

Gaussian Macro-Finance Term Structure Models with Lags

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Abstract

This paper develops a new family of Gaussian macro-dynamic term structure models (*MTSMs*) in which bond yields follow a low-dimensional factor structure and the historical distribution of bond yields and macroeconomic variables is characterized by a vector-autoregression with order $p > 1$. Most formulations of *MTSMs* with $p > 1$ are shown to imply a much higher dimensional factor structure for yields than what is called for by historical data. In contrast, our “asymmetric” arbitrage-free *MTSM* gives modelers the flexibility to match historical lag distributions with $p > 1$ while maintaining a parsimonious factor representation of yields. Using our canonical family of *MTSMs* we revisit: (i) the impact of no-arbitrage restrictions on the joint distribution of bond yields and macro risks, comparing models with and without the restriction that macro risks are spanned by yield curve information; and (ii) the identification of the policy parameters in Taylor-style monetary policy rules within *MTSMs* with macro risk factors and lags.

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1 Introduction

Dynamic term structure models in which a subset of the pricing factors are macroeconomic variables (*MTSMs*) often have bond yields depending on lags of these factors.¹ As typically parameterized, such *MTSMs* imply that the cross-section of bond yields is described by a high-dimensional factor model, one that seems over-parameterized relative to the low-dimension suggested by principal component analyses (Litterman and Scheinkman (1991)). Consider, for example, a *MTSM* in which the \mathcal{N} risk factors Z_t follow a vector-autoregression of order q under and risk-neutral (pricing) distribution. Then, by logic similar to the spanning arguments in Joslin, Le, and Singleton (2011) (JLS), this *MTSM* is observationally equivalent to one with $\mathcal{N}q$ portfolios of *contemporaneous* yields as risk factors. Most *MTSMs* assume that \mathcal{N} is three or four. With $\mathcal{N} = 3$ and $q = 12$ (annual lags in monthly data), modelers are effectively adopting a thirty-six factor representation of the cross-section of bond yields!

In fact, using economic growth and inflation as macro factors and U.S. Treasury bond yields, and assuming that \mathcal{N} is three or four, we provide model-free evidence that the factor structure of yields calls for a first-order vector-autoregressive (VAR) model for Z_t under the risk-neutral distribution (\mathbb{Q}). These observations are not incompatible with evidence that the historical (\mathbb{P}) conditional mean of Z_t follows a $\text{VAR}^{\mathbb{P}}(p)$ with $p > 1$. Rather, it amounts to indirect evidence that the market prices of Z_t induce a lower-dimensional VAR model for these risk factors under the pricing measure.

Motivated by these theoretical and empirical observations, this paper develops a new family of *MTSMs* in which Z_t follows a $\text{VAR}^{\mathbb{P}}(p)$ (for arbitrary $p < \infty$) under the historical distribution, and a $\text{VAR}^{\mathbb{Q}}(1)$ ($q = 1$) under the pricing distribution. With \mathcal{M} macro variables M_t included in Z_t , a *MTSM* can potentially capture the rich cyclical comovements of bond yields and the macroeconomy, while preserving the low-dimensional factor structure of yields that is so prevalent in the extant literature. Building upon JLS, the canonical form for this family of *MTSMs* has Z_t normalized so that the first \mathcal{M} factors are M_t and the remaining risk factors are the first $\mathcal{L} = \mathcal{N} - \mathcal{M}$ principal components (*PCs*) of bond yields. In this setting bond yields are affine functions of Z_t , and forecasts of future yields (and hence risk premiums) are based on a $\text{VAR}^{\mathbb{P}}(p)$ model of Z_t . Using this framework we explore two empirical questions about the properties of *MTSMs*.

First, affine *MTSMs* are often connected to the large literature on dynamic Taylor rules followed by monetary authorities.² This connection is natural given that the affine dependence of the short-term interest rate on output growth and inflation as well as other latent shocks

¹See Ang and Piazzesi (2003), Ang, Dong, and Piazzesi (2007), and Jardet, Monfort, and Pegoraro (2012), among others. Monfort and Pegoraro (2007) allow for higher-order Markov processes in standard setting without macro pricing factors. This is in contrast to the vast majority of econometric studies of classical dynamic term structure models in which the latent pricing factors follow first-order Markov processes. The first-order Markov model can be traced back at least to Vasicek (1977) and Cox, Ingersoll, and Ross (1985), and this structure is adopted in Dai and Singleton (2000)'s characterization of affine models. Much of the recent literature on arbitrage-free term structure models extends the latter characterization while maintaining the first-order Markov specification.

²See, for examples, Ang, Dong, and Piazzesi (2007), Bikbov and Chernov (2010), Smith and Taylor (2009), and Ang, Boivin, Dong, and Loo-Kung (2011).

resembles standard formulations of Taylor rules. However, using a benchmark formulation of this short-rate dependence on M_t , and allowing for a flexible $\text{VAR}^{\mathbb{P}}(p)$ specification of Z_t , we argue that the parameters of a Taylor rule are not econometrically identified within reduced-form *MTSMs*. This is illustrated with several *observationally equivalent* representations of the short rate that have very different weights on output growth and inflation.

Second, as emphasized by Joslin, Priebsch, and Singleton (2011) (JPS), *MTSMs* that include M_t as risk factors imply that these macro variables are fully contemporaneously spanned by the low-order *PCs* of bond yields. In fact, there are large components of many macro factors that are unspanned by yields and that also have predictive power for excess returns (Ludvigson and Ng (2010), JPS). This implies that the impulse responses (*IRs*) of M_t to yield-curve shocks and vice versa are more accurately portrayed in *MTSMs* that accommodate unspanned macro risks. Accordingly, of central importance to the literature that enforces macro-spanning is whether the joint distribution of bond yields, macro factors, and risk premiums are distorted by this constraint. To address this issue we compare the model-implied *IRs* within canonical *MTSMs* with $p = 1$ and $p > 1$ and macro-spanning enforced to those implied by *MTSMs* in which the macro-spanning constraint is relaxed.

The remainder of this paper is organized as follows. Section 2 provides model-free evidence, based on commonly studied macro risk factors and bond yields, suggesting that the factor structure of affine *MTSMs* is best described by a low-dimensional Z_t following a $\text{VAR}^{\mathbb{Q}}(1)$. Our canonical *MTSM* with lags under the \mathbb{P} distribution and a $\text{VAR}^{\mathbb{Q}}(1)$ for Z_t under \mathbb{Q} , is developed in Section 3. In this section we also compare our framework to several complementary modeling strategies in the literature. The issue of whether insights about the structure of Taylor rules can be reliably extracted from *MTSMs* is taken up in Section 4. The impact of no-arbitrage restrictions on the fitted impulse response functions from *MTSMs* with and without macro-spanning enforced is explored empirically in Section 5.

To fix notation, suppose that a *MTSM* is evaluated using a set of J yields $y_t = (y_t^{m_1}, \dots, y_t^{m_J})'$ with maturities (m_1, \dots, m_J) with $J \geq \mathcal{N}$. We introduce a fixed, full-rank matrix of portfolio weights $W \in \mathbb{R}^{J \times J}$ and define the “portfolios” of yields $\mathcal{P}_t = W y_t$ and, for any $j \leq J$, we let \mathcal{P}_t^j and W^j denote the first j portfolios and their associated weights. The modeler’s choice of W will determine which portfolios of yields enter the *MTSM* as risk factors and which additional portfolios are used in estimation. We use the notation y_t^o to differentiate between observed yields and their theoretical counterpart y_t .

2 The Lag Structure of Affine Bond Pricing Models

Before exploring the properties of specific *MTSMs* with higher order lags, we review the spanning condition for the macro factors M_t derived in JLS for first-order Markov Z_t under \mathbb{Q} , and the observational equivalence of this class of models to one in which yield portfolios are used as risk factors. Next, we examine the potential impact on spanning of introducing higher order lags. Finally, we examine some model-free evidence on the order of the $\text{VAR}^{\mathbb{Q}}$ called for by the standard risk factors in *MTSMs*. Throughout most of this section we are agnostic about the order of the Markov process Z_t under \mathbb{P} as our focus is on the spanning

properties of *MTSMs* implied by arbitrage-free pricing. Indeed, as we discuss in [Section 3](#), within a canonical setting, the order of the $\text{VAR}^{\mathbb{P}(p)}$ representation of Z_t is unconstrained by a researcher's chosen order of the $\text{VAR}^{\mathbb{Q}(q)}$ process for Z_t .

