



Local identification in DSGE models[☆]

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ABSTRACT

The issue of identification arises whenever structural models are estimated. Lack of identification means that the empirical implications of some model parameters are either undetectable or indistinguishable from the implications of other parameters. Therefore, identifiability must be verified prior to estimation. This paper provides a simple method for conducting local identification analysis in linearized DSGE models, estimated in both full and limited information settings. In addition to establishing which parameters are locally identified and which are not, researchers can determine whether the identification failures are due to data limitations, such as lack of observations for some variables, or whether they are intrinsic to the structure of the model. The methodology is illustrated using a medium-scale DSGE model.

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1. Introduction

Structural macroeconomic models are one of the important tools available to economic policy-makers. However, insofar as the quantitative implications of the models are of interest, it is crucial that the inputs to the models—their parameter values, have empirical credibility. This has prompted a very active research effort aimed at the estimation and empirical evaluation of DSGE models.

The empirical implications of DSGE models come from the restrictions they impose on the joint probability distribution of observed macroeconomic variables. Both full information methods, which use all of the restrictions, and limited information methods, which use only some of them, are applied in the literature to estimate the parameters of DSGE models. Whether consistent estimation is possible in the first place is typically not verified formally. Examples of unidentified DSGE models can be found in Kim (2003), Beyer and Farmer (2004) and Cochrane (2007). While it has been recognized that the lack of identification is potentially a serious problem, the issue is rarely addressed in the empirical literature. This is partly due to the widespread use of Bayesian methods, which, as Canova and Sala (2009) point out, may serve to conceal existing identification problems. Another reason is that few models allow for direct verification of identifiability, as in the studies cited above. Larger models can only be solved numerically, thus making it impossible to

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derive explicitly the relationship between the deep parameters and the statistical model used to estimate them. As a result it appears that parameter identification can only be assessed indirectly and with the heavy use of numerical methods.

This paper shows that local identification analysis is possible and indeed straightforward in the general class of linearized DSGE models estimated using first and second order moments of observables. This covers most econometric approaches used in the empirical DSGE literature, including full information likelihood-based methods under normality, and various limited information methods, such as impulse response matching. The analysis consists of evaluating the ranks of Jacobian matrices for which explicit analytical expressions are provided in the paper. Using the proposed necessary and sufficient conditions, researchers can determine which model parameters are identified and which are not, and whether identification failures are due to data limitations or are intrinsic to the structural model. Furthermore, the role of each observable variable and its moments for the identification of any parameter of interest may be assessed, thus providing guidance on the choice of observables to use for estimation.

In this paper I adhere to the classical treatment of identification as a “yes-or-no” problem. Establishing that a model is locally identified in this sense is important as it shows that consistent estimation is possible, and that the estimators possess the usual asymptotic properties. Lack of local identification, on the other hand, may point at issues with overparameterization in the structural model, which should be dealt with before estimation. Another important aspect of identification, not considered here, is the identification strength. A large recent literature has shown that weak identification may be a serious problem in structural estimation, affecting the accuracy of estimation and the reliability of asymptotic inference in finite samples. Weak identification issues in DSGE models are discussed by [Canova and Sala \(2009\)](#), in the context of estimation by impulse response matching, and by [Iskrev \(2009\)](#), in the context of maximum likelihood estimation under normality. These papers do not provide a general method for determining the identifiability of parameters in DSGE models.

Local identification in linearized DSGE models is studied also in [Komunjer and Ng \(2009\)](#). They present a rank condition for identification of the deep parameters from the spectrum of all endogenous variables in the model. Identification from the spectrum, which involves an infinite number of autocovariances, is necessary but not sufficient for identification with a finite number of second order moments. The condition in [Komunjer and Ng \(2009\)](#) is also not sufficient for identification when some of the endogenous variables in the model are not observable, which is common in practice. Furthermore, identifiability from the autocovariances is not necessary for identification when both first and second order moments are available. Thus, deep parameters that affect only the mean, or both the mean and the autocovariances of the observables, may be unidentifiable from the spectrum. In contrast, using the approach in the present paper one can determine if the deep parameters can be identified from all, or a subset of, the means and any finite number of autocovariances of observable variables.¹

The remainder of the paper is organized as follows. Section 2 introduces the class of linearized DSGE models and the notation used throughout the paper. It also describes the relationship between the deep parameters and the first and second order moments of the observables. The main results of the paper are in Section 3, where the necessary and sufficient rank conditions for local identification are presented, and it is explained how the required Jacobian matrices can be computed analytically. Section 3 also discusses how to determine whether identification failures are inherent in the structure of the economic model, or are due to the fact that some model variables are not observed, and are therefore excluded from the analysis. In Section 4, the rank condition is extended to estimation methods based on transformations of the first and second order moments, using impulse response matching and VAR-based estimation as examples. Section 5 applies the methodology to the medium-scale DSGE model estimated by [Smets and Wouters \(2007\)](#). Concluding comments are given in Section 6.

2. DSGE models

This section provides a brief discussion of the class of linearized DSGE models and the restrictions they imply on the first and second order moments of the observed variables.

2.1. Structural model and reduced form

A DSGE model is summarized by a system of non-linear equations. Currently, most studies involving either simulation or estimation of DSGE models use linear approximations of the original models. That is, the model is first expressed in terms of stationary variables, and then linearized around the steady-state values of these variables. Let $\hat{\mathbf{z}}_t$ be a m -dimensional vector of the stationary variables, and $\hat{\mathbf{z}}^*$ be the steady state value of $\hat{\mathbf{z}}_t$. Once linearized, most DSGE models can be written in the following form

$$\Gamma_0(\theta)\mathbf{z}_t = \Gamma_1(\theta)\mathbf{E}_t\mathbf{z}_{t+1} + \Gamma_2(\theta)\mathbf{z}_{t-1} + \Gamma_3(\theta)\mathbf{u}_t \quad (1)$$

¹ Local identification from the means and an infinite number of autocovariances can be established by verifying the positive definiteness of the asymptotic expected Fisher information matrix. For linear Gaussian DSGE models the information matrix can be computed analytically as shown in [Iskrev \(2008\)](#).

where $\mathbf{z}_t = \hat{\mathbf{z}}_t - \hat{\mathbf{z}}^*$, and \mathbf{u}_t an n -dimensional random vector of structural shocks with $\mathbf{E}\mathbf{u}_t = \mathbf{0}$, $\mathbf{E}\mathbf{u}_t\mathbf{u}_t' = \mathbf{I}_n$. The elements of the matrices $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 are functions of a k -dimensional vector of deep parameters θ , where θ is a point in $\Theta \subset \mathbb{R}^k$. The parameter space Θ is defined as the set of all theoretically admissible values of θ .

There are several algorithms for solving linear rational expectations models (see for instance Blanchard and Kahn, 1980; Anderson and Moore, 1985; King and Watson, 1998; Klein, 2000; Christiano, 2002; Sims, 2002).² Depending on the value of θ , there may exist zero, one, or many stable solutions. Assuming that a unique solution exists, it can be cast in the following form

$$\mathbf{z}_t = \mathbf{A}(\theta)\mathbf{z}_{t-1} + \mathbf{B}(\theta)\mathbf{u}_t \quad (2)$$

where the $m \times m$ matrix \mathbf{A} and the $m \times n$ matrix \mathbf{B} are functions of θ .

For a given value of θ , the matrices \mathbf{A} , $\mathbf{\Omega} := \mathbf{B}\mathbf{B}'$, and $\hat{\mathbf{z}}^*$ completely characterize the equilibrium dynamics and steady state properties of all endogenous variables in the linearized model. Typically, some elements of these matrices are constant, i.e. independent of θ . For instance, if the steady state of some variables is zero, the corresponding elements of $\hat{\mathbf{z}}^*$ will be zero as well. Furthermore, if there are exogenous autoregressive (AR) shocks in the model, the matrix \mathbf{A} will have rows composed of zeros and the AR coefficients. As a practical matter, it is useful to separate the solution parameters that depend on θ from those that do not. I will use τ to denote the vector collecting the non-constant elements of $\hat{\mathbf{z}}^*$, \mathbf{A} , and $\mathbf{\Omega}$, i.e. $\tau := [\tau_z', \tau_A', \tau_\Omega']$, where τ_z , τ_A , and τ_Ω denote the elements of $\hat{\mathbf{z}}^*$, $\text{vec}(\mathbf{A})$ and $\text{vech}(\mathbf{\Omega})$ that depend on θ .³

In most applications the model in (2) cannot be taken to the data directly since some of the variables in \mathbf{z}_t are not observed. Instead, the solution of the DSGE model is expressed in a state space form, with transition equation given by (2), and a measurement equation

$$\mathbf{x}_t = \mathbf{s}(\theta) + \mathbf{C}\mathbf{z}_t \quad (3)$$

where \mathbf{x}_t is a l -dimensional vector of observed variables, $\mathbf{s}(\theta) := \mathbf{C}\hat{\mathbf{z}}^*(\theta)$ is a l -dimensional vector, and \mathbf{C} is a $l \times m$ matrix selecting the observed among all variables in \mathbf{z}_t .

