable eliminates a lot of relevant information relevant for estimating transition probabilities. In short, by normalizing the time-series we obfuscate the difference between regimes and consequently the estimated filtered probabilities are almost equivalent. Given that our primary objective is to estimate transition probabilities to predict future direction of the VIX, we proceed by fitting the model to the raw weekly VIX data instead.

4.1 Prediction

The model is implemented in Python via the *statsmodels* [12] package, which provides an API to access a variety of econometric and time-series models. We estimate an MSAR model with the weekly VIX as the endogenous variable, the matrix of transformed variables as our exogenous variables, two regimes, a constant trend, assumed an autoregressive order of one, and non-switching autoregressive parameters. Table 1 describes the summary results from the fitted model

The Markov Switching Model results reveal two distinct regimes with significantly different baseline levels, as evidenced by the intercepts for Regime 0 (14.8930) and Regime 1 (19.8009). The non-switching parameters, particu-

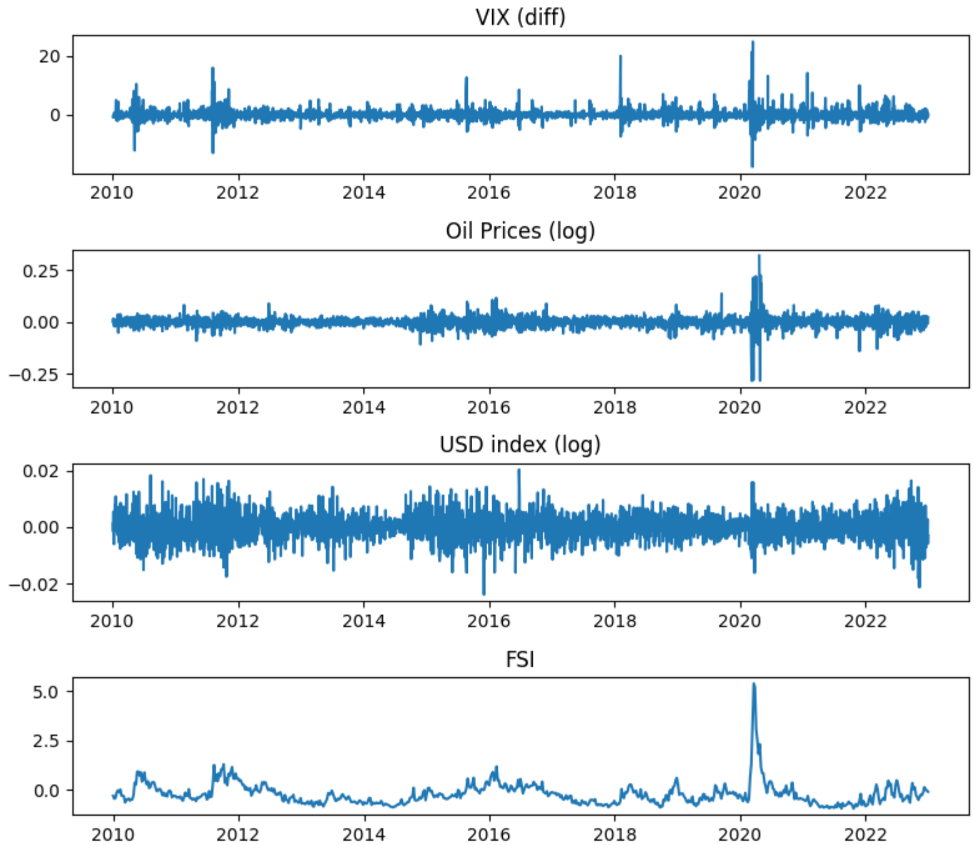

Figure 2: Transformed time-series

larly x3, show a substantial impact across both regimes, indicating that USD index has a significant influence on the dependent variable. The autoregressive component, ar.L1, with a coefficient of 0.83 suggests that the close value of the VIX is highly influenced by its immediate past value, a common characteristic in financial time series data. We believe that our large sample size (677) and careful selection of covariates help explain the strong statistical significance of the exogenous macroeconomic variables.

Figure 3 shows the in-sample predictions of our model. It is clear from the plot that the MSAR model is able to captured the dynamics of the VIX's time series, but in-sample predictions only provide a limited understanding of the model's capability to generalize.