2.1 Theoretical Spanning and Observational Equivalence for First Order Markov Models

Suppose that \mathcal{M} macroeconomic variables M_t enter a *MTSM* as risk factors and that the one-period interest rate r_t is an affine function of M_t and an additional \mathcal{L} pricing factors $\mathcal{P}_t^{\mathcal{L}}$,

$$r_t = \rho_0 + \rho_{1M} \cdot M_t + \rho_{1\mathcal{P}} \cdot \mathcal{P}_t^{\mathcal{L}} \equiv \rho_0 + \rho_1 \cdot Z_t, \quad Z_t = (M_t', \mathcal{P}_t^{\mathcal{L}'}'). \quad (1)$$

The risk factors Z_t are assumed to follow a Gaussian $\text{VAR}^{\mathbb{Q}}(1)$ process:

$$Z_t = K_{0Z}^{\mathbb{Q}} + K_{1Z}^{\mathbb{Q}} Z_{t-1} + \sqrt{\Sigma_Z} \epsilon_t^{\mathbb{Q}}, \quad \epsilon_t^{\mathbb{Q}} \sim N(0, I). \quad (2)$$

Absent arbitrage opportunities in this bond market, (1) and (2) imply affine pricing of bonds of all maturities ([Duffie and Kan \(1996\)](#)). The yield portfolios \mathcal{P}_t can be expressed as

$$\mathcal{P}_t = A_{TS} + B_{TS} Z_t, \quad (3)$$

where the loadings (A_{TS}, B_{TS}) are known functions of the parameters $(K_{0Z}^{\mathbb{Q}}, K_{1Z}^{\mathbb{Q}}, \rho_0, \rho_1, \Sigma_Z)$ governing the risk neutral distribution of yields in the term structure ("TS") model.

A key implication of (3) is that, within any *MTSM* that includes M_t as pricing factors in (1), these macro factors must be spanned by $\mathcal{P}_t^{\mathcal{N}}$:

$$M_t = \gamma_0 + \gamma_1 \mathcal{P}_t^{\mathcal{N}}, \quad (4)$$

for some conformable γ_0 and γ_1 that implicitly depend on W . This follows immediately from inversion of the first \mathcal{N} rows of (3) to express Z_t as an affine function $\mathcal{P}_t^{\mathcal{N}}$:

$$Z_t = \begin{pmatrix} M_t \\ \mathcal{P}_t^{\mathcal{L}} \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\mathcal{L}} & \gamma_1 \\ 0_{\mathcal{L} \times (\mathcal{N} - \mathcal{L})} \end{pmatrix} \mathcal{P}_t^{\mathcal{N}}. \quad (5)$$

Starting from the canonical form for yield-based \mathcal{N} -factor models of [Joslin, Singleton, and Zhu \(2011\)](#) (JSZ), (5) leads to a canonical form for this *MTSM* in which the risk factors are M_t and $\mathcal{P}_t^{\mathcal{L}}$, r_t satisfies (1), and Z_t follows the Gaussian \mathbb{Q} process (2) (see JLS).

2.2 Theoretical Spanning and Observational Equivalence for Higher-Order Markov Models Under \mathbb{Q}

To extend this discussion to higher-order Markov models, suppose that the \mathcal{N} risk factors Z_t follow the $\text{VAR}^{\mathbb{Q}}(q)$ process

$$Z_t = K_{0Z}^{\mathbb{Q}} + K_{1Z}^{\mathbb{Q}}(L; q) Z_{t-1} + \sqrt{\Sigma_Z} \epsilon_t^{\mathbb{Q}}, \quad (6)$$

where $K_{1Z}^{\mathbb{Q}}(L; q)$ is a lag polynomial of order $\infty > q \geq 1$, and r_t is an affine function of the q -history $\mathcal{Z}_t^q \equiv (Z_t, Z_{t-1}, \dots, Z_{t-q+1})$. Here Σ_Z has full rank so \mathcal{N} is minimal in the sense of being the minimal number of shocks underlying variation of bond yields under the \mathbb{Q} measure.

In this extended setting y_t is again affine in the pricing factors, though now with q lags:³

$$y_t = A_Z + B_Z(L; q)Z_t. \quad (7)$$

As long as $J \geq q\mathcal{N}$ —the number of bonds in the cross-section is at least as large as the number of current and lagged Z in (7)—and given A_Z and $B_Z(L; q)$, the expression for Wy_t implied by (7) can be solved for \mathcal{Z}_t^q using any full-rank $q\mathcal{N} \times J$ portfolio weight matrix W . Moreover, the model-implied Z_t inverted from Wy_t or from any Wy_{t+s} , for $0 \leq s \leq q-1$, are identical. It follows that \mathcal{Z}_t^q is fully spanned by the *contemporaneous* cross-section of yields y_t . This is the generalization of the spanning relation (5) that we derived for the case of $q = 1$. In particular, any macro factors in Z_t , as well as their q lags that may appear in (7), remain fully spanned by contemporaneous bond yields. This is true regardless of whether the \mathcal{L} non-macro factors in Z_t are latent or observed portfolios of yields.

Though this formulation retains theoretical spanning, with $q > 1$ it is no longer possible to apply simple rotations to models with latent factors to obtain a counterpart to JLS's canonical form for the case of $q = 1$. For suppose that we start with a *MTSM* in which the risk factors X_t are a mix of latent and macro variables and satisfy (6). Then, in general, it is not possible to find an $\mathcal{N} \times J$ portfolio matrix W such that $\mathcal{P}_t^{\mathcal{N}} = Wy_t$ can be substituted for X_t in (6). Instead, premultiplying (7) by W and inverting $B_Z(L; q)$, we can express X_t as an infinite-order distributed lag of $\mathcal{P}_t^{\mathcal{N}}$, and when this expression is substituted into (6) the resulting time-series model for $\mathcal{P}_t^{\mathcal{N}}$ inherits this infinite order.⁴

An important practical implication of this observation is that two *MTSMs* with identical $(\mathcal{M}, \mathcal{L})$ and macro factors M , one with \mathcal{L} latent pricing factors ℓ_t and the other with \mathcal{L} yield portfolios $\mathcal{P}_t^{\mathcal{L}}$ as pricing factors, are not equivalent when $q > 1$. In particular, the *MTSMs* studied by [Jardet, Monfort, and Pegoraro \(2012\)](#) and [Ang, Dong, and Piazzesi \(2007\)](#), when matched with identical $(\mathcal{M}, \mathcal{L})$ and M_t , are not equivalent theoretical models.

One special case where theoretical equivalence carries over to the presence of lags is when these lags appear only on the macro factors M_t . Specifically, consider the *MTSM* in which $Z_t' = (M_t', \ell_t')$ and r_t is an affine function of ℓ_t and the q -history \mathcal{M}_t^q of the macro factors:

$$r_t = \rho_0 + \rho_{\mathcal{M}} \cdot \mathcal{M}_t^q + \rho_{\mathcal{L}} \cdot \ell_t; \quad (8)$$

and $Z_t = (M_t', \ell_t')'$ follows the Gaussian process

$$\begin{pmatrix} M_t \\ \ell_t \end{pmatrix} = \begin{pmatrix} K_{0M}^{\mathbb{Q}} \\ K_{0\ell}^{\mathbb{Q}} \end{pmatrix} + \begin{pmatrix} K_{1MM}^{\mathbb{Q}} & K_{1M\ell}^{\mathbb{Q}} \\ K_{1\ell M}^{\mathbb{Q}} & K_{1\ell\ell}^{\mathbb{Q}} \end{pmatrix} \begin{pmatrix} \mathcal{M}_{t-1}^q \\ \ell_{t-1} \end{pmatrix} + \sqrt{\Sigma_X} \epsilon_t^{\mathbb{Q}}, \quad (9)$$

³Generically, $B(L; q) = B_1 + B_2L + \dots + B_qL^{q-1}$ generates weighted, summed q -histories.

