Applied mathematics

F.E. Semester-2

Index:-

1.Basic Integration Formula & standard results

Set1

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$	(9) $\int secxtanx$dx=secx+c	(17) $\int \frac{1}{x\sqrt{x^2-1}}$dx=sec$^{-1}x + c = -cosec^{-1}x$+c
(2) $\int \frac{1}{x} dx = \log\|x\| + c$	(10) $\int$ cosecxcotxdx=$-cosecx$+c	(18) $\int \frac{1}{a^2+x^2}$dx=$\frac{1}{a} tan^{-1}(\frac{x}{a})$+c
(3) $\int e^x dx = e^x + c)$	(11) $\int tanx$dx=log $\|secx\|$+c	(19) $\int \frac{1}{a^2-x^2}$dx=$\frac{1}{2a}\log(\frac{x+a}{x-a})$+c
(4) $\int a^x dx = \frac{a^x}{loga} + c$	(12) $\int cotx$dx= log $\|sinx\|$++c	(20) $\int \frac{1}{x^2-a^2}$dx=$\frac{1}{2a}\log(\frac{x-a}{x+a})$+c
(5) $\int sinx$dx=$-cosx$+c	(13) $\int secx$dx= log $\|secx + tanx\|$+c	(21) $\int \frac{1}{\sqrt{x^2+a^2}}$dx=log $\|x + \sqrt{a^2+x^2}\|$+c
(6) $\int cosx$dx=sinx+c	(14) $\int cosex$dx= log $\|cosecx - cotx\|$++c	(22) $\int \frac{1}{\sqrt{x^2-a^2}}$dx= log $\|x + \sqrt{x^2-a^2}\|$ +c
(7) $\int sec^2x$dx=tanx+c	(15) $\int \frac{1}{\sqrt{1-x^2}}$dx=$sin^{-1}x + c = -cos^{-1}x$+c	(23) $\int \frac{1}{\sqrt{a^2-x^2}}$dx=$sin^{-1}(\frac{x}{a})$+c
(8) $\int cosec^2 x$dx=$-cotx$+c	(16) $\int \frac{1}{1+x^2}$dx=tan$^{-1}x + c = -cot^{-1}x$+c	(24) $\int \frac{x}{x\sqrt{x^2-a^2}}$dx=$\frac{1}{a} sec^{-1}(\frac{x}{a})$+c

(25) $\int \sqrt{x^2+a^2}$dx=$\frac{x}{2}\sqrt{x^2+a^2}$+$\frac{a^2}{2}\log[x + \sqrt{x^2+a^2}]$
(26) $\int \sqrt{x^2-a^2}$dx=$\frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log[x + \sqrt{x^2+a^2}]$
(27) $\int e^{ax} sinbxdx = \frac{e^{ax}}{a^2+b^2}(asinbx - bcosbx)$
(28) $\int e^{ax} cosbxbxdx = \frac{e^{ax}}{a^2+b^2}(acosbx + bsinbx)$
(29) $\int \frac{x}{x^2+a^2} dx = \log\|x + \sqrt{x^2-a^2}\|$

Set2

If in case of substitution →

Expansion	Substitution
$\sqrt{a^2 - x^2}$	$x = $ asinθ $or\ x = acos\theta$
$\sqrt{x^2 + a^2}$	$x = $ atan $\theta\ or\ x = acot\theta$
$\sqrt{x^2 - a^2}$	$x = $ asecθ $or\ x = acosec\theta$

Set 3

Standard results,

(1) $\int \frac{1}{x^n} dx = \frac{-1}{(n-1)x^{n-1}} + c$

(2) $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^n}{n+1} + c$

(3) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

(4) $\int \frac{f'(x)}{f(x)} dx = logf(x) + c$

(5) $\int e^{f(x)} f'(x)dx = e^{f(x)} + c$

(6) → (a) $\int_0^{\frac{\pi}{2}} sin^n xdx\ \ or\ \int_0^{\frac{\pi}{2}} cos^n x$dx

$= \frac{(n-1)(n-3)(n-5)...}{n(n-2)(n-4)...} \times \frac{\pi}{2} \rightarrow only\ if\ n\ is\ even$

(b) $\int_0^{\frac{\pi}{2}} sin^m xcos^n xdx$

$= \frac{[(m-1)(m-3)...][(n-1)(n-3)...]}{[(m+n)(m+n-2)(m+n-4)...]} \times \frac{\pi}{2} \rightarrow only\ if\ n\&m\ both\ are\ even$

Set 4

u.v rule in integration

type1→ $\int uvdx = uv' - u_1v' + u_2v'' - \cdots$

where u should be algebraic and v can be anything other then log or inverse

type1→ $\int uvdx = u\int vdx + \int [\frac{d}{dx}(u\int vdx)]dx$ {hint: remember FIS-$\int DFIS$}

u and v are decided by using LIATE rule where , L= logarithmic , I = inverse ,A=algebraic ,T=trigonometric and E=exponential

Set5

Rules of definite integration,

(A)

If $f(2a - x) = f(x)$ $then$ $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$

If $f(2\pi - x) = f(x)$ $then$ $\int_0^{2\pi} f(x)dx = 2\int_0^{\pi} f(x)dx$

If $f(\pi - x) = f(x)$ $then$ $\int_0^{\pi} f(x)dx = 2\int_0^{\frac{\pi}{2}} f(x)dx$

(B)

$f(-x) = f(x) \to [f(x) is\ even]$

$f(-x) = -f(x) \to [f(x) is\ odd]$

Now $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \{if\ f(x) is\ even\} or = 0\ \{if\ f(x) is\ even\}$

2.Differential equation of 1st order & 1st degree

Step 1: every differential equation can be put in the form mdx+ndy=0 , where m and n both are functions of x and y and constants.

Step2: prove -> $\frac{dm}{dy} = \frac{dn}{dx}$ (result)-> equation will be said to have exact solution.

Step3: Then find its solution which is given by the formula -> $\int mdx + \int ndy = c$, where m->with respect to x treating y constant and n-> with respect to y independent of x.

Examples:

Example 1:(1+$e^{\frac{x}{y}}$)dx+$e^{\frac{x}{y}}$(1-$\frac{x}{y}$)dy=0

<u>Solution :</u> on comparing with mdx+ndy=0 , m= 1+$e^{\frac{x}{y}}$ $,n = e^{\frac{x}{y}} - e^{\frac{x}{y}}\frac{x}{y}$

Now $\frac{dm}{dy} = -e^{\frac{x}{y}}\frac{x}{y^2}$ and $\frac{dn}{dx} = e^{\frac{x}{y}}\frac{1}{y} - \left[e^{\frac{x}{y}}\frac{1}{y} + e^{\frac{x}{y}}\frac{x}{y^2}\right] = -e^{\frac{x}{y}}\frac{x}{y^2}$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$,equation is exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int (1 + e^{\frac{x}{y}}) + \int 0dy = c$$

X+$\frac{e^{\frac{x}{y}}}{\frac{1}{y}} + c1 = c$

X+$ye^{\frac{x}{y}} = k \quad (where\ k = c - c1)$

Example 2:$\frac{dy}{dx} + \frac{ycosx+siny\ y}{sinx+xcosy+x} = 0$

<u>Solution :</u>

$$\frac{dy}{dx} = -[\frac{ycosx + siny + y}{sinx + xcosy + x}]$$

(ycosx+siny+BBC y)dx+(sinx+xcosy+x)dy=0

On comparing with mdx+ndy=0

m=ycosx+siny+y and n=sinx+xcosy+x

now $\frac{dm}{dx} = cosx + cosy + 1and$ $\frac{dn}{dy} = cosx + cosy + 1$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$,equation is exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int (ycosx + siny + y)dx + \int 0dy = c$$

Ysinx+xsiny+xy+c1=c

Ysinx+xsiny+xy=k $(where\ k = c - c1)$

Example 3: [log($x^2 + y^2$) + $\frac{2x^2}{x^2+y^2}$]dx + [$\frac{2xy}{x^2+y^2}$]dy=0

Solution: on comparing with mdx+ndy=0

m=log($x^2 + y^2$)+$\frac{2x^2}{x^2+y^2}$

now $\frac{dm}{dx} = \frac{2y}{x^2+y^2} - \frac{4yx^2}{(x^2+y^2)^2}$ and $\frac{dn}{dy} = \frac{(x^2+y^2)2y-2xy(2x)}{(x^2+y^2)^2}$=$\frac{2y}{x^2+y^2} - \frac{4yx^2}{(x^2+y^2)^2}$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$,equation is exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int [\log(x^2 + y^2) + \frac{2x^2}{x^2 + y^2}]dx + \int 0dy = c$$

$\int \log(x^2 + y^2)$dx + $\int [\frac{2x^2}{x^2+y^2}]\ dx$=k

$\log(x^2 + y^2) \int 1$ dx-$\int [\frac{d}{dx}\log(x^2 + y^2) \int 1$ dx]+ $\int [\frac{2x^2}{x^2+y^2}]\ dx$=k

xlog($x^2 + y^2$)- $\int [\frac{2x^2}{x^2+y^2}]\ dx + \int [\frac{2x^2}{x^2+y^2}]\ dx = k$

xlog($x^2 + y^2$)=k

3.Making differential equation exact by multiplying I.F.

I.F-> integrating factor

If $\frac{dm}{dy} \neq \frac{dn}{dx}$ then given differential equation is not exact and its solution can not be found.

But equation can be made be exact by multiplying I.F throughout.

[note: we usually drop constants from I.F]

Type 1:

If the equation can be put in form f1(xy)ydx+f2(xy)xdy=0 then,

I.F.=$\frac{1}{mx-ny}$

Type 2:

If in a differential equation whose combined power of x and y is constant throughout [i.e homogenous] then,

I.F=$\frac{1}{mx+ny}$

Type 3:

If type 1 and type 2 conditions are not satisfied then try,

(3A)->$\frac{\frac{dm}{dy}-\frac{dn}{dx}}{n}$=f(x) or constant ,then I.F=$e^{\int f(x)dx}$

(3B)->$\frac{\frac{dn}{dx}-\frac{dm}{dy}}{m}$=f(y) or constant , then I.F=$e^{\int f(y)dy}$

[Note: In this type , the simpler term is written in the denominator and accordingly the numerator is written]

[type1, type2, type 3A, and type 3B are the most important formula of I.F to be remembered]

Example 1: (xy^2+y)dx+($x+yx^2+x^3y^2$)dy=0

Solution given equation is in the form mdx+ndy=0

$$\frac{dm}{dy}=2xy+1\ and\ \frac{dn}{dx}=1+2xy+3x^2y^2$$

Therefore $\frac{dm}{dy}\neq\frac{dn}{dx}$ i.e not exact

But we can write equation in form y(xy+1)dx+x(1+xy+x^2y^2)dy=0

So I.F.=$\frac{1}{mx-ny}=\frac{1}{x^2y^2+xy-xy-x^2y^2-x^3y^3}=\frac{-1}{x^3y^3}$

on multiplying I.F. throughout given equation,

[$\frac{1}{yx^2}+\frac{1}{x^3y^2}$]dx+[$\frac{1}{x^2y^3}+\frac{1}{xy^2}+\frac{1}{y}$]dy=0

equation is in the form mdx+ndy=0

so $\frac{dm}{dy}=-\frac{1}{x^2y^2}-\frac{2}{x^3y^3}\ and\ \frac{dn}{dx}=-\frac{2}{x^3y^3}-\frac{1}{x^2y^2}+0$

Therefore $\frac{dm}{dy}=\frac{dn}{dx}$ i.e exact

Now solution is given by $\int mdx+\int ndy=c$

$$\int\left(\frac{1}{yx^2}+\frac{1}{x^3y^2}\right)dx+\int\left(\frac{1}{y}\right)dy=c$$

-$\frac{1}{xy}-\frac{1}{y^22x^2}+logy=c$

Example 2: ydx+x(1-3x^2y^2)dy=0

Solution given equation is in the form mdx+ndy=0

$$\frac{dm}{dy} = 1 \; and \; \frac{dn}{dx} = 1 - 3x^2 3y^2$$

Therefore $\frac{dm}{dy} \neq \frac{dn}{dx}$ i.e not exact

But it is in form f1(xy)ydx+f2(xy)xdy=0

So I.F.= $\frac{1}{mx-ny} = \frac{1}{xy-xy+3x^3y^3} = \frac{1}{3x^3y^3}$

on multiplying I.F. throughout given equation,

$(\frac{1}{x^3y^2})$dx+$(\frac{1}{x^2y^3} - \frac{3}{y})$dy=0

equation is in the form mdx+ndy=0

so $\frac{dm}{dy} = \frac{-2}{x^3y^3} \; and \; \frac{dn}{dx} = \frac{-2}{x^3y^3} + 0$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$ i.e exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int \left(\frac{1}{x^3y^2}\right) dx + \int \left(\frac{-3}{y}\right) dy = c$$

$$-\frac{1}{2x^2y^2} - 3logy = c$$

Example 3: $(x^2 - 3xy + 2y^2)$dx+x(3x-2y)dy=0

<u>Solution</u> given equation is in the form mdx+ndy=0

$$\frac{dm}{dy} = -3x + 4y \; and \; \frac{dn}{dx} = 6x - 2y$$

Therefore $\frac{dm}{dy} \neq \frac{dn}{dx}$ i.e not exact

However equation is homogenous in x and y

So I.F.= $\frac{1}{mx+ny} = \frac{1}{x^3-3x^2y+2xy^2+3x^2y-2xy^2} = \frac{1}{x^3}$

on multiplying I.F. throughout given equation,

$$\left(\frac{1}{x} - \frac{3y}{x^2} + \frac{2y^2}{x^3}\right) dx + \left(\frac{3}{x} - \frac{2y}{x^2}\right) dy = 0$$

equation is in the form mdx+ndy=0

so $\frac{dm}{dy} = \frac{-3}{x^2} + \frac{4y}{x^3} \; and \; \frac{dn}{dx} = \frac{-3}{x^2} + \frac{4y}{x^3}$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$ i.e exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int\left(\frac{1}{x}-\frac{3y}{x^2}+\frac{2y^2}{x^3}\right)dx+\int 0dy=c$$

$logx+\frac{3y}{x}-\frac{y^2}{x^2}$=c

Example 4: (3xy^2-y^3)dx+(xy^2-2x^2y)dy=0

<u>Solution</u> given equation is in the form mdx+ndy=0

$$\frac{dm}{dy}=6xy-3y^2\ \ and\ \frac{dn}{dx}=y^2-4xy$$

Therefore $\frac{dm}{dy}\neq\frac{dn}{dx}$ i.e not exact

However equation is homogenous in x and y

So I.F.=$\frac{1}{mx+ny}=\frac{1}{3x^2y^2-xy^3+xy^3-2x^2y^2}=\frac{1}{x^2y^2}$

on multiplying I.F. throughout given equation,

($\frac{3}{x}-\frac{y}{x^2}$)dx+($\frac{1}{x}-\frac{2}{y}$)dy=0

equation is in the form mdx+ndy=0

so $\frac{dm}{dy}=-\frac{1}{x^2}and\ \frac{dn}{dx}=-\frac{1}{x^2}$

Therefore $\frac{dm}{dy}=\frac{dn}{dx}$ i.e exact

Now solution is given by $\int mdx+\int ndy=c$

$$\int(\frac{3}{x}-\frac{y}{x^2})\text{dx}+\int(-\frac{2}{y})\text{dy}=c$$

3logx+$\frac{y}{x}$-2logy=c

Example 5:$(1-xy+x^2y^2)dx+(yx^3-x^2)dy=0$

<u>Solution</u> given equation is in the form mdx+ndy=0

$$\frac{dm}{dy}=0-x+2yx^2\ \ and\ \frac{dn}{dx}=3yx^2-2x$$

Therefore $\frac{dm}{dy}\neq\frac{dn}{dx}$ i.e not exact

$$\frac{\frac{dm}{dy}-\frac{dn}{dx}}{n}=\frac{x-x^2y}{x^3y-x^2}=\frac{-x(-1+y)}{x^2(y-1)}=\frac{-1}{x}=f(x)$$

Thus I.F.= $e^{\int f(x)dx}$=$e^{\int -\frac{1}{x}dx}$=$e^{-\log x}=\frac{1}{x}$

on multiplying I.F. throughout given equation,

($\frac{1}{x}-y+xy^2$)dx+(x^2y-x)dy+0

equation is in the form mdx+ndy=0

so $\frac{dm}{dy} = -1 + 2xyand\ \frac{dn}{dx} = -1 + 2xy$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$ i.e exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int (\frac{1}{x} - y + xy^2)dx + \int 0dy = c$$

$$logx - xy + \frac{x^2y^2}{2} + c1 = c$$

$$logx - xy + \frac{x^2y^2}{2} = k$$

Example 6: ylogydx+(x-logy)dy=0

Solution given equation is in the form mdx+ndy=0

$$\frac{dm}{dy} = 1 + logy\ and\ \frac{dn}{dx} = 1$$

Therefore $\frac{dm}{dy} \neq \frac{dn}{dx}$ i.e not exact

$$\frac{\frac{dn}{dx} - \frac{dm}{dy}}{m} = \frac{1-(1+logy)}{ylogy} = \frac{-1}{y} = f(y)$$

Thus I.F.= $e^{\int f(y)dy}$=$e^{\int -\frac{1}{y}dy}$=$e^{-logy} = \frac{1}{y}$

on multiplying I.F. throughout given equation,

(logy)dx+($\frac{x-logy}{y}$)dy=0

equation is in the form mdx+ndy=0

so $\frac{dm}{dy} = \frac{1}{y} and\ \frac{dn}{dx} = \frac{1}{y}$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$ i.e exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int (\text{logy})\text{dx} + \int (\frac{-logy}{y})\text{dy} = c$$

$$xlogy - \int \frac{1}{y} logydy = c$$

$$\text{xlogy} - \frac{(logy)^2}{2} = c$$

Example 7:(1+y^2)dx+$\left(x - e^{tan^{-1}y}\right)dy = 0$

Solution given equation is in the form mdx+ndy=0

$$\frac{dm}{dy} = 2y \text{ and } \frac{dn}{dx} = 1$$

Therefore $\frac{dm}{dy} \neq \frac{dn}{dx}$ i.e not exact

$$\frac{\frac{dn}{dx} - \frac{dm}{dy}}{m} = \frac{1-2y}{1+y^2} = f(y)$$

Thus I.F.= $e^{\int f(y)dy}$=$e^{\int \frac{1}{1+y^2} - \frac{2y}{1+y^2}dy}$=$e^{[tan^{-1}y - \log(1+y^2)]}$=$\frac{e^{tan^{-1}y}}{1+y^2}$

on multiplying I.F. throughout given equation,

$(e^{tan^{-1}y})$dx+($\frac{xe^{tan^{-1}y}}{1+y^2} - \frac{e^{2tan^{-1}y}}{1+y^2}$)dy=0

equation is in the form mdx+ndy=0

so $\frac{dm}{dy} = \frac{e^{tan^{-1}y}}{1+y^2}$ and $\frac{dn}{dx} = \frac{e^{tan^{-1}y}}{1+y^2}$

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$ i.e exact

Now solution is given by $\int mdx + \int ndy = c$

$$\int (e^{tan^{-1}y})\mathrm{dx} + \int \left(-\frac{e^{2tan^{-1}y}}{1+y^2}\right)\mathrm{dy} = c$$

$$xe^{tan^{-1}y} - \frac{1}{2}e^{2tan^{-1}y} = c$$

Example 8:If x^p is the I.F. of $(5x^2 + 12xy - 3y^2)dx + (3x^2 - 2xy)dy = 0$, $then\ find\ its\ solution$.

Solution

on multiplying I.F. throughout given equation,

$$(5x^{p+2} + 12x^{p+1}y - 3x^p y^2)dx + (3x^{p+2} - 2x^{p+1}y)dy = 0$$

equation is in the form mdx+ndy=0

Therefore $\frac{dm}{dy} = \frac{dn}{dx}$ and exact

so $\frac{dm}{dy} = 12x^{p+1} - 6x^p y$ and $\frac{dn}{dx} = 3(p+2)x^{p+1} - 2(p+1)x^p y$

now $12x^{p+1} - 6x^p y = 3(p+2)x^{p+1} - 2(p+1)x^p y$

on comparing

3(p+2)=12

P=2

Now solution is given by $\int mdx + \int ndy = c$

$$\int (5x^4 + 12x^3y - 3x^2y^2)\,dx + \int 0\,dy = c$$

$x^5 + 3x^4y - x^3y^2 + c1 = c$

$x^5 + 3x^4y - x^3y^2 = k$

4.Linear differential equations

The equation is always in the form $\frac{dy}{dx} + py = q$

Where p and q are functions of constant .