One limitation we found with statsmodels implementation of MSAR is

Dep. Variable:	Close	**No. Observations:**	676
Model:	MarkovAutoregression	**Log Likelihood**	-1740.139
Date:	Mon, 11 Dec 2023	**AIC**	3500.277
Time:	12:20:15	**BIC**	3545.439
Sample:	01-17-2010 - 01-01-2023	**HQIC**	3517.763
Covariance Type:	approx		

	coef	**std err**	**z**	**P> |z|**	**[0.025**	**0.975]**
const	14.8930	1.257	11.852	0.000	12.430	17.356
	coef	**std err**	**z**	**P> |z|**	**[0.025**	**0.975]**
const	19.8009	0.675	29.330	0.000	18.478	21.124
	coef	**std err**	**z**	**P> |z|**	**[0.025**	**0.975]**
x1	16.9873	5.496	3.091	0.002	6.215	27.760
x2	-5.8513	2.375	-2.463	0.014	-10.507	-1.196
x3	160.9426	26.129	6.159	0.000	109.730	212.155
x4	7.6144	0.483	15.773	0.000	6.668	8.561
sigma2	7.9047	0.778	10.156	0.000	6.379	9.430
ar.L1	0.8310	0.026	32.436	0.000	0.781	0.881

Table 1: Markov Switching Model Results

that they do not support out-of-sample predictions, thus requiring a manual implementation. To do so, we follow the expected maximization (EM) approach to estimating future movements of the dependent variable via maximum likelihood. Alternative approaches include Markov Chain Monte Carlo (MCMC), which uses simulation integration, but due to time constraints and unclear relative benefits we opted for the more straightforward implementation of EM by estimating

$$E[y_t] = \lambda_1(\hat{\alpha_1} + \hat{\beta}_1 \cdot E(\hat{X_{t-1}}) + \lambda_2(\hat{\alpha_2} + \hat{\beta}_2 \cdot E(\hat{X_{t-1}}) \tag{5}$$

where λ_i denotes the filtered probabilities of each regime.

Filtered probabilities are used to estimate the likelihood of the system being in a particular state at a given time, based on the observed data up to that point. They are essential for predicting the next state in the Markov Switching framework. The estimated coefficients are extracted from the summary