In the absence of a structural model it would, in general, be impossible to fully recover the properties of \mathbf{z}_t from observing only \mathbf{x}_t . Having the model in (1) makes this possible by imposing restrictions, through (2) and (3), on the joint probability distribution of the observables. The model-implied restrictions on the first and second order moments of the \mathbf{x}_t are discussed next.

2.2. Theoretical first and second moments

From (2)–(3) it follows that the unconditional first and second moments of \mathbf{x}_t are given by

$$\mathbf{E}\mathbf{x}_t := \boldsymbol{\mu}_x = \mathbf{s} \quad (4)$$

$$\text{cov}(\mathbf{x}_{t+i}, \mathbf{x}_t') := \Sigma_x(i) = \begin{cases} \mathbf{C}\Sigma_z(0)\mathbf{C}' & \text{if } i = 0 \\ \mathbf{C}\mathbf{A}^i\Sigma_z(0)\mathbf{C}' & \text{if } i > 0 \end{cases} \quad (5)$$

where $\Sigma_z(0) := \mathbf{E}\mathbf{z}_t\mathbf{z}_t'$ solves the matrix equation

$$\Sigma_z(0) = \mathbf{A}\Sigma_z(0)\mathbf{A}' + \mathbf{\Omega} \quad (6)$$

Denote the observed data with $\mathbf{X}_T := [\mathbf{x}_1', \dots, \mathbf{x}_T']'$, and let Σ_T be its covariance matrix, i.e.

$$\Sigma_T := \mathbf{E}\mathbf{X}_T\mathbf{X}_T' = \begin{pmatrix} \Sigma_x(0), & \Sigma_x(1)', & \dots, & \Sigma_x(T-1)' \\ \Sigma_x(1), & \Sigma_x(0), & \dots, & \Sigma_x(T-2)' \\ \dots & \dots & \dots & \dots \\ \Sigma_x(T-1), & \Sigma_x(T-2), & \dots, & \Sigma_x(0) \end{pmatrix} \quad (7)$$

Let $\boldsymbol{\sigma}_T$ be a vector collecting the unique elements of Σ_T , i.e.

$$\boldsymbol{\sigma}_T := [\text{vech}(\Sigma_x(0))', \text{vec}(\Sigma_x(1))', \dots, \text{vec}(\Sigma_x(T-1))']'$$

Furthermore, let $\mathbf{m}_T := [\boldsymbol{\mu}', \boldsymbol{\sigma}_T']'$ be a $(T-1)^2 + l(l+3)/2$ -dimensional vector collecting the parameters that determine the first two moments of the data. Assuming that the linearized DSGE model is determined everywhere in Θ , i.e. τ is unique for each admissible value of θ , it follows that \mathbf{m}_T is a function of θ . If either \mathbf{u}_t is Gaussian, or there are no distributional assumptions about the structural shocks, the model-implied restrictions on \mathbf{m}_T contain all information that can be used for

² Although these algorithms use different representations of the linearized model and of the solution, it is not difficult to convert one representation into another. See the appendix in Anderson (2008) for some examples.

³ The number of constants in the solution matrices may also depend on the solution algorithm one uses. For instance, to write the model in the form used by Sims (2002) procedure, one may have to include in \mathbf{z}_t redundant state variables; this will increase the size of the solution matrices and the number of zeros in them. Removing the redundant states and excluding the constant elements from τ is not necessary, but has practical advantages in terms of speed and numerical accuracy of the calculations. See also footnote 5.

the estimation of θ . The identifiability of θ depends on whether that information is sufficient or not. This is the subject of the next section.

3. Identification

This section explains the role of the Jacobian matrix of the mapping from θ to \mathbf{m}_T for identification, and shows how it can be computed analytically.

3.1. The rank condition

The probability density function of the data contains all available sample information about the value of the parameter vector of interest θ . Thus, a basic prerequisite for making inference about θ is that distinct values of θ imply distinct values of the density function. This is known as the identification condition.

Definition 1. Let $\theta \in \Theta \subset \mathbb{R}^k$ be the parameter vector of interest, and suppose that inference about θ is made on the basis of T observations of a random vector \mathbf{x} with a known joint probability density function $f(\mathbf{X}; \theta)$, where $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_T]$. A point $\theta_0 \in \Theta$ is said to be globally identified if

$$f(\mathbf{X}; \tilde{\theta}) = f(\mathbf{X}; \theta_0) \quad \text{with probability } 1 \Rightarrow \tilde{\theta} = \theta_0 \quad (8)$$

for any $\tilde{\theta} \in \Theta$. If (8) is true only for values $\tilde{\theta}$ in an open neighborhood of θ_0 , then θ_0 is said to be locally identified.

In most applications the distribution of \mathbf{X} is unknown or assumed to be Gaussian. Thus, the estimation of θ is usually based on the first two moments of the data. If the data is not normally distributed, higher-order moments may provide additional information about θ , not contained in the first two moments. Therefore, identification based on the mean and the variance of \mathbf{X} is only sufficient but not necessary for identification with the complete distribution. Using the notation introduced in the previous section, we have the following result (see, e.g., Hsiao, 1983 and the references therein)

Theorem 1. Suppose that the data \mathbf{X}_T is generated by the model (2)–(3) with parameter vector θ_0 . Then θ_0 is globally identified if

$$\mathbf{m}_T(\tilde{\theta}) = \mathbf{m}_T(\theta_0) \Leftrightarrow \tilde{\theta} = \theta_0 \quad (9)$$

for any $\tilde{\theta} \in \Theta$. If (9) is true only for values $\tilde{\theta}$ in an open neighborhood of θ_0 , the identification of θ_0 is local. If the structural shocks are normally distributed, then the condition in (9) is also necessary for identification.

The condition in (9) requires that the mapping from the population moments of the sample— $\mathbf{m}_T(\theta)$, to θ is unique. If this is not the case, there exist different values of θ that result in the same value of the population moments, and the true value of θ cannot be determined even with an infinite number of observations. In general, there are no known global conditions for unique solutions of systems of non-linear equations, and it is therefore difficult to establish the global identifiability of θ . Local identification, on the other hand, can be verified with the help of the following condition:

Theorem 2. Suppose that \mathbf{m}_T is a continuously differentiable function of θ . Then θ_0 is locally identifiable if the Jacobian matrix $J(q) := \partial \mathbf{m}_q / \partial \theta'$ has a full column rank at θ_0 for $q \leq T$. This condition is both necessary and sufficient when $q = T$ if \mathbf{u}_t is normally distributed.

This result follows from the implicit function theorem, and can be found, among others, in Fisher (1966) and Rothenberg (1971).⁴ Note that, even though $J(T)$ having full rank is not necessary for local identification in the sense of Definition 1, it is necessary for identification from the first and second order moments. Therefore, when the rank of $J(T)$ is less than k , θ_0 is said to be unidentifiable from a model that utilizes only the mean and the variance of \mathbf{X}_T . A necessary condition for identification in that sense is that the number of deep parameters does not exceed the dimension of \mathbf{m}_T , i.e. $k \leq (T-1)l^2 + l(l+3)/2$. A stronger necessary condition for identification will be presented in the next section.

The local identifiability of a point θ_0 can be established by verifying that the Jacobian matrix $J(T)$ has full column rank when evaluated at θ_0 . Local identification at one point in Θ , however, does not guarantee that the model is locally identified everywhere in the parameter space. There may be some points where the model is locally identified, and others where it is not. Moreover, local identifiability everywhere in Θ is necessary but not sufficient to ensure global identification. Nevertheless, it is important to know if a model is locally identified or not for the following two reasons. First, local identification makes possible the consistent estimation of θ , and is sufficient for the estimator to have the usual asymptotic properties (see Florens et al., 2008). Second, and perhaps more important in the context of DSGE models is that with the help of the Jacobian matrix we can detect problems that are a common cause for identification failures in these

⁴ Both Fisher (1966) and Rothenberg (1971) makes the additional assumption that θ_0 is a regular point of $J(T)$, which means that if it belongs to an open neighborhood where the rank of the matrix does not change. Without this assumption the rank condition in Theorem 2 is only sufficient for local identification under normality. Although it is possible to construct examples where regularity does not hold (see Shapiro and Browne, 1983), typically the set of irregular points is of measure zero (see Bekker and Pollock, 1986).

models. If, for instance, a deep parameter θ_j does not affect the solution of the model, it will be unidentifiable since its value is irrelevant for the statistical properties of the data generated by the model, and the first and second moments in particular. Consequently, $\partial \mathbf{m}_T / \partial \theta_j$ —the column of $J(T)$ corresponding to θ_j , will be a vector of zeros for any T , and the rank condition for identification will fail. Another type of identification failure occurs when two or more parameters enter in the solution in a manner which makes them indistinguishable, e.g. as a product or a ratio. As a result it will be impossible to identify the parameters separately, and some of the columns of the Jacobian matrix will be linearly dependent. An example of the first problem is the unidentifiability of the Taylor rule coefficients in a simple New Keynesian model pointed out in [Cochrane \(2007\)](#). An example of the second is the equivalence between the intertemporal and multisectoral investment adjustment cost parameters in [Kim \(2003\)](#). In these papers the problems are discovered by solving the models explicitly in terms of the deep parameters. That approach, however, is not feasible for larger models, which can only be solved numerically. As will be shown next, the Jacobian matrix in Theorem 2 is straightforward to compute analytically for linearized models of any size or complexity.

