A single-item capacitated lot-sizing problem with stochastic demand under service level

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1 Joint chance-constrained programming model

We consider the single-item single capacitated lot-sizing problem. We present the deterministic variant of this problem which can be formulated as a mixed integer linear program (MILP). We model the problem using the general model and the aggregate formulation. We then consider a stochastic variants of this problem where the customer demand to be satisfied is subject to uncertainty and introduce the joint chance constrained programming formulation studied in the present paper.

2 Mathematical models

2.1 Deterministic models

We wish to plan production for a single product to be processed on a single capacitated resource over a planning horizon involving T periods indexed by $t = 1...T$. All problem parameters are assumed to be deterministically known at the time when the production plan is built. f_t denotes the fixed setup cost to be paid if production occurs on the resource in period t, h_t the inventory holding cost per unit held in stock at the end of period t and c_t the production capacity available in period t. d_t is the demand to be satisfied at the end of each period t and $dc_t = \sum_{t'=1}^{t} d'_t$ is the cumulative demand over interval $[1; t]$. The initial inventory level, I_0, is set to 0 without loss of generality. We introduce the following decision variables:

- x_t : the quantity produced in period t.
- $y_t \in \{0, 1\}$: the resource setup state in period t. $y_t = 1$ if a setup occurs in period t, 0 otherwise.
- I_t: the inventory at the end of period t

The first deterministic single-item single-resource capacitated lot-sizing problem can be formulated as follows (M_1):

$$min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} h_t I_t \tag{1}$$

$$x_t \leq C y_t \qquad \forall t \in 1, ..., T \tag{2}$$

$$I_{t-1} + x_t = d_t + I_t \qquad \forall t \in 1, ..., T \tag{3}$$

$$x_t, I_t \geq 0 \qquad \forall t \in 1, ..., T \tag{4}$$

$$y_t \in \{0, 1\} \qquad \forall t \in 1, ..., T \tag{5}$$

The second model using the aggregate formulation as follows (M_2):

$$min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} h_t \left(\sum_{t'=1}^{t} x'_t - dc_t \right) \tag{6}$$

$$x_t \leq C y_t \qquad \forall t \in 1, ..., T \tag{7}$$

$$\sum_{t'=1}^{t} x'_t = d_t \qquad \forall t \in 1, ..., T \tag{8}$$

$$x_t \geq 0 \qquad \forall t \in 1, ..., T \tag{9}$$

$$y_t \in \{0, 1\} \qquad \forall t \in 1, ..., T \tag{10}$$

2.2 Stochastic models

2.2.1 Classical formulation

$$min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} h_t I_t \tag{11}$$

$$x_t \leq C y_t \qquad \forall t \in 1, ..., T \tag{12}$$

$$Pr(I_{t-1} + x_t - I_t \geq d_t; \quad \forall t \in 1, ..., T) \geq 1 - \alpha \tag{13}$$

$$x_t, I_t \geq 0 \qquad \forall t \in 1, ..., T \tag{14}$$

$$y_t \in \{0, 1\} \qquad \forall t \in 1, ..., T \tag{15}$$

The obtained stochastic linear model (SM_1):

$$min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} h_t I_t \tag{16}$$

$$x_t \leq C y_t \qquad \forall t \tag{17}$$

$$I_{t-1} + x_t - I_t \geq z_t; \qquad \forall t \tag{18}$$

$$\sum_{t} \sum_{j=d_{t(min)}}^{d_t(max)} z_{jt} log(F_t(j)) \geq log(1 - \alpha) \tag{19}$$

$$\sum_{j=d_{t(min)}}^{d_t(max)} z_{jt} = 1, \qquad \forall t \tag{20}$$

$$z_t = \sum_{j=l_t}^{u_t} j z_{jt}, \qquad \forall t \tag{21}$$

$$x_t, I_t \geq 0 \qquad \forall t \tag{22}$$

$$y_t \in \{0,1\} \qquad \forall t \tag{23}$$

$$z_{jt} \in \{0,1\} \quad \forall t \forall j \in d_{t(min)}, ..., d_t(max) \tag{24}$$

2.2.2 The aggregate stochastic formulation

$$min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} h_t max \left(\left(\sum_{t'=1}^{t} x_{t'} - Dc_t \right), 0 \right) \tag{25}$$

$$x_t \leq C y_t \qquad \forall t \in 1, ..., T \tag{26}$$

$$Pr \left(\sum_{t'=1}^{t} x_{t'} \geq Dc_t; \forall t \in 1, ..., T \right) \geq 1 - \alpha \tag{27}$$

$$x_t \geq 0 \qquad \forall t \in 1, ..., T \tag{28}$$

$$y_t \in \{0,1\} \qquad \forall t \in 1, ..., T \tag{29}$$

A Linearization of SM1

$$\text{Define the set } Z_\alpha = \{ z \in [0,1]^T | Pr(d \leq z) \geq 1 - \alpha \} \tag{30}$$

$$\text{where } d = (d_1, ..., d_T)$$

As d is the vector of independent random variables

$$Pr(d \leq z) = \Pi_t (Pr(d_t \leq z_t)) \tag{31}$$

Equation (13) can be rewritten as follows:

$$I_{t-1} + x_t - I_t \geq z_t; \quad \forall t \in 1, ..., T \tag{32}$$

$$\Pi_t (Pr[d_t \leq z_t)] \geq 1 - \alpha \tag{33}$$

In order to linearize equation (33) first we apply log and then we introduce the binary variables z_{jt} as follows:

$$\sum_{j=l_t}^{u_t} z_{jt} = 1, \forall t \tag{34}$$

$$z_t = \sum_{j=l_t}^{u_t} j z_{jt} \tag{35}$$

$$\sum_t log(F_t(z_t)) \geq log(1 - \alpha) \tag{36}$$

$$\sum_t log(F_t(\sum_{j=l_t}^{u_t} jz_{jt})) \geq log(1-\alpha)$$

Finally we can linearize equation (36) as follows:

$$\sum_t \sum_{j=l_t}^{u_t} z_{jt} log(F_t(j)) \geq log(1-\alpha) \tag{37}$$

B Linearization of SM2

The objective function (25) and the constraint (27) are random variables.

References