⁴A simple illustration of this point is as follows: consider a one-factor model with r_t being the factor, and suppose that $[r_t, r_{t-1}]$ is first-order Markov. Then in general the two-period bond yield will take the form $y_t^2 = a + b_1r_t + b_2r_{t-1}$ and this, in turn, implies that the \mathbb{Q} -forecast of $[y_t^2, y_{t-1}^2]$ will depend on $[r_t, r_{t-1}, r_{t-2}]$. Thus, bond yields do not follow first-order Markov processes.

where $\epsilon_t^{\mathbb{Q}}$ is distributed as $N(0, I)$. This formulation nests the Macro Lag *MTSM* of [Ang and Piazzesi \(2003\)](#) (it is the special case with $q = 12$).

Given this structure, bonds yields satisfy

$$y_t = A_X(\Theta^{\mathbb{Q}}) + B_M(\Theta^{\mathbb{Q}})\mathcal{M}_t^q + B_\ell(\Theta^{\mathbb{Q}})\ell_t, \quad (10)$$

where $\Theta^{\mathbb{Q}}$ denotes the set of all risk-neutral parameters that govern (8) and (9). Theoretical spanning implies ℓ_t must be spanned by *any* \mathcal{L} portfolios of yields, $\mathcal{P}_t^{\mathcal{L}}$, and \mathcal{M}_t^q . Applying the weights $W^{\mathcal{L}}$ of $\mathcal{P}_t^{\mathcal{L}}$ to (10) and inverting for ℓ_t , we can write:

$$\ell_t = (W^{\mathcal{L}}B_\ell)^{-1}(\mathcal{P}_t^{\mathcal{L}} - P^{\mathcal{L}}A_X - P^{\mathcal{L}}B_M\mathcal{M}_t^q). \quad (11)$$

Substituting (11) in (8) and (9), we recover an equivalent model where $\mathcal{P}_t^{\mathcal{L}}$ replaces ℓ_t in the vector of risk factors $Z_t' = (M_t', \mathcal{P}_t^{\mathcal{L}'})$:

$$r_t = \rho_0 + \rho_{\mathcal{M}} \cdot \mathcal{M}_t^q + \rho_{\mathcal{L}} \cdot \mathcal{P}_t^{\mathcal{L}}, \text{ and} \quad (12)$$

$$\begin{pmatrix} M_t \\ \mathcal{P}_t^{\mathcal{L}} \end{pmatrix} = \begin{pmatrix} K_{0M}^{\mathbb{Q}} \\ K_{0P}^{\mathbb{Q}} \end{pmatrix} + \begin{pmatrix} K_{1MM}^{\mathbb{Q}} & K_{1MP}^{\mathbb{Q}} \\ K_{1PM}^{\mathbb{Q}} & K_{1PP}^{\mathbb{Q}} \end{pmatrix} \begin{pmatrix} \mathcal{M}_{t-1}^q \\ \mathcal{P}_{t-1}^{\mathcal{L}} \end{pmatrix} + \sqrt{\Sigma_Z}\epsilon_t^{\mathbb{Q}} \text{ under } \mathbb{Q}. \quad (13)$$

Thus, in this setting, without loss of generality we can rotate the factors in a *MTSM* so that any latent factors are replaced by portfolios of bond yields of the modeler's choosing. This representation is canonical in the sense of being maximally flexible within this family of *MTSMs* with q lags of M_t and one lag of either ℓ_t or $\mathcal{P}_t^{\mathcal{L}}$ under \mathbb{Q} .

We emphasize that, in obtaining theoretical equivalence, the key transition from (10) to (11) relies critically on the assumption that Z_t depends only on a single lag of ℓ_t under \mathbb{Q} . If instead there is dependence on a history of ℓ_t , this transition will typically break down since the right hand side of (11) would depend on an infinite number of lags of $\mathcal{P}_t^{\mathcal{L}}$.

2.3 Empirical Evidence on the Factor Structure of Bond Yields

If Z_t follows a $\text{VAR}^{\mathbb{Q}}(q)$ under the pricing measure, then it is no longer the case that any \mathcal{N} portfolios of yields formed with a full rank weight matrix $W^{\mathcal{N}}$ span M_t or any latent factors in Z_t . However, outside of knife-edge cases, it will be the case that the macro and latent factors are spanned by $\mathcal{N}q$ portfolios of yields (see [Section 2.2](#)). It follows that setting $q > 1$ under the pricing distribution effectively increases the number of pricing factors from \mathcal{N} to $\mathcal{N}q$. In many *MTSMs* the choice of $q > 1$ under \mathbb{Q} arises as a consequence of the assumptions that Z_t follows a $\text{VAR}^{\mathbb{P}}(p)$ under \mathbb{P} with $p > 1$ in the presence of flexible market prices of risk. The question of whether the data call for $q > 1$ is often not addressed directly.

Within the family of *MTSMs*, guidance on the lag structure of the \mathbb{Q} distribution of (r_t, Z_t) is provided by the projections of yields onto current and lagged values of Z^o . The null that Z_t follows the first-order $\text{VAR}^{\mathbb{Q}}(1)$ in (2) implies that projections of yields onto current and lagged values of Z_t^o do not improve the explained variation in yields relative to

the contemporaneous projections of y_t^o onto Z_t^o . On the other hand, evidence of improved fits would suggest that the bond data call for setting $q > 1$ under \mathbb{Q} .

Now, strictly speaking, these projections only address variants of models in which the observed yield portfolios $\mathcal{P}_t^{\mathcal{L}^o}$ and their theoretical model counterparts $\mathcal{P}_t^{\mathcal{L}}$ are identical. However, for our empirical analysis we set $\mathcal{L} = 2$ and choose the weight matrix W so that \mathcal{P}_t^2 is comprised of the first two PC s of y_t^o , $\mathcal{C}_t^{2'} = (PC1_t, PC2_t)$. For the reasons given in JLS, the filtered $(PC1_t^f, PC2_t^f)$ are nearly identical to $(PC1_t^o, PC2_t^o)$ in the variants of our models that accommodate measurement errors ($\mathcal{C}_t^{2'o} \neq \mathcal{C}_t^2$). Intuitively, the reason for this is that, even if individual bonds are measured with substantial errors, the diversification effect from the formation of portfolios implies accurate pricing of the low-order PC s. Therefore, we expect the following results to be robust to pricing errors on individual bond yields.

To address the order under \mathbb{Q} empirically we consider several variants of $MTSM$ s with M_t comprised of a measure of real economic growth (g_t) and inflation (π). We follow [Ang and Piazzesi \(2003\)](#) and use the first PC of the help wanted index, unemployment, the growth rate of employment, and the growth rate of industrial production ($REALPC$) as our measure of g , and the first PC of measures of inflation based on the CPI, the PPI of finished goods, and the spot market commodity prices ($INFPC$) for π . The monthly zero yields are the unsmoothed Fama-Bliss series for maturities three- and six-months, and one through ten years ($J = 12$) over the sample period 1972 through 2003.

Model $GM_3(g)$ has $(\mathcal{N} = 3, \mathcal{M} = 1)$; model $GM_3(g, \pi)$ has $(\mathcal{N} = 3, \mathcal{M} = 2)$; and model $GM_4(g, \pi)$ has $(\mathcal{N} = 4, \mathcal{M} = 2)$. Their associated sets of macro risk factors M_t of Z_t are given in [Table 1](#). This table also presents the root mean-squared projection errors in basis points, for lag lengths $q = 1, 6, 12$ months (i.e., $q = 12$ is one year). For two of the three cases, the improvements in fit from setting $q > 1$ are tiny, at most one or two basis points. The exception is $GM_3(g, \pi)$ with state vector $(REALPC, INFPC, PC1)$. In this case the fit is very bad, with root-mean-squared errors ($RMSE$ s) as large as sixty basis points. Consequently, adding lags under \mathbb{Q} improves the $RMSE$ s by up to eight basis points. In all cases, AIC and BIC model selection criteria select the lag length $q = 1$.