3.2. Computing the Jacobian matrix

The simplest method for computing the Jacobian matrix of the mapping from θ to \mathbf{m}_T is by numerical differentiation. The problem with this approach is that numerical derivatives tend to be inaccurate for highly non-linear functions. In the present context this may lead to wrong conclusions concerning the rank of the Jacobian matrix and the identifiability of the parameters in the model. Some evidence to that effect are discussed in Section 5. For this reason, it is better to use analytical derivatives whenever possible. Below I outline the main steps in obtaining the Jacobian matrix analytically leaving the full derivation to the online appendix.

It helps to consider the mapping from θ to \mathbf{m}_T as comprising two steps: (1) a transformation from θ to τ ; (2) a transformation from τ to \mathbf{m}_T . Thus, the Jacobian matrix can be expressed as

$$J(T) = \frac{\partial \mathbf{m}_T}{\partial \tau'} \frac{\partial \tau}{\partial \theta'} \quad (10)$$

The derivation of the first term on the right-hand side is straightforward since the function mapping τ into \mathbf{m}_T is available explicitly (see the definition of τ and Eqs. (4)–(6)); thus the Jacobian matrix $J_1(T) := \partial \mathbf{m}_T / \partial \tau'$ may be obtained by direct differentiation. Closed-form expressions for computing $J_1(T)$ are provided in the online appendix.

The elements of the second term, the Jacobian of the transformation from θ to τ , can be divided into three groups corresponding to the three blocks of τ : τ_z , τ_A and τ_Ω . Since \hat{z}^* is a known function of θ , implied by the steady state of the model, the derivative of τ_z can be computed by direct differentiation. The derivatives of τ_A and τ_Ω can be obtained from the derivatives of $\text{vec}(\mathbf{A})$ and $\text{vech}(\Omega)$, by removing the zeros corresponding to the constant elements of \mathbf{A} and Ω .⁵ The derivative of $\text{vec}(\mathbf{A})$ can be computed using the implicit function theorem as suggested in [Iskrev \(2008\)](#). An implicit function of θ and $\text{vec}(\mathbf{A})$ is provided by the restrictions the structural model (1) imposes on the reduced form (2). In particular, from (2) we have $E_t \mathbf{z}_{t+1} = \mathbf{A} \mathbf{z}_t$, and substituting in (1) yields

$$(\Gamma_0 - \Gamma_1 \mathbf{A}) \mathbf{z}_t = \Gamma_2 \mathbf{z}_{t-1} + \Gamma_3 \mathbf{u}_t \quad (11)$$

Combining the last equation with Eq. (2) gives to the following matrix equation

$$\mathbf{F}(\theta, \text{vec}(\mathbf{A})) := (\Gamma_0(\theta) - \Gamma_1(\theta) \mathbf{A}) \mathbf{A} - \Gamma_2(\theta) = \mathbf{O} \quad (12)$$

Vectorizing (12) and applying the implicit function theorem gives

$$\frac{\partial \text{vec}(\mathbf{A})}{\partial \theta'} = - \left(\frac{\partial \text{vec}(\mathbf{F})}{\partial \text{vec}(\mathbf{A})'} \right)^{-1} \frac{\partial \text{vec}(\mathbf{F})}{\partial \theta'} \quad (13)$$

Closed-form expressions for computing the derivatives in (13) are provided in the online appendix.

Using $\Omega = \mathbf{B} \mathbf{B}'$, the differential of Ω is given by

$$d\Omega = d\mathbf{B} \mathbf{B}' + \mathbf{B} d\mathbf{B}' \quad (14)$$

Having $d\Omega$ in terms of $d\mathbf{B}$ is convenient since it shows how to obtain the derivative of Ω from that of \mathbf{B} . Note that from Eqs. (11) and (2) we have

$$(\Gamma_0 - \Gamma_1 \mathbf{A}) \mathbf{B} = \Gamma_3 \Rightarrow \mathbf{B} = (\Gamma_0 - \Gamma_1 \mathbf{A})^{-1} \Gamma_3 \quad (15)$$

Thus, once $\partial \text{vec}(\mathbf{A}) / \partial \theta'$ is available, it is straightforward to compute, first $\partial \text{vec}(\mathbf{B}) / \partial \theta'$ and $\partial \text{vech}(\Omega) / \partial \theta'$, and then $\partial \tau_A / \partial \theta'$ and $\partial \tau_\Omega / \partial \theta'$. For more details see the online appendix.

⁵ In principle one should obtain the same value of the Jacobian $J(T)$ whether the constants in the solution matrices are included in τ or not. In practice, however, there may be differences due to: (1) the finite precision of the computer arithmetic, and (2) numerical inaccuracies inherent in the solution algorithms for solving linear rational expectations models (see [Anderson, 2008](#) for details). As a result, terms in the Jacobian matrix that should be equal to zero may be numerically different from zero. This may lead to wrong conclusion regarding the rank of the Jacobian matrix and the identifiability of the parameters. See Section 5.4 for a related discussion of the use of numerical derivatives.

3.3. Discussion

It should be emphasized that the conditions for (local) identification given in Definition 1 and Theorem 2, involve the true values of the moments and reflect the understanding of identification as a population, not a finite sample issue.⁶ Thus, the question whether a model is identified or not can in principle be addressed prior to confronting the model with a particular set of data. In practice, since the number of points in Θ is infinite, one can only check whether the rank condition holds in some parts of the parameter space. A procedure for doing this, using many random draws from Θ , is described in Section 3.4.

Although the identifiability of θ does not depend on the characteristics of a particular set of observations, it does depend on how many and which of the variables in the model are observed, as well as on the sample size. There must be at least as many moments as there are deep parameters, in order to satisfy the order condition. In general, having more moments may help to identify a model which is otherwise unidentifiable. Since the number of available moments is limited by the sample size, the latter is one aspect of the data that is relevant for identification. Furthermore, how many and which of the variables in the model are observed, are features of the data that are also relevant for identification. Again, more observed variables is generally better than fewer, although having more variables is not necessarily better than having a smaller number of different variables. For instance, observing capital may be more useful for the identification of real business cycle model than observing several endogenous jump variables instead of capital.

One can easily check whether having more data would help with identification. A larger sample size would increase the value of T and the dimension of \mathbf{m}_T . The effect of having additional observed variables can be investigated by changing the dimension of \mathbf{C} , the matrix which selects the observed among all model variables (see Eq. (3)). For instance, to find out the effect of observing capital on the identification of θ , one has to increase the number of rows in \mathbf{C} with a row vector that has 1 in the position of capital in \mathbf{z}_t , and zeros everywhere else.

In some models identification may fail for purely model-related reasons, not because of data limitations. This happens, for instance, if there are parameters that play no role in the equilibrium of the model, or if the effect of a parameter cannot be distinguished from that of other parameters. As was indicated in Section 3, such problems are common in the DSGE literature, and are not always easy to detect by inspecting the equations of a model. The factorization of the Jacobian matrix in Eq. (10) provides a simple method for doing that. Note that the second term, $J_2 := \partial\tau/\partial\theta$, captures the effect of perturbations in θ on the parameters characterizing the equilibrium of the model. The matrix will have less than full column rank if either: (1) $\partial\tau/\partial\theta_i = \mathbf{0}$, or (2) $\partial\tau/\partial\theta_i = \sum_{j \neq i} a_j \partial\tau/\partial\theta_j$, where at least one of the constants a_j is not zero. In either case some parameters are unidentified in the model, and therefore cannot be identified when the model is taken to the data, even if all state variables were observed. This implies the following corollary to Theorem 1,

Corollary 1. *The point θ_0 is locally identifiable only if the rank of J_2 at θ_0 is equal to k .*

The condition is necessary because the distribution of \mathbf{X}_T depends on θ only through τ , irrespectively of the distribution of \mathbf{u}_t . It is not sufficient since, unless all state variables are observed, τ may be unidentifiable.⁷

The discussion so far has focused on the use of the Jacobian matrix to study parameter identification. However, there are other purposes for which the formulas presented in this paper may be useful. For instance, we can use an analytical Jacobian of transformation to construct priors about deep parameters on the basis of beliefs about the moments of the data. This method for eliciting priors for the parameters in DSGE models was developed in [Del Negro and Schorfheide \(2008\)](#) using dummy observations instead of the analytical change of variables. Having analytical derivatives is also very convenient when DSGE models are estimated using gradient-based optimization methods, or when inference is based on the usual first-order approximations. From an economic modeling perspective, one may wish to know the sign and size of the effect of small changes in some deep parameters on properties of the equilibrium outcomes, such as volatility or persistence of some endogenous variables. Such questions can be answered directly by using the expressions for analytical derivatives derived here.