Thus we need to find I.F. by I.F.=$e^{\int pdx}$

And so the solution is given by $y(I.F.) = \int q(I.F.)dx + c$

Note : If $\frac{dx}{dy} + px = q$ $then\ I.F. =\ e^{\int pdY}\ and\ thus\ solution\ is\ \ x(I.F.) = \int q(I.F.)dy + c$

$$\frac{dy}{dx} + py = q \rightarrow (general\ form) \rightarrow\ f'(y)\frac{dy}{dx} + pf(y) = q$$

$\downarrow$

$put\ f(y) = u\ , so\ on\ differentiating\ f'(y)\frac{dy}{dx} = \frac{du}{dx}$ →So solution is given by $u(I.F.) = \int q(I.F.)dy + c$

Example 1 :$[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}]\frac{dx}{dy} = 1$

Solution $\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}=\frac{dy}{dx}$

$\frac{dy}{dx}+\frac{1}{\sqrt{x}}y=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Therefore on comparing we get p and q

I.F.= $e^{\int pdx}=e^{\int\frac{1}{\sqrt{X}}dx} = e^{2\sqrt{X}}$

So solution is given by $y(I.F.) = \int q(I.F.)dx + c$

y$e^{2\sqrt{X}}=\int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}e^{2\sqrt{X}}dx + c$

y$e^{2\sqrt{X}} = 2\sqrt{X} + c$

Example 2 : $(1 + siny)\frac{dx}{dy} = 2ycosy - x(\text{secy} + tany)$

Solution $\frac{dx}{dy} = \frac{2ycosy}{(1+siny)} - x(\frac{1}{cosy} + \frac{siny}{cosy})(\frac{1}{(1+siny)})$

$$\frac{dx}{dy} = \frac{2ycosy}{(1 + siny)} - x\left(\frac{1 + siny}{cosy}\right)\left(\frac{1}{1 + siny}\right)$$

$$\frac{dx}{dy}+xsecy=\frac{2ycosy}{(1+siny)}$$

Therefore on comparing we get p and q

I.F.= $e^{\int pdy}$=$e^{\int secydy}=e^{\log(secy+tany)}$=secy+tany

So solution is given by $x(I.F.)=\int q(I.F.)dy+c$

x$(secy+tany)=\int\frac{2ycosy}{(1+siny)}(\text{secy}+\text{tany})dy+c$

x$(secy+tany)=\int 2\,ydy+c$

x$(secy+tany)=y^2+c$

x$(secy+tany)-y^2=c$

Example 3: $y^4dx=\left(x^{-\frac{3}{4}}-y^3x\right)dy$

Solution $\frac{dx}{dy}=\frac{x^{-\frac{3}{4}}}{y^4}-\frac{y^3x}{y^4}$

$$\frac{dx}{dy}+\frac{1}{y}x=\frac{x^{-\frac{3}{4}}}{y^4}$$

$x^{\frac{3}{4}}\frac{dx}{dy}+\frac{1}{y}x^{\frac{7}{4}}$=$\frac{1}{y^4}$

[put $x^{\frac{7}{4}}=u$, so after differentiation

$\frac{7}{4}x^{\frac{3}{4}}\frac{dx}{dy}$=$\frac{du}{dy}$]

$\frac{4}{7}\frac{du}{dy}+\frac{1}{y}u$=$\frac{1}{y^4}$

$\frac{du}{dy}+\frac{7}{4y}u$=$\frac{7}{4y^4}$

Therefore on comparing we get p and q

I.F.= $e^{\int pdy}$=$e^{\int\frac{7}{4y}dy}=e^{\frac{7}{4}logy}$=$y^{\frac{7}{4}}$

So solution is given by $u(I.F.)=\int q(I.F.)dy+c$

$$(xy)^{\frac{7}{4}}=\frac{7}{4}\int\frac{y^{\frac{7}{4}}}{y^4}dy+c$$

$$(xy)^{\frac{7}{4}}=\frac{7}{4}\int y^{\frac{-9}{4}}dy+c$$

$$(xy)^{\frac{7}{4}}=\frac{7}{4}\frac{y^{\frac{-5}{4}}}{\frac{-5}{4}}+c$$

$$(xy)^{\frac{7}{4}}+\frac{7}{5}y^{\frac{-5}{4}}=c$$

Example 4 : $\frac{dy}{dx} + x^3 sin^2 y + xsin2y = x^3$

<u>Solution</u> $\frac{dy}{dx} + xsin2y = x^3(1 - sin^2 y)$

$$\frac{dy}{dx} + x2sinycosy = x^3 cos^2 y$$

$$sec^2 y \frac{dy}{dx} + 2xtany = x^3$$

[put tany=u , so after differentiation

$sec^2 y \frac{dy}{dx} = \frac{du}{dx}$]

$$\frac{du}{dx} + 2xu = x^3$$

Therefore on comparing we get p and q

I.F.= $e^{\int pdx}$=$e^{\int 2xdx} = e^{x^2}$

So solution is given by $tanye^{x^2} = \int x^2 e^{x^2} xdx + c$

[put $x^2 = t \ so \ 2xdx = dt$]

$tanye^{x^2}$=$\frac{1}{2}\int te^t dt + c$___{by $\int uv$}

$tanye^{x^2}$=$\frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$

$tanye^{x^2}$=$\frac{e^{x^2}}{2}(x^2 - 1) + c$

Example 5: $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

<u>Solution</u> $\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y)$

$$e^y \frac{dy}{dx} = e^{2x} - e^x e^y$$

$$e^y \frac{dy}{dx} + e^x e^y = e^{2x}$$

[put $e^y = u$, so after differentiation

$e^y \frac{dy}{dx} = du$]

du+u$e^x = e^{2x}$

Therefore on comparing we get p and q

I.F.= $e^{\int pdx}$=$e^{\int e^x dx} = e^{e^x}$

So solution is given by $u(I.F.) = \int q(I.F.)dx + c$

$$e^y e^{e^x} = \int e^x e^{e^x} e^x \, dx + c$$

[put $e^x = t \; so \; e^x dx = dt$]

$e^y e^{e^x} = \int te^t dt + c$

$e^y e^{e^x} = e^t(t-1) + c$

$e^y e^{e^x} = e^{e^x}(e^x - 1) + c$

5.Substitution of differentials

1. $xdx + ydy \rightarrow put \; x^2 + y^2 = u^2 \;, so \; xdx + ydy = udu$
2. $xdy + ydx \rightarrow put \; xy = u \;, so \; xdy + ydx = du$
3. $xdy - ydx \rightarrow put \; \frac{y}{x} = u \;, so \; \frac{xdy-ydx}{x^2} = du$
4. $ydx - xdy \rightarrow put \; \frac{x}{y} = u \;, so \; \frac{ydx-xdy}{y^2} = du$

Note : special case i.e when $xdx + ydy$ and $xdy - ydx$ comes together, put $x = rcos\theta \; and \; y = rsin\theta$

$$1. x^2 + y^2 = u^2 \; so \; xdx + ydy = rdr$$

$$2. \frac{y}{x} = tan\theta \; so \; \frac{xdy - ydx}{x^2} = sec^2\theta d\theta$$

$$xdy - ydx = \frac{x^2}{cos^2\theta} d\theta = rdr$$

Example 1: $\frac{ydx-xdy}{(x-y)^2} = \frac{dx}{\sqrt{(1-x^2)}}$

<u>*solution*</u>

$$\frac{\frac{ydx - xdy}{y^2}}{\frac{(x-y)^2}{y^2}} = \frac{dx}{\sqrt{(1-x^2)}}$$

$\frac{\frac{ydx-xdy}{y^2}}{(\frac{x}{y}-1)^2} = \frac{dx}{\sqrt{1-x^2}}$ [put $\frac{x}{y} = u$]

$\frac{du}{(u-1)^2} = \frac{dx}{\sqrt{1-x^2}}$

Now on integrating both sides

$$\int \frac{du}{(u-1)^2} = \int \frac{dx}{\sqrt{1-x^2}}$$

$$\frac{-1}{u-1} = sin^{-1}x + c$$

$$\frac{-1}{\frac{x}{y}-1} - sin^{-1}x = c$$

Example 2: $(ysinxy + xy^2cosxy)$dx+$(xsinxy + x^2ycosxy)$dy=0

Solution

y$(sinxy + xycosxy)$dx+x$(sinxy + xycosxy)$dy=0

$(ydx + xdy)(sinxy + xycosxy) = 0$

[put xy=u so ydx+xdy=du]

du$(sinu + ucosu)$=0

now on integrating

$$\int (sinu + ucosu)\mathrm{du} = c$$

$-cosu + usinu + cosu = c$

$xysinxy = c$

Example 3 : $(\frac{y}{x}cos\frac{y}{x})$dx-$(\frac{x}{y}sin\frac{y}{x} + cos\frac{y}{x})$dy=0

Solution

$[put\ \frac{y}{x} = u\ \ so\ y = xu\ so\ \frac{dy}{dx} = u + x\frac{du}{dx}]$

$$(ucosu)dx - \left(\frac{1}{u}sinu + cosu\right)dy = 0$$

$$(ucosu) - \left(\frac{1}{u}sinu + cosu\right)\frac{dy}{dx} = 0$$

$$(ucosu) - \left(\frac{1}{u}sinu + cosu\right)(u + x\frac{du}{dx}) = 0$$

$$ucosu - ucosu - sinu - (\frac{1}{u}sinu + cosu)(\ x\frac{du}{dx}) = 0$$

$$-(\frac{1}{u}sinu + cosu)(\ x\frac{du}{dx}) =\ sinu$$

$$-(\frac{sinu + ucosu}{usinu})du = \frac{1}{x}dx$$

$$\frac{1}{x}dx + \left(\frac{sinu+ucosu}{usinu}\right)du = 0$$

Now on integrating

$$\int \frac{1}{x}dx + \int \left(\frac{sinu + ucosu}{usinu}\right)du = c$$

$$logx + \log(sinu + ucosu) = iogc$$

$$x(sinu + ucosu) = c$$

$$x\left(sin\frac{y}{x}+\frac{y}{x}cos\frac{y}{x}\right)=c$$

Example 4: $(x+y)^2\left(x\frac{dy}{dx}+y\right)=x(1+\frac{dy}{dx})$

Solution

[hint : when two differentials appear with there derivatives, put one as u and another as v]

$$put\ x+y=u\ and\ xy=v$$

So $x\frac{dy}{dx}+y=\frac{dv}{dx}\ and\ 1+\frac{dy}{dx}=\frac{du}{dx}$

So $u^2\frac{dv}{dx}=v\frac{du}{dx}$

$$\frac{1}{v}\frac{dv}{dx}=\frac{1}{u^2}\frac{du}{dx}$$

Now on integrating

$$\int\frac{1}{v}dv=\int\frac{1}{u^2}du+c$$

$$logv=-\frac{1}{u}+c$$

$$logxy+\frac{1}{x+y}=c$$

Example 5: $\frac{dy}{dx}+x(x+y)=x^3(x+y)^3-1$

Solution

[hint : in differential equation if a particular term repeats more then once then that term can be put as u]

$$put\ x+y=u$$

$$so\ 1+\frac{dy}{dx}=\frac{du}{dx}$$

Therefore given equation becomes

$$\frac{du}{dx}+xu=x^3u^3$$

$$\frac{1}{u^3}\frac{du}{dx}+\frac{1}{u^2}x=x^3$$

[put $\frac{1}{u^2}=v$ so on differentiating w.r.t x $\frac{-2}{u^3}\frac{du}{dx}=\frac{dv}{dx}$]

$$-\frac{1}{2}\frac{dv}{dx}+xv=x^3$$

$$\frac{dv}{dx}-2xv=-2x^3$$

So I.F.=$e^{\int -2xdx}$=e^{-x^2}

Now solution is given by v(I.F.)=$\int q$(I.F.)dx+c

v$e^{-x^2} = \int -2x^3\, e^{-x^2} dx + c$

$\frac{1}{u^2} e^{-x^2} = \int (-2x\, e^{-x^2}) x^2 dx + c$

$\frac{1}{(x+y)^2} e^{-x^2}$=$x^2 \int (-2x\, e^{-x^2}) x^2 dx - \int [\frac{d}{dx}(x^2 \int (-2x\, e^{-x^2}) x^2 dx)] dx$+c

$\frac{1}{(x+y)^2} e^{-x^2} = x^2 e^{-x^2} - 2x e^{-x^2} + c$

$\frac{1}{(x+y)^2} e^{-x^2} = e^{-x^2}(x^2 - 2x) + c$

6.Higher Order-Differential equation with constant coefficient

Higher order differential equation will be for example

$\frac{d^2y}{dx}+5\frac{dy}{dx}+6y=x$

Step 1 : put $\frac{d}{dx}$ = D → (according to Taylor operator)

Step 2 : thus equation will become : $(D^2+5D+6)\ y = x$

f(D).y = x

where f(D) = 0 → Auxiliary equation

Step 3 : find the ratio of Auxiliary equation

$\therefore D^2+5D+6 = 0$

$\therefore (D+2)(D+3) = 0$

$\therefore D = -2, -3$

❖ Rules of finding Complementary function →(CF / y_c)

Case 1: Roots are real and distinct

CF / $y_c = C_1e^{m_1x}+C_2e^{m_2x}+\ldots\ldots$

Where m_1, m_2 → roots

C_1, C_2 →constants

Case 2: Roots are real and repeated

CF / $y_c = (C_1+C_2x+C_3x^2+\ldots\ldots C_nx^{n-1})e^{m_1x}$

Where m → repeated root

C_1, C_2 → constants

Case 3: Roots are complex and distinct

[If roots are α+iβ and α-iβ]

CF / y_c = $[e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)] + [e^{\alpha x}(C_3\cos\beta x - C_4\sin\beta x)]$

Case 4: <u>Roots are complex and repeated</u>

[If α+iβ root is repeated twice]

CF / y_c = $e^{\alpha x}[(C_1 + C_2)\cos\beta x + (C_3 + C_4)\sin\beta x]$

Example 1: - $9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$

Sol:- replace $\frac{d}{dx}$ = D

∴ $9D^2 + 18D - 16 = 0$

∴ D = $\frac{2}{3}$, $\frac{-8}{3}$

$$\therefore \mathrm{CF} = C_1 e^{\frac{2}{3}x} + C_2 e^{\frac{-8}{3}x}$$

Example 2: - $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

Sol:- replace $\frac{d}{dx}$ = D

∴ $D^4 + 8D^2 + 16 = 0$

∴ roots are D = $\pm 2i, \pm 2i$

$$\therefore \mathrm{CF} = e^0[(C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x]$$

❖ <u>Rules of finding Particular Integral →(PI / y_p)</u>

Type 1: where x=e^{ax}

PI / y_p = $\frac{1}{f(D)}$.[x] = $\frac{1}{f(D)}$.[e^{ax}]

Always write $f(D)$ into factorized form.

Replace every D by a provided the term does not become zero.

If on replacing D by a the term becomes zero, we will use the following formula.

$$\frac{1}{(D-a)^n}.[\,e^{ax}]=\frac{x^n}{n!}.e^{ax}$$

Note: if x=a^x , then x=$e^{(loga)(x)}$

Note: D - means differentiation and $\frac{1}{D}$ – means integration

Example 1: - $\frac{d^3y}{dx^3}$ - 2 $\frac{d^2y}{dx^2}$ - 5 $\frac{dy}{dx}$ + 6y = $(e^{2x}+3)^2$

Solution:- ∴ on replacing $\frac{d}{dx}$ = D

$\therefore f(D) =$ D^3 – 2D² -5D+6 = 0

And x = $(e^{2x}+3)^2$

∴roots are D = $-2,3,1$

∴PI / y_p = $\frac{1}{f(D)}$.[x] =$\frac{1}{D^3-2D^2-5D+6}$. $(e^{2x}+3)^2$

=$\frac{1}{(D+2)(D-3)(D-1)}$.[$e^{4x}+9+6e^{2x}$]

∴PI= $\frac{e^{4x}}{(D+2)(D-3)(D-1)}$ $\frac{9e^{0x}}{(D+2)(D-3)(D-1)}$ $\frac{6e^{2x}}{(D+2)(D-3)(D-1)}$

∴PI= $\frac{e^{4x}}{18}-\frac{6e^{2x}}{4}+\frac{9}{6}$

Example 2: - $\frac{d^3y}{dx^3}$ - 16$\frac{dy}{dx}$ = 2cos h^2 $2x$

Solution:- On replacing $\frac{d}{dx}$ = D

$\therefore f(D) =$ D^3 - 16D = 0

And x = 2cos h^2 $2x$

∴roots are D = $\pm 4,0$

∴PI / y_p = $\frac{1}{f(D)}$.[x] =$\frac{1}{D^3-16D}$. 2cos h^2 $2x$

= $\frac{1}{D(D-4)(D+4)}$.[2($\frac{2+e^{4x}+e^{-4x}}{4}$)]

=

$\frac{1}{2}\left(\frac{1}{D(D-4)(D+4)}\right).[\,e^{4x}]+\frac{1}{2}\left(\frac{1}{D(D-4)(D+4)}\right).[\,e^{-4x}]+\frac{2}{2}\left(\frac{1}{D(D-4)(D+4)}\right).[\,1.e^{0x}]$

$\therefore$PI = $\frac{1}{2}.\frac{1}{4.8}.\frac{x^1}{1!}.e^{4x}+\frac{1}{2}.\frac{1}{(-4).(-8)}.\frac{x^1}{1!}.e^{-4x}+\frac{1}{(4).(-4)}.\frac{x^1}{1!}.1$

$\therefore$PI = $\frac{xe^{4x}}{32.2}+\frac{xe^{-4x}}{32.2}$ - $\frac{x}{16}$

PI = $\frac{x}{32}.\frac{cosh4x}{1}$ - $\frac{x}{16}$

PI = $\frac{x}{16}.\left[\frac{cosh4x}{2}-1\right]$

Example 3:- $(D^4-16)y = coshx.sinhx$

Solution:-

Auxiliary Equation=$(D^4-16)=0=f(D)$

Roots=> $\pm 2, \pm 2i$

$\therefore$ PI=$\frac{1}{f(D)}$[x]

$\therefore$ PI=$\left(\frac{1}{D^4-16}\right)[sinhx.coshx]$

$\therefore$PI=$\left(\frac{1}{D^4-16}\right)\left[\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)\right]$

$\therefore$PI=$\left(\frac{1}{D^4-16}\right)\left[\left(\frac{e^{2x}-e^{-2x}}{4}\right)\right]$

$\therefore$PI=$\left(\frac{1}{D^4-16}\right)\left[\frac{e^{2x}}{4}\right]-\left(\frac{1}{D^4-16}\right)\left[\frac{e^{-2x}}{4}\right]$

$\therefore$PI=$\frac{1}{(D^2+4)(D-2)(D+2)}\left[\frac{e^{2x}}{4}\right]-\frac{1}{(D^2+4)(D-2)(D+2)}\left[\frac{e^{-2x}}{4}\right]$

$\therefore \mathrm{PI} = \frac{1}{32} \cdot \frac{x}{1!} \left(\frac{e^{2x}}{4}\right) + \frac{1}{32} \cdot \frac{x}{1!} \left(\frac{e^{-2x}}{4}\right)$

$\therefore \mathrm{PI} = \frac{x}{64} \left(\frac{e^{2x} - e^{-2x}}{2}\right)$

$\therefore \mathrm{PI} = \frac{x}{64} cosh2x$

Type 2: where x=$sinax/cosax$

PI / y_p = $\frac{1}{f(D)}$.[x] = $\frac{1}{f(D)}$.$[sinax/cosax]$

Always write $f(D)$ into factorized form.

Replace every D^2 by $-a^2$provided the term does not become zero.

If on replacing the term becomes zero, we will use the formula of rationalisation and other important option multiply numerator by x and simultaneously differentiate denominator i.e $f(D)$ once.