This evidence is period and bond-market specific. For a different sample period spanning 1964 - 1995, [Monfort and Pegoraro \(2007\)](#) found that $q > 1$ within a parametric regime-switching VAR model was useful for fitting the dramatic changes in the distributions of bond yields that arose from changes in the Federal Reserve's policy rules. Our empirical analysis focuses on what is effectively one of the post-war regimes (see [Dai, Singleton, and Yang \(2007\)](#)). For this regime, $q = 1$ appears adequate.

3 $MTSM$ s With Lags under \mathbb{P}

Supported by the evidence in [Table 1](#), we proceed to explore asymmetric formulations of $MTSM$ s in which Z_t follows a $VAR^{\mathbb{P}}(p)$ under \mathbb{P} , and a $VAR^{\mathbb{Q}}(1)$ under \mathbb{Q} , the family $MTSM_{\mathcal{M}}^{\mathcal{N}}(p)$.⁵ Since Z_t follows the first-order Markov process (2) under \mathbb{Q} we adopt the

⁵While, as we have seen, we could accommodate lags in M under \mathbb{Q} , we restrict our attention to the case of $q = 1$ based on the evidence in [Section 2](#).

Model	M_t	q	$y_t^{0.5yr}$	y_t^{1yr}	y_t^{2yr}	y_t^{5yr}	y_t^{7yr}	y_t^{10yr}
$GM_3(g)$	<i>REALPC</i>	1	10	15	17	10	10	18
$GM_3(g)$	<i>REALPC</i>	6	9	14	16	10	10	18
$GM_3(g)$	<i>REALPC</i>	12	9	14	16	9	9	17
$GM_3(g, \pi)$	<i>REALPC, INFPC</i>	1	60	45	22	25	36	47
$GM_3(g, \pi)$	<i>REALPC, INFPC</i>	6	57	42	20	24	34	45
$GM_3(g, \pi)$	<i>REALPC, INFPC</i>	12	53	37	18	22	31	40
$GM_4(g, \pi)$	<i>REALPC, INFPC</i>	1	9	15	17	10	10	18
$GM_4(g, \pi)$	<i>REALPC, INFPC</i>	6	8	14	16	9	10	17
$GM_4(g, \pi)$	<i>REALPC, INFPC</i>	12	8	13	15	9	9	16

Table 1: Root-mean-squared fitting errors, measured in basis points, from projections of bond yields onto current and lagged values of the risk factors Z_t^o . For given q , the conditioning information is $(Z_t, Z_{t-1}, \dots, Z_{t-q+1})$.

JLS normalization under which $Z_t' = (M_t', \mathcal{C}_t^{\mathcal{L}'})$, so $\mathcal{P}_t^{\mathcal{L}}$ is rotated to the first \mathcal{L} PCs of y_t^o , M_t continues to satisfy the spanning condition (4), and r_t is given by (1). Furthermore, building upon JSZ, the parameters governing (1) and (2) are expressed as explicit functions of $\Theta_{TS}^{\mathbb{Q}} = (r_{\infty}^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \gamma_0, \gamma_1, \Sigma)$, where $r_{\infty}^{\mathbb{Q}}$ is the long-run \mathbb{Q} -mean of r_t and (γ_0, γ_1) are the weights in the macro spanning condition (4).

Where our *MTSM* with lags differs from the standard models discussed in Dai and Singleton (2000), Collin-Dufresne, Goldstein, and Jones (2008), and JSZ is in the specification of the \mathbb{P} -dynamics of Z_t . Instead of their first-order Markov models, we allow the conditional \mathbb{P} distribution of Z_t to be governed by the $\text{VAR}^{\mathbb{P}}(p)$ process

$$Z_t = K_{0Z}^{\mathbb{P}} + K_{1Z}^{\mathbb{P}} Z_{t-1}^{\mathbb{P}} + \sqrt{\Sigma_Z} \epsilon_t^{\mathbb{P}}. \quad (14)$$

This formulation nests the \mathbb{P} distributions studied by Ang and Piazzesi (2003), Ang, Dong, and Piazzesi (2007), and Jardet, Monfort, and Pegoraro (2012), among others.

Note that having Z_t follow a $\text{VAR}^{\mathbb{Q}}(1)$ process under the pricing measure and a $\text{VAR}^{\mathbb{P}}(p)$ process under the data-generating process are fully compatible assumptions. In fact, given $q \geq 1$, we are free to set p in the $\text{VAR}^{\mathbb{P}}(p)$ model for Z_t to any non-negative integer value through our choice of the market prices of Z_t risks. To illustrate this point, consider the case where Z_t includes the first \mathcal{N} PCs of bond yields and y_t is an affine function of Z_t ($q = 1$). Since $Z_t = W y_t = \mathcal{C}_t^{\mathcal{N}}$, for suitably chosen portfolio matrix W , it follows that the loading matrices in (3) satisfy $W A_{TS} = 0$ and $W B_{TS} = I_{\mathcal{N}}$. However, the $\text{VAR}^{\mathbb{P}}$ structure of Z remains unconstrained, since the market prices of risk may depend on multiple lagged values of Z_t , in which case so will the \mathbb{P} distribution of Z_t . Similar reasoning applies to *MTSMs* in which some of the risk factors are macro variables.

3.1 A Canonical *MTSM* with Lags Under \mathbb{P}

Specializing (12) and (13) to $q = 1$ gives

$$r_t = \rho_0 + \rho_{\mathcal{M}} \cdot M_t + \rho_{\mathcal{L}} \cdot \mathcal{P}_t^{\mathcal{L}}, \quad (15)$$

$$\begin{pmatrix} M_t \\ \mathcal{P}_t^{\mathcal{L}} \end{pmatrix} = \begin{pmatrix} K_{0M}^{\mathbb{Q}} \\ K_{0\mathcal{P}}^{\mathbb{Q}} \end{pmatrix} + \begin{pmatrix} K_{1MM}^{\mathbb{Q}} & K_{1M\mathcal{P}}^{\mathbb{Q}} \\ K_{1\mathcal{P}M}^{\mathbb{Q}} & K_{1\mathcal{P}\mathcal{P}}^{\mathbb{Q}} \end{pmatrix} \begin{pmatrix} M_{t-1} \\ \mathcal{P}_{t-1}^{\mathcal{L}} \end{pmatrix} + \sqrt{\Sigma_Z} \epsilon_t^{\mathbb{Q}} \text{ under } \mathbb{Q}. \quad (16)$$

We complete our specification by assuming that Z_t follows the $VAR^{\mathbb{P}}(p)$ process under \mathbb{P} .⁶

$$\begin{pmatrix} M_t \\ \mathcal{P}_t^{\mathcal{L}} \end{pmatrix} = \begin{pmatrix} K_{0M}^{\mathbb{P}} \\ K_{0\mathcal{P}}^{\mathbb{P}} \end{pmatrix} + \begin{pmatrix} K_{1MM}^{\mathbb{P}} & K_{1M\mathcal{P}}^{\mathbb{P}} \\ K_{1\mathcal{P}M}^{\mathbb{P}} & K_{1\mathcal{P}\mathcal{P}}^{\mathbb{P}} \end{pmatrix} \begin{pmatrix} \mathcal{M}_{t-1}^p \\ \mathcal{P}_{t-1}^{\mathcal{L}^p} \end{pmatrix} + \sqrt{\Sigma_Z} \epsilon_t^{\mathbb{P}}. \quad (17)$$

To ensure that we end up with a canonical and econometrically identified *MTSM*, normalizations are required. Here we follow the literature and impose normalizations under \mathbb{Q} and, since $q = 1$, our pricing distribution is identical to the one examined in JLS. That is, (15) and (16) constitute a standard \mathbb{Q} -affine pricing model in which (effectively, as with essentially all \mathcal{N} -factor *MTSMs* without lags under \mathbb{Q}) the priced risks are the first \mathcal{N} yield portfolios $\mathcal{P}^{\mathcal{N}}$.