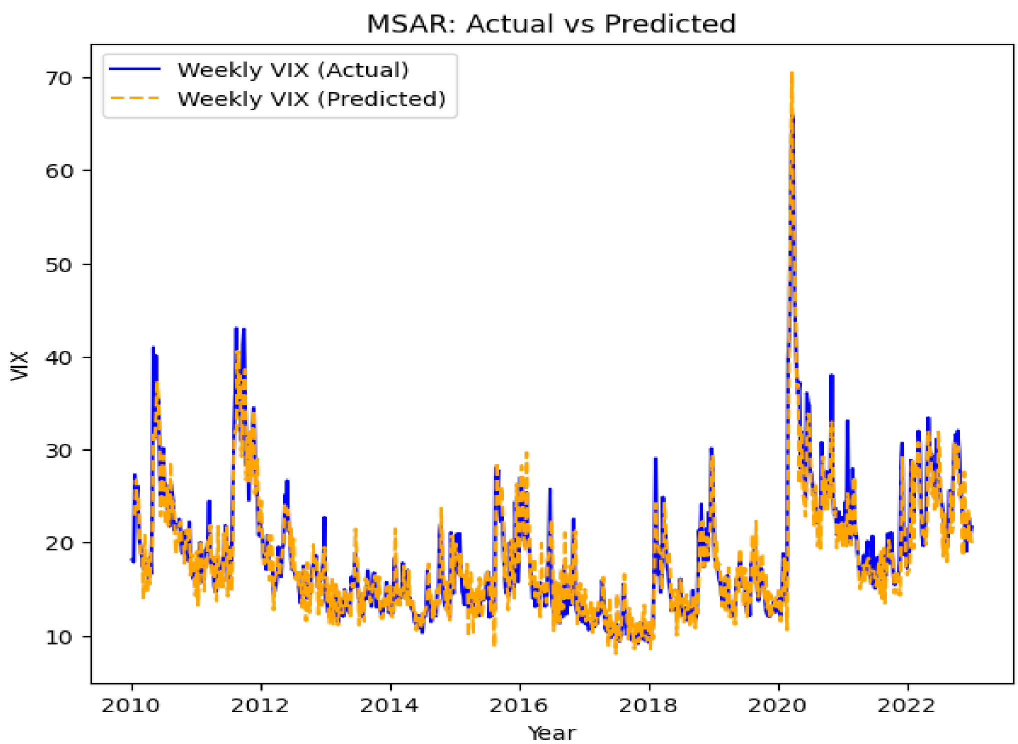

Figure 3: MSAR in-sample predictions

statistics in Table 1, and the filtered transition probabilities are estimated by taking the dot product of the transition matrix with the filtered probabilities of the last observation (i.e., 2023-01-01); both parameters readily available from the MSAR model. The predicted state probabilities for the next period are:

$$\begin{bmatrix} 0 & 0.0721 \\ 1 & 0.9279 \end{bmatrix} \cdot \begin{bmatrix} 0.005339 \\ 0.994661 \end{bmatrix} = \begin{bmatrix} 0.07176427 \\ 0.92823573 \end{bmatrix}$$

The high probability in the second regime (0.994661) for the last observation suggests that the model is quite confident about the regime the series was in at the end of the sample. The predicted state probabilities for the next period (0.07176427 for regime 0 and 0.92823573 for regime 1) indicate the model's expectation that the series is more likely to stay in the same regime (regime 1). We can now leverage these estimated probabilities to forecast the value of the VIX at future time steps. A custom python function is developed for this purposes, available to the reader in the Appendix and supplementary materials. The obtained forecasts are as follows:

Time Step	Forecasted VIX
1	18.0565
2	17.7305
3	17.7540
4	17.7523
5	17.7524

Table 2: Forecasted VIX Values for Future Time Steps

The MSAR model forecasts a drop in the value of the VIX in upcoming weeks, this means that we expect a downward movement of implied volatility in the markets but still remain in the same high-volatility regime. The model's prediction of a continued presence in Regime 1 aligns with current market expectation of an unsettled market, potentially driven by macroeconomic factors, geopolitical tensions, or other systemic risks. Hence, these results can be interpreted as a potential future decrease in Implied Volatility, making its way to the low-volatility regime.

In light of these predictions, investors might adopt more conservative strategies, prioritizing assets that are traditionally seen as less volatile or hedging their portfolios to mitigate potential risks. For options traders, this could mean increased demand for hedging instruments, leading to higher premiums on options contracts. On a broader scale, these forecasts could influence financial institutions and policy makers in their risk assessment and management strategies. However, it's important to acknowledge the model's limitations and the inherently unpredictable nature of financial markets. External shocks and unforeseen events can rapidly alter market dynamics, underscoring the need for continual monitoring and adaptation of predictive models to new data and changing conditions.

5 Conclusion & Further Thoughts

This paper embarked on a detailed exploration of the financial markets through the lens of the Markov Switching Autoregressive (MSAR) model, with a specific focus on forecasting the direction of the Volatility Index (VIX). The MSAR model's capacity to identify distinct regimes offered a nuanced understanding of market fluctuations, a key element in our analytical framework. The integration of macroeconomic variables and the examination of their influence on the VIX further enriched our analysis, providing a multifaceted view of market behavior.

The findings of our study are both significant and timely. The identification of two regimes, one characterized by higher volatility, presents a critical insight into the market's underlying mechanisms. Our model forecasts an upsurge in market volatility, suggesting that we are likely entering a period marked by increased economic uncertainty. This projection is particularly relevant given the current global economic landscape, which is fraught with challenges ranging from geopolitical tensions to shifting monetary policies. For investors and market analysts, this implies a need for heightened vigilance and a potential reevaluation of risk management strategies. It also underscores the value of predictive modeling in navigating the complex and often turbulent financial markets.

While our study contributes valuable perspectives to the field of financial forecasting, it is not without its limitations. The MSAR model, despite its robustness, is inherently sensitive to the parameters and assumptions underpinning it. Our reliance on historical data, while extensive, cannot fully account for the capricious nature of financial markets, where unforeseen events can dramatically alter trajectories. Additionally, the lack of out-of-sample predictions in our current framework points to an area ripe for further development. Future research could focus on enhancing the model's predictive power, perhaps by incorporating real-time data analysis or exploring alternative econometric techniques. The continuous evolution of financial markets necessitates an adaptable and forward-looking approach to modeling and analysis.