3.4. Identification analysis procedure

The point of departure in the analysis of identification consists of two elements: a system of equations describing the linearized model, and a parameter space. One difficulty with determining Θ for DSGE models is that it is usually impossible to know, before solving the model, for which values of θ there are either zero or many solutions. Such points are typically deemed as inadmissible, and therefore have to be excluded from Θ . A second problem arises from the fact that there are infinitely many points in Θ , and it is not feasible to check the identification condition for all of them. In view of these difficulties, one possible approach is to check the rank condition at many randomly drawn points from Θ' , where $\Theta \subset \Theta'$,

⁶ In the words of [Koopmans and Reiersøl \(1950, p. 170\)](#) “the study of identifiability proceeds from a hypothetical exact knowledge of the probability distribution of observed variables rather than from a finite sample of observations.”

⁷ The identifiability of the solution parameters τ depends on the number of observables on one hand, and on the number and the location of the constant elements of \mathbf{A} , $\mathbf{\Omega}$, and \mathbf{z}^* , on the other. It is easy to show that, if \mathbf{S} is an invertible matrix satisfying $\mathbf{C} = \mathbf{CS}^{-1}$, then $\tilde{\tau}$ defined by $\tilde{\mathbf{A}} = \mathbf{SAS}^{-1}$, $\tilde{\mathbf{\Omega}} = \mathbf{S\Omega S}^{-1}$, and $\tilde{\mathbf{z}}^* = \mathbf{S z}^*$ is observationally equivalent to τ . In order to have $\tilde{\tau} = \tau$, there must be sufficiently many restricted elements of the solution matrices, so that the only \mathbf{S} which satisfies the restrictions is the identity matrix.

discarding values of θ that do not imply a unique solution. The set Θ' contains all values of θ that are theoretically plausible, and may be constructed by specifying a lower and an upper bound for each element of θ . Such bounds are usually easy to come by from the economic meaning of the parameters. After specifying a distribution for θ with support on Θ' , one can obtain points from Θ by drawing from Θ' and removing draws for which the model is either indetermined or does not have a solution.⁸ Conditions for existence and uniqueness can be found in Sims (2002), and are automatically checked by most computer algorithms for solving linear rational expectations models. The identifiability of each draw θ^j is then established using the necessary and sufficient conditions discussed above. Finding that matrix J_2 is rank deficient at θ^j implies that this particular point in Θ is unidentifiable in the model. Finding that J_2 has full rank but $J(T)$ does not, means that θ^j cannot be identified given the set of observed variables and the number of observations. On the other hand, if θ is identified at all, it would typically suffice to check the rank condition for a small number of moments, since $J(q)$ is likely to have full rank for q much smaller than T . According to Theorem 2 this is sufficient for identification; moreover, the smaller matrix may be much easier to evaluate than the Jacobian matrix for all available moments. A good candidate to try first is the smallest q for which the order condition is satisfied, and then increase the number of moments if the rank condition fails.

4. Extensions

This section discusses extensions of the rank condition for identification to estimation methods based on transformations of the first and second order moments of \mathbf{X}_T . The particular examples considered are the VAR-based indirect inference approach proposed by Smith (1993), and the impulse response matching estimator, used, among others, in Rotemberg and Woodford (1997) and Christiano et al. (2005).

Suppose that ξ is a r -dimensional vector such that $\xi = \mathbf{g}(\mathbf{m}_T)$, and that the functions $g_1(\mathbf{m}_T), \dots, g_r(\mathbf{m}_T)$ are continuously differentiable. Then, in parallel to Theorem 2, a sufficient condition for local identification of $\theta_0 \in \Theta$ is that $\partial \xi / \partial \theta'$ has full column rank at $\xi_0 = \mathbf{g}(\mathbf{m}_T(\theta_0))$. Using the chain rule, we have

$$\frac{\partial \xi}{\partial \theta'} = \frac{\partial \xi}{\partial \mathbf{m}_T'} \frac{\partial \mathbf{m}_T}{\partial \theta'} \quad (16)$$

where the second term in the product on the right-hand side has already been derived in Section 3.2.

To illustrate the derivation of $\partial \xi / \partial \mathbf{m}_T'$, suppose that the auxiliary model used to estimate θ is a VAR with p lags, i.e.

$$\mathbf{x}_t = \varphi^{(p)} + \sum_{i=1}^p \Phi_i^{(p)} \mathbf{x}_{t-i} + \varepsilon_t^{(p)} \quad (17)$$

where $\varepsilon_t^{(p)}$ is uncorrelated with \mathbf{x}_{t-i} , $i \geq 1$. Let $\Phi^{(p)} := [\Phi_1^{(p)'}, \dots, \Phi_p^{(p)'}]'$ and $\Omega^{(p)} := E \varepsilon_t^{(p)} \varepsilon_t^{(p)'}.$ Then we have

$$\xi = [\varphi^{(p)'}, \text{vec}(\Phi^{(p)})', \text{vech}(\Omega^{(p)})']'$$

where $\varphi^{(p)}$, $\Phi^{(p)}$ and $\Omega^{(p)}$ are given by

$$\varphi^{(p)} = \left(\mathbf{I}_l - \sum_{i=1}^p \Phi_i^{(p)} \right) \mathbf{s}$$

$$\Phi^{(p)} = \begin{bmatrix} \Sigma_{\mathbf{x}}(0) & \Sigma_{\mathbf{x}}(1) & \dots & \Sigma_{\mathbf{x}}(p-1) \\ \Sigma_{\mathbf{x}}(1)' & \Sigma_{\mathbf{x}}(0) & \dots & \Sigma_{\mathbf{x}}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{\mathbf{x}}(p-1)' & \Sigma_{\mathbf{x}}(p-2)' & \dots & \Sigma_{\mathbf{x}}(0) \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{\mathbf{x}}(1) \\ \Sigma_{\mathbf{x}}(2) \\ \vdots \\ \Sigma_{\mathbf{x}}(p) \end{bmatrix}$$

and

$$\Omega^{(p)} = \Sigma_{\mathbf{x}}(0) - \begin{bmatrix} \Sigma_{\mathbf{x}}(1)' \\ \Sigma_{\mathbf{x}}(2)' \\ \vdots \\ \Sigma_{\mathbf{x}}(p)' \end{bmatrix} \begin{bmatrix} \Sigma_{\mathbf{x}}(0) & \Sigma_{\mathbf{x}}(1) & \dots & \Sigma_{\mathbf{x}}(p-1) \\ \Sigma_{\mathbf{x}}(1)' & \Sigma_{\mathbf{x}}(0) & \dots & \Sigma_{\mathbf{x}}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{\mathbf{x}}(p-1)' & \Sigma_{\mathbf{x}}(p-2)' & \dots & \Sigma_{\mathbf{x}}(0) \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{\mathbf{x}}(1) \\ \Sigma_{\mathbf{x}}(2) \\ \vdots \\ \Sigma_{\mathbf{x}}(p) \end{bmatrix}$$

Differentiating $\varphi^{(p)}$, $\text{vec}(\Phi^{(p)})$ and $\text{vech}(\Omega^{(p)})$ with respect to \mathbf{m}_T gives $\partial \xi / \partial \mathbf{m}_T'$.⁹

Matching impulse responses to identified structural shocks is another estimation approach that has been used in the empirical DSGE literature. To check the rank condition for identification one could compute the Jacobian matrix of the mapping from the VAR or moving average representation of \mathbf{x}_t to the implied impulse responses one wishes to match. However, it is simpler to use directly the relationship between the impulse responses and the parameters of the reduced form solution. Note that from the state space representation (1)–(3) we have the following expression for the h -step ahead

⁸ The choice of distribution for θ may follow the logic of specifying a prior distribution for a Bayesian estimation of DSGE models.