Example1:- $(D^2 - 5D + 6)y$ =Sin3x

Solution:-

$\therefore$ PI=$\frac{1}{f(D)}$[x]

$\therefore \mathrm{PI} = \left(\frac{1}{D^2 - 5D + 6}\right) [sin3x]$

Put $D^2 = -9$

$\therefore \mathrm{PI} = \left(\frac{1}{-3-5D}\right) [sin3x]$

$\therefore \mathrm{PI} = \left(\frac{-3+5D}{9-25D^2}\right) [sin3x]$

Put $D^2 = -9$

$\therefore \mathrm{PI} = \left(\frac{-3+5D}{234}\right) [sin3x]$

$\therefore \mathrm{PI} = \left(\frac{-3}{234}\right) [sin3x] + \left(\frac{5D}{234}\right) [sin3x]$

$\therefore \mathrm{PI} = \left(\frac{-3}{234}\right) [sin3x] + \left(\frac{15}{234}\right) [cos3x]$

Example2:- $(D^4 - 1)y = e^x + cosx.cos3x$

Solution:-

$\therefore$ PI=$\frac{1}{f(D)}$[x]

$\therefore$ PI=$\left(\frac{1}{(D^4-1)}\right)[e^x + cosx.cos3x]$

$\therefore$ PI=$\left(\frac{1}{(D^4-1)}\right)[e^x + \frac{1}{2}(cos4x + cos2x)]$

$\therefore$PI=$\frac{1}{2}\left(\frac{1}{(D^2+1)(D-1)(D+1)}\right)[cos4x]+\frac{1}{2}\left(\frac{1}{(D^2+1)(D-1)(D+1)}\right)[cos2x]+$ $\frac{1}{2}\left(\frac{1}{(D^2+1)(D-1)(D+1)}\right)[e^x]$

[since roots of AE are$\pm1,\pm$i|

$\therefore$PI= $\frac{x}{1!}(\frac{1.e^x}{4}) + \frac{1}{2}\left(\frac{1}{D^{2^2}-1}\right)[cos4x] + \frac{1}{2}\left(\frac{1}{D^{2^2}-1}\right)[cos2x]$

$\therefore$PI= $(\frac{x.e^x}{4}) + \frac{1}{2}\left(\frac{1}{256-1}\right)[cos4x] + \frac{1}{2}\left(\frac{1}{16-1}\right)[cos2x]$

$\therefore$PI= $(\frac{x.e^x}{4}) + \left(\frac{1}{510}\right)[cos4x] + \left(\frac{1}{30}\right)[cos2x]$

Example 3: -(D^3 + D)y = sinx

Solution:-

PI=$\frac{1}{f(D)}$[x]= $\frac{1}{(D^3 + \mathrm{D})}$[sinx]

$\therefore$ PI=$\frac{1}{\mathrm{D}(D^2+1)}$[sinx]

$\therefore$ PI=$\frac{1..x}{\mathrm{D}(2\mathrm{D})}$[sinx]

$\therefore$ PI=$\frac{.x}{2D^2}$[sinx]

$\therefore$ PI=$\frac{.x}{2(-1)}$[sinx] =$\frac{-.x}{2}$sinx

Type 3: where x=x^m

PI / y_p = $\frac{1}{f(D)}$.[x^m]

Here $f(D)$ =1$\pm\phi(D)$

Example: $f(D) = D^4+D^3 + D^2$ +4 = 4(1+$\frac{D^4+D^3+D^2}{4}$)

We use $(1+t)^{-1}$=1-t+t^2-t^3 + - - -

$(1-t)^{-1}$=1+t+t^2+t^3 + - - - i.e Formulas in this type

So PI / y_p = $\frac{x^m}{1\pm\varphi(D)}$ = $(1\pm\varphi(D))^{-1}$.[x^m]

Example 1: $(D^3 + 2D^2 + D)y$ = $[x^2 + x]$

Solution:- PI / y_p = $\frac{1}{f(D)}$.[x]

$\therefore$ PI=$\frac{1}{((D^3+2D^2+D)}[x^2+x]$

$\therefore$ PI=$\frac{1}{((D^3+2D^2+D)}[x^2] + \frac{1}{((D^3+2D^2+D)}[x]$

$\therefore$ PI=$\frac{1}{D}.(1+2D+D^2)^{-1}.[x^2] + \frac{1}{D}.(1+2D+D^2)^{-1}.[x]$

$\therefore$ PI=$\frac{1}{D}$[1-$(2D+D^2)$+ $(2D+D^2)^2 - - -].[x^2]$ +$\frac{1}{D}$[1-$(2D+D^2)$+ $- - -$][x]

$\therefore$ PI=$\frac{1}{D}$[1-$2D - D^2$+ $(4D^2)^2.[x^2] + \frac{1}{D}$[1-$2D - D^2$][x]

$\therefore$ PI = $\frac{1}{D}$[1-x^2 - $4x - 2 + 4(2)$]+ $\frac{1}{D}$[x$-$2]

$\therefore$ PI =$\frac{x^3}{3} - \frac{4x^2}{2} + 6x + \frac{x^2}{2} - 2x$

$\therefore$ PI =$\frac{x^3}{3} - \frac{3x^2}{2} + 4x$

Example 2: $(D^3 - 2D + 4)y$ = $[3x^2 - 5x + 2]$

Solution:- PI / y_p = $\frac{1}{f(D)}$.[x]

$\therefore \text{PI}=\frac{1}{(D^3-2D+4)}[3x^2-5x+2]$

$\therefore \text{PI}=\frac{1}{4(1+\frac{(D^3-2D)}{4})}[3x^2]-\frac{1}{4(1+\frac{(D^3-2D)}{4})}[5x]+\frac{1}{(D^3-2D+4)}[2e^0]$

$\therefore \text{PI}=\frac{3}{4}.(1+\frac{(D^3-2D)}{4})^{-1}[x^2]-\frac{5}{4}.(1+\frac{(D^3-2D)}{4})^{-1}[x]+\frac{2}{4}$

$\therefore \text{PI}=\frac{3}{4}.(1-\frac{(D^3-2D)}{4}+\left[\frac{D^3-2D}{4}\right]^2)[x^2]-\frac{5}{4}.(1+\frac{(D^3-2D)}{4})[x]+\frac{2}{4}$

$\therefore \text{PI}=\frac{3}{4}.(1+\frac{2D}{4}+\frac{4D^2}{16})[x^2]-\frac{5}{4}.(1+\frac{2D}{4})[x]+\frac{2}{4}$

$\therefore \text{PI}=\frac{3}{4}.(x^2+x+\frac{1}{2})-\frac{5}{4}.(x+\frac{1}{2})+\frac{2}{4}$

$\therefore \text{PI}=\frac{3}{4}x^2+\frac{3}{4}x+\frac{3}{8}-\frac{5}{4}x-\frac{5}{8}+\frac{4}{8}$

$\therefore \text{PI}=\frac{3}{4}x^2-\frac{2}{4}x+\frac{2}{8}$

$\therefore \text{PI}=\frac{1}{4}.(3x^2-2x+\frac{2}{2})$

$\therefore \text{PI}=\frac{1}{4}.(3x^2-2x+1)$

Type 4: where x= e^{ax}.V

Where V= x^m,sinax,---

PI / y_p = $\frac{1}{f(D)}$.[x] = $\frac{e^{ax}.\text{V}}{f(D)}$

Put D=D+a

$\therefore \text{PI}=\frac{e^{ax}.\text{V}}{f(D+a)}$

Example 1: $(D^2-3D+2)y = x^2.e^{2x}$

Solution:- PI / y_p = $\frac{1}{f(D)}$.[x]

$\therefore$ PI=$\frac{1}{(D^2-3D+2)}\,[x^2.\,e^{2x}]$

Put D=D+2

$\therefore$ PI= $e^{2x}.\frac{1}{(D+2)^2-3(D+2)+2}.\,[x^2]$

$\therefore$ PI= $e^{2x}.\frac{1}{D^2+4+4D-3D+2-6}.\,[x^2]$

$\therefore$ PI= $e^{2x}.\frac{1}{D^2+D}.\,[x^2]$

$\therefore$ PI= $e^{2x}\left[\frac{1}{D(D+1)}\right][x^2]$

$\therefore$ PI= $e^{2x}\left[\frac{1}{D}(1+D)^{-1}\right][x^2]$

$\therefore$ PI= $e^{2x}.\left[\frac{1}{D}(1-D+D^2)\right][x^2]$

$\therefore$ PI= $e^{2x}.\left[\frac{1}{D}(x^2-2x+2)\right]$

$\therefore$ PI= $e^{2x}.\left[\frac{x^3}{3}-\frac{2x^2}{2}+2\right]$

Example 2: $(D^2-3D+2)y$ = $2e^x\cos\frac{x}{2}$

Solution:- PI / y_p = $\frac{1}{f(D)}$.[x]

$\therefore$ PI=$\frac{1}{(D^2-3D+2)}\,[2e^x\cos\frac{x}{2}]$

Put D=D+2

$\therefore$ PI=$2e^x.\frac{1}{(D+1)^2-3(D+1)+2}.\,[\cos\frac{x}{2}]$

$\therefore$ PI=$2e^x.\frac{1}{D^2+1+2D-3D+2-3}.\,[\cos\frac{x}{2}]$

$\therefore$ PI=$2e^x.\frac{1}{D^2-D}.\,[\cos\frac{x}{2}]$

Put D^2=$-\frac{1}{4}$

$\therefore$ PI=$2e^x.\frac{1}{-\frac{1}{4}-D}.\,[\cos\frac{x}{2}]$

$\therefore$ PI=$8e^x.\frac{1}{-1-4D}.[\cos\frac{x}{2}]$

$\therefore$ PI=$-8e^x.\frac{(1-4D)}{(1+4D)\ (1-4D)}.[\cos\frac{x}{2}]$

$\therefore$ PI=$-8e^x.\frac{(1-4D)}{(1-16D^2)}.[\cos\frac{x}{2}]$

Put D^2=$-\frac{1}{4}$

$\therefore$ PI=$-8e^x.\frac{(\cos\frac{x}{2}+\frac{4}{8}sin\frac{x}{2})}{(1+4)}$

$\therefore$ PI= $-\frac{8}{5}e^x(\cos\frac{x}{2}+2sin\frac{x}{2})$

Example based on mix of all types:-

Example:- (D^3 - D) y = 1- 4cosx +2x +2^x find solution

Solution:- A.E. is D^3 - D = 0

$\therefore$ roots are D = $1, -1, 0$

$\therefore$ CF $= C_1e^x + C_2e^{-x} + C_3$ —(i)

Now$\therefore$ PI=$\frac{1}{(D^3-D)}[1.e^0]$- $\frac{1}{(D^3-D)}[4cosx]$ +$\frac{1}{(D^3-D)}[2x]$+ $\boxed{\frac{1}{(D^3-D)}[e^{xlog2}]}$

$\rightarrow D^2$=xlog2

$\therefore$ PI=$\frac{x}{(3D^4-1)}[e^0]$ - $\frac{1}{D(D^2-1)}[4cosx]$ +$\frac{1}{D(D^2-2)}[2x]$+ $\frac{1}{(log2^3-log2)}[e^{xlog2}]$

$\therefore$ PI=- x - $\frac{[4cosx]}{D(-2)}$ + $\frac{1}{-D(1-D^2)}[2x]$+ $\frac{2^x}{(log2^3-log2)}$

$\therefore$ PI=- x +2sinx - $\frac{1}{D}(1-D^2)^{-1}[2x]$+ $\frac{2^x}{(log2^3-log2)}$

$\therefore$ PI=- x +2sinx - $\frac{1}{D}(1+D^2---)[2x]$+ $\frac{2^x}{(log2^3-log2)}$

$\therefore$ PI=- x +2sinx - $\frac{1}{D}2x]$+ $\frac{2^x}{(log2^3-log2)}$

$\therefore$ PI=- x +2sinx - x^2 + $\frac{2^x}{(log2^3-log2)}$ ---(ii)

$\therefore$from (i) and (ii)

Solution is y= $y_c + y_p$ /CF+PI

Type 5: Variation of parameter method

where x=x.V

step 1 : find CF

step 2 : replace e^{m_1x}&e^{m_2x} as y_1&y_2

step 3 : D= $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

step 4 : Now PI=u y_1+v y_2

Where U=$\int \frac{-y_2 x}{D}$dx

& V=$\int \frac{y_1 x}{D}$dx

Thus on adding CF and PI we get the solution.

Example 1: $(D^2 + 3D + 2)y = e^{e^x}$

Solution:-8 A.E. is $D^2 + 3D + 2 = 0$

$\therefore$ roots are D = $-1, 2$

$\therefore \text{CF} = C_1 e^{-x} + C_2 e^{-2x}$—(i)

Now D= $\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -e^{-2x} \end{vmatrix}$

$\therefore$ D=$-e^{-3x}$

Now PI=u e^{-x}+v e^{-2x}

$\therefore$V=$\int \frac{y_1 x}{D}$dx = $-\int \frac{e^{-x}.e^{e^x}}{e^{-3x}}$dx

$\therefore$V=$-\int e^{-2x}.e^{e^x}$dx

put. e^x=t

$\therefore. e^x \text{dx} = \text{dt}$

$\therefore$ V=$-\int t.e^t$dt

$\therefore$ V=$-$[$t\int e^t - \int(\frac{d}{dx} + \int.e^t$dt) dt]

$\therefore$V=$-e^t(t-1)$

$\therefore$V=$-e^{e^x}(1-e^x)$

$\therefore$U=$\int \frac{e^{-2x}e^{e^x}}{-e^{-3x}}$dx

$\therefore$U=$\int e^x e^{e^x}$dx

Put $e^x = t$

$\therefore e^x$dx=dt

$\therefore$U=$\int e^t$dt =e^t

$\therefore$U= e^{e^x}

$\therefore$ PI=$e^{e^x}.e^{-x} - e^{e^x}(1-e^x)e^{-2x}$ –(ii)

$\therefore$from (i) and (ii)

Solution is y= CF+PI

Example 2: $(D^2 - 2D + 2)y = e^x$tanx

Solution:- A.E. is $D^2 - 2D + 2$= 0

$\therefore$roots are D = $1, \pm i$

$\therefore$CF =$e^x(\ C_1 cosx + C_2 sinx)$—(i)

Now D= $\begin{vmatrix} e^x\text{cosx} & e^x\text{sinx} \\ e^x(\text{cosx} - \text{sinx}) & e^x(sinx + cosx) \end{vmatrix}$

$\therefore$ D = $e^{2x}(sinxcosx + cos^2x)$- $e^{2x}(sinxcosx + sin^2x)$

$\therefore$ D = $e^{2x}(sin^2x + cos^2x)$

$\therefore$ D=e^{2x}

Now PI=u e^xcosx + v e^xsinx

$\therefore$ V=$\int \frac{y_1 x}{D}$dx = $-\int \frac{e^x \text{cosx} . e^x \text{tanx}}{e^{2x}}$dx = $\int sinx$dx

$\therefore$ V=$-cosx$

U=$\int \frac{-y_2 x}{D}$dx

$\therefore$ U=$-\int \frac{sin^2 x}{cosx}$dx

$\therefore$ U=$\int \frac{1-cos^2 x}{cosx}$dx

$\therefore$ U=$-\int secx$dx+$\int cosx$dx

$\therefore$ U= $-\log(secx + tanx)$ + sinx

$\therefore$ PI=$-\log(secx + tanx)e^x\text{cosx} + \text{sinx}(e^x\text{cosx}) - \text{sinx}(e^x\text{cosx})$ –(ii)

$\therefore$from (i) and (ii)

Solution is y=CF+PI

7.Numerical Differentiation

1. Eulers Method:-

Steps → some function f(x,y) will be given

→ $[y_{x_0} = y_0]$ identify x_0 and y_0 by comparing

→ $find\ h\ by\ formula\ h = \frac{x - x_0}{n}$

→ $Find\ y\ at\ given\ x\ by\ formula\ y_{n+1} = y_n + hf(x_n, y_n)$

→ $x_n = x_{n+1} + h$

Example 1: $\frac{dy}{dx} = \frac{y-x}{x}$ & $y_1 = 2\ taking\ h = 0.2\ , find\ y\ at\ x = 2$

Solution On comparing $y_{x_0} = y_0\ , x_0 = 1\ , y_0 = 2\ , h = 0.2$

f(x,y)= $\frac{y-x}{x}$ so

x_n	y_n	$y_{n+1} = y_n + hf(x_n, y_n)$
x_0=1	y_0=2	$y_1 = y_0 + hf(x_0, y_0)$ =.2.2
x_1=1.2	y_1=2.2	$y_2 = y_1 + hf(x_1, y_1)$ = 2.366
x_2=1.4	y_2=2.306	$y_3 = y_2 + hf(x_2, y_2)$ =2.504
x_3=1.6	y_3=2.504	$y_4 = y_3 + hf(x_3, y_3)$ = 2.617
x_4=1.8	y_4=2.617	$y_5 = y_4 + hf(x_4, y_4)$ = 2.707
x_5=2	y_5=2.707	

So $y\ at\ x = 2\ is\ 2.707$

Example 2: $\frac{dy}{dx} = \frac{y-x}{y+x}$ & $y_0 = 1\ \ in\ 5\ steps\ , find\ aproximate\ value\ at\ x = 0.1$

Solution On comparing $y_{x_0} = y_0\ , x_0 = 0\ , y_0 = 1$

$$h = \frac{0.1 - 0}{5} = 0.02$$

x_n	y_n	$y_{n+1} = y_n + hf(x_n, y_n)$
x_0=0	y_0=1	$y_1 = y_0 + hf(x_0, y_0)$ =1.02
x_1=0.02	y_1=1.02	$y_2 = y_1 + hf(x_1, y_1)$ = 1.0392
x_2=0.04	y_2=1.0392	$y_3 = y_2 + hf(x_2, y_2)$ =1.0577
x_3=0.06	y_3=1.0577	$y_4 = y_3 + hf(x_3, y_3)$ = 1.0755
x_4=0.08	y_4=1.0755	$y_5 = y_4 + hf(x_4, y_4)$ = 1.0927
x_5=0.1	y_5=1.0927	

So y=1.0927 is the approximate value

2. Eulers modified Method / RK method of 2nd order :

Steps → some function f(x,y) will be given

→ $[y_{x_0} = y_0\,]$ identify x_0 and y_0 by comparing

→ $find\ h\ by\ formula\ h = \dfrac{x - x_0}{n}$

→ $to\ find\ solution\ at\ x_1\ and\ x_2\ ,$

Solution at $\boldsymbol{x_1}$	Solution at $\boldsymbol{x_2}$
Find $\boldsymbol{y_1 = y_0 + hf(x_0, y_0)}$	Find $y_2 = y_1 + hf(x_1, y_1)$
$\boldsymbol{y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]}$	$\boldsymbol{y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]}$
$\boldsymbol{y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)]}$	$\boldsymbol{y_2^{(2)} = y_2 + \frac{h}{2}[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)]}$
↓	↓
Till answers starts repeating	**Till answers starts repeating**

Example 1: $\frac{dy}{dx} = log(x + y)\ \boldsymbol{at} y_1 = 2$. find the value of y for x=1.2 and x=1.4 ; taking h=0.2

Solution

On comparing $y_{x_0} = y_0\ ,\ x_0 = 1\ ,\ y_0 = 2\ ,\ h = 0.2$

Solution at $\boldsymbol{x_1}$=1.2	Solution at $\boldsymbol{x_2}$=1.4
Find $\boldsymbol{y_1 = y_0 + hf(x_0, y_0)}$=2.2197	Find $y_2 = y_1 + hf(x_1, y_1)$=2.4799
$\boldsymbol{y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]}$=2.2328	$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$=2.4921
$\boldsymbol{y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)]}$=2.2332	$y_2^{(2)} = y_2 + \frac{h}{2}[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)]$=2.4924
$\boldsymbol{y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)]}$=2.2332	$y_2^{(3)} = y_2 + \frac{h}{2}[f(x_1, y_1) + f\left(x_2, y_2^{(2)}\right)]$=2.4924

So $y_1 = 2.2332\ and\ y_1 = 2.4924$

Example 2: $\frac{dy}{dx} = y - \frac{2x}{y}\ taking\ h = 0.1\ , find\ y\ at\ x = 0.1$