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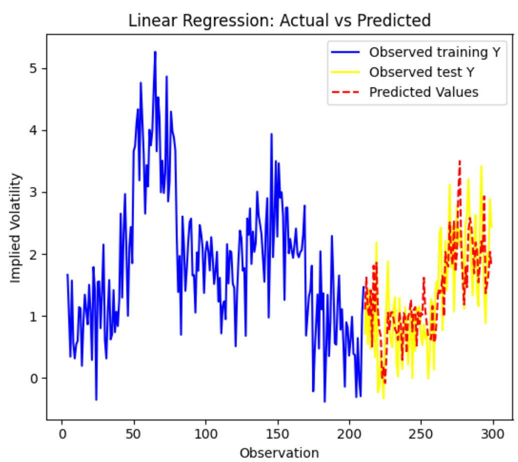
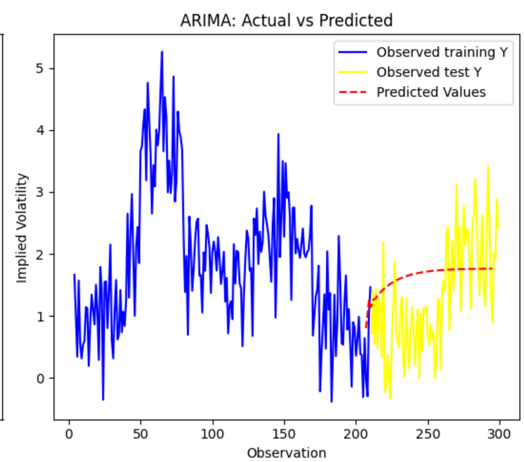
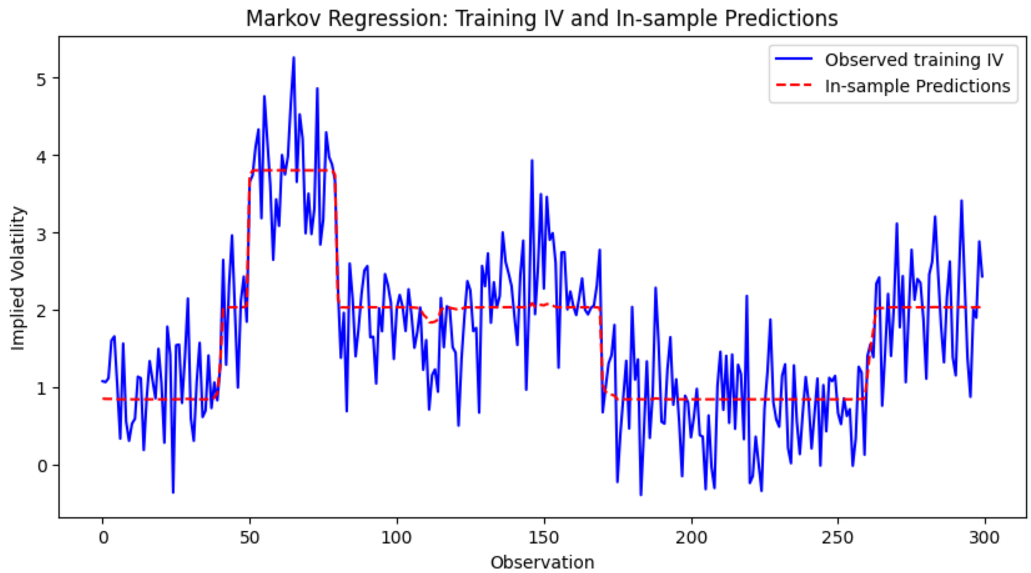
Simulations

The simulated dataset was crafted based on an Autoregressive (AR) process of order 4 (AR(4)), with a white noise error component characterized by a standard deviation of 0.5. This dataset incorporates two covariates, x1 and x2, both of which follow a normal distribution. Central to our simulation is the inclusion of three distinct regimes, each signifying a change in the mean value of the dependent variable. The 'y' variable in our dataset is thus a product of an autoregressive process that is influenced by its four preceding values (or four lags). This process is further modulated by the covariates x1 and x2, and is subject to shifts in mean value, contingent upon the time index, while maintaining a fixed variance. This simulation framework is designed to test candidate models, providing an insightful understanding of their capability to model time-series with properties that characterize Implied Volatility.