⁹ Note that here ξ is a function of the mean and the first p second moments and of \mathbf{x}_t . Thus $\partial \xi / \partial \theta' = \partial \xi / \partial \mathbf{m}_p' \partial \mathbf{m}_p / \partial \theta'$.

response of the variables in \mathbf{x} to an impulse in period t

$$\xi^h := E_t \mathbf{x}_{t+h} - E_{t-1} \mathbf{x}_{t+h} = \mathbf{CA}^h \mathbf{B}, \quad h \geq 0 \quad (18)$$

The h -step ahead response of the i -th observable variable to an impulse in the j -th structural shock is given by the (i, j) -th element of ξ^h .

An impulse response function matching estimator minimizes the distance between the vector ξ , collecting all theoretical impulse response of interest, and its empirical counterpart. As before, the rank condition for local identification is that the rank of the Jacobian matrix $\partial \xi / \partial \theta'$ equals the dimension of θ . The Jacobian matrix may be constructed as in (16) by using the expression for ξ^h in (18) from which it is straightforward to compute $\partial \xi_{ij}^h / \partial \theta_j$ for any horizon h . Details are provided in the online appendix. Note that this estimator does not utilize the mean of \mathbf{x}_t , and therefore does not account for some of the steady state-implied restrictions that may be useful for identifying some elements of θ .

5. Application: identification in the Smets and Wouters (2007) model

This section applies the rank conditions developed above to a medium-scale DSGE model estimated in Smets and Wouters (2007) (SW07 henceforth). An outline of the main components of the model is presented first, and then the identification of the parameters is analyzed.¹⁰

5.1. The model

The model, based on the work of Smets and Wouters (2003) and Christiano et al. (2005), is an extension of the standard RBC model featuring a number of nominal frictions and real rigidities. These include: monopolistic competition in goods and labor markets, sticky prices and wages, partial indexation of prices and wages, investments adjustment costs, habit persistence and variable capacity utilization. The economy evolves along a balanced growth path, driven by a deterministic labor-augmenting technological progress. The endogenous variables in the model, expressed as log-deviations from steady state, are: output (y_t), consumption (c_t), investment (i_t), utilized and installed capital (k_t^p, k_t), capacity utilization (z_t), rental rate of capital (r_t^k), Tobin's q (q_t), price and wage markup (μ_t^p, μ_t^w), inflation rate (π_t), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). The log-linearized equilibrium conditions for these variables are presented in Table 1. The last equation in the table gives the policy rule followed by the central bank, which sets the nominal interest rate in response to inflation and the deviation of output from its potential level. To determine potential output, defined as the level of output that would prevail in the absence of the price and wage mark-up shocks, the set of equations in Table 1 is extended with their flexible price and wage version (see Table 2). The the business cycle dynamics of the model is driven by seven stationary shocks. Five of them—total factor productivity (ε_t^a), investment-specific technology (ε_t^i), government purchases (ε_t^g), risk premium (ε_t^b), and monetary policy (ε_t^r)—follow AR(1) processes; the remaining two shocks—to wage and price markup (ε_t^w and ε_t^p)—follow ARMA(1,1) processes.

The model is estimated using data on seven variables: output growth, consumption growth, investment growth, real wage growth, inflation, hours worked and the nominal interest rate. Thus, the vector of observables is given by

$$\mathbf{x}_t = [y_t - y_{t-1}, c_t - c_{t-1}, i_t - i_{t-1}, w_t - w_{t-1}, \pi_t, l_t, r_t]' \quad (19)$$

and the constant term in the measurement Eq. (3) is given by

$$\mathbf{s} = [\bar{\gamma}, \bar{\gamma}, \bar{\gamma}, \bar{\gamma}, \bar{\pi}, \bar{l}, \bar{r}]' \quad (20)$$

where $\bar{\gamma}$ is the growth rate of output, consumption, investment and wages, $\bar{\pi}$ is the steady state rate of inflation, \bar{l} is the steady state level of hours worked and \bar{r} is the steady state nominal interest rate.

The deep parameters of the model are collected in a 41-dimensional vector θ given by¹¹

$$\theta = [\delta, \lambda_w, g_y, \varepsilon_p, \varepsilon_w, \rho_{ga}, \beta, \mu_w, \mu_p, \alpha, \psi, \phi, \sigma_c, \lambda, \Phi, l_w, \xi_w, l_p, \xi_p, \sigma_l, r_\pi, r_{\Delta y}, r_y, \rho, \rho_a, \rho_b, \rho_g, \rho_l, \rho_r, \rho_p, \rho_w, \gamma, \sigma_a, \sigma_b, \sigma_g, \sigma_l, \sigma_r, \sigma_p, \sigma_w, \bar{\pi}, \bar{l}]' \quad (21)$$

In the remainder of this section I study the identifiability of θ , starting with the same setting as in SW07, which I refer to as full information setting. Then I consider identification in several types of limited information settings.

5.2. Identification analysis: full information setting

I begin the identification analysis of the SW07 model assuming the same information setting as in original article. That is, I assume that the model is estimated using all available first and second order moments of the seven variables listed in (19). The analysis proceeds along the lines described in Section 3.4, using 1 million draws from the prior distribution in SW07 (see Table 3).

¹⁰ Replication programs are available at the JME Science Direct web page.

¹¹ Note that by definition $\bar{\gamma} = 100(\gamma - 1)$, and \bar{r} is determined from the values of β, σ_c, γ and $\bar{\pi}$ from $\bar{r} = \pi \gamma^{\sigma_c} / \beta + 100(\gamma^{\sigma_c} / \beta - 1)$.

Table 1

Log-linearized equations of the SW07 model (sticky-price-wage economy).

(1)	$y_t = c_y c_t + i_y i_t + r^{kss} k_y z_t + e_t^g$
(2)	$c_t = \frac{\lambda/\gamma}{1+\lambda/\gamma} c_{t-1} + \frac{1}{1+\lambda/\gamma} E_t c_{t+1} + \frac{w^{ss} \sigma_c (\sigma_c - 1)}{c^{ss} \sigma_c (1+\lambda/\gamma)} (l_t - E_t l_{t+1})$ $-\frac{1-\lambda/\gamma}{(1+\lambda/\gamma)\sigma_c} (r_t - E_t \pi_{t+1}) - \frac{1-\lambda/\gamma}{(1+\lambda/\gamma)\sigma_c} e_t^b$
(3)	$i_t = \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} i_{t-1} + \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}} E_t i_{t+1} + \frac{1}{\phi\gamma^2(1+\beta\gamma^{(1-\sigma_c)})} q_t + e_t^i$
(4)	$q_t = \beta(1-\delta)\gamma^{-\sigma_c} E_t q_{t+1} - r_t + E_t \pi_{t+1} + (1-\beta(1-\delta)\gamma^{-\sigma_c}) E_t r_{t+1}^k - e_t^b$
(5)	$y_t = \phi_p (\alpha k_t^g + (1-\alpha) l_t + e_t^a)$
(6)	$k_t^g = k_{t-1}^g + z_t$
(7)	$z_t = \frac{1-\psi}{\psi} r_t^k$
(8)	$k_t = (1-\delta)/\gamma k_{t-1} + (1-(1-\delta)/\gamma) i_t + (1-(1-\delta)/\gamma) \phi\gamma^2(1+\beta\gamma^{(1-\sigma_c)}) e_t^i$
(9)	$\mu_t^p = \alpha(k_t^g - l_t) - w_t + e_t^g$
(10)	$\pi_t = \frac{\beta\gamma^{(1-\sigma_c)}}{1+l_p\beta\gamma^{(1-\sigma_c)}} E_t \pi_{t+1} + \frac{l_p}{1+\beta\gamma^{(1-\sigma_c)} l_p} \pi_{t-1} - \frac{(1-\beta\gamma^{(1-\sigma_c)}) \xi_p (1-\xi_p)}{(1+l_p\beta\gamma^{(1-\sigma_c)})(1+(\phi_p-1)e_p)\xi_p} \mu_t^p + e_t^p$
(11)	$r_t^k = l_t + w_t - k_t$
(12)	$\mu_t^w = w_t - \sigma_l l_t - \frac{1}{1-\lambda/\gamma} (c_t - \lambda/\gamma c_{t-1})$
(13)	$w_t = \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}} (E_t w_{t+1} + E_t \pi_{t+1}) + \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} (w_{t-1} + l_w \pi_{t-1}) - \frac{1+\beta\gamma^{(1-\sigma_c)} l_w}{1+\beta\gamma^{(1-\sigma_c)}} \pi_t$ $-\frac{(1-\beta\gamma^{(1-\sigma_c)}) \xi_w (1-\xi_w)}{(1+\beta\gamma^{(1-\sigma_c)})(1+(\phi_w-1)e_w)\xi_w} \mu_t^w + e_t^w$
(14)	$r_t = \rho r_{t-1} + (1-\rho)(r_\pi \pi_t + r_y (y_t - y_t^*) + r_{\Delta y} ((y_t - y_t^*) - (y_{t-1} - y_{t-1}^*))) + e_t^r$
(15)	$e_t^a = \rho_a e_{t-1}^a + \eta_t^a$
(16)	$e_t^b = \rho_b e_{t-1}^b + \eta_t^b$
(17)	$e_t^g = \rho_g e_{t-1}^g + \rho_{ga} \eta_t^a + \eta_t^g$
(18)	$e_t^i = \rho_i e_{t-1}^i + \eta_t^i$
(19)	$e_t^r = \rho_r e_{t-1}^r + \eta_t^r$
(20)	$e_t^p = \rho_p e_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p$
(21)	$e_t^w = \rho_w e_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$