Solution at $\boldsymbol{x_1}$=0.1

Find $\boldsymbol{y_1 = y_0 + hf(x_0, y_0)}$=1.1

$\boldsymbol{y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]}$=1.0959

$\boldsymbol{y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)]}$=1.0957

$\boldsymbol{y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)]}$=1.0957

So y=1.0957 at x=0.1

3. RK method of 3rd order :

Steps → some function f(x,y) will be given

→ $[y_{x_0} = y_0]$ identify x_0 and y_0 by comparing

→ $find\ h\ by\ formula\ h = \frac{x - x_0}{n}$

→ $find\ y\ at\ given\ x$

Method $first\ find\ k1, k2, k3, k4\ by$

$k1 = hf(x_0, y_0)$

$k2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k1}{2}\right)$

$k3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k2}{2}\right)$

$k4 = hf\left(x_0 + \frac{h}{1}, y_0 + \frac{k3}{1}\right)$

Then find $k = \frac{1}{6}[k1 + 2(k2) + 2(k3) + k4]$

$so\ y_0 + k = y_1 \rightarrow y_n + k = y_{n+1}$

Example 1:$\frac{dy}{dx} = xy\ at\ y_1 = 1\ taking\ h = 0.1\ find\ y_1\ for x_1\ and\ y_2\ for\ x_2$

Solution $x_1 = x_0 + h = 1.1\ and\ x_2 = x_1 + h = 1.2$

For $x_1 = 1.1,$

$k1 = hf(x_0, y_0)$=0.1

$k2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k1}{2}\right)$=0.1103

$k3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k2}{2}\right)$=0.1108

$k4 = hf\left(x_0 + \frac{h}{1}, y_0 + \frac{k3}{1}\right)$=0.1221

Then find $k = \frac{1}{6}[k1 + 2(k2) + 2(k3) + k4]$=0.1107

So y_1=$y_0 + k = 1.1107$

For $x_2 = 1.2,$

$k1 = hf(x_1, y_1)$=0.1222

$k2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k1}{2}\right)$=0.1348

$k3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k2}{2}\right)$=0.1355

$k4 = hf\left(x_1 + \frac{h}{1}, y_1 + \frac{k3}{1}\right)$=0.1495

Then find $k = \frac{1}{6}[k1 + 2(k2) + 2(k3) + k4]$=0.1355

So y_2=$y_1 + k = 1.2462$

Example 2: $\frac{dy}{dx} = x + y^2$ $at\ y_0 = 1\ taking\ h = 0.1\ finf\ value\ of\ y\ at\ x = 0.1$

Solution $x_1 = x_0 + h$ =0+0.1=0.1

Now solution for $x_1 = 0.1$ is,

$k1 = hf(x_0, y_0)$=0.1

$k2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k1}{2}\right)$=0.11525

$k3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k2}{2}\right)$=0.11686

$k4 = hf\left(x_0 + \frac{h}{1}, y_0 + \frac{k3}{1}\right)$=0.13474

Then find $k = \frac{1}{6}[k1 + 2(k2) + 2(k3) + k4]$=0.1165

So y_1=$y_0 + k = 1.1165$

4. Taylors Series :

Steps → some function f(x,y) will be given

→ $[y_{x_0} = y_0]$ identify x_0 and y_0 by comparing & we will have to find y at given x

$$\rightarrow in\ taylor\ series\ h = x - x_0\ and\ y\ at\ given\ x\ is\ given\ by\ ,$$

$$y = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\prime v}$$

Note : check $\frac{h^n}{n!}y_0^n$ periodically to find the answers upto specified decimal places

Example : using Taylor's series method solve $\frac{dy}{dx} = 2y + 3e^x$ $with\ y_0 = 0\ when\ x_0 = 0\ for\ x = 0.1,0.2$

Solution

Taylor series is given by

$y = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\prime v}$+.......

We have $x_0 = 0$ &$y_0 = 0$

$y' = 2y + 3e^x$ so y_0'=3

$y'' = 2y' + 3e^x$ so $y_0'' = 9$

$y''' = 2y'' + 3e^x$ so y_0'''=21

$y^{\prime v} = 2y''' + 3e^x$ so $y_0^{\prime v}$=45

Therefore on substituting $y = 0 + 3x + 4.5x^2 + 4.5x^3 + 1.875x^4 + \cdots$

(a) When x=0.1 , y=0.34869 &
(b) When x=0.2 , y=0.8110

8.Numerical Integration

$$I = \int_{x=a}^{x=b} [y = f(x)]dx$$

↓

We divide interval [a…..b] into n-equal sub intervals and width of interval is given by $h = \frac{x_n - x_a}{n}$

↓

It also means there will be n+1 ordinates

↓

$x_1 = x_0 + h$

$x_2 = x_1 + h$ ≫≫≫ by substituting in $y = f(x)$ *we get* y_0, y_1, y_2, y_3 ….

$x_3 = x_2 + h$

……

Value of I given by different methods are :-

<1> trapezoidal method:-

$$\mathrm{I} = \frac{h}{2}[(y_0 + y_n) + 2(remaining\ ordinate)]$$

$$\mathrm{I} = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots . + y_{n-1})]$$

<2> sympsons 1/3rd rule:-

$$\mathrm{I} = \frac{h}{3}[(y_0 + y_n) + 2(even\ ordinate) + 4(odd\ ordinate)]$$

$$\mathrm{I} = \frac{h}{2}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + \cdots)]$$

<3> sympsons 3/8rd rule:-

$$\mathrm{I} = \frac{3h}{8}[(y_0 + y_n) + 2(multiple\ of\ 3\ ordinate) + 3(remaining\ ordinate)]$$

$$\mathrm{I} = \frac{h}{2}[(y_0 + y_n) + 2(y_3 + y_6 + \cdots) + 3(y_1 + y_2 + y_4 + \cdots)]$$

Example 1: the water under portion of a water tank is divided by horizontal planes one meter apart into the following areas given in table . Use trapezoidal rule to find volume in cubic meters to extreme ends

Distance (S)	1	2	3	4	5	6	7	8	9
Area(A)	472	398	302	198	116	60	34	12	4

Solution: h=1 ,n=9

Therefore volume by trapezoidal rule is

$$\mathrm{I} = \frac{h}{2}[(y_0 + y_n) + 2(remaining\ ordinate)]$$

$$\mathrm{I} = \frac{h}{2}[(472 + 4) + 2(398 + 302 + 198 + 116 + 60 + 34 + 12)]$$

$$\mathrm{I} = \frac{1}{2}[2716] = 1358\ cubic\ units$$

Example 2: the velocity of train which starts from rest is given by following table, the time being recorded in minuets from start and speed in km/hr.

Time	0	2	4	6	8	10	12	14	16	18	20	
peed	0	10	18	25	29	32	30	11	5	2	0	
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	

Find total distance run in 20 minuets by sympsons 1/3rd rule

Solution: h=$\frac{2}{60} = \frac{1}{30}$

Therefore by Simpson's 1/3rd rule

$$\mathrm{S} = \frac{h}{3}[(y_0 + y_n) + 2(even\ ordinate) + 4(odd\ ordinate)]$$

$$\mathrm{S} = \frac{1}{90}[(y_0 + y_{10}) + 2(even\ ordinate) + 4(odd\ ordinate)]$$

$\mathrm{S} = \frac{1}{90}[0 + 164 + 320]$=5.378 kms

Example 3:

Time	0	3	6	9	12	15	18
Velocity	0	22	29	31	20	4	0
ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

From the above table find the approximate-distance covered in 18 minutes by Simpson's 3/8th rule

Solution: h=$\frac{3}{60} = \frac{1}{20} hrs$

Therefore according to sympsons 3/8rd rule:-

$$\mathrm{I} = \frac{3h}{8}[(y_0 + y_n) + 2(multiple\ of\ 3\ ordinate) + 3(remaining\ ordinate)]$$

$\mathrm{I} = \frac{3}{160}[(0) + 2(31) + 3(22 + 29 + 20 + 4)]$=5.38 meter.

Example 4: A curve is given by the table . find the area by Simpson's 3/8th rule.

X	0	1	2	3	4	5	6
Y	0	2	2.5	2.3	2	1.7	1.5
ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Solution:

according to sympsons 3/8rd rule:-

$$\mathrm{I} = \frac{3(h)}{8}[(y_0 + y_n) + 2(multiple\ of\ 3\ ordinate) + 3(remaining\ ordinate)]$$

$$\mathrm{I} = \frac{3(1)}{8}[(1.5) + 2(2.3) + 3(8.2)]\text{=11.5125 } unit^2$$

Example 5: $\int_0^1 \frac{dx}{1+x^2}$ where h=1/4 (using)→ trapezoidal method, sympsons 1/3rd rule, sympsons 3/8rd rule.

Solution : a=$x_0 = 0\ and\ b = x_n = 1$

$x_1 = x_0 + h$=1/4 $\quad y_0 = \frac{1}{1+x_0^2} = 1$

$x_2 = x_1 + h$ =1/2 $\quad y_1 = \frac{1}{1+x_1^2} = 16/17$

$x_3 = x_2 + h$=3/4 $\quad y_2 = \frac{1}{1+x_2^2}$ =4/5

$x_4 = x_3 + h$ =1 $\quad y_3 = \frac{1}{1+x_3^2}$ =16/25

$y_4 = \frac{1}{1+x_4^2}$ =1/2

trapezoidal method:-

$$\mathrm{I} = \frac{h}{2}[(y_0 + y_n) + 2(remaining\ ordinate)]\text{= 0.7828}$$

sympsons 1/3rd rule:-

$$\mathrm{I} = \frac{h}{3}[(y_0 + y_n) + 2(even\ ordinate) + 4(odd\ ordinate)]\text{=0.7854}$$

sympsons 3/8rd rule:-

$$\mathrm{I} = \frac{3h}{8}[(y_0 + y_n) + 2(multiple\ of\ 3\ ordinate) + 3(remaining\ ordinate)]\text{=0.7503}$$

Example 6: $\int_0^6 \frac{dx}{1+3x}$ where h=1 (using)→ trapezoidal method, sympsons 1/3rd rule, sympsons 3/8rd rule.

Solution : a=$x_0 = 0 , b = x_n = 6$ and $y = \frac{1}{1+3x}$

	x	$y = \frac{1}{1+3x}$	
x_0	0	1	y_0
x_1	1	0.25	y_1
x_2	2	0.1428	y_2

x_3	3	0.1	y_3
x_4	4	0.0769	y_4
x_5	5	0.0625	y_5
x_6	6	0.0526	y_6

trapezoidal method:-

I $= \frac{h}{2}[(y_0 + y_n) + 2(remaining\ ordinate)]$= 1.1585

sympsons 1/3rd rule:-

I $= \frac{h}{3}[(y_0 + y_n) + 2(even\ ordinate) + 4(odd\ ordinate)]$=1.0473

sympsons 3/8rd rule:-

I $= \frac{3h}{8}[(y_0 + y_n) + 2(multiple\ of\ 3\ ordinate) + 3(remaining\ ordinate)]$=1.0685

9.Beta gama function

Definition: $\Gamma(n) = \int_{0}^{\infty} e^{-x}x^{n-1}dx$

Properties:-

(1) $\Gamma(n+1) = n\Gamma(n) \; or \; \Gamma(n) = ((n-1)\Gamma(n)$

(2) $\Gamma(n) = n! \;\; [only \; if \; n \; is \; a \; whole \; number]$

(3) $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

(4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(5) $\Gamma(n)\Gamma(m) = \frac{\pi}{sinm\pi \;\; or \;\; sinn\pi}$ [such that m+n=1 ; take the smaller term i.e m or in the denomitor]

(6) $\beta(m,n) = \beta(n,m)$

(7) $\beta(m,n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$

Examples:-

$\Gamma(3) = 2!$

$\Gamma\left(\frac{5}{2}\right) = (\frac{3}{2})(\frac{1}{2})\;\Gamma(\frac{1}{2})$

$\Gamma\left(\frac{7}{4}\right) = (\frac{3}{4})\;\Gamma(\frac{3}{4})$

Type 1:- for $\int_0^\infty e^{-ax^n}dx \;\; put \; ax^n = t \;\; so \; k \int_0^\infty e^{-t}t^{(n-1)}dt = k \;\; \Gamma(n)$

Examples

Example 1: I=$\int_0^\infty \sqrt{x}e^{-x^2}dx$

Solution:- $put \; x^2 = t \; so \; dx = \frac{1}{2}t^{-\frac{1}{2}}dt$

X	0	∞
t	0	∞

So I=$\frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{4}}dt = (\frac{1}{2})\;\Gamma(\frac{3}{4})$

Example 2: I=$[\int_0^\infty xe^{-x^8}dx] \times [\int_0^\infty x^2e^{-x^4}dx] = \frac{\pi}{16\sqrt{2}}$

Solution:- $I_1 = \int_0^\infty xe^{-x^8}dx \; put \; x^8 = t \; so \; dx = \frac{1}{8}t^{-\frac{7}{8}}dt$

$$I_1 = \frac{1}{8}\int_0^\infty t^{\frac{1}{8}}e^{-t}t^{-\frac{7}{8}}dt = \left(\frac{1}{8}\right)\Gamma\left(\frac{2}{8}\right) = \left(\frac{1}{8}\right)\Gamma\left(\frac{1}{4}\right)$$

X	0	∞
t	0	∞

$I_2 = \int_0^\infty x^2e^{-x^4}dx \;\;\; \text{put } x^4 = t \; so \; dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

X	0	∞
t	0	∞

$$I_2 = \frac{1}{4}\int_0^\infty t^{\frac{2}{4}} e^{-t} t^{-\frac{3}{4}} dt = \left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

(1) So now $I_1 \times I_2 = \left(\frac{1}{8}\right)\Gamma\left(\frac{1}{4}\right) \times \left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{32}\frac{\pi}{\sin\frac{\pi}{4}}$.. ________ $\{since\ \Gamma(n)\Gamma(m) = \frac{\pi}{sinm\pi\ or\ sinn\pi}$ [such that m+n=1 ; take the smaller term i.e m or in the denomitor]

}

$$I_1 \times I_2 = \frac{1}{32}\sqrt{2}\pi = \frac{\pi}{16\sqrt{2}}$$

Example 3: I=$\int_0^\infty (x^2-4)^2 e^{-3x^2} dx$

Solution:-put $3x^2 = t \quad x = \frac{t^{\frac{1}{2}}}{\sqrt{3}}$ so $dx = \frac{1}{2}\frac{t^{-\frac{1}{2}}}{\sqrt{3}}\ dt$

X	0	∞
t	0	∞

I=$\int_0^\infty (\frac{t}{3} - 4)^2 e^{-t} \frac{1}{2\sqrt{3}} t^{-\frac{1}{2}} dt = \frac{1}{2\sqrt{3}}\int_0^\infty (\frac{t^2}{9} - \frac{8t}{3} + 16) \cdot t^{-\frac{1}{2}} e^{-t}$dt

I=$\frac{1}{2\sqrt{3}}\int_0^\infty (\frac{t^{\frac{3}{2}}}{9} - \frac{8t^{\frac{1}{2}}}{3} + 16t^{-\frac{1}{2}}) \cdot t^{-\frac{1}{2}} e^{-t}$dt=$\frac{1}{2\sqrt{3}}[\left(\frac{1}{9}\right)\Gamma\left(\frac{5}{2}\right) - \left(\frac{8}{3}\right)\Gamma\left(\frac{3}{2}\right) + (16)\Gamma\left(\frac{1}{2}\right)]$

I=$\frac{1}{2\sqrt{3}}\ \Gamma\left(\frac{1}{2}\right)[\frac{1}{12} - \frac{4}{3} + 16]$

I=$\frac{\sqrt{\pi}}{2\sqrt{3}}\left[\frac{59}{8}\right]$

Type 2:- If $\int_0^1 (logx)^m x^n dx\ or\ \int_0^1 (log\frac{1}{x})^m x^n dx$

Then put logx=$-t$ and $log\frac{1}{x} = t$

So $\downarrow \qquad \downarrow$

$x = e^{-t} \qquad \frac{1}{x} = e^{-t}$

Thus it will get converted as $\int_0^\infty e^{-ax^t} t^m dx$

Thus put at=u and so on …

Note:- (a)$log0 = -\infty$

(b)$log1 = 0$

Example : I=$\int_0^1 (xlogx)^4 dx$

Solution :- put $logx = -t\ \ so\ x = e^{-t}\ \ so\ dx = -e^{-t}dt$

So $I = \int_\infty^0 e^{-4t}(-t)^4(-e^{-t})dt$

X	0	1
t	∞	0

$$I = -\int_\infty^0 e^{-5t} t^4 dt$$

$$= \int_0^\infty e^{-5t} t^4 dt$$

t	0	∞
U	0	∞

Put $5t = u\ \ so\ \ t = \frac{u}{5}\quad dt = \frac{du}{5}$

$$I = \int_0^\infty e^{-u}\left(\frac{u}{5}\right)^4 \frac{du}{5}$$

$$I = \left(\frac{1}{5}\right)^5 \int_0^\infty e^{-u}u^4 du = \frac{\Gamma(5)}{(5)^5} = \frac{4!}{3125} = \frac{24}{3125}$$

Type 3:-

IF	Then
I=$\int_0^\infty e^{-ax}cosbxx^{n-1}dx$	RP$\int_0^\infty e^{-ax}x^{n-1}e^{ibx}$
I=$\int_0^\infty e^{-ax}sinbxx^{n-1}dx$	IP$\int_0^\infty e^{-ax}x^{n-1}e^{ibx}$
I=$\int_0^\infty cosbxx^{n-1}dx$	RP$\int_0^\infty e^{-ibx}x^{n-1}dx$
I=$\int_0^\infty sinbxx^{n-1}dx$	IP$\int_0^\infty e^{-ibx}x^{n-1}dx$
I=$\int_0^\infty \sin(cosax^{\frac{1}{n}})dx$	(put angle=t)

Example 1: I=$\int_0^\infty e^{-ax}x^{n-1}cosbxdx$

Solution :- I=RP of I_1

$$I_1 = RP\int_0^\infty e^{-ax}x^{n-1}e^{ibx}\,dx = RP\int_0^\infty e^{-(ax-ibx)}x^{n-1}dx$$

Put (a-ib)x=t → $x = \frac{t}{(a-ib)}$ $dx = \frac{dt}{(a-ib)}$

X	0	∞
t	0	∞

$$I_1 = \int_0^\infty e^{-t}\left(\frac{t}{a-ib}\right)^{n-1}\frac{dt}{(a-ib)}$$

$$I_1 = \int_0^\infty e^{-t}(t)^{n-1}\frac{1}{(a-ib)^n}dt$$

$$I_1 = \frac{\Gamma(n)}{(a-ib)^n}$$

[for converting it into polar form , put a=rcosθ anb b=rsinθ]

[r=$\sqrt{a^2+b^2}$ and $\theta = tan^{-1}(\frac{b}{a})$]

$$I_1 = \frac{\Gamma(n)}{r^n}[cosn\theta + isinn\theta]$$

$I_1 = \frac{\Gamma(n)}{r^n}[\cos(ntan^{-1}\left(\frac{b}{a}\right)) + isin(ntan^{-1}\left(\frac{b}{a}\right))]$ __[where $r^n = (\sqrt{a^2+b^2}\ )^n$]

Thus on substituting

I=RP of $I_1 = \frac{\Gamma(n)}{(\sqrt{a^2+b^2}\ )^n}[\cos\left(ntan^{-1}\left(\frac{b}{a}\right)\right)]$

Note :- I=$\int_0^\infty e^{-ax} x cosbx dx$

Then solution:- I=RP ofI_1

$$I_1 = \frac{\Gamma(2)}{(a-ib)^2} = \frac{1!(a+ib)^2}{(a-ib)^2(a+ib)^2} = \frac{a^2+i^2b^2+2abi}{[a^2-(bi)^2]^2} = \frac{(a^2-b^2)+2abi}{(a^2+b^2)^2} = \frac{(a^2-b^2)}{(a^2+b^2)^2}$$

I=RP of$I_1 = \frac{(a^2-b^2)}{(a^2+b^2)^2}$

Example 2 : show that I=$\int_0^\infty \cos\left(ax^{\frac{1}{n}}\right) dx = \frac{1}{a^n}\Gamma(n+1)\cos\left(\frac{n\pi}{2}\right)$