Given this pricing equivalence, we proceed by adopting the computationally convenient JLS normalization scheme. Specifically, consider the latent \mathcal{N} -factor model with

$$r_t = \rho_{0X} + \rho_{1X} \cdot X_t, \quad (18)$$

$$X_t = K_{0X}^{\mathbb{Q}} + K_{1X}^{\mathbb{Q}} X_{t-1} + \sqrt{\Sigma_X} \epsilon_t^{\mathbb{Q}}, \quad \epsilon_t^{\mathbb{Q}} \sim N(0, I). \quad (19)$$

We assume that the eigenvalues of $K_{1X}^{\mathbb{Q}}$ are non-zero, real and distinct,⁷ whence we impose the normalizations

$$K_{0X}^{\mathbb{Q}} = 0, \quad K_{1X}^{\mathbb{Q}} = \text{diag}(\lambda^{\mathbb{Q}}), \quad \rho_{1X} = (1, \dots, 1)', \quad \rho_{0X} = r_{\infty}^{\mathbb{Q}}, \quad (20)$$

where $\lambda^{\mathbb{Q}}$ is the \mathcal{N} -vector of eigenvalues of $K_{1X}^{\mathbb{Q}}$ and $r_{\infty}^{\mathbb{Q}}$ is the long-run \mathbb{Q} -mean of r_t . Under these normalizations we can rotate X_t so that the \mathcal{N} factors are $\mathcal{C}_t^{\mathcal{N}}$, the first \mathcal{N} *PCs* of bond yields (see JSZ). The loadings A_{TS} and B_{TS} in (3) are fully determined by the parameters $(r_{\infty}^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_{CC})$ and the weights $W^{\mathcal{N}}$, where Σ_{CC} is the conditional covariance matrix of $\mathcal{C}_t^{\mathcal{N}}$. The pricing piece of our *MTSM* is then completed by enforcing the macro spanning condition (4), the incremental economic content of a *MTSM*. The full set of parameters governing the \mathbb{Q} distribution of $Z'_t = (M'_t, \mathcal{C}_t^{\mathcal{L}'})$ is $\Theta^{\mathbb{Q}} = (r_{\infty}^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_Z, \gamma_0, \gamma_1)$.

Implicit in this construction is the assumption that the *MTSMs* are non-degenerate in the sense that there is no transformation such that the effective number of risk factors is less than \mathcal{N} . In terms of the parameters of our canonical form, we require that none of the eigenvectors of the risk-neutral feedback matrix $K_{1X}^{\mathbb{Q}}$ is orthogonal to the loadings ρ_{1X} for the short rate (see Joslin (2007)). Finally, to maintain valid transformations between alternative choices of risk factors, we require that the matrices $W^{\mathcal{N}} B_{TS}$ and γ_1 be full rank.

We summarize the above in the following theorem:

⁶With some slight abuse of our notation, $\mathcal{P}_t^{\mathcal{L}^p}$ denotes the p -history of $\mathcal{P}_t^{\mathcal{L}}$. Extension to the case where the number of lags of \mathcal{P}_t and M_t are different in (17) is straightforward.

⁷These assumptions can be relaxed along the lines of the analysis in JSZ.

Theorem 1. Fix a full-rank portfolio matrix $W \in \mathbb{R}^{J \times J}$, and let $\mathcal{P}_t = Wy_t$. Any canonical form for the family of models $MTSM_{\mathcal{M}}^{\mathcal{N}}(p)$ is observationally equivalent to a unique member of $MTSM_{\mathcal{M}}^{\mathcal{N}}(p)$ in which the first \mathcal{M} components of the pricing factors are the macro variables M_t , and the remaining \mathcal{L} components are $\mathcal{C}_t^{\mathcal{L}}$; r_t is given by (15); M_t is spanned by $\mathcal{C}_t^{\mathcal{N}}$ as long as $J \geq \mathcal{M} + \mathcal{L}$; the risk factors follow the Gaussian process (16) under \mathbb{Q} and (17) under \mathbb{P} , where $K_{0Z}^{\mathbb{Q}}$, $K_{1Z}^{\mathbb{Q}}$, Σ_{cc} , ρ_0 , $\rho_{\mathcal{M}}$, and $\rho_{\mathcal{L}}$ are explicit functions of $\Theta^{\mathbb{Q}}$. For given W , our canonical form is parametrized by $\Theta^{TS} = (r_{\infty}^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_Z, \gamma_0, \gamma_1, K_{0Z}^{\mathbb{P}}, K_{1Z}^{\mathbb{P}})$.

The proof follows the development in JLS for the case of $p = 1$, suitably adjusted to accommodate the lags in \mathcal{Z}_{t-1}^p .

Though we have structured our canonical form to have Z_t following a $\text{VAR}^{\mathbb{Q}}(1)$, from the discussion in Section 2.2 it is evident that our canonical form is easily extended to the case of (9) where Z_t depends only on the first lag of the yield portfolios $\mathcal{P}_t^{\mathcal{L}}$ (equivalently, on the first lag of a set of latent factors), but on $q > 1$ lags of the macro factors M_t . In this extended model we would normalize $K_{1\mathcal{P}\mathcal{P}}^{\mathbb{Q}}$ to be in Jordan form and the coefficient matrices on the lagged values of M_t under \mathbb{Q} become additional free parameters of the pricing distribution.

Analogously to the construction in JLS, our canonical form reveals the essential difference between term structure models based entirely on yield-based pricing factors $\mathcal{P}_t^{\mathcal{N}}$ and those that include macro risk factors. Relative to the JSZ canonical form with pricing factors $\mathcal{P}_t^{\mathcal{N}}$, a *MTSM* adds the spanning property (4) with its $\mathcal{M}(\mathcal{N} + 1)$ free parameters (γ_0, γ_1) . Thus, any canonical \mathcal{N} -factor *MTSM* with macro factors M_t gains $\mathcal{M}(\mathcal{N} + 1)$ free parameters relative to pure latent-factor Gaussian models. This is true *regardless of the modeler's choice of p* , the lag structure under \mathbb{P} , because we have set $q = 1$ under the pricing distribution.

3.2 The Conditional Distribution of the Pricing Factors

In taking the model to the data, we accommodate the fact that the observed data $\{M_t^o, \mathcal{C}_t^o\}$ are not perfectly matched by a theoretical no-arbitrage model by supposing that the observed yield portfolios \mathcal{C}_t^o are equal to their theoretical values plus a mean-zero measurement error. Following standard practice in the literature, we presume that the measurement errors are *i.i.d.* normal, thereby giving rise to a Kalman filtering problem.

The \mathbb{P} distribution of Z implied by Theorem 1 is

$$f^{\mathbb{P}}(Z_t | \mathcal{Z}_{t-1}^p; K_{1Z}^{\mathbb{P}}, K_{0Z}^{\mathbb{P}}, \Sigma_Z) = (2\pi)^{-\mathcal{N}/2} |\Sigma_Z|^{-1/2} \exp \left(-\frac{1}{2} \|\Sigma_Z^{-1/2} (Z_t - E_{t-1}^{\mathbb{P}}[Z_t])\|_2^2 \right), \quad (21)$$

where $E_{t-1}^{\mathbb{P}}[Z_t] = K_{0Z}^{\mathbb{P}} + K_{1Z}^{\mathbb{P}} \mathcal{Z}_{t-1}^p$. Importantly, the only parameters from the set $\Theta^{\mathbb{Q}}$ that appear in (21) are those in Σ_Z . This distinctive feature of our canonical form, inherited from the JSZ normalization, will play a prominent role in our discussion of the implications of no-arbitrage for the conditional distributions of bond yields.