Note: The model variables are: output (y_t), consumption (c_t), investment (i_t), utilized and installed capital (k_t^g, k_t), capacity utilization (z_t), rental rate of capital (r_t^k), Tobin's q (q_t), price and wage markup (μ_t^p, μ_t^w), inflation rate (π_t), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). The shocks are: total factor productivity (e_t^a), investment-specific technology (e_t^i), government purchases (e_t^g), risk premium (e_t^b), monetary policy (e_t^r), wage markup (e_t^p) and price markup (e_t^w).

Table 2

Log-linearized equations of the SW07 model (flexible-price-wage economy).

(1)	$y_t^* = c_y c_t^* + i_y i_t^* + r^{kss} k_y z_t^* + e_t^g$
(2)	$c_t^* = \frac{\lambda/\gamma}{1+\lambda/\gamma} c_{t-1}^* + \frac{1}{1+\lambda/\gamma} E_t c_{t+1}^* + \frac{w^{ss} \sigma_c (\sigma_c - 1)}{c^{ss} \sigma_c (1+\lambda/\gamma)} (l_t^* - E_t l_{t+1}^*)$ $-\frac{1-\lambda/\gamma}{(1+\lambda/\gamma)\sigma_c} r_t^* - \frac{1-\lambda/\gamma}{(1+\lambda/\gamma)\sigma_c} e_t^b$
(3)	$i_t^* = \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} i_{t-1}^* + \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}} E_t i_{t+1}^* + \frac{1}{\phi\gamma^2(1+\beta\gamma^{(1-\sigma_c)})} q_t^* + e_t^i$
(4)	$q_t^* = \beta(1-\delta)\gamma^{-\sigma_c} E_t q_{t+1}^* - r_t^* + (1-\beta(1-\delta)\gamma^{-\sigma_c}) E_t r_{t+1}^{k*} - e_t^b$
(5)	$y_t^* = \phi_p (\alpha k_t^{g*} + (1-\alpha) l_t^* + e_t^a)$
(6)	$k_t^{g*} = k_{t-1}^{g*} + z_t^*$
(7)	$z_t^* = \frac{1-\psi}{\psi} r_t^{k*}$
(8)	$k_t^* = (1-\delta)/\gamma k_{t-1}^* + (1-(1-\delta)/\gamma) i_t^* + (1-(1-\delta)/\gamma) \phi\gamma^2(1+\beta\gamma^{(1-\sigma_c)}) e_t^i$
(9)	$\mu_t^{p*} = \alpha(k_t^{g*} - l_t^*) - w_t^* + e_t^g$
(10)	$\mu_t^{p*} = 1$
(11)	$r_t^{k*} = l_t^* + w_t^* - k_t^*$
(12)	$\mu_t^{w*} = -\sigma_l l_t^* - \frac{1}{1-\lambda/\gamma} (c_t^* + \lambda/\gamma c_{t-1}^*)$
(13)	$w_t^* = \mu_t^{w*}$

Note: The model variables are: output (y_t^*), consumption (c_t^*), investment (i_t^*), utilized and installed capital (k_t^{g*}, k_t^*), capacity utilization (z_t^*), rental rate of capital (r_t^{k*}), Tobin's q (q_t^*), price and wage markup (μ_t^{p*}, μ_t^{w*}), real wage (w_t^*), and total hours worked (l_t^*).

Table 3

Prior Distribution and posterior mean.

Parameter	Prior					Posterior
	distr.	mean	std.	lb	ub	
ρ_{ga}	\mathcal{B}	0.5000	0.2500	0.0100	2.0000	0.5211
\bar{l}	\mathcal{N}	0.0000	2.0000	-10.0000	10.0000	0.5416
$\bar{\pi}$	\mathcal{G}	0.6250	0.1000	0.1000	2.0000	0.7852
$100(\beta^{-1}-1)$	\mathcal{G}	0.2500	0.1000	0.0100	2.0000	0.1661
μ_w	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.8414
μ_p	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.6988
α	\mathcal{N}	0.3000	0.0500	0.0100	1.0000	0.1906
ψ	\mathcal{B}	0.5000	0.1500	0.0100	1.0000	0.5462
ϕ	\mathcal{N}	4.0000	1.5000	2.0000	15.0000	5.7439
σ_c	\mathcal{N}	1.5000	0.3750	0.2500	3.0000	1.3803
λ	\mathcal{B}	0.7000	0.1000	0.0010	0.9900	0.7140
Φ	\mathcal{N}	1.2500	0.1250	1.0000	3.0000	1.6043
l_w	\mathcal{B}	0.5000	0.1500	0.0100	0.9900	0.5891
ξ_w	\mathcal{B}	0.5000	0.1000	0.3000	0.9500	0.7007
l_p	\mathcal{B}	0.5000	0.1500	0.0100	0.9900	0.2437
ξ_p	\mathcal{B}	0.5000	0.1000	0.5000	0.9500	0.6503
σ_l	\mathcal{N}	2.0000	0.7500	0.2500	10.0000	1.8365
r_π	\mathcal{N}	1.5000	0.2500	1.0000	3.0000	2.0454
$r_{\Delta y}$	\mathcal{N}	0.1250	0.0500	0.0010	0.5000	0.2237
r_y	\mathcal{N}	0.1250	0.0500	0.0010	0.5000	0.0876
ρ	\mathcal{B}	0.7500	0.1000	0.5000	0.9750	0.8084
ρ_a	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.9577
ρ_b	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.2167
ρ_g	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.9764
ρ_l	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.7106
ρ_r	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.1513
ρ_p	\mathcal{B}	0.5000	0.2000	0.0100	0.9999	0.8914
ρ_w	\mathcal{B}	0.5000	0.2000	0.0010	0.9999	0.9682
γ	\mathcal{N}	0.4000	0.1000	0.1000	0.8000	0.4310
σ_a	\mathcal{IG}	0.1000	2.0000	0.0100	3.0000	0.4595
σ_b	\mathcal{IG}	0.1000	2.0000	0.0250	5.0000	0.2405
σ_g	\mathcal{IG}	0.1000	2.0000	0.0100	3.0000	0.5289
σ_l	\mathcal{IG}	0.1000	2.0000	0.0100	3.0000	0.4532
σ_r	\mathcal{IG}	0.1000	2.0000	0.0100	3.0000	0.2453
σ_p	\mathcal{IG}	0.1000	2.0000	0.0100	3.0000	0.1399
σ_w	\mathcal{IG}	0.1000	2.0000	0.0100	3.0000	0.2443
δ	\mathcal{B}	0.0250	0.0050	0.0100	0.4000	0.0250
λ_w	\mathcal{N}	1.5000	0.2500	1.0000	2.0000	1.5000
g_y	\mathcal{N}	0.1800	0.0500	0.1500	0.2500	0.1800

Note: \mathcal{N} is Normal distribution, \mathcal{B} is Beta-distribution, \mathcal{G} is Gamma distribution, \mathcal{IG} is Inverse Gamma distribution.

As a preliminary step, the Jacobian matrix J_2 is computed at a few points in the parameter space, namely, the prior mean and the posterior mean and median reported in SW07. A rank deficient J_2 indicates that some deep parameters are unidentifiable for reasons that are inherent in the structure of the model. For the set of all deep parameters (see (21)), J_2 has 41 columns, while the rank is 39 at all points where it was evaluated. This rank deficiency is caused by linear dependence among the columns of J_2 corresponding to ε_p and ξ_p on one hand, and ε_w and ξ_w , on the other. The lack of separate identification can be explained with the very similar roles the two curvature parameters— ε_p and ε_w , and the two Calvo parameters— ξ_p and ξ_w , play in the model. A high value of ε_p , for instance, implies that the elasticity of demand increases rapidly when a firm's relative price increases. This implies that it is optimal for the firm to increase its price by a smaller amount, compared to the case when ε_p is low. As a result, prices are adjusted less rapidly. The same outcome is observed when ξ_p —the probability that a firm is not able to adjust its price to the optimal level, is large. It should be noted, however, that, though similar, these parameters are not necessarily equivalent in the original model, as they become after linearization. The same applies to the wage parameters ε_w and ξ_w . As in SW07, I assume that the curvature parameters ε_w , and ε_p are known and are both equal to 10. Hence, in the following analysis I study the identification of a 39-dimensional vector θ , obtained by removing ε_w , and ε_p from the list of parameters in (21).