Solution:- put $ax^{\frac{1}{n}} = t \ \rightarrow x^{\frac{1}{n}} = \frac{t}{a} \rightarrow x = \frac{t^n}{a^n} \quad so\ dx = \frac{nt^{n-1}}{a^n} dt$

X	0	∞
t	0	∞

$$I = \int_0^\infty \text{cost}\left(\frac{nt^{n-1}}{a^n}\right) dt$$

I= $\frac{n}{a^n}\int_0^\infty \text{cost}\,(t^{n-1}) dt$

So I= $\frac{n}{a^n} RP \int_0^\infty e^{-it} t^{n-1} dt$

Let $I_1 = \int_0^\infty e^{-it} t^{n-1} dt$ put it=u so t=$\frac{u}{i}$ $\rightarrow dt = -idu$

$$I_1 = \int_0^\infty e^{-u}\left(\frac{u}{i}\right)^{n-1}(-i) du$$

t	0	∞
u	0	∞

$$I_1 = (-i)^n \int_0^\infty e^{-u} u^{n-1} du = (-i)^n\, \Gamma(n)$$

$$I_1 = \left(cos\frac{\pi}{2} - isin\frac{\pi}{2}\right)^n \Gamma(n) = \left(cos\frac{n\pi}{2} - isin\frac{n\pi}{2}\right)\Gamma(n)$$

So on substituting I=$\frac{n}{a^n} RP I_1 = \frac{n}{a^n} cos\frac{n\pi}{2}\ \Gamma(n)$=$\frac{1}{a^n}\Gamma(n+1)\cos\left(\frac{n\pi}{2}\right)$

Type 4:-

(A)$\int_0^1 x^{m-1}(1-x)^{n-1} dx = \beta(m,n)$

(B)$\int_0^\infty \frac{x^{m-1}}{(1-x)^{m+n}} dx = \beta(m,n)$

(C)$\int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta \, d\theta = \frac{1}{2}\beta(\frac{p+1}{2},\frac{q+1}{2})$

Example 1:- I=$\int_0^1 \sqrt{1-x^4}\, dx$

Solution :- I=$\int_0^1 (1-x^4)^{\frac{1}{2}}dx$ $\quad$ $put\ x^4 = t \to x = t^{\frac{1}{4}}\ so\ dx = \frac{1}{4}t^{-\frac{3}{4}}\, dt$

X	0	1
t	0	1

I=$\frac{1}{4}\int_0^1 (1-t)^{\frac{1}{2}}\ t^{-\frac{3}{4}}\, dt$

I=$\frac{1}{4}\beta\left(\frac{3}{2},\frac{1}{4}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} = \frac{1}{4}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)} = \sqrt{\pi}\frac{1}{6}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{6}\frac{(\Gamma\left(\frac{1}{4}\right))^2}{\frac{\pi}{sin\frac{\pi}{4}}} = \frac{1}{12}\sqrt{\frac{2}{\pi}}(\Gamma\left(\frac{1}{4}\right))^2$

Example 2:- I=$\int_0^2 x\ \sqrt[3]{8-x^3}dx = \frac{16\pi}{9\sqrt{3}}$

Solution :- I=$\int_0^2 x\ (8-x^3)^{\frac{1}{3}}dx$ $\quad$ $put\ x^3 = 8t\ \to x = 2t^{\frac{1}{3}}\ so\ \ dx = 2-\frac{1}{3}t^{-\frac{2}{3}}dt$

X	0	2
t	0	1

I=$\int_0^1 2t^{\frac{1}{3}}(8-8t)^{\frac{1}{3}}\frac{2}{3}t^{-\frac{2}{3}}dt$

I=$\frac{8}{3}\int_0^1 t^{\frac{1}{3}}(1-t)^{\frac{1}{3}}t^{-\frac{2}{3}}dt$

I=$\frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right) = \frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)} = \frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{1!} = \frac{8}{9}\frac{\pi}{sin\frac{\pi}{3}} = \frac{16\pi}{9\sqrt{3}}$

Example 3:-I= $\int_0^\infty \frac{x^6}{(1+x^2)^6}dx$

Solution :- I=$\int_0^\infty \frac{x^6}{(1+x^2)^6}dx$ $\quad$ $put\ x^2 = t\ \to x = t^{\frac{1}{2}}\ so\ dx = \frac{1}{2}t^{-\frac{1}{2}}dt$

X	0	∞
t	0	∞

I=$\frac{1}{2}\int_0^\infty \frac{t^3 t^{-\frac{1}{2}}}{(1+t)^6}dt = \frac{1}{2}\int_0^\infty \frac{t^{\frac{5}{2}}}{(1+t)^6}dt = \frac{1}{2}\beta\left(\frac{7}{2},\frac{5}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(6)} = \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{5!} = \frac{3\pi}{512}$

Example 4:-I= $\int_0^\infty \frac{x^6}{(2+3x)^{15}}dx$

Solution:- $put\ 3x = 2t\ \to x = \frac{2}{3}t\ \ so\ dx = \frac{2}{3}dt$

X	0	∞
t	0	∞

I=$\frac{2}{3}\int_0^\infty \frac{(\frac{2}{3}t)^6}{2^{15}(1+t)^{15}}dt = \frac{1}{2^8.3^7}\beta(7,8) = \frac{6!7!}{2^8.3^7} = \frac{175}{27}$

Example 5:-I= $\int_0^\infty \frac{x^8-x^5}{(1+x^3)^5}dx$

X	0	∞
t	0	∞

Solution :- put $x^3 = t \;\rightarrow x = t^{\frac{1}{3}}\; so\; dx = \frac{1}{3}t^{-\frac{2}{3}}dt$

I=$\frac{1}{3}\int_0^\infty \frac{(t^{\frac{8}{3}}-t^{\frac{5}{3}})}{(1+t)^5}t^{-\frac{2}{3}}dt = \frac{1}{3}\int_0^\infty \frac{t^2}{(1+t)^5}t^{-\frac{2}{3}}dt - \frac{1}{3}\int_0^\infty \frac{t}{(1+t)^5}dt = \frac{1}{3}[\beta(3,2)-\beta(2,3)] = 0$

Example 6:- prove that $\int_0^{\frac{\pi}{2}} tan^n x dx = \frac{\pi}{2} sec\frac{n\pi}{2}$

Solution :-I=$\int_0^{\frac{\pi}{2}} \frac{sin^n x}{cos^n x}dx = \int_0^{\frac{\pi}{2}} sin^n x cos^{-n} x dx = \frac{1}{2}\beta(\frac{n+1}{2},\frac{1-n}{2})$

I=$\frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(1)}$=$\frac{1}{2}\frac{\pi}{sin\frac{(n-1)\pi}{2}} = \frac{1}{2}\frac{\pi}{\sin(\frac{n\pi}{2}-\frac{\pi}{2})} = \frac{\pi}{2}\frac{1}{cos\frac{n\pi}{2}} = \frac{\pi}{2}\; sec\frac{n\pi}{2}$

Example 7:-I=$\int_0^\pi \frac{sin^4\theta}{(1+cos\theta)^2}d\theta$

Solution :-

I=$\int_0^\pi \frac{(2sin\frac{\theta}{2}cos\frac{\theta}{2})^4}{(2cos^2\frac{\theta}{2})^2}d\theta$=4$\int_0^\pi sin^4\left(\frac{\theta}{2}\right)d\theta$

θ	0	π
t	0	$\frac{\pi}{2}$

put$\frac{\theta}{2} = t \;\rightarrow \theta = 2t\; so\; d\theta = 2dt$

I=8$\int_0^{\frac{\pi}{2}} sin^4(t)dt = 8\frac{1}{2}\beta\left(\frac{5}{2},\frac{1}{2}\right) = 4\frac{\frac{3}{2}\cdot\frac{1}{2}\pi}{2!} = \frac{3\pi}{2}$

Example 8:-$\int_0^{\frac{\pi}{6}}(sin6\theta)^2(cos3\theta)^3 d\theta$

Solution :-put $3\theta = t \rightarrow \theta = \frac{t}{3}\; so\; d\theta = \frac{dt}{3}$

θ	0	$\frac{\pi}{6}$
t	0	$\frac{\pi}{2}$

I=$\frac{1}{3}\int_0^{\frac{\pi}{2}}(sin2t)^2(cost)^3dt$

I=$\frac{1}{3}\int_0^{\frac{\pi}{2}}(2sintcost)^2(cost)^3dt$

I=$\frac{4}{3}\int_0^{\frac{\pi}{2}}(sint)^2(cost)^5dt$

I=$\frac{4}{3}.\frac{1}{2}\beta\left(\frac{3}{2},\frac{6}{2}\right) = \frac{2}{3}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{6}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = \frac{32}{315}$

Miscellaneous examples:-

Example 1:- P.T. $\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}} = \frac{\beta(p,q)}{(a+b)^p a^q}$

Solution :- put $t = \frac{(a+b)x}{(a+bx)} \to at + btx = ax + bx \to at = (a+b-bt)x \to x = \frac{at}{a+b-bt}$ _{1}

So $dx = \frac{(a+b-bt)a - at(-b)dt}{(a+b-bt)^2} = \frac{(a+b)adt}{(a+b-bt)^2}$ _{2}

Now from {1}

$1 - x = 1 - \frac{at}{a+b-bt} = \frac{(a+b)-(a+b)t}{a+b-bt} = \frac{(a+b)(1-t)}{a+b-bt}$ _{3}

Now $a + bx = a + b(\frac{at}{a+b-bt}) = \frac{a(a+b)-abt+abt}{a+b-bt} = \frac{a(a+b)}{a+b-bt}$ _{4}

So on substituting {1},{2},{3}&{4} in given question ,

I=$\int_0^1 \frac{(\frac{at}{a+b-bt})^{p-1}(\frac{(a+b)(1-t)}{a+b-bt})^{q-1}}{(\frac{a(a+b)}{a+b-bt})^{p+q}} \frac{(a+b)a}{(a+b-bt)^2} dt = \frac{a^p(a+b)^q}{a^{p+q}(a+b)^{p+q}} \int_0^1 t^{p+1}(1-t)^{q-1} dt = \frac{\beta(p,q)}{(a+b)^p a^q}$

Example 2:- P.T. I=$\int_0^\pi \frac{\sqrt{sinx}}{(5+3cosx)^{\frac{3}{2}}} dx = \frac{1}{2\sqrt{2\pi}} (\Gamma(\frac{3}{4}))^2$

Solution:- put $\tan\frac{x}{2} = t \to x = 2tan^{-1}t$ so $dx = \frac{2dt}{1+t^2}$ _{1}

x	0	π
t	0	∞

Now $sinx = \frac{2\tan\frac{x}{2}}{1+(\tan\frac{x}{2})^2} = \frac{2t}{1+t^2}$ _{2}

$cosx = \frac{1-(\tan\frac{x}{2})^2}{1+(\tan\frac{x}{2})^2} = \frac{1-t^2}{1+t^2}$ _{3}

$5+3cosx = 5 + 3\left(\frac{1-t^2}{1+t^2}\right) = \frac{2(4+t^2)}{1+t^2}$ _{4}

So on substituting {1},{2},{3}&{4} in given question ,

I=$\int_0^\infty \frac{(\frac{2t}{1+t^2})^{\frac{1}{2}}}{(\frac{2(4+t^2)}{1+t^2})^{\frac{3}{2}}} \frac{2dt}{1+t^2} = \int_0^\infty \frac{t^{\frac{1}{2}}}{(4+t^2)^{\frac{3}{2}}} dt$

t	0	∞
u	0	∞

Put $t^2 = 4u \to t = 2u^{\frac{1}{2}}$ so $dt = u^{-\frac{1}{2}} du$

I=$\int_0^\infty \frac{\left(2u^{\frac{1}{2}}\right)^{\frac{1}{2}} u^{-\frac{1}{2}} du}{(4)^{\frac{3}{2}}(1+u)^{\frac{3}{2}}}$

$I = \frac{1}{2^{\frac{5}{2}}} \int_0^\infty \frac{u^{-\frac{1}{4}}du}{(1+u)^{\frac{3}{2}}} = \frac{1}{2^{\frac{5}{2}}} \beta(\frac{3}{4}, \frac{3}{4}) = \frac{1}{2^{\frac{5}{2}}} \frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{2})} = \frac{1}{2\sqrt{2\pi}} (\Gamma(\frac{3}{4}))^2$

Example 3:-evaluate $\int_a^b (x-a)^m (b-x)^n dx$

Solution:- put $(x-a)=(b-a)t$ ___{1}

$\rightarrow x=a+(b-a)t\ so\ dx=(b-a)dt$ ___{2}

x	a	b
t	0	1

Now $(b-x)=b-[a+(b-a)t]=(b-a)+[-(b-a)t]=(b-a)(1-t)$ ___{3}

So on substituting {1},{2}&{3} in given question ,

I= $\int_0^1 [(b-a)t]^m[(b-a)(1-t)]^n\,(b-a)dt$

I= $(a+b)^{m+n+1}\int_0^1 (t)^m(1-t)^n dt=(a+b)^{m+n+1}\beta[(m+1),(n+1)]$

Example 4:- $prove\ that\ \int_0^\infty \frac{1}{(e^x+e^{-x})^n}dx=\frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)\ and\ hence\ \int_0^\infty sech^8 x dx$

Solution:- $\int_0^\infty \frac{1}{(e^x+e^{-x})^n}dx=\int_0^\infty \frac{dx}{(e^x+\frac{1}{e^x})^n}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(\frac{1+e^{2x}}{e^x})^n}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{(e^x)^n dx}{(1+e^{2x})^n}$

Put $e^{2x}=t\rightarrow e^x=t^{\frac{1}{2}}\rightarrow x=\frac{1}{2}logt\ \ so\ dx=\frac{1}{2t}dt$

x	$-\infty$	∞
t	0	∞

I=$\frac{1}{2}\int_0^\infty \frac{t^{\frac{n}{2}}}{(1+t)^n}(\frac{1}{2t})dt=\frac{1}{4}\int_0^\infty \frac{t^{(\frac{n}{2}-1)}}{(1+t)^n}dt=\frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)$

Hence I=$\int_0^\infty sech^8 x dx=\int_0^\infty \frac{2^8}{(e^x+e^{-x})^8}dx=\frac{2^8}{4}\beta(4,4)=\frac{16}{35}$

Example 5:- prove that $\int_1^\infty \frac{x^{(\frac{n}{2}-1)}}{(x+1)^n}dx=\frac{1}{2}\beta\left(\frac{n}{2},\frac{n}{2}\right)$

Solution:-I=$\int_1^\infty \frac{x^{(\frac{n}{2}-1)}}{(x+1)^n}dx$ ___{1}

put $x=\frac{1}{t}\ so\ dx=-\frac{1}{t^2}dt$

x	1	∞
t	1	0

So I=$\int_1^0 \frac{\frac{1}{t}^{(\frac{n}{2}-1)}}{(\frac{1}{t}+1)^n}(-\frac{1}{t^2})$dt=$\int_0^1 \frac{\frac{1}{t}^{(\frac{n}{2}-1)}}{(\frac{1+t}{t})^n}(t^{-2})$dt=$\int_0^1 \frac{t^{(\frac{n}{2}-1)}}{(1+t)^n}$dt=$\int_0^1 \frac{x^{(\frac{n}{2}-1)}}{(1+x)^n}$dx ___{2}

Adding {1}&{2} I+I=$\int_1^\infty \frac{x^{(\frac{n}{2}-1)}}{(x+1)^n}dx+\int_0^1 \frac{x^{(\frac{n}{2}-1)}}{(1+x)^n}$dx $=\int_0^\infty \frac{x^{(\frac{n}{2}-1)}}{(1+x)^n}$dx $=\ \beta\left(\frac{n}{2},\frac{n}{2}\right)$

So 2I= $\beta\left(\frac{n}{2},\frac{n}{2}\right)$ →I=$\frac{1}{2}\beta\left(\frac{n}{2},\frac{n}{2}\right)$

Example 6:- show that $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}\Gamma(2m)}{2^{2m-1}}$

Solution:- $\int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta\, d\theta = \frac{1}{2}\beta(\frac{p+1}{2},\frac{q+1}{2})$

Put p=q on both sides

so $\int_0^{\frac{\pi}{2}} sin^p\theta cos^p\theta\, d\theta = \frac{1}{2}\beta(\frac{p+1}{2},\frac{p+1}{2})$

$\frac{1}{2^p}\int_0^{\frac{\pi}{2}}(2sin\theta cos\theta)^p\, d\theta = \frac{1}{2}\beta(\frac{p+1}{2},\frac{p+1}{2})$

$$\frac{1}{2^p}\int_0^{\frac{\pi}{2}} sin^p 2\theta\, d\theta = \frac{1}{2}\beta(\frac{p+1}{2},\frac{p+1}{2})$$

Put $2\theta = t \ \rightarrow x = \frac{t}{2}$

θ	0	$\frac{\pi}{2}$
t	0	π

$$\frac{1}{2^p}\int_0^{\pi} sin^p t\frac{dt}{2} = \frac{1}{2}\beta(\frac{p+1}{2},\frac{p+1}{2})$$

$$\frac{2}{2^p}\int_0^{\frac{\pi}{2}} sin^p t\, dt = \beta(\frac{p+1}{2},\frac{p+1}{2})$$

$$\frac{2}{2^p}\times\frac{1}{2}\,\beta\left(\frac{p+1}{2},\frac{1}{2}\right) = \beta(\frac{p+1}{2},\frac{p+1}{2})$$

$$\frac{1}{2^p}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{2p+2}{2}\right)}$$

$$\frac{\sqrt{\pi}}{2^p}.\frac{1}{\Gamma\left(\frac{p+2}{2}\right)}=\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)}$$

[Let $p = 2m - 1\ so\ m = \frac{p+1}{2}$]

So $\frac{\sqrt{\pi}\Gamma(2m)}{2^{2m-1}} = \Gamma(m)\Gamma\left(m+\frac{1}{2}\right)$

10.DUIS {DIFFERENTIAL UNDER INTEGRAL SIGN}

Method:-

We will be given an integral I $= \int_{\alpha}^{\beta} f(x,a)dx$

We evaluate it by DUIS by following the steps :-

(1)assume given integral as I(a)

(2)differentiate both sides wrt a

(3)Find $\mathrm{I}'(a) = \phi(a)$

(4)I(a) $= \int \phi(a)da + c$

(5)to find c we give some arbitrary value to a

Note:- If "evaluate …and hence evaluate…"question comes , then 1st integrate using standard formulas and then use DUIS

EXAMPLES:-

Example 1:- prove that $I = \int_0^{\infty} \frac{\log(1+ax^2)}{x^2} dx = \pi\sqrt{a}\,; a > 0$

Solution:- let $I(a) = \int_0^{\infty} \frac{\log(1+ax^2)}{x^2} dx$ ___$\{1\}$

So on differentiating both sides wrt a,

$$\mathrm{I}'(a) = \int_0^{\infty} \left(\frac{1}{x^2} \times \frac{x^2}{1+ax^2}\right)dx = \int_0^{\infty} \frac{1}{1+ax^2}dx = \frac{1}{\sqrt{a}}[\tan^{-1}\sqrt{a}x]_0^{\infty} = \frac{1}{\sqrt{a}} \times \frac{\pi}{2}$$

So $I(a) = \int \frac{1}{\sqrt{a}} \times \frac{\pi}{2}\, da + c = \pi\sqrt{a} + c$ __$\{2\}$

Put a=0 in {1}&{2}

$I(a) = 0 \;\; in \{1\}\, and \;\; I(a) = 0 + c \;\; in \{2\}\, so\; c = 0$

Thus $I = \pi\sqrt{a}$

Example 2:-prove that $\int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx = \pi[\sqrt{a+1} - 1]$

Solution:- let $I(a) = \int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx$ _$\{1\}$

So on differentiating both sides wrt a,

$$\mathrm{I}'(a) = \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x} \times \frac{\sin^2 x}{1+a\sin^2 x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+a\sin^2 x}dx = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{\frac{1+a\sin^2 x}{\cos^2 x}}dx$$

$$I'(a) = \int_0^{\frac{\pi}{2}} \frac{sec^2 xdx}{sec^2 x + atan^2 x} = \int_0^{\frac{\pi}{2}} \frac{sec^2 xdx}{1 + tan^2 x + atan^2 x} = \int_0^{\frac{\pi}{2}} \frac{sec^2 xdx}{1 + tan^2 x(1+a)}$$