Initially, suppose that the first \mathcal{L} entries of \mathcal{C}_t are priced perfectly by the model ($\mathcal{C}_t^{o\mathcal{L}} = \mathcal{C}_t^{\mathcal{L}}$), and the last $J - \mathcal{L}$ entries, say \mathcal{C}_t^e , are priced up to the *i.i.d.* measurement errors $e_t = \mathcal{C}_t^{eo} - \mathcal{C}_t^e$,

$e_t \sim N(0, \Sigma_e)$.⁸ In this case, the joint (conditional) density of the data can be written as:

$$f^{\mathbb{P}}(Z_t^o, \mathcal{C}_t^{eo} | \mathcal{Z}_{t-1}^p; \Theta^{TS}, \Sigma_e) = f^{\mathbb{P}}(\mathcal{C}_t^{eo} | Z_t^o; \Theta^{\mathbb{Q}}, \Sigma_e) \times f^{\mathbb{P}}(Z_t^o | \mathcal{Z}_{t-1}^p; K_{0Z}^{\mathbb{P}}, K_{1Z}^{\mathbb{P}}, \Sigma_Z). \quad (22)$$

Since $Z_t^o = Z_t$, the conditional density $f^{\mathbb{P}}(Z_t | \tilde{\mathcal{Z}}_{t-1}^p; \Theta^{TS})$ is given by (21). It depends on $(K_{0Z}^{\mathbb{P}}, K_{1Z}^{\mathbb{P}}, \Sigma_Z)$, but not on $(\lambda^{\mathbb{Q}}, r_{\infty}^{\mathbb{Q}}, \gamma_0, \gamma_1)$. On the other hand, the density of \mathcal{C}_t^{eo} is

$$f^{\mathbb{P}}(\mathcal{C}_t^{eo} | Z_t^o; \Theta^{\mathbb{Q}}, \Sigma_e) = (2\pi)^{-(J-\mathcal{L})/2} |\Sigma_e|^{-1/2} \exp\left(-\frac{1}{2} e_t (\Theta^{\mathbb{Q}})' \Sigma_e^{-1} e_t (\Theta^{\mathbb{Q}})\right), \quad (23)$$

and it depends only on Σ_e and the risk-neutral parameters $\Theta^{\mathbb{Q}}$.

Optimization of (22) proceeds from initial starting values of Θ^{TS} as follows. Given $\Theta^{\mathbb{Q}}$ we compute $K_{0Z}^{\mathbb{Q}}, K_{1Z}^{\mathbb{Q}}, \rho_{0Z}$, and ρ_{1Z} using the known mappings from JSZ and the invariant transformation (5). These parameters are then used to compute bond yields as functions of the *observed* state Z_t^o using standard affine pricing formulas. Finally, the model-implied yields along with the portfolio matrix W are used to compute the conditional density $f^{\mathbb{P}}(\mathcal{P}_t^{eo} | Z_t^o; \Theta^{\mathbb{Q}}, \Sigma_e)$. As will be discussed as part of our subsequent empirical analyses, reliable starting values are easily obtained for most of the parameters in Θ^{TS} . We have found in practice that our search algorithm converges to the global optimum extremely quickly, usually in just a few seconds.

It is becoming standard practice to relax the exact pricing assumption $\mathcal{P}_t^o = \mathcal{P}_t$ by introducing *i.i.d.* measurement errors for the entire vector of yields y_t^o . With the addition of measurement errors, *ML* estimation involves the use of the Kalman filter. To set up the Kalman filtering problem we start with a given set of portfolio weights $W \in \mathbb{R}^{J \times J}$ and our canonical normalization with theoretical pricing factors $Z_t' = (M_t, \mathcal{C}_t^{\mathcal{L}'})$. From W and $\Theta^{\mathbb{Q}}$, we construct $(K_{0Z}^{\mathbb{Q}}, K_{1Z}^{\mathbb{Q}}, \rho_{0Z}, \rho_{1Z})$. Based on the no-arbitrage pricing of bonds we then construct $A_{TS}(\Theta_{TS}^{\mathbb{Q}}) \in \mathbb{R}^J$ and $B_{TS}(\Theta_{TS}^{\mathbb{Q}}) \in \mathbb{R}^{J \times \mathcal{N}}$ with $\mathcal{P}_t = A_{TS} + B_{TS} Z_t$. The observation equation is then (3) adjusted for measurement errors:

$$\mathcal{C}_t^o = A_{TS}(\Theta^{\mathbb{Q}}) + B_{TS}(\Theta^{\mathbb{Q}}) Z_t + e_t, \quad e_t \sim N(0, \Sigma_e), \quad (24)$$

and the state equation is (17). Together (17) and (24) comprise the state space representation of the *MTSM*. Note that the dimension of e_t is now J , and not $J - \mathcal{L}$, because all J yields y_t^o are presumed to be measured with error.

Consistent with the literature, we assume always that the observed macro factors M_t^o coincide with their theoretical counterparts M_t , though this assumption can be relaxed as in JLS. We do not pursue this relaxation here, because JLS found that introducing measurement errors on the macro factors effectively led to M_t being driven out of the model. That is, the filtered M_t^f bore only very weak resemblance to their historical counterparts M_t^o . Enforcing $M_t = M_t^o$ allows us to preserve a central role for the macro factors in pricing and, equally importantly, it aligns our empirical analysis with the extant literature on *MTSMs*.

⁸In the literature on *DTSMs* this assumption was introduced by [Chen and Scott \(1993\)](#) and it has been adopted in many empirical studies of latent-factor term structure models. It is also maintained by [Ang, Piazzesi, and Wei \(2006\)](#) and [Jardet, Monfort, and Pegoraro \(2012\)](#), among others, in studies of *MTSMs*.

Though estimation with filtering can be challenging in terms of finding a global optimum to the likelihood function, the factorization of the likelihood function under our normalization leads directly to quite accurate starting values. First, the parameters governing the conditional \mathbb{Q} -mean of Z_t enter only through the density of the pricing errors. Therefore, the conditional \mathbb{P} -mean of Z_t is invariant to the imposition of restrictions on the conditional \mathbb{Q} -mean of the risk factors (see also JSZ). In other words, ML estimates of the \mathbb{P} -mean parameters ($K_{0Z}^{\mathbb{P}}, K_{1Z}^{\mathbb{P}}$) can be obtained from a regression of Z_t^o on Z_{t-1}^p , regardless of Σ_Z or other constraints on the \mathbb{Q} dynamics. Second, with or without lags, Σ_Z only affects the level of yields and not their factor loadings.⁹ Consequently, an estimate of Σ_Z based on the residuals from OLS estimation of a $\text{VAR}^{\mathbb{P}}(p)$ model for Z_t usually gives starting values that are very close to the ML estimates at the global optimum.

A complementary approach to parameterizing arbitrage-free factor models for excess returns on bonds has been proposed by [Adrian, Crump, and Moench \(2012\)](#) (ACM). The computational tractability of the (JSZ, JPS) approach is inherited by the ACM baseline parameterizations, because they normalize their pricing factors to be portfolios of yields. However, the ACM implementation of this normalization does not yield a logically consistent pricing model. There is an explicit link between the eigenvalues $\lambda^{\mathbb{Q}}$ of the risk-neutral feedback matrix in the $\text{VAR}^{\mathbb{Q}}(1)$ representation of $\mathcal{C}_t^{\mathcal{N}}$ and the loadings B_{TS} on Z_t that we enforce. On the other hand, ACM leave the parameters of their B_{TS} free and, as such, the model implied $\mathcal{C}_t^{\mathcal{N}}$ will not in general match the historical PC s even though their model and estimation strategy presume such a match. Put differently, they have more free parameters than in the maximally flexible, canonical Gaussian model ([Dai and Singleton \(2000\)](#)), and this induces an inconsistency between actual and model-implied prices of certain portfolios of yields.