A second preliminary step is to determine the minimum number of moments for which the Jacobian matrix $J(q)$ may have full rank. Since there are seven observables and 39 deep parameters, the order condition is satisfied as long as $q \geq 2$. Evaluating $J(2)$ at the prior mean and the posterior mean and median shows that the rank condition is satisfied. In the

following analysis I start by first computing the rank of $J(2)$, and then add additional moments when it is of less than full rank.

The results can be summarized as follows. Approximately 96.8% of the 1 million draws from Θ' are admissible, amounting to 968,318 points from Θ . All but one of these points are identified in the model, i.e. result in J_2 with full rank. Two additional points do not pass the rank condition for identification for any number of moments up to T . In short, almost all of the points from parameter space are locally identifiable with any estimation method that utilizes the mean and at least the first two second order moments of the variables listed in (19).

It is interesting to analyze which variables fail the identification conditions and why. In all three cases the rank conditions fail because of linear dependence between the columns of the Jacobian matrix that correspond to the wage markup parameter λ_w , and the wage stickiness parameter ξ_w . In SW07 the authors maintain that λ_w cannot be identified, and therefore do not estimate it. The evidence presented here does not support that assertion. Even at the few points where the identification conditions fail, this is sensitive to the method used for determining the rank of a matrix. Without going into great details, in Matlab, which is used in the study, the rank of a matrix is determined as the number of singular values that exceed a certain tolerance value. Using the default value of 8.9×10^{-4} results in rank deficiency.¹² Using the smaller threshold value of 1.7×10^{-4} results in Jacobian matrices with full rank. For comparison, the tolerance value must be set to less than 4.7×10^{-16} in order to overturn the conclusion regarding the lack of identification of either one of the two pairs of parameters discussed above. In the light of these considerations, it appears more reasonable to conclude that λ_w is locally identified, but its identification is very weak in some parts of the parameter space.

5.3. Identification analysis: limited information setting

The conclusion from the analysis so far is that, if e_w and e_p are assumed known, all other parameters in the SW07 model are identifiable from the first and second order moments of the seven observables used in their study. I next investigate whether the same parameters can be identified in less informative settings. Three examples are considered: (1) using only second order moments of the seven observables; (2) using first and second order moments of fewer observables; (3) using impulse responses of the seven observables to only one shock at a time.

First, suppose that only second order moments are used to estimate θ . Applying the rank condition reveals that 36 of the 39 parameters can be identified as long as at least two of the second order moments are utilized. Two of the unidentifiable parameters are \bar{l} , $\bar{\pi}$, which affect only the mean of \mathbf{x} ; in addition, there is a set of five parameters, namely δ , β , ϕ , λ and γ , any four of which can be identified only if the fifth one is known. This is due to an exact linear dependence among the columns of the Jacobian matrix corresponding to these five parameters. Consequently, the effect on the second order moments of any one of them can be replicated, at least locally, by the effect of the remaining four.

Next, consider a setting with fewer observables. It is a priori clear that \bar{l} is not identifiable unless data on hours worked (l_t) is used, and $\bar{\pi}$ is unidentified unless data on either inflation (π_t) or interest rate (r_t) is used. With these two exceptions, applying the rank condition shows that all other parameters are locally identified as long as one includes at least two observables and at least nine second order moments, in addition to the means of the included variables. Using only one observable always results in a Jacobian matrix with rank of at most 25, irrespectively of the number of autocovariances used. This means that even though the order condition is satisfied when sufficiently many moments are included, after some point increasing the number of included autocovariances does not add any additional information, and the rank condition fails. It should be stressed that this failure is not due to having variables whose moments are unaffected by some structural parameters. This can be seen from Table 4, which shows the sensitivity of the moments of each observable to each one of the 41 parameters. The sensitivity is computed as the Euclidian norm of the vector of elasticities of the mean, variance and first order autocovariance to each structural parameter. It is clear that all parameters except \bar{l} and $\bar{\pi}$ affect the moments of all observables.

Finally, consider a model that matches impulse responses of all seven variables to a single structural shock. Let $\xi = [\xi_j^0, \xi_j^1, \dots, \xi_j^H]'$ be the vector of the used impulse responses. Since ξ does not depend on the mean of \mathbf{x}_t , the parameters that were found to be unidentified in the first example are again unidentified. For the same reason the parameters which characterize the stochastic properties of the excluded structural shocks are also unidentifiable. The remaining parameters are identified as long $H \geq 3$ when impulse responses to: TFP shock (e_t^a), wage markup shock (e_t^w) or price markup shock (e_t^p) are matched, and $H \geq 2$ if impulse responses to: government purchases shock (e_t^g), investment-specific technology shock (e_t^i), risk premium shock (e_t^b), or monetary policy shock (e_t^r) are matched. The difference in the minimum required number of impulse responses arises because of the different number of shock parameters that can be identified. A TFP shock activates the government purchases process, thus identifying ρ_{ga} and ρ_g (see Eq. (21) in Table 1) in addition to 21 other parameters. The wage and price markups shocks follow ARMA(1,1) processes, leading to 22 identified parameters. The remaining 4 shocks identify 21 parameters.

¹² The default tolerance depends on the properties of the matrix—its dimension and largest singular value.

Table 4

Sensitivity of the moments of each variable with respect to the parameters.

Parameter	y	c	i	l	π	w	r
ρ_{ga}	0.1477	0.0704	0.0309	0.0193	0.0196	0.0045	0.0693
\bar{l}	0	0	0	1.0000	0	0	0
$\bar{\pi}$	0	0	0	0	1.0000	0	0.5092
$100(\beta^{-1}-1)$	0.0334	0.0032	0.0274	0.0234	0.0176	0.0082	0.1102
μ_w	3.2739	5.7321	0.7856	12.5593	10.8743	5.8593	7.0003
μ_p	0.9188	0.7486	0.4339	0.5447	1.7503	3.5144	0.6029
α	0.6929	0.0923	0.7711	0.2912	0.1103	0.3094	0.2528
ψ	0.1907	0.1075	0.1724	0.0031	0.0860	0.0327	0.0372
ϕ	0.0690	0.0901	0.3464	0.0212	0.0868	0.0549	0.3202
σ_c	0.1577	0.1410	0.7739	0.2232	0.1366	0.4003	0.9957
λ	0.2805	0.8403	0.3482	0.3264	0.1113	0.3158	1.4912
Φ	0.7212	0.4689	0.7329	1.7599	0.7539	1.7639	0.4004
l_w	0.0727	0.1094	0.0272	0.1078	0.1899	0.4327	0.1069
ξ_w	3.0732	4.9363	1.0169	10.8410	8.8165	0.3753	5.6934
l_p	0.1602	0.1497	0.0553	0.0879	0.3024	0.3627	0.1166
ξ_p	0.1774	0.3339	0.4174	0.0323	0.5012	4.7867	0.4240
σ_l	0.3851	0.3700	0.1867	1.0786	0.7164	0.2449	0.3395
r_π	0.2926	0.5607	0.0268	0.3539	2.9403	0.2077	1.3719
$r_{\Delta y}$	0.4584	0.4894	0.1950	0.2321	0.2947	0.0024	0.0660
r_y	0.1039	0.0857	0.0627	0.1436	1.1066	0.1459	0.8719
ρ	0.0788	0.5354	0.2118	0.4000	1.3024	1.7316	3.2097
ρ_a	1.8993	2.8982	0.4592	0.0530	0.1901	3.0039	1.0344
ρ_b	0.2038	0.4518	0.0231	0.0322	0.0128	0.0348	0.1136
ρ_g	0.7960	0.5831	0.1751	5.2908	0.0768	0.3948	0.1354
ρ_i	1.0722	0.1832	4.2887	0.5091	0.3589	0.3744	1.6136
ρ_r	0.0523	0.0790	0.0129	0.0157	0.0284	0.0284	0.0247
ρ_p	2.0683	1.5058	1.0828	1.6404	3.8649	8.6089	1.4282
ρ_w	6.9632	15.1517	0.5612	50.3417	36.9290	3.6684	27.1295
γ	1.0001	1.0011	1.0000	0.0363	0.0230	1.0000	0.3875
σ_a	0.5433	0.2402	0.1909	0.0444	0.0997	0.2996	0.2734
σ_b	0.4496	1.1987	0.0479	0.0665	0.0210	0.0524	0.2130
σ_g	0.5862	0.1048	0.0256	0.2829	0.0265	0.0081	0.1022
σ_l	0.6579	0.0452	2.2523	0.2119	0.1053	0.1717	0.5381
σ_r	0.2590	0.4079	0.0608	0.0768	0.1307	0.1341	0.3579
σ_p	0.3052	0.2624	0.1378	0.1714	0.6864	0.9357	0.2253
σ_w	0.5518	0.9503	0.1322	1.9757	1.7702	1.3637	1.1244
δ	0.1152	0.2408	0.2029	0.0885	0.0355	0.1057	0.0402
λ_w	1.3657	1.9919	0.7370	4.8708	4.0517	0.0604	2.4749
g_y	0.1566	0.2793	0.0497	0.2238	0.1316	0.0097	0.0049
ε_p	0.0324	0.0609	0.0762	0.0059	0.0915	0.8735	0.0774
ε_w	0.4527	0.7272	0.1498	1.5971	1.2988	0.0553	0.8387