$put\ tanx = t\ so\ sec^2 xdx = dt$

x	0	$\frac{\pi}{2}$
t	0	∞

$$I'(a) = \int_0^{\infty} \frac{dt}{1 + (1+a)t^2}$$

$$I'(a) = \int_0^{\infty} \frac{dt}{1 + \left(\sqrt{1+a}t\right)^2} = \frac{1}{\sqrt{1+a}}[tan^{-1}(\sqrt{1+a}t)]_0^{\infty} = \frac{\pi}{2\sqrt{1+a}}$$

So $I(a) = \int \frac{\pi}{2\sqrt{1+a}} da + c = \pi\sqrt{1+a} + c$ ___{2}

Put a=0 in {1}&{2}

$I(a) = 0\ \ in\ \{1\}\ and\ \ I(a) = \pi + c\ \ in\ \{2\}\ so\ c = -\pi$

Thus $I = \pi\sqrt{1+a} - \pi = \pi[\sqrt{a+1} - 1]$

Example 3:- prove that $\int_0^1 \frac{x^a - 1}{logx} dx = \log(1+a)$; And hence evaluate $\int_0^1 \frac{a^7 - 1}{log7} dx$

Solution:- let $I(a) = \int_0^1 \frac{x^a - 1}{logx} dx$ ___{1}

So on differentiating both sides wrt a,

$$I'(a) = \int_0^1 \frac{1}{logx} x^a logx dx = \int_0^1 x^a dx = [\frac{x^a}{a+1}]_0^1 = \frac{1}{a+1}$$

So $I(a) = \int \frac{1}{a+1} da = \log(a+1) + c$ ___{2}

On putting a=0 in {1}&{2} $c = 0$

Thus $I = \log(1+a)$

Thus $\int_0^1 \frac{a^7 - 1}{log7} dx = log8$

Example 4:- prove that $I = \int_0^{\infty} \frac{1 - cosax}{x^2} dx = \frac{\pi a}{2}$

Solution:- let $I(a) = \int_0^{\infty} \frac{1 - cosax}{x^2} dx$ ___{1}

$$I(a) = \int_0^{\infty} \frac{1}{x^2} - \frac{cosax}{x^2} dx$$

So on differentiating both sides wrt a,

$I'(a) = \int_0^{\infty} \frac{sinax}{x^2} xdx = \int_0^{\infty} \frac{sinax}{x} dx$ [put ax=t so x=$\frac{t}{a}$; dx=$\frac{dt}{a}$]

$I'(a) = \int_0^{\infty} \frac{sint}{\frac{t}{a}} \frac{dt}{a} = \int_0^{\infty} \frac{sint}{t} dt = \frac{\pi}{2}$ [since $\int_0^{\infty} \frac{sinx}{x} dx = \frac{\pi}{2}, a\ standerd\ result$]

So $I(a) = \int \frac{\pi}{2} da = \frac{\pi a}{2} + c$ ___{2}

On putting a=0 in {1}&{2} $c = 0$

So $I = \frac{\pi a}{2}$

Example 5:-evaluate $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+acos^2x}$; and hence evaluate $\int_0^{\frac{\pi}{2}} \frac{cos^2x}{(3+cos^2x)^2} dx = \frac{\pi\sqrt{3}}{96}$

Solution:-let $I(a) = \int_0^{\frac{\pi}{2}} \frac{dx}{1+acos^2x} = \int_0^{\frac{\pi}{2}} \frac{sec^2xdx}{sec^2x+a} = \int_0^{\frac{\pi}{2}} \frac{sec^2xdx}{tan^2x+(1+a)}$

on putting tanx=t so $sec^2xdx = dt$

x	0	$\frac{\pi}{2}$
t	0	∞

so $I(a) = \int_0^{\infty} \frac{dt}{(1+a)+t^2}$

$$I(a) = \frac{1}{\sqrt{1+a}}[tan^{-1}\left(\frac{t}{\sqrt{1+a}}\right)]_0^{\infty} = \frac{\pi}{2\sqrt{1+a}}$$

Therefore on differentiating both sides;

$I'(a) = \frac{\pi}{2}(-\frac{1}{2})(1+a)^{-\frac{3}{2}}$ ___{1}

But $I(a) = \int_0^{\frac{\pi}{2}} \frac{dx}{1+acos^2x}$

So on differentiating both sides wrt a,

$$I'(a) = \int_0^{\frac{\pi}{2}} -\frac{1}{(1+acos^2x)^2} \times cos^2xdx \text{ ___{2}}$$

So from {1}&{2}

$$\frac{\pi}{2}(-\frac{1}{2})(1+a)^{-\frac{3}{2}} = -\int_0^{\frac{\pi}{2}} \frac{cos^2xdx}{(1+acos^2x)^2}$$

Now put a=$\frac{1}{3}$ [for required solution]

So $\int_0^{\frac{\pi}{2}} \frac{cos^2x}{(3+cos^2x)^2} dx = \frac{\pi}{4}(\frac{3}{4})^{\frac{3}{2}} = \frac{\pi\sqrt{3}}{96}$

IMPORTANT FORMULAS IN THIS CHAPTER

(a)$\int e^{ax} sinbxdx = \frac{e^{ax}}{a^2+b^2}(asinbx - bcosbx)$
(b)$\int e^{ax} cosbxbxdx = \frac{e^{ax}}{a^2+b^2}(acosbx + bsinbx)$

Example 6:- evaluate $\int_0^{\pi} \frac{1}{a-cosx} dx \; and \; hence \; evaluate \; \int_0^{\pi} \frac{1}{(2-cosx)^2} dx$

Solution :- let $I(a) = \int_0^{\pi} \frac{1}{a-cosx} dx$ __{1}

since $cosx = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}$ so $I(a) = \int_0^{\pi} . \frac{dx}{a-\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$

x	0	π
t	0	∞

Now put $tan\frac{x}{2} = t \;\; \rightarrow x = 2tan^{-1}t \; so \; dx = \frac{2}{1+t^2}dt$

$$I(a) = \int_0^{\infty} \frac{1}{a - \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = 2\int_0^{\infty} \frac{dt}{a + at^2 - 1 + t^2} = 2\int_0^{\infty} \frac{dt}{(a-1)+(a+1)t^2}$$

$$I(a) = 2\int_{0.}^{\infty} \frac{1}{(a+1)[\frac{a-1}{a+1} + t^2]} dt = \frac{2}{a+1}\sqrt{\frac{a+1}{a-1}}[tan^{-1}\frac{t}{\frac{a-1}{a+1}}]_0^{\infty}$$

$I(a) = \frac{2}{\sqrt{a+1}\sqrt{a-1}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{\sqrt{a^2-1}}$ ____{2}

Now $I(a) = \int_0^{\pi} \frac{1}{a-cosx} dx$; So on wrt a,

$I'(a) = -\int_0^{\pi} \frac{1}{(a-cosx)^2} dx = \pi(a^2-1)^{-\frac{3}{2}}(\frac{-1}{2})(2a)$ ___[from {1} &{2}]

Now put a=2 [for required solution]

So $\int_0^{\pi} \frac{1}{(2-cosx)^2} dx = \frac{\pi.2}{(4-1)^{\frac{3}{2}}} = \frac{2\pi}{3\sqrt{3}}$

Example 7:- prove that $\int_0^{\infty} \frac{e^{-x}-e^{-ax}}{xsecx} dx = \frac{1}{2} iog(\frac{a^2+1}{2})$

Solution:- let $I(a) = \int_0^{\infty} \frac{e^{-x}-e^{-ax}}{xsecx} dx$ __{1}

So on differentiating both sides wrt a,

$$I'(a) = \int_0^{\infty} \frac{e^{-ax}.x}{xsecx} dx = \int_0^{\infty} e^{-ax} cosxdx = [\frac{e^{-ax}}{a^2+1}(-acosx + sinx)]_0^{\infty}$$

$$I'(a) = \frac{a}{a^2+1}$$

So $I(a) = \int \frac{a}{a^2+1} da = \frac{1}{2}\log(a^2+1) + c$ __{2}

Put a=1 in {1}&{2}

$I(a) = 0 \;\; in \, \{1\} \, and \;\; I(a) = \frac{1}{2}\log(2) + c \;\; in \, \{2\} \, so \; c = -\frac{1}{2}\log(2)$

So $I = \frac{1}{2} iog(\frac{a^2+1}{2})$

Example 8:- prove that $\int_0^{\infty} \frac{cos\lambda x}{x}(e^{-ax} - e^{-bx})dx = \frac{1}{2}\log(\frac{b^2+\lambda^2}{a^2+\lambda^2})$

Solution:- let$I(a) = \int_0^\infty \frac{cos\lambda x}{x}(e^{-ax} - e^{-bx})dx$ ___{1}

So on differentiating both sides wrt a,

$$I'(a) = \int_0^\infty \frac{cos\lambda x}{x}(-xe^{-ax})dx = -\int_0^\infty cos\lambda x(e^{-ax})dx$$

$$I'(a) = -[\frac{e^{-ax}}{a^2+\lambda^2}(-acos\lambda x + \lambda sin\lambda x)]_0^\infty = -\frac{a}{a^2+\lambda^2}$$

so$I(a) = -\int \frac{a}{a^2+\lambda^2}da = -\frac{1}{2}\int \frac{2a}{a^2+\lambda^2}da = -\frac{1}{2}\log(a^2+\lambda^2) + c$ ___{2}

Put a=b in {1}&{2}

$I(a) = 0 \ \ in \{1\} \ and \ \ I(a) = -\frac{1}{2}\log(b^2+\lambda^2) + c \ \ in \{2\} \ so \ c = \frac{1}{2}\log(b^2+\lambda^2)$

So $I = -\frac{1}{2}\log(a^2+\lambda^2) + \frac{1}{2}\log(b^2+\lambda^2) = \frac{1}{2}\log(\frac{b^2+\lambda^2}{a^2+\lambda^2})$

Example 9:-$evaluate$ $\int_0^\pi \frac{dx}{a+bcosx}$ hence evaluate $\int_0^\pi \frac{dx}{(a+bcosx)^2} + \int_0^\pi \frac{cosxdx}{(a+bcosx)^2}$

Solution:-$I = \int_0^\pi \frac{dx}{a+bcosx}$ since $cosx = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}$ so $I(a) = \int_0^\pi . \frac{dx}{a+b(\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}})}$

Now put $tan\frac{x}{2} = t \ \ \rightarrow x = 2tan^{-1}t \ so \ dx = \frac{2}{1+t^2}dt$

x	0	π
t	0	∞

So

$$I(a) = \int_0^\infty \frac{1}{a+b(\frac{1-t^2}{1+t^2})}\frac{2}{1+t^2}d = 2\int_0^\infty \frac{dt}{a+at^2+b-bt^2} = 2\int_0^\infty \frac{dt}{(a+b)+(a-b)t^2}$$

$$I(a) = 2\int_{0.}^\infty \frac{1}{(a-b)[\frac{a+b}{a-b}+t^2]}dt = \frac{2}{a-b}\frac{1}{[\sqrt{\frac{a+1}{a-1}}]}[tan^{-1}\frac{t}{\sqrt{\frac{a+1}{a-1}}}]_0^\infty$$

$I(a) = \frac{2}{\sqrt{a-b}\sqrt{a+b}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{\sqrt{a^2-b^2}}$

So $\int_0^\pi \frac{dx}{a+bcosx} = \frac{\pi}{\sqrt{a^2-b^2}}$ ___{1}

On differentiating equation {1} both sides wrt to a ,

$\int_0^\pi \frac{dx}{(a+bcosx)^2} = \frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}$ ___{2}

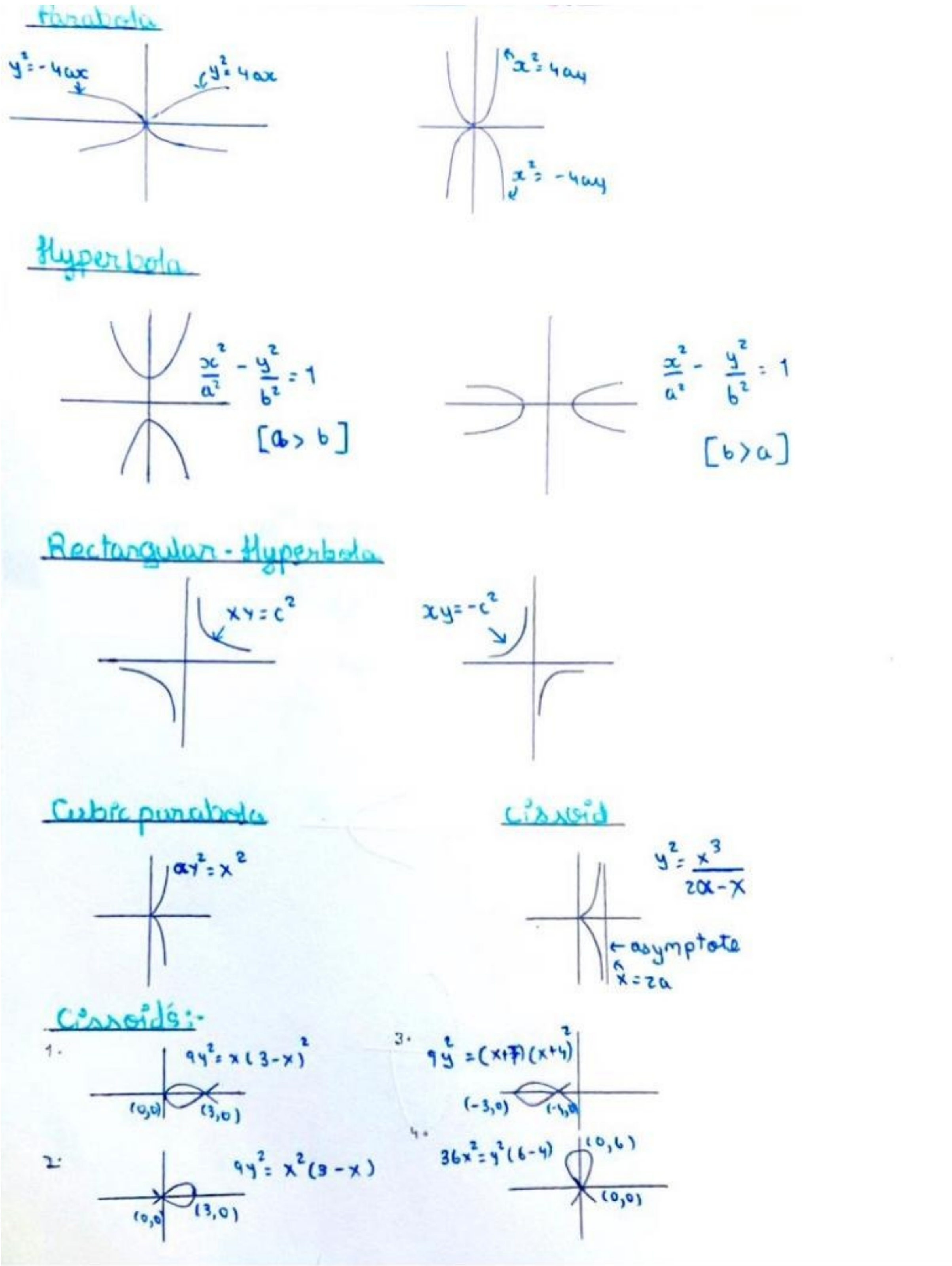
On differentiating equation {1} both sides wrt to b ,

$\int_0^\pi \frac{cosxdx}{(a+bcosx)^2} = \frac{-\pi b}{(a^2-b^2)^{\frac{3}{2}}}$ ___{3}

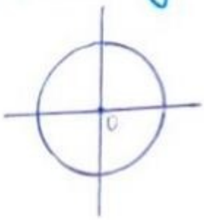
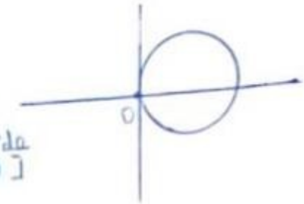
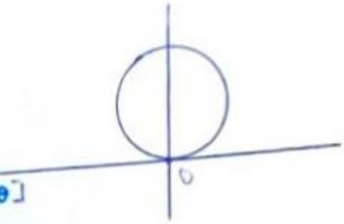
So on adding {2} & {3}

$$\int_0^{\pi} \frac{dx}{(a + bcosx)^2} + \int_0^{\pi} \frac{cosxdx}{(a + bcosx)^2} = \frac{\pi(a - b)}{(a^2 - b^2)^{\frac{3}{2}}}$$

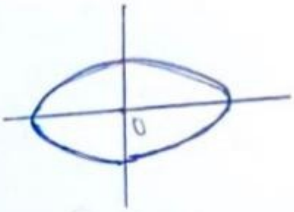
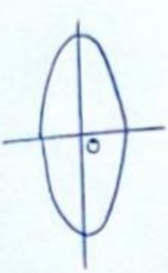
11.standard curves{for sem-2 maths}



Circle

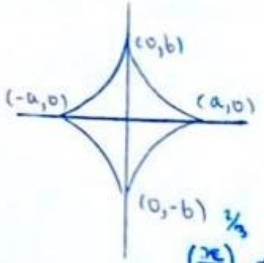
Equation	Polar form	Tracing
1. $x^2+y^2=a^2$	$r=a$	
2. $x^2+y^2=2ax$	$r=2a\cos\theta$ [if on negative side $r=-2a\cos\theta$]	
3. $x^2+y^2=2ay$	$r=2a\sin\theta$ [if on negative axis $r=-2a\sin\theta$]	

Ellipse

Equation	Polar form	Tracing
1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$r=1$ [a>b]	
2. $\frac{x^2}{a}+\frac{y^2}{b^2}=1$	$r=1$ [b>a]	

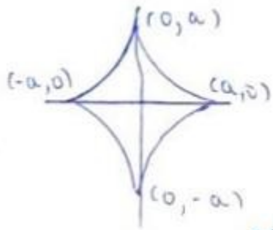
note :- O → origin

Astroid



$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

Hypocycloid



$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

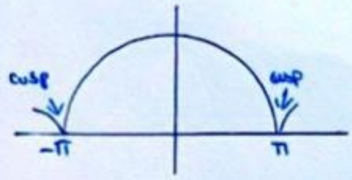
$$\therefore\ x^{2/3} + y^{2/3} = a^{2/3}$$

Astroid & hypocycloid parametric is

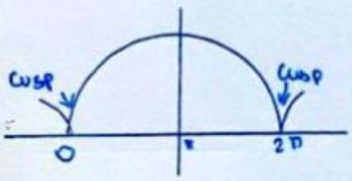
$$x = a\cos^3\theta \qquad \& \qquad x = a\cos^3\theta$$
$$y = b\cos^3\theta \qquad\qquad\ \ y = a\sin^3\theta$$

respectively

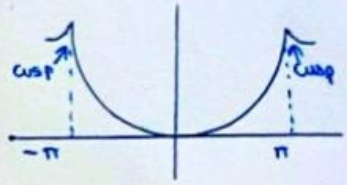
Cycloids



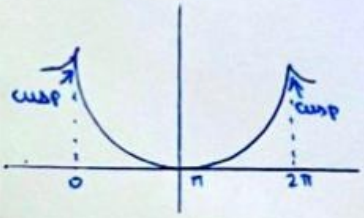
$$x = a(\theta + \sin\theta)$$
$$y = a(1 + \cos\theta)$$



$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$



$$x = a(\theta + \sin\theta)$$
$$y = a(1 - \cos\theta)$$



$$x = a(\theta - \sin\theta)$$
$$y = a(1 + \cos\theta)$$

Note:- Hint

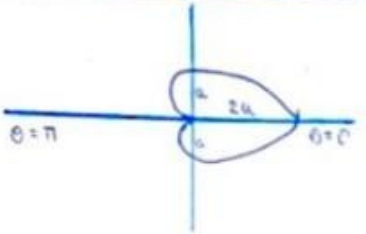
If $\theta + \sin\theta \rightarrow -\pi$ to π

If $\theta - \sin\theta \rightarrow 0$ to 2π

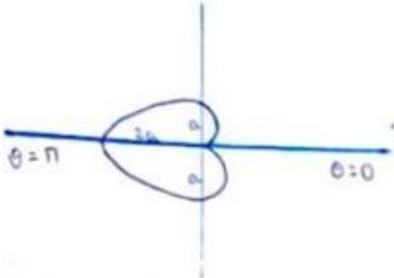
Same sign $\rightarrow$

Opposit sign $\rightarrow$

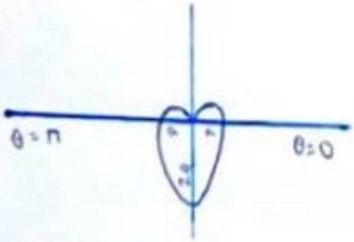
Cardiodes



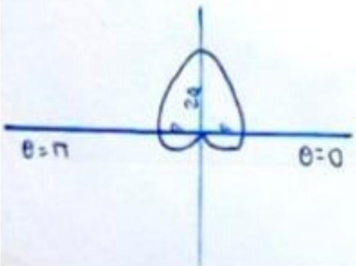
$r = a(1+\cos\theta)$



$r = a(1-\cos\theta)$



$r = a(1-\sin\theta)$



$r = a(1+\sin\theta)$

Lemniscate

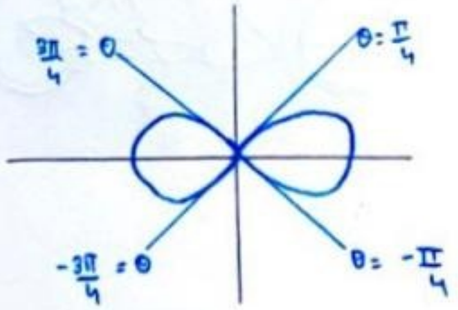
$(x^2+y^2)^2 = a^2(x^2-y^2)$

↓ where a=1, it is called bernoullis lemniscale.