There is another important sense in which ACM follow a “limited information” estimation strategy relative to the known structure of their affine pricing models. By focusing on projections of excess returns onto the risk factors $\mathcal{P}_t^{\mathcal{N}}$, ACM do not enforce the known link between the level of bond yields and the parameters of their model. That is, they do not enforce the model-implied constraints on the conditional mean of their pricing kernel. Consequently, their estimators will in general be inefficient relative to the full-information maximum likelihood estimators implemented in JSZ and in our analysis.

Our framework and the ACM approach offer equal flexibility in modeling the pricing errors on individual and portfolios of yields. In particular, the introduction of serially correlated measurement errors on bond yields, which alters the implied degree of serial correlation in the errors in measured holding period returns, introduces no new conceptual or practical issues for econometric analysis of our canonical model. We illustrate this point in [Section 3.4](#).

3.3 State-Space Formulations Under Alternative Hypotheses

Throughout our subsequent analysis we compare the $MTSM$ s characterized in [Theorem 1](#) to their “unconstrained alternatives.” Since a $MTSM$ involves multiple over-identifying restrictions, the relevant alternative model depends on which of these restrictions one is

⁹Only $K_{1Z}^{\mathbb{Q}}$ and $\rho_{\mathcal{M}}$ determine the loadings of risk-factors on yields.

interested in relaxing. We follow JLS and distinguish between three alternative formulations. The arbitrage-free “null model” for the case where all yields are measured with errors is denoted by TS^f , with the superscript f meaning Kalman filter.

The FV^f (“factor-VAR”) alternative maintains the state equation (17), but generalizes the observation equation to

$$\mathcal{P}_t^o = A_{FV} + B_{FV} Z_t + e_t, \quad (25)$$

for conformable matrices A_{FV} and B_{FV} , with e_t normally distributed from the same family as the $MTSM$. For identification we normalize the first \mathcal{L} entries of A_{FV} to zero and the first \mathcal{L} rows of B_{FV} to the corresponding standard basis vectors. Except for this, A_{FV} and B_{FV} are free from any restrictions. The full parameter set $\Theta^{FV} = (A_{FV}, B_{FV}, K_{0Z}^{\mathbb{P}}, K_{1Z}^{\mathbb{P}}, \Sigma_Z, \Sigma_e)$ is estimated using the Kalman filter.

Special cases of models TS and FV that are also of interest are when the state yield portfolios are measured perfectly. We distinguish these special cases by the notation TS^n and FV^n (for *no* pricing errors on the risk factors). The Kalman filtering problem then simplifies to conventional ML estimation. In particular, for the FV^n model, estimation conveniently reduces to two sets of OLS regressions: a VAR for the observed risk factors Z_t^o gives the parameters in (17),¹⁰ and an OLS regression of \mathcal{C}_t^o on Z_t^o recovers the parameters characterizing (25).

Relative to model TS^f , model FV^f relaxes the over-identifying restrictions implied by the assumption of no arbitrage, but maintains the low-dimensional factor structure of returns and the presumption of measurement errors on bond yields. Thus, in assessing whether these two models imply nearly identical joint distributions for (y_t, M_t) , the focus is on whether the no arbitrage restrictions induce a difference. On the other hand, differences between the TS^f and TS^n models, which both maintain a similar no-arbitrage structure, should arise mainly out of the different treatments of measurement errors of the pricing factors. Finally, in moving from model TS^f to model FV^n one is relaxing both the no arbitrage restrictions and the presumption that the state yield portfolios are measured without errors ($\mathcal{C}_t^{\mathcal{L}} = \mathcal{C}_t^{\mathcal{L}^o}$ in model FV^n), while again maintaining the low-dimensional factor structure.

3.4 No-Arbitrage Restrictions in $MTSMs$ with Lags

Prior to delving into various substantive economic and modeling issues, we address the basic question of whether the no-arbitrage restrictions of a $MTSM$ impact the joint distributions of the macro factors and yield portfolios $\mathcal{P}_t^{\mathcal{L}}$. JLS found, in the context of canonical $MTSMs$ with $p = 1$, that the impulse response functions among these factors were virtually identical to those obtained from an unconstrained factor-VAR, model FV^n . We address this same question in the context of a $\mathcal{N} = 4$ factor $MTSM$ with $p = 3$, model $GM_{3,4}(g, \pi)$. The risk factors are ($INFPC, REALPC, PC1, PC2$), and impulse response (IR) functions are computed using this ordering. To arrive at our choice of p we fit VAR models over a wide

¹⁰The ML estimators of $K_{0Z}^{\mathbb{P}}$ and $K_{1Z}^{\mathbb{P}}$ are the standard OLS estimators, and the ML estimator of Σ_Z is the usual sample covariance matrix based on the OLS residuals.

	Model $GM_{3,4}(g, \pi)$				Corresponding Model FV			
No Filtering	0.0138	-	-	-	0.0138	-	-	-
	0.169	0.0052	-	-	0.172	0.00519	-	-
	0.0728	-0.0516	0.201	-	0.072	-0.0509	0.201	-
	0.204	0.0424	-0.0136	0.233	0.204	0.043	-0.0137	0.233
Filtering	0.0136	-	-	-	0.0136	-	-	-
	0.185	0.00478	-	-	0.19	0.00472	-	-
	0.0737	-0.0556	0.201	-	0.073	-0.0523	0.201	-
	0.207	0.0466	-0.014	0.233	0.209	0.0375	-0.0142	0.233

Table 2: Estimated conditional standard deviations along the diagonals and correlations along the off-diagonals. The risk factors PC1, PC2, INFPC, and REALPC follow a $\text{VAR}^{\mathbb{P}}(3)$ process and estimation is by Kalman filtering.

range of p and evaluated the optimal lag length using standard model selection criteria. The optimal p chosen by the BIC (AIC) model selection criterion was 2 (3), so we chose $p = 3$ to offer maximal flexibility consistent with these criteria.

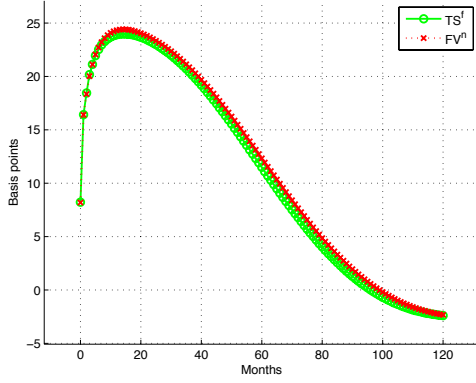
The $MTSMs$ and factor- $VARs$ give very similar estimates for the parameters governing the conditional mean¹¹ (not shown) and conditional covariance (Table 2) of the risk factors, even when allowing for measurement errors on all yields. Estimates of the standard deviations and correlations implied by the covariance matrix Σ_Z were computed for model $GM_{3,4}(g, \pi)$ estimated with and without filtering on \mathcal{P}_t^2 and for corresponding VAR models FV^f and FV^n . The estimates across all four models are very similar, consistent with our priors that Kalman filter estimates of model $GM_{3,4}(g, \pi)$ would be very close to those of model FV^n even though the latter model ignores the no-arbitrage restrictions and the measurement errors on the yields.

Though the estimates of each individual element of the parameter set $(K_{0Z}^{\mathbb{P}}, K_{1Z}^{\mathbb{P}}, \Sigma_Z)$ are close across models with and without arbitrage restrictions and with and without filtering, it remains to document that any small differences do not compound to impact the IRs . Figure 1 displays the IRs implied by models TS and FV^n within family $GM_{3,4}(g, \pi)$.¹² There it is seen that the imposition of no arbitrage and the use of filtering is virtually inconsequential for how shocks to macro factors impact the yield curve. This finding demonstrates that the results on the irrelevance of no-arbitrage restrictions of IRs for the case of $p = 1$ carry over to models in which the \mathbb{P} distribution of the risk factors is governed by a higher-order VAR .