Note: Sensitivity is defined as the norm of the vector of elasticities of the moments of x_t with respect to θ_i . The moments used are the mean, the variance and the first order autocovariance of x_t . The elasticities are evaluated at the posterior mean of θ .

5.4. Summary and discussion

The preceding analysis shows that it is straightforward to establish the local identification of parameters in a fairly large and sophisticated DSGE model, as well as to determine the causes leading to unidentifiability of some parameters. Given a statistical model and a set of observables, a parameter will be unidentifiable if its empirical implications are either undetectable or indistinguishable from those of other parameters. Examples of the first situation are the steady state levels of hours worked or inflation—when the means of these variables are not used, and the structural shock parameters—when impulse responses to other shocks are matched. Examples of the second situation are the curvature parameters— ε_p and ε_w , and the Calvo parameters— ξ_p and ξ_w , which are locally indistinguishable in the linearized model, and the five intertemporal substitution parameters— δ , β , ϕ , λ and γ , each one of which is locally indistinguishable from the others when only second order moments are accounted for.

The choice and number of observables was found to be of little consequence for the identifiability of most parameters. This is due to the fact that the moments of all variables are affected by, and are therefore informative about, most deep parameters. However, as can be seen from Table 4, the variables differ in the sensitivity of their moments to the parameters. This implies that the choice of observables would have consequences for the precision with which different parameters may be estimated. The same observation can be made about the informativeness of the impulse responses to different shocks. Table 5 reports the sensitivities of the impulse response functions with respect to the deep parameters.

Table 5

Sensitivity of the impulse response functions with respect to the parameters.

Parameter	ε_t^a	ε_t^b	ε_t^g	ε_t^i	ε_t^r	ε_t^p	ε_t^w
ρ_{ga}	0.3788	0	0	0	0	0	0
\bar{I}	0	0	0	0	0	0	0
$\bar{\pi}$	0	0	0	0	0	0	0
$100(\beta^{-1}-1)$	0.0104	0.0009	0.0063	0.0294	0.0110	0.0015	0.0015
μ_w	0	0	0	0	0	0	2.1470
μ_p	0	0	0	0	0	0.5732	0
α	0.1719	0.0209	0.8067	0.1816	0.2785	0.0326	0.0523
ψ	0.0652	0.0108	0.6673	0.4899	0.0576	0.0276	0.0226
ϕ	0.1730	0.0679	0.3025	0.1539	0.3082	0.0311	0.0178
σ_c	0.4252	0.3340	1.4160	2.5252	1.7648	0.0833	0.0899
λ	0.7375	0.6005	1.5370	1.4935	4.4995	0.1546	0.1863
Φ	1.3645	0.2056	0.6608	0.6577	1.3289	0.3059	0.3142
l_w	0.0562	0.1090	0.6808	0.0366	0.1312	0.0601	0.1356
ξ_w	0.3896	1.1014	1.3167	0.8801	1.9373	0.1336	1.4532
l_p	0.0466	0.0549	0.3160	0.0227	0.1193	0.0632	0.0718
ξ_p	0.8535	0.4663	2.5348	0.6350	1.8055	0.5366	0.6417
σ_l	0.0431	0.0437	0.3644	0.4102	0.2739	0.0401	0.0988
r_π	0.3552	0.1742	1.2139	0.6475	0.7325	0.1252	0.3287
$r_{\Delta y}$	0.0636	0.1287	0.2954	0.4471	0.5698	0.0507	0.1428
r_y	0.0526	0.0879	0.3345	0.1359	0.2162	0.0397	0.0447
ρ	0.4305	0.0365	1.1342	1.3246	4.7259	0.4073	0.8438
ρ_a	1.6588	0	0	0	0	0	0
ρ_b	0	0.1668	0	0	0	0	0
ρ_g	2.3697	0	19.4669	0	0	0	0
ρ_l	0	0	0	2.5889	0	0	0
ρ_r	0	0	0	0	0.1052	0	0
ρ_p	0	0	0	0	0	1.7131	0
ρ_w	0	0	0	0	0	0	2.8822
γ	0.0165	0.0041	0.0469	0.1393	0.0175	0.0021	0.0083
σ_a	0.1543	0	0	0	0	0	0
σ_b	0	0.1543	0	0	0	0	0
σ_g	0	0	0.1543	0	0	0	0
σ_l	0	0	0	0.1543	0	0	0
σ_r	0	0	0	0	0.1543	0	0
σ_p	0	0	0	0	0	0.1543	0
σ_w	0	0	0	0	0	0	0.1543
δ	0.0483	0.0098	0.2892	0.7082	0.0730	0.0104	0.0398
λ_w	0.2934	0.4460	0.2560	0.2309	0.9258	0.0788	0.6347
g_y	0.1207	0.0247	0.0145	0.2727	0.1423	0.0111	0.0262
ε_p	0.1558	0.0851	0.4626	0.1159	0.3295	0.0979	0.1171
ε_w	0.0574	0.1623	0.1940	0.1297	0.2854	0.0197	0.2141

Note: Sensitivity is defined as the norm of the vector of the elasticities of the impulse responses ($H = 5$) of all variables with respect to θ_i . The elasticities are evaluated at the posterior mean of θ .

Again, although most parameters are identified and can be estimated from impulse responses to any shock, the estimation precision would certainly be different.

Before concluding this section, it is worth commenting on the advantages of using analytical instead of numerical derivatives in the analysis of identification. When the Jacobian matrix of the first and second order moments of all seven variables (i.e. the full information setting) is computed numerically,¹³ the rank of the matrix is equal to 41, while the true Jacobian matrix has rank 39. Similarly, when the derivatives of the impulse response functions to a TFP shock are computed numerically, the rank of the Jacobian matrix is 33, while the rank of the matrix obtained using analytical derivatives is 23. In both cases the numerical derivatives with respect to ε_p and ξ_p , on one hand, and ε_w and ξ_w , on the other, are not perfectly collinear, as the true derivatives are. The same type of numerical differences make δ , β , ϕ , λ and γ to appear separately identifiable from the impulse response functions. Furthermore, the derivatives of the impulse response functions with respect to μ_p , μ_w , ρ_b , ρ_l , ρ_r , ρ_p and ρ_w are numerically small but different from zero, implying that these parameters are identified from the impulse responses to a TFP shock, while in fact they are not. One way of avoiding this sort of problems is to allow for a much larger tolerance value in the numerical computation of the rank (see footnote 12). However, as was indicated earlier, this would make it difficult to discriminate between problems of weak identification

¹³ I used a two-sided finite difference method implemented in the function `fdjac.m` from the COMPECON toolbox (see Miranda and Fackler, 2002).

and complete identification failure. Alternatively, one could use a more sophisticated algorithm for numerical differentiation, which would inevitably come at a greater computational burden.

6. Conclusion

Econometricians have long been aware that structural models cannot be consistently estimated unless they are identified. For this reason identification should be, whenever possible, verified prior to confronting a model with the data. This paper provides a simple and yet general framework for investigating local identification in linearized DSGE models. Using the conditions developed in the paper, researchers can establish which model parameters are identified and which are not, and whether identification failures are due to data limitations, such as lack of observations for some variables, or whether they are intrinsic to the structural model. As the later are not uncommon in linearized DSGE models, performing such analysis may also provide useful insights to researchers who are not interested in estimation.

Appendix A. Supplementary data

Supplementary data associated with this article can be found in the online version at doi:[10.1016/j.jmoneco.2009.12.007](https://doi.org/10.1016/j.jmoneco.2009.12.007).

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