Put $x = r\cos\theta$ & $y = r\sin\theta$

$\therefore r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$

$\therefore r^4 = a^2 r^2 \cos 2\theta$ → polar. Eq



12.Rctification

Type 1:- Cartesian curves

$ds=\sqrt{1+(\frac{dy}{dx})^2}dx \to$ if $y=f(x)$

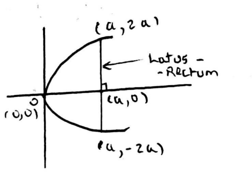
$ds=\sqrt{1+(\frac{dx}{dy})^2}dy \to$ if $x=f(y)$

Length of curve is given by $s=\int ds$

Example 1:- S.T length of arc of parabola $y^2=4ax$ from the vertex to one end of its latus-rectum is

$\to$ $a[\sqrt{2}+\log(1+\sqrt{2})]$

Solution:-



$y^2=4ax$ so $x=\frac{y^2}{4a}$ $\to$ $\frac{dx}{dy}=\frac{2y}{4a}$

So $1+(\frac{dx}{dy})^2=1+\frac{y^2}{4a^2}=\frac{y^2+4a^2}{4a^2}$

$ds=\sqrt{1+(\frac{dx}{dy})^2}dy=\sqrt{\frac{y^2+4a^2}{4a^2}}dy=\frac{\sqrt{y^2+4a^2}}{2a}dy$

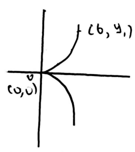
So now length $S=\int_0^L ds=\int_{y=0}^{y=2a}\frac{\sqrt{y^2+4a^2}}{2a}dy=\frac{1}{2a}[\frac{y}{2}\sqrt{y^2+4a^2}+\frac{4a^2}{2}\log(y+\sqrt{y^2+4a^2})]_{y=0}^{y=2a}$

$S=\frac{1}{2a}[2a\sqrt{2}a+2a^2\log(2a+2\sqrt{2}a)-2a^2log2a]=\frac{2a^2}{2a}[\sqrt{2}+\log(\frac{2+2\sqrt{2}a}{2a})]$

$S=a[\sqrt{2}+\log(1+\sqrt{2})]$

Example 2:- S.T length of curve of $ay^2=x^3$ from vertex to the point whose absicca is b , is $\frac{1}{27\sqrt{a}}(9b+2a)^{\frac{3}{2}}-\frac{8a}{27}$

Solution:-



$y^2=\frac{x^3}{a}\to y=\frac{x^{\frac{3}{2}}}{\sqrt{a}}$ so $\frac{dy}{dx}=\frac{1}{\sqrt{a}}\frac{3}{2}x^{\frac{1}{2}}=\frac{3\sqrt{x}}{2\sqrt{a}}$

Now $1+(\frac{dy}{dx})^2=1+\frac{9x}{4a}$

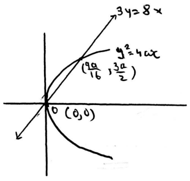
$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{\frac{4a+9x}{4a}}\,dx = \frac{1}{\sqrt{4a}}(4a+9x)^{\frac{1}{2}}\text{dx}$$

So now length $S = \int_0^b \frac{1}{\sqrt{4a}}(4a+9x)^{\frac{1}{2}}\,dx = \frac{1}{\sqrt{4a}}\int_0^b (4a+9x)^{\frac{1}{2}}dx = \frac{1}{\sqrt{4a}}\left[\frac{(4a+9x)^{\frac{3}{2}}}{\frac{27}{2}} - \frac{(4a)^{\frac{3}{2}}}{\frac{27}{2}}\right]$

$$S = \frac{1}{27\sqrt{a}}(9b+2a)^{\frac{3}{2}} - \frac{8a}{27}$$

Example 3:- S.T length of arc of parabola $y^2 = 4ax$ cut off by line $3y = 8x$ is $a[log2 + \frac{15}{16}]$

Solution:-



$y^2 = 4ax \,\&\, y = \frac{8x}{3}$ so $\frac{64x^2}{9} = 4ax \rightarrow \frac{16x^2}{9} - ax = 0 \rightarrow 16x^2 - 9ax = 0 \rightarrow x(16x-9a) = 0$

When $x = 0\,, y = 0$ $and\ when\ x = \frac{9a}{16}, y = \frac{3a}{2}$ therefore we get (0,0) and $(\frac{9a}{16}, \frac{3a}{2})$

Now $y^2 = 4ax \;\rightarrow x = \frac{y^2}{4a}$ so $\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$

So $1+(\frac{dx}{dy})^2 = 1 + \frac{y^2}{4a^2} = \frac{y^2+4a^2}{4a^2}$

$$ds = \sqrt{1+(\frac{dx}{dy})^2}dy = \sqrt{\frac{y^2+4a^2}{4a^2}}dy = \frac{\sqrt{y^2+4a^2}}{2a}dy$$

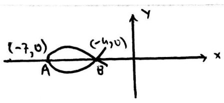
So now length $S = \int_{y=0}^{y=\frac{3a}{2}} \frac{\sqrt{y^2+4a^2}}{2a}dy = \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right]_{y=0}^{y=\frac{3a}{2}}$

$$S = \frac{1}{2a}\left[\frac{1}{2}\frac{3a}{2}\frac{5a}{2} + 2a^2(log4a - log2a)\right] = \frac{a^2}{2a}\left[\frac{15}{8} + 2\log(2)\right]$$

$$S = a[log2 + \frac{15}{16}]$$

Example 4:-Find the total length of loop of the curve $9y^2 = (x+7)(x+4)^2$

Solution:-



total length = S =2 $\int_A^B ds$

now $9y^2 = (x+7)(x+4)^2$

so $y = \frac{(x+4)\sqrt{x+7}}{3} \;\rightarrow \frac{dy}{dx} = \frac{1}{3}\left[\frac{x+4}{2\sqrt{x+7}} + \sqrt{x+7}\right] = \frac{1}{6}\left[\frac{3x+18}{\sqrt{x+7}}\right] = \frac{x+6}{2\sqrt{x+7}}$

now $1 + [\frac{dy}{dx}]^2 = 1 + [\frac{x+6}{2\sqrt{x+7}}]^2 = \frac{[x+8]^2}{4[x+7]}$

so $ds = \sqrt{\frac{[x+8]^2}{4[x+7]}} = \frac{x+8}{2\sqrt{x+7}}$ ___[from -7 to -4]

therefore total length $S=2\int_{-7}^{-4} \frac{x+8}{2\sqrt{x+7}} dx = \int_{-7}^{-4} \left(\frac{x+7+1}{\sqrt{x+7}}\right) dx = \int_{-7}^{-4} \left(\frac{x+7}{\sqrt{x+7}} + \frac{1}{\sqrt{x+7}}\right) dx$

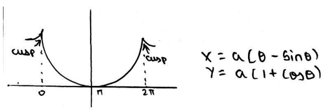
$S=\int_{-7}^{-4} [(x+7)^{\frac{1}{2}} + (x+7)^{-\frac{1}{2}}]\, dx = [\frac{(x+7)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+7)^{\frac{1}{2}}}{\frac{1}{2}}]_{-7}^{-4} = 4\sqrt{3}$

Type 2:-parametric curves

If $x = f(t)$& y=$f(t)$ then $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$; $thus\ S = \int ds$

Example 1:- find the length of one arc of cycloid $x = a(\theta - sin\theta)$ & $y = a(1 + cos\theta)$;draw the graph.

Solution:-



$\frac{dx}{d\theta} = a(1 - cos\theta)$ & $\frac{dy}{d\theta} = asin\theta$

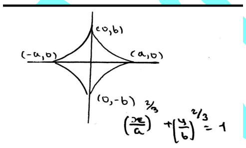
S= $\int_0^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}d\theta = \int_0^{2\pi} \sqrt{a^2(1 - 2cos\theta + cos^2\theta) + a^2sin^2\theta}\, d\theta$

S= $\int_0^{2\pi} a\sqrt{(2 - 2cos\theta)}d\theta = \int_0^{2\pi} 2asin\frac{\theta}{2}d\theta = 2a.\frac{1}{\frac{1}{2}}[-cos\frac{\theta}{2}]_0^{2\pi}$

S= $-4a(cos\pi - cos0) = -4a(-1 - 1) = 8a$

Example 2:-Find the total length of curve $(\frac{x}{a})^{\frac{2}{3}} + (\frac{y}{b})^{\frac{2}{3}} = 1$

Solution:-



$(\frac{x}{a})^{\frac{2}{3}} + (\frac{y}{b})^{\frac{2}{3}} = 1$ is equation of cycloid.

So parametric form is

$x = acos^3\theta$ & $y = bsin^3\theta$

$\frac{dx}{d\theta} = -3acos^2\theta sin\theta$ & $\frac{dy}{d\theta} = 3bsin^2\theta cos\theta$

So $(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = 9cos^2\theta sin^2\theta(a^2cos^2\theta + b^2sin^2\theta)$

S= $4\int_0^{\frac{\pi}{2}}\sqrt{(\frac{dx}{d\theta})^2+(\frac{dy}{d\theta})^2}d\theta = 4\int_0^{\frac{\pi}{2}} 3sin\theta cos\theta\sqrt{a^2cos^2\theta+b^2sin^2\theta}d\theta$

__{put $a^2cos^2\theta+b^2sin^2\theta$ =t^2 ; 2sin$\theta cos\theta(b^2-a^2)d\theta = 2tdt$

sin$\theta cos\theta d\theta = \frac{t}{b^2-a^2}dt$}

θ	0	$\frac{\pi}{2}$
t	a	b

S= $12\int_a^b \sqrt{t^2}\frac{t}{b^2-a^2}dt = \frac{12}{b^2-a^2}[\frac{t^3}{3}]_a^b = \frac{4(b^3-a^3)}{b^2-a^2}$

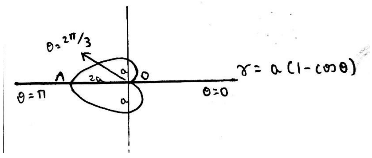
Type 2:-Polar curves

For polar curves $ds = [\sqrt{(r)^2+(\frac{dr}{d\theta})^2}]d\theta$ if $r=f(\theta)$ or $ds = [\sqrt{1+(r)^2(\frac{d\theta}{dr})^2}]dr$ if $\theta = f(r)$

Thus S=$\int ds$

Example 1:-Find the perimeter of the car-diode $r = a(1-cos\ \theta)$ & prove that the line $\theta = \frac{2\pi}{3}$ bisects upper half of car-diode .

Solution:-



$r = a(1-cos\theta)$ $\frac{dr}{d\theta} = asin\theta$

So $(r)^2+(\frac{dr}{d\theta})^2$=$a^2(1-cos\theta)^2+a^2sin^2\theta = a^2(1-2cos\theta+1) = a^2(2-2cos\theta) = 2a^22sin^2\frac{\theta}{2}$

Therefore $ds = \sqrt{(r)^2+(\frac{dr}{d\theta})^2}d\theta = 2asin\frac{\theta}{2}d\theta$

Total length S= $2\int_o^a ds$ ____{at o=$\theta = 0$ & $at\ a = \theta = \pi$}

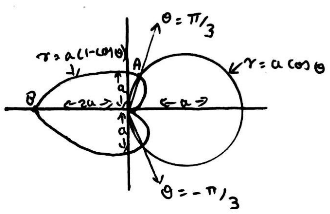
S=$2\int_0^\pi 2asin\frac{\theta}{2}d\theta = 4a[\frac{-cos\frac{\theta}{2}}{\frac{1}{2}}]_0^\pi = \left(-cos\frac{\pi}{2}+cos0\right) = 8a$

For line OP , S=$\int_0^{\frac{2\pi}{3}} ds = \int_0^{\frac{2\pi}{3}} 2asin\frac{\theta}{2}d\theta = 2a[\frac{-cos\frac{\theta}{2}}{\frac{1}{2}}]_0^{\frac{2\pi}{3}} = 4a\left(\frac{-1}{2}+1\right) = 2a$

Since length of upper half =$4a$; therefore $\theta = \frac{2\pi}{3}$ bisects upper half of car-diode .

Example 2:- Find the length of car-diode $r = a(1-cos\ \theta)$ lying outside the circle of $r = acos\theta$

Solution:-



for point of intersection, $a(1 - cos\ \theta) = \ acos\theta \rightarrow cos\theta = \frac{1}{2}$ so $\theta = \pm\frac{\pi}{3}$

Now $r = a(1 - cos\ \theta) \rightarrow \frac{dr}{d\theta} = asin\theta$

$$ds = \sqrt{(r)^2 + (\frac{dr}{d\theta})^2 d\theta} = \sqrt{a^2(1 - cos\theta)^2 + a^2 sin^2\theta d\theta} = \sqrt{a^2(1 - 2cos\theta + 1)d\theta}$$

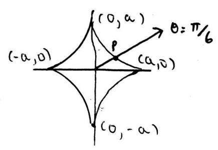
$$ds = \sqrt{a^2(2 - 2cos\theta)d\theta} = \sqrt{2a^2 2sin^2\frac{\theta}{2}d\theta} = 2asin\frac{\theta}{2}d\theta$$

So $S=\int ds = 2\int_{\pi}^{\frac{\pi}{3}} 2asin\frac{\theta}{2}d\theta = 4a[\frac{-cos\frac{\theta}{2}}{\frac{1}{2}}]_{\pi}^{\frac{\pi}{3}} = 8a\left(0 + \frac{\sqrt{3}}{2}\right) = 4\sqrt{3}a$

Miscellaneous Examples:-

Example 1:-Prove that from astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the line $\theta = \frac{\pi}{6}$ devide the arc in 1st quadrant in ratio of 1:3

Solution:-



we call astroid $(\frac{x}{a})^{\frac{2}{3}} + (\frac{x}{b})^{\frac{2}{3}} = 1$ as hypocycloid when $a = b$, i.e $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Parametric form is,
$x = acos^3\theta\ \&\ y = asin^3\theta$

$$\frac{dx}{d\theta} = -3acos^2\theta sin\theta\ \&\ \frac{dy}{d\theta} = 3asin^2\theta cos\theta$$

So $(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = 9a^2cos^2\theta sin^2\theta(cos^2\theta + sin^2\theta) = 9a^2cos^2\theta sin^2\theta$

$$ds = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 d\theta} = \sqrt{9a^2cos^2\theta sin^2\theta d\theta} = 3acos\theta sin\theta d\theta$$

Now S= $\int_0^{\frac{\pi}{2}} 3acos\theta sin\theta d\theta = \frac{3a}{2}\int_0^{\frac{\pi}{2}} sin2\theta d\theta = \frac{3a}{2}[\frac{-cos\theta}{2}]_0^{\frac{\pi}{2}} = \frac{3a}{4}(1 + 1) = \frac{3a}{2}$units

And length of arc till point P is

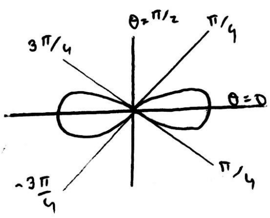
S= $\int_0^{\frac{\pi}{6}} 3acos\theta sin\theta d\theta = \frac{3a}{2}\int_0^{\frac{\pi}{6}} sin2\theta d\theta = \frac{3a}{2}[\frac{-cos\theta}{2}]_0^{\frac{\pi}{6}} = \frac{3a}{4}\left(1 - \frac{1}{2}\right) = \frac{3a}{8}$units

Therefore $\frac{3a}{2} - \frac{3a}{8} = \frac{9a}{8}$

So ratio is $\frac{3a}{8} \Big/ \frac{9a}{8} = 1:3$

Example 2:- show that length of perimeter of lemniscate $(x^2+y^2)^2 = a^2(x^2-y^2)$ is $\frac{a}{\sqrt{2\pi}}[\Gamma\left(\frac{1}{4}\right)]^2$

Solution:-



For converting it into polar form put $x = r\cos\theta$ and $y = r\sin\theta$

So $r^2 = a^2\cos2\theta \rightarrow r = a\sqrt{\cos2\theta}$

So $\frac{dr}{d\theta} = \frac{a(-2\sin2\theta)}{2\sqrt{\cos2\theta}} = \frac{-a\sin2\theta}{\sqrt{\cos2\theta}}$

$$ds = \sqrt{(r)^2 + (\frac{dr}{d\theta})^2}d\theta = \sqrt{a^2\cos2\theta + \frac{a^2\sin^2 2\theta}{\cos2\theta}}d\theta = a\sqrt{\frac{1}{\cos2\theta}}d\theta$$

θ	0	$\frac{\pi}{4}$
t	0	$\frac{\pi}{2}$

Now S= $4\int_0^{\frac{\pi}{4}} ds = 4\int_0^{\frac{\pi}{4}} a(\cos2\theta)^{-\frac{1}{2}}d\theta$ ___{put $2\theta = t \rightarrow \theta = \frac{t}{2}$ so $d\theta = \frac{dt}{2}$}

S=$4a\int_0^{\frac{\pi}{2}} \cos t^{-\frac{1}{2}}\frac{dt}{2} = 2a.\frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{4}\right) = a\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \times \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4})} = \frac{a}{\sqrt{2\pi}}[\Gamma\left(\frac{1}{4}\right)]^2$

13.Multiple integration

Type 1:-Direct evaluation of double and triple integration

Double/triple integration in form $\iiint f(x,y)dxdydz$ with some limits will be given.

Always start with the innermost integral and observe the limits .

In case of triple integration , the innermost integral has to be evaluated wrt z.

In case of double integration , we observe the limits $\rightarrow$ (A)limits in term of x belong to y [$\int wrt\ y$]

(B)limits in terms of y belong to x [$\int wrt$ x]

(C)limits in terms of $\theta\ belong\ to\ r$

We always integrate from inside to outside .

Outermost limit is always constant .

Note :- {1}If two differentials are dr and dθ ,then order is fixed [innermost is always wrt r & outermost is always wrt θ].

{2}If two limits are constant and equal , we integrate according to our convenience.

{3}If limits are constant but not equal , then order of integration must be given.

{4}If two limits are constant and order is known , then integrations can be solved independently.