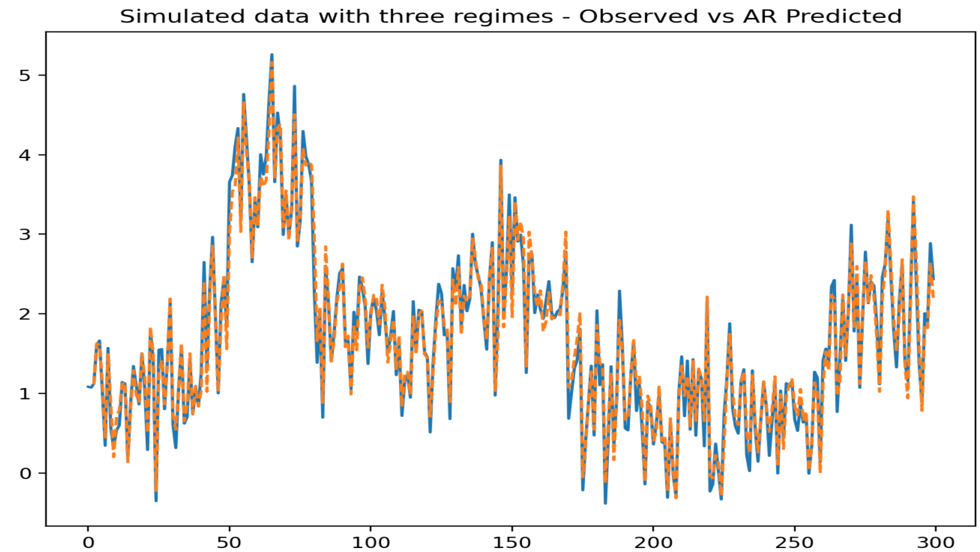




Multivariate regressions present challenges similar to those from machine learning models, they fit the data better but are hard to interpret. Still, despite the better fit, it is still not capable of capturing the regime transitions which is our main interest. On the other hand, ARIMA models fall short in capturing the non-linear properties common in time-series describing implied volatility. Thus demanding transformations of the data that, as shown in section 4.1, obfuscate the regime transitions characterizing the time-series of interest.



Simple Markov Switching Regression (MSR) provides an improvement over traditional linear models by allowing the researcher to estimate a set of linear equations conditional on the regime. This type of conditional mean model allows us to accurately capture the change in expected value of our regressions, which serves in estimating transition probabilities, but fails to capture the variance within the regimes. We observe this limitation in the in-sample predictions of the model.



Finally, Markov Switching Autoregression (MSAR) is able to capture the complex behavior of the time-series describing the variable of interest within each regime. This upgrade is observed by the superior in-sample fit of the MSAR model compared to all other models we experimented with. Because of this superior performance on the simulated scenario, we decided to implement it in our main analysis to model future direction of the VIX.

Forecasting

```
def MSAR_forecast(model, data, steps=1):
    """
    Forecast using a Markov Switching Autoregression model.

    :param model: A fitted MSAR model.
    :param data: DataFrame containing the exogenous variables
                 for the last observation.
    :param steps: Number of steps to forecast.
    :return: A list containing the forecast for each step.
    """
    # Initialize predictions
```

```python
predictions = []

# Extract parameters from the model
mu1 = model.params['const[0]']
mu2 = model.params['const[1]']

# estimated coefficients of exogenous variables
betas = model.params.drop(['const[0]', 'const[1]',
                           'sigma2', 'ar.L1',
                           'p[0->0]', 'p[1->0]']).values
transMat = model.regime_transition

# Latest observation for exogenous variables used
last_obs = data.iloc[-1, [1, 3, 4, 6]].values

# Initial regime probabilities
state_probs = model.filtered_marginal_probabilities.iloc[-1]

reshaped_transition_matrix = transMat.reshape(2, 2)

# Forecasting loop
for _ in range(steps):
    # Calculate regime predictions
    regime1_pred = mu1 + np.dot(betas, last_obs)
    regime2_pred = mu2 + np.dot(betas, last_obs)

    # Weighted sum of regime predictions
    weighted_pred = regime1_pred * state_probs[0] + \
                    regime2_pred * state_probs[1]
    predictions.append(weighted_pred)

    # Update state probabilities
    state_probs = np.dot(reshaped_transition_matrix, state_probs)

return predictions
```