Combinatorial aspects of surface Cluster algebras and applications to Frobenius' conjecture

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Introduction

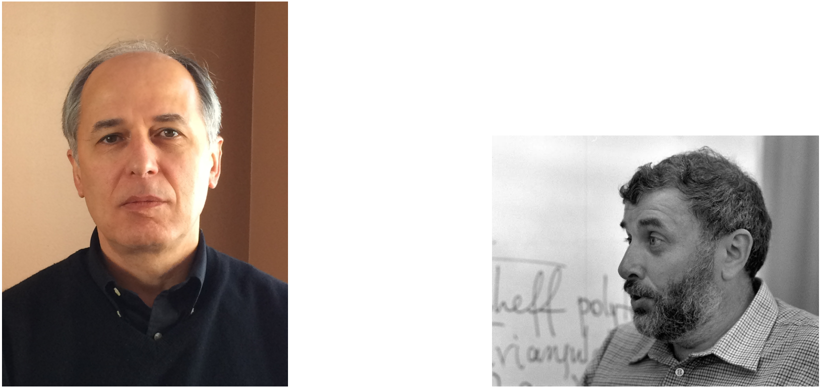

Figure: Cluster algebra was first introduced, in 2002, by Sergey Fomin and Andrei Zelevinsky.

Introduction

Where are Cluster algebras encountered?

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- Frieze patterns,
- Triangulated surfaces,
- Coxeter groups,
- Tropical geometry,
- Hyperbolic geometry,
- Quiver representations,
- Poisson geometry...

Cluster algebras

A *quiver* $\mathcal{Q}$ is a directed graph; i.e. a 4-tuple $(\mathcal{Q}_0, \mathcal{Q}_1, h, t)$.

- $\mathcal{Q}_0$ is the set of vertices $\{1, 2, \ldots, n\}$;
- $\mathcal{Q}_1$ is the set of arrows,
- $h, t : \mathcal{Q}_0 \to \mathcal{Q}_1$, where h maps the head of the arrow and t maps the tail, each to the appropriate vertex.

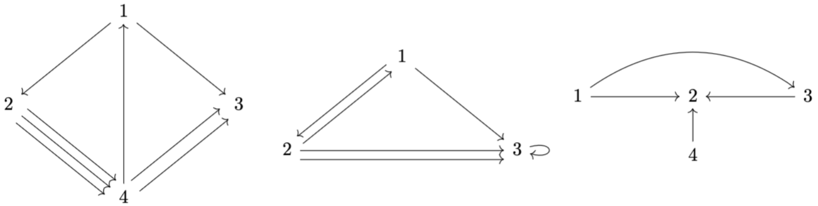

Figure: Example of 2 *cluster quiver* (left and rightmost) and a non-cluster quiver (middle)

Cluster algebras

A *seed* $(\mathbf{x}, \mathbf{y}, \mathcal{Q})$ determines the corresponding Cluster algebras through a few rules; where

- $\mathbf{x} = (x_1, \ldots, x_n)$ is the n-tuple of variables, called the *initial cluster*; e.g. in the setting of Conway-Coxeter Frieze patterns, all these variables are equal to 1.
- $\mathbf{y} = (y_1, \ldots, y_n)$ is the n-tuple of generators, called the *initial coefficients*; e.g. in the setting of triangulated polygons, these are precisely the diagonals of the triangulation.
- $\mathcal{Q}$ a cluster quiver.

Cluster algebras

The Cluster algebra is generated by a applying a recursive method to the initial seed, called a *cluster mutation*. A *cluster mutation* μ_k acts on $\mathbf{x}$ as follows;

$\mu_k((x_1, \ldots, x_k, \ldots, x_n)) = (x_1, \ldots, x_k', \ldots, x_n)$, where;

$$x_k' = \frac{\prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i}{x_k}.$$

Cluster algebras

A mutation μ_k also yields a new quiver $\mathcal{Q}'$ in the following way:

1. For any path $i \to k \to j$ add an arrow $i \to j$;
2. Invert all arrows going into, or coming out from, vertex k;
3. Cancel any 2-cycle.

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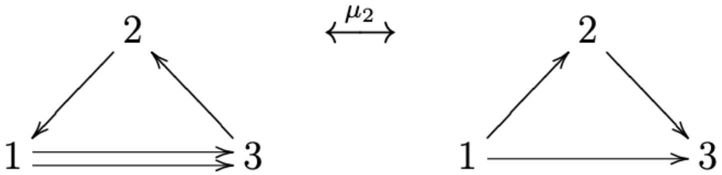

Figure: Example of mutation.

Frobenius' Conjecture

Markov's equation:

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e.g. $(1, 5, 13), (5, 13, 194), (1, 89, 233), (5, 29, 433)$.

Frobenius' Conjecture

Consider any Markov triple (a, b, c); we permute them to (x, y, z), such that $x \leq y \leq z$. Then, Frobenius' conjecture states:

Theorem

Given any two ordered Markov triples (a_1, a_2, τ), (b_1, b_2, τ), *then* $a_1 = b_1$, *and* $a_2 = b_2$.

Connection?

How does Cluster algebra factor in when thinking about Frobenius' conjecture?

Connection?

Consider the torus $\mathbb{T}^2$, as the quotient space

$$\mathbb{T}^2 = I \times I / \sim_{vert} / \sim_{side},$$

where $I = [0, 1] \subset \mathbb{R}$ and $\sim_{vert}$ and $\sim_{side}$ is the equivalence relation $(x, 1) \sim_{vert} (x, 0))$, and $(0, y) \sim_{side} (1, y)$...

The End

Thank you for listening!