Examples:-

Example 1:- I=$\int_0^{log2}\int_0^{x}\int_0^{x+y} e^{x+y+z}dxdydz$

Solution:-I=$\int_0^{log2}\int_0^{x}[e^{x+y+z}]_{z=0}^{z=x+y}$dxdy

I=$\int_0^{log2}\int_0^{x}[e^{2x+2y}-e^{x+y}]$dxdy

I=$\int_0^{log2}[\frac{e^{2x+2y}}{2}-e^{x+y}]_{y=0}^{y=x}$dx

I=$\int_0^{log2}[\frac{e^{4x}}{2}-e^{2x}-\frac{e^{2x}}{2}+e^{x}]$dx

I=$\left[\frac{e^{4x}}{8}-\frac{e^{2x}}{2}-\frac{e^{2x}}{4}+e^{x}\right]_0^{log2}$

I=$\left[\frac{e^{4log2}}{8}-\frac{e^{2log2}}{2}-\frac{e^{2log2}}{4}+e^{log2}\right]-\frac{1}{8}+\frac{1}{4}+\frac{1}{2}-1$

I=$\frac{16}{8}-\frac{4}{2}-\frac{4}{4}+2-\frac{1}{8}+\frac{1}{4}+\frac{1}{2}-1$=$\frac{5}{8}$

Example 2:-I=$\int_0^{\infty}\int_0^{\infty} e^{-x^2(1+y^2)}xdydx$

Solution:-I=$\int_0^{\infty}\int_0^{\infty} e^{-x^2(1+y^2)}\frac{(-2x)[e^{-x^2(1+y^2)}]}{[-2(1+y^2)]}dydx$

I=$-\int_0^{\infty}\frac{1}{2(1+y^2)}\int_0^{\infty} e^{-x^2(1+y^2)}(-2x)[e^{-x^2(1+y^2)}]dxdy$

I=$-\int_0^{\infty}\frac{1}{2(1+y^2)}[e^{-x^2(1+y^2)}]_0^{\infty}dy$

I=$-\int_0^{\infty}\frac{1}{2(1+y^2)}(0-1)dy$

I=$\int_0^{\infty}\frac{1}{2(1+y^2)}dy$=$\frac{1}{2}(tan^{-1}y)_0^{\infty}=\frac{1}{2}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{4}$

Example 3:-I=$\int_0^{\frac{\pi}{2}}\int_0^{3(1-cosy)}x^2sinydxdy$

Solution:-$I=\int_0^{\frac{\pi}{2}}siny(\frac{x^3}{3})_{x=0}^{x=3(1-cosy)}dy$

I=$\int_0^{\frac{\pi}{2}}siny[\frac{27(1-cosy)^3}{3}]dy$

I=9 $\int_0^{\frac{\pi}{2}}siny\,(1-cosy)^3dy$

I=9$[\frac{(1-cosy)^4}{4}]_0^{\frac{\pi}{2}}$

I=$\frac{9}{4}[1-0]$=$\frac{9}{4}$

Example 4:-I=$\int_0^{\frac{\pi}{2}}\int_0^{acos\theta}\int_0^{\sqrt{a^2-r^2}}rdzdrd\theta$

Solution:-I=$\int_0^{\frac{\pi}{2}}\int_0^{acos\theta}r[z]_{z=0}^{\sqrt{a^2-r^2}}drd\theta$

I=$\int_0^{\frac{\pi}{2}}\int_0^{acos\theta}r[\sqrt{a^2-r^2}]drd\theta$

I=$\int_0^{\frac{\pi}{2}}(-\frac{1}{2})\int_0^{acos\theta}(-2r)[a^2-r^2]^{\frac{1}{2}}drd\theta$

I=$-\frac{1}{2}\int_0^{\frac{\pi}{2}}[\frac{(a^2-r^2)^{\frac{3}{2}}}{\frac{3}{2}}]_{r=0}^{r=acos\theta}d\theta$

I=$-\frac{1}{2}\times\frac{2}{3}\int_0^{\frac{\pi}{2}}[a^2-a^2cos^2\theta)^{\frac{3}{2}}-(a^2)^{\frac{3}{2}}]\,d\theta$

I=$\frac{-1}{3}\int_0^{\frac{\pi}{2}}[a^3[(1-cos^2\theta)^{\frac{3}{2}}-a^3]d\theta$

I=$\frac{-a^3}{3}\int_0^{\frac{\pi}{2}}[(sin^2\theta)^{\frac{3}{2}}-1]d\theta$

I=$\frac{-a^3}{3}\int_0^{\frac{\pi}{2}}[(sin\theta)^3-1]d\theta$

I=$\frac{-a^3}{3}[\frac{2}{3.1}-\frac{\pi}{2}]$

Example 5:-I=$\int_0^{a\sqrt{3}}\int_0^{\sqrt{x^2+a^2}}\frac{x}{x^2+y^2+a^2}dxdy$

Solution:- I=$\int_0^{a\sqrt{3}}\int_0^{\sqrt{x^2+a^2}}[x\times\frac{1}{y^2+(\sqrt{x^2+a^2})^2}]dxdy$

I=$\int_0^{a\sqrt{3}}x.\frac{1}{\sqrt{x^2+a^2}}[tan^{-1}\frac{y}{\sqrt{x^2+a^2}}]_{y=0}^{y=\sqrt{x^2+a^2}}dx$

I=$\int_0^{a\sqrt{3}} x.\frac{1}{\sqrt{x^2+a^2}}[tan^{-1}(1)-tan^{-1}(0)]dx$=$\frac{\pi}{4}\int_0^{a\sqrt{3}} x.\frac{1}{\sqrt{x^2+a^2}}dx$=$\frac{\pi}{8}\int_0^{a\sqrt{3}} 2x.\frac{1}{\sqrt{x^2+a^2}}dx$

I=$\frac{\pi}{8}[2\sqrt{x^2+a^2}]_0^{a\sqrt{3}}$=$\frac{\pi}{8}[2(2a-a)]=\frac{\pi a}{4}$

Example 6:-I=$\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}}\frac{r}{(1+r^2)^2}drd\theta$

Solution:- I=$\int_0^{\frac{\pi}{4}}d\theta\int_0^{\sqrt{\cos 2\theta}}\frac{r}{(1+r^2)^2}dr$ [put $1+r^2=t$ so $2rdr=dt$]

I=$\int_0^{\frac{\pi}{4}}d\theta\int_1^{2cos^2\theta}\frac{dt}{(2t)^2}$

r	$\sqrt{\cos 2\theta}$	0
t	$2cos^2\theta$	1

I=$\int_0^{\frac{\pi}{4}}\frac{d\theta}{2}[\frac{t^{-1}}{-1}]_1^{2cos^2\theta}$

I=$\int_0^{\frac{\pi}{4}}\frac{-d\theta}{2}[\frac{1}{2cos^2\theta}-1]$=$-\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}sec^2\theta-1\right)d\theta=\frac{-1}{2}[-\theta+\frac{1}{2}tan\theta]_0^{\frac{\pi}{4}}$=$\frac{-1}{2}[-\frac{\pi}{4}+\frac{1}{2}]$=$\frac{\pi-2}{8}$

Example 7:-I= $\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}xydxdy$

Solution:- I= $\int_{\frac{1}{2}}^{1}[\frac{xy^2}{2}]_{\frac{1}{2}}^{1}dx=\int_{\frac{1}{2}}^{1}\frac{x}{2}\left[1-\frac{1}{4}\right]dx$

I= $\int_{\frac{1}{2}}^{1}\frac{3x}{8}dx=\frac{3}{8}[\frac{x^2}{2}]_{\frac{1}{2}}^{1}=\frac{9}{64}$

Example 8:-I= $\int_0^1\int_0^x x^2dxdy$

Solution:- I= $\int_0^1[x^2y]_{y=0}^{y=x}dx$

I= $\int_0^1(x^3-0)dx=[\frac{x^4}{4}]_0^1=\frac{1}{4}$

Type 2:- Integration over a given area

A double integration in form of $\iint_R f(x,y)dxdy$ will be given where R is the region of integration.

Method:- (1)Trace all the given curves and all the given lines and get region of integration R

(2)Decide whether we 1st want to integrate wrt to X or wrt Y

→If we want to integrate wrt Y , draw strip parallel to Y-axis in region R[inner limits will be y=f_1x to y=f_2x and outer limits will be x=constant to x=constant]

→If we want to integrate wrt X , draw strip parallel to X-axis in region R[inner limits will be x=f_1y to x=f_2y and outer limits will be y=constant to y=constant]

Note:- (1) for limits X → left to right

Y →bottom to top

(2)Outermost limits always tells about movement of strip and innermost limits represents those curves where end of strip lies.

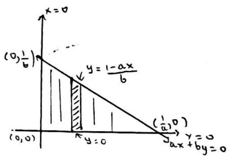
(3)Order of integration is decided by number of variables present inside integral sign.

[If X is comparatively lesser in number then we first integrate wrt to X or vice-versa. If both are same in number then priority is given 1st wrt Y and then wrt X]

Example 1:- $I = \iint_R \sin[\pi(ax+by)]\,dxdy$

where R is region bounded by $x = 0$ *to* $y = 0$ *and* $ax + by = 1$

Solution:-



let us integrate 1st wrt Y and then wrt X ;hence we take a strip parallel to Y-axis in region R

$$I = \int_{x=0}^{x=\frac{1}{a}} \int_{y=0}^{y=\frac{1-ax}{b}} \sin[\pi(ax+by)]\,dxdy$$

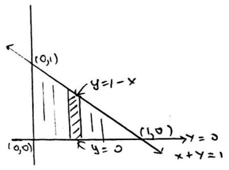
$$I = \int_{x=0}^{x=\frac{1}{a}} \left[\frac{-\cos[\pi(ax+by)]}{\pi b}\right]_{y=0}^{y=\frac{1-ax}{b}} dx = \frac{1}{\pi b}\int_0^{\frac{1}{a}} \{-\cos\,[\pi ax + \pi\,(1-ax)] + \cos\,(\pi ax)\}dx$$

$$I = \frac{1}{\pi b}\int_0^{\frac{1}{a}} [-\cos\pi + cos\pi ax]dx = I = \frac{1}{\pi b}\int_0^{\frac{1}{a}} [1 + cos\pi ax]dx = \frac{1}{\pi b}[x + \frac{sin\pi ax}{\pi a}]_0^{\frac{1}{a}}$$

$$I = \frac{1}{\pi b}\left[\frac{1}{a} + 0\right] = \frac{1}{\pi ab}$$

Example 2:-$I = \iint_R \sqrt{xy(1-x-y)}\,dxdy$ over region R given by ven by $x \geq 0, y \geq 0\,\&\, x + y \leq 1$

Solution:-



let us integrate 1st wrt Y and then wrt X ;hence we take a strip parallel to Y-axis in region R

$I = \int_{x=0}^{x=1} \sqrt{x} \int_{y=0}^{y=1-x} \sqrt{y}\sqrt{1-x-y}dxdy$ __put $y = (1-x)t$ *so* $dy = (1-x)dt$

y	0	1-x
t	0	1

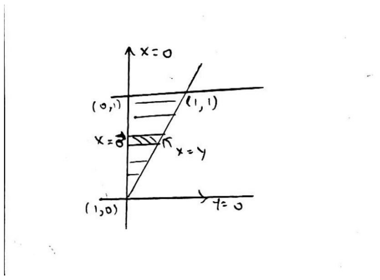
$$I = \int_{x=0}^{x=1} \sqrt{x}dx \int_{t=0}^{t=1} [(1-x)t]^{\frac{1}{2}}[(1-x)-(1-x)t]^{\frac{1}{2}}dt = \int_{x=0}^{x=1} \sqrt{x}(1-x)^2dx \int_{t=0}^{t=1} (1-t)^{\frac{1}{2}}t^{\frac{1}{2}}dt$$

$$I = \beta\left[\frac{3}{2},3\right]\beta\left[\frac{3}{2},\frac{3}{2}\right] = \frac{2\pi}{105}$$

Example 3:-

$\iint_R \frac{2xy^5}{\sqrt{1-y^4+x^2y^2}}dxdy$ *where R is the region bounded by triangle whose vertices are* $(0,0), (0,1), (1,0).$

Solution:-



let us integrate 1[st] wrt X and then wrt Y ;hence we take a strip parallel to X-axis in region R

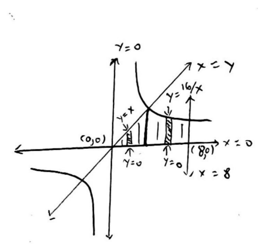
$$I=\int_{y=0}^{y=1} y^3 \int_{x=0}^{x=y} \frac{2xy^2}{\sqrt{1-y^4+x^2y^2}}dxdy=\int_{y=0}^{y=1} 2y^3\left(\sqrt{1-y^4+x^2y^2}\right)_{x=0}^{x=y}dy$$

$$I=\int_{y=0}^{y=1} 2y^3[1-\sqrt{1-y^4}]dy=\int_{y=0}^{y=1}[(2y^3)-\left(2y^3\sqrt{1-y^4}\right)]dy$$

$$I=2[\frac{y^4}{4}]_0^1+\frac{1}{2}\int_{y=0}^{y=1}\left(-4y^3\sqrt{1-y^4}\right)]dy=\frac{1}{2}[\frac{(1-y^4)^{\frac{3}{2}}}{\frac{3}{2}}]_0^1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

Example 4:-$I=\iint_R x^2dxdy \text{ where } R \rightarrow xy=16, y=x \text{ and } x=8$

Solution:-



let us integrate 1[st] wrt Y and then wrt X ;hence we take a strip parallel to Y-axis in region R

We divide the region for simplification.

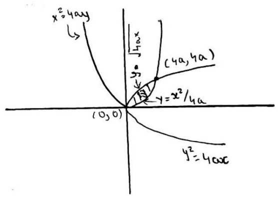
$$I_1=\int_0^4\int_{y=0}^{y=x} x^2dxdy=\int_0^4[x^2y]_{y=0}^{y=x}dx=\int_0^4 x^3dx=[\frac{x^4}{4}]_0^4=64$$

$$I_2=\int_4^8\int_{y=0}^{y=\frac{16}{x}} x^2dydx=\int_4^8[x^2y]_{y=0}^{y=\frac{16}{x}}dx=\int_4^8 16xdx=16[\frac{x^2}{2}]_{x=4}^{x=8}=384$$

So $I_1+I_2=64+384=448$

Example 5:-$I=\iint_R xydxdy \quad \text{where } R \text{ is region bounded by parabola } y^2=4ax \;\&\; x^2=4ay$

Solution:-



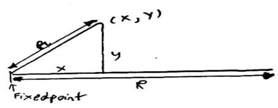
let us integrate 1st wrt Y and then wrt X ;hence we take a strip parallel to Y-axis in region R

$$I=\int_{x=0}^{x=4a} x\int_{y=\frac{x^2}{4a}}^{y=\sqrt{4ax}} ydxdy=\int_0^{4a} x^2\left[\frac{y^2}{2}\right]_{y=\frac{x^2}{4a}}^{y=\sqrt{4ax}}=\frac{1}{2}\int_0^{4a} x\left[4ax-\frac{x^4}{(4a)^2}\right]dx$$

$$I=\frac{1}{2}\left[4a\frac{x^3}{3}-\frac{x^6}{6(4a)^2}\right]_{x=0}^{x=4a}=\frac{1}{2}\left[\frac{(4a)^4}{3}-\frac{(4a)^4}{6}\right]=\frac{64a^4}{3}$$

Type 3:-Polar-form based questions:-

Whenever a region is bounded by circle or ellipse we use polar form or if the given integration contains x^2+y^2 term , then also we use polar form.



$cos\theta=\frac{x}{r}$ & $sin\theta=\frac{y}{r}$ so $x=rcos\theta$ & $y=rsin\theta$

i.e for converting into polar form we put $x=rcos\theta$ & $y=rsin\theta$ & also $x^2+y^2=r^2$ &

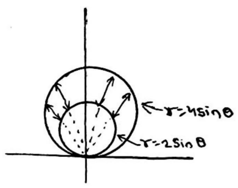
$dxdy=rdrd\theta$

Note:- If region is bounded by only ellipse , then we use elliptical polar given by

$x=arcos\theta$ & $y=brsin\theta$ and
$dxdy=abrdrd\theta$

Example 1:- evaluate $\iint r^3drd\theta$ over the area between two circles $r=2sin\theta$ & $r=4sin\theta$

Solution :-



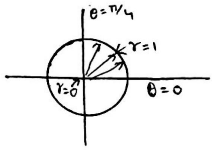
Limits :- $r=2sin\theta\ to\ r=4sin\theta$ and $\theta=0\ to\ \pi$

So $I=\int_{\theta=0}^{\theta=\pi}\int_{r=2sin\theta}^{r=4sin\theta} r^3drd\theta=\int_0^{\pi}\left[\frac{r^4}{4}\right]_{r=2sin\theta}^{r=4sin\theta}d\theta=\frac{1}{4}\int_0^{\pi}(256sin^4\theta-16sin^4\theta)d\theta$

$$I=\frac{240}{4}\int_0^{\pi} sin^4\theta d\theta=60\int_0^{\pi} sin^4\theta d\theta=120\int_0^{\frac{\pi}{2}} sin^4\theta d\theta=120\left(\frac{3.1}{4.2}\times\frac{\pi}{2}\right)=\frac{45\pi}{2}$$

Example 2:-Evaluate $\iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy$ where R is positive quadrant of circle $x^2+y^2=1$

Solution:-



put $x = rcos\theta$ & $y = rsin\theta$,so $x^2 + y^2 = r^2$

Limits :- $r = 0\ to\ r = 1$ and $\theta = 0\ to\ \frac{\pi}{2}$

r	0	1
t	0	$\frac{\pi}{2}$

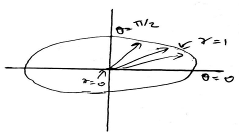
So $I = \int_{\theta=0}^{\theta=\frac{\pi}{2}} d\theta \int_{r=1}^{r=0} \sqrt{\frac{1-r^2}{1+r^2}}\, rdr = \frac{\pi}{2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, rdr$ ___[put $r^2 = cost\ so\ rdr = -\frac{1}{2}sintdt$]

$$I = \frac{\pi}{2}\int_{\frac{\pi}{2}}^{0} \sqrt{\frac{1-cost}{1+cost}}.(-\frac{1}{2}sintdt) = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \sqrt{\frac{2sin^2\frac{t}{2}}{2cos^2\frac{t}{2}}}\left(2sin\frac{t}{2}cos\frac{t}{2}\right)dt$$

$$I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} 2sin^2\frac{t}{2}dt = \frac{\pi}{4}\int_0^{\frac{\pi}{2}}(1-cost)dt = \frac{\pi}{4}\left(\frac{\pi}{2}-1\right) = \frac{\pi(\pi-2)}{8}$$

Example 3:-Evaluate $\iint_R xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dxdy$ where R is positive quadrent of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Solution:-



put $x = arcos\theta$ & $y = brsin\theta$ and
$dxdy = abrdrd\theta$

Limits :- $r = 0\ to\ r = 1$ and $\theta = 0\ to\ \frac{\pi}{2}$

So $I = \iint_R (arcos\theta)(arsin\theta)(r^2)^{\frac{n}{2}}(abrdrd\theta) = a^2b^2\int_{\theta=0}^{\theta=\frac{\pi}{2}} cos\theta sin\theta d\theta \int_{r=0}^{r=1} r^{n+3}dr = a^2b^2$

$$I = a^2b^2[\frac{1.1}{2}][\frac{r^{n+4}}{n+4}]_{r=0}^{r=1} = \frac{a^2b^2}{2(n+4)}$$

Now changing the order of integration:-

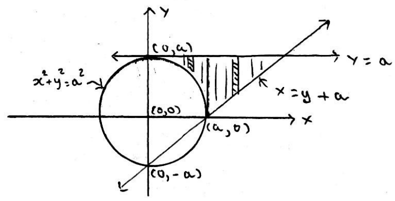
Let us understand with an example:- $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)dxdy$

Solution:- so given integration is 1st wrt X and then wrt Y

i.e $x = \sqrt{a^2-y^2}$ to $x = y + a$

therefore $x^2 + y^2 = a^2$ & $x + y = a$

And we get shaded region R as:-



Now we want to integrate 1st wrt Y and then wrt to X (i.e change the order), hence strip parallel to Y-axis in region R.

Region R is divided into R_1 &R_2 for simplisit.

Limits of $R_1 \rightarrow\ y = \sqrt{a^2 - x^2}\ \ to\ y = 0$ and $x = 0\ to\ x = a$

Limits of $R_2 \rightarrow\ y = x - a\ to\ y = a$ and $x = 0\ to\ x = 2a$

Therefore change of order of integration we have is ,

$$I = \int_{x=0}^{x=a} \int_{y=\sqrt{x^2-a^2}}^{y=a} f(x,y)dxdy + \int_{x=0}^{x=2a} \int_{y=x-a}^{y=a} f(x,y)dxdy$$