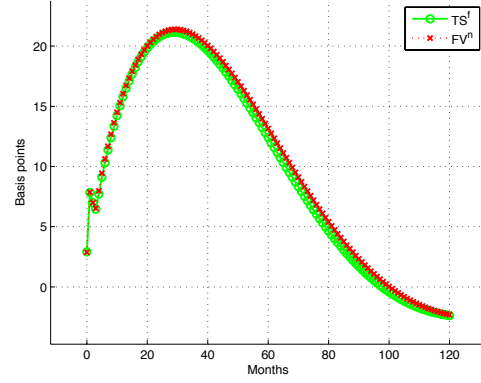
Up to this point we have assumed the measurement errors on the higher order PCs

¹¹ This confirms, for this family of canonical models, that VAR estimates of $(K_{0Z}^{\mathbb{P}}, K_{1Z}^{\mathbb{P}})$ are likely to be good starting values for ML estimation of $MTSMs$.

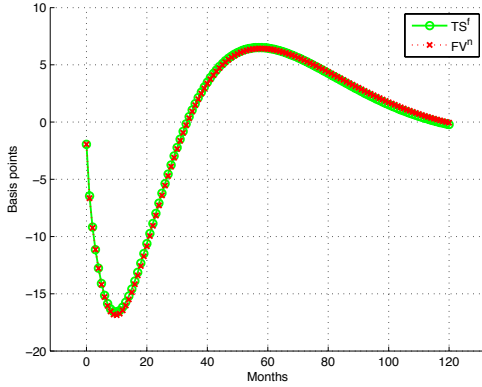
¹² The choppy behavior over short horizons for some of the responses in Figure 1 is also evident in the IRs reported in Ang and Piazzesi (2003) for their $MTSM$ with lags.



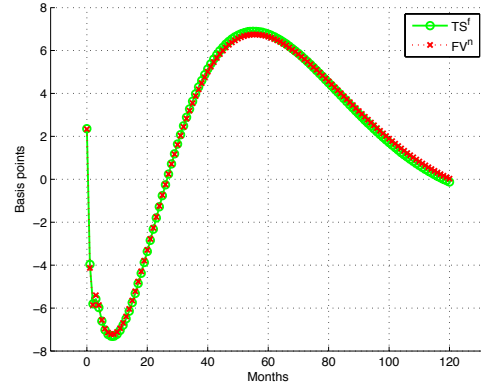
(a) Response of $PC1$ to $REALPC$



(b) Response of $PC1$ to $INFPC$



(c) Response of $PC2$ to $REALPC$

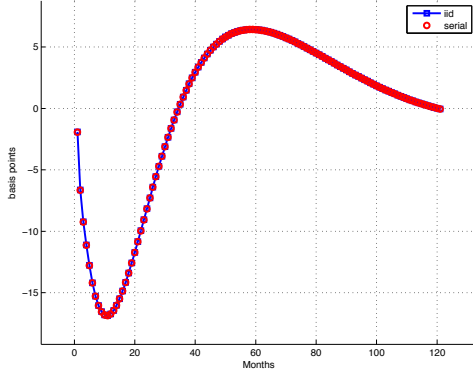


(d) Response of $PC2$ to $INFPC$

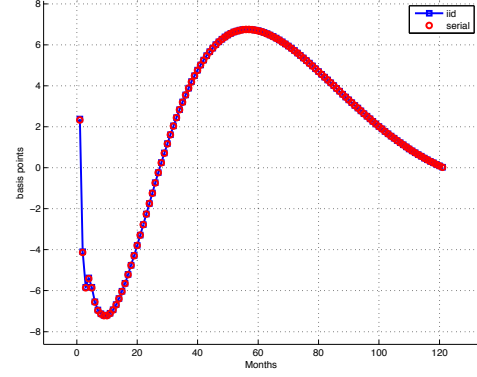
Figure 1: Impulse responses of $PC1$ and $PC2$ to innovations in $REALPC$ and $INFPC$ based on ML estimates of model $GM_{3,4}(g, \pi)$.

appearing in (24) are serially uncorrelated. Though this assumption is commonplace in the literature, it implies that the measurement errors on holding period returns, which are quasi-differences of the yields, are persistent (ACM). Our prior, informed by the results of Dai and Singleton (2000) for yield-only models, is that the introduction of serial correlation will have little impact on the dynamic properties of the model-implied yields in fitted $MTSMs$.

We examine this issue in model $GM_{3,4}(g, \pi)$ by introducing first-order autoregressive measurement errors in (24). The fitted errors on the higher order PCs are in fact persistent—typical estimates of the autocorrelation coefficients are around 0.75. Figure 2 displays the responses of $PC2$ to shocks to $REALPC$ and $INFPC$, the counterparts to panels (c) and (d) of Figure 1. It is immediately apparent that the IRs from the models with and without serially correlated measurement errors are virtually identical. Though not shown, this is true of the other cases displayed in Figure 1, as well as the responses of the individual bond yields to innovations in the macro variables. We conclude that the $MTSM$ -implied IRs are robust



(a) Response of $PC2$ to $REALPC$



(b) Response of $PC2$ to $INFPC$

Figure 2: Impulse responses of $PC2$ to innovations in $REALPC$ and $INFPC$ based on ML estimates of model $GM_{3,4}(g, \pi)$, with measurement errors following $AR(1)$ processes.

to alternative assumptions about the persistence of the measurement errors.

4 Taylor-Style Rules and $MTSMs$

Several recent studies interpret their versions of the short-rate equation (1) as a Taylor-style rule. This is premised, we suspect, on the virtually identical appearance of special cases of (1) with $M'_t = (g_t, \pi_t)$ and $\mathcal{L} = 1$,

$$r_t = \rho_0 + \rho_\pi \pi_t + \rho_g g_t + \rho_\ell \ell_t, \quad (26)$$

and policy rules that have central banks setting the short rate according to targets for inflation and the output gap. Furthermore, just as in standard structural formulations of a Taylor rule, there is also a latent shock ℓ_t to the short rate. However, without imposing additional *economic* structure on a $MTSM$, the parameters (ρ_π, ρ_g) are not meaningfully interpretable as the reaction coefficients of a central bank.

To see this in the simplest, benchmark case we focus on the family $GM_3(g, \pi)$ with the short-rate given by (26). As in the $MTSMs$ of [Ang, Dong, and Piazzesi \(2007\)](#) and [Chernov and Mueller \(2009\)](#), for example, the latent shock ℓ_t is allowed to be correlated with M_t . [Cochrane \(2007\)](#) argues that it is an inherent feature of new-Keynesian models that the policy shock “jumps” in response to changes in inflation or output gaps.

Within this setting, an immediate implication of [Theorem 1](#) is that each choice of full-rank portfolio matrix $W \in \mathbb{R}^{N \times J}$ leads to a canonical model for $GM_3(g, \pi)$ in which r_t is given by

$$r_t = \rho_0^W + \rho_\pi^W \pi_t + \rho_g^W g_t + \rho_{\mathcal{P}^1}^W \mathcal{P}_t^{1W}. \quad (27)$$

One choice of W places the first PC (the “level” factor) in place of ℓ_t in (26)–sets $\mathcal{P}_t^1 = PC1_t$. Another choice sets $\mathcal{P}_t^1 = PC2_t$ (the “slope” factor), etc. All choices give rise to *theoretically*

equivalent representations of the short-term rate and, through the no-arbitrage term structure model, identical bond yields at all maturities. Moreover, when all yields are measured with errors, all choices of W give rise to *observationally equivalent* versions of (27). Absent the imposition of additional economic structure, there appears to be no basis for interpreting any one of these equivalent rotations (27) as *the* Taylor rule of a structural model.

Put differently, when modelers include a latent factor ℓ_t in their short-rate specification, then the normalizations they choose select one among an uncountable infinity of equivalent normalizations. A particular normalization may lead to estimates of the loadings ρ_π and ρ_g that bear some resemblance to estimates of the policy parameters in the literature on fitting structural Taylor rules. However, this would be coincidental. Further, even in this case, ℓ_t is representable as a portfolio of bond yields and the plausibility of this portfolio being the central bank’s policy shock warrants evaluation.

For instance, Ang, Dong, and Piazzesi (2007) estimate a benchmark *MTSM* with a short-rate equation similar to (26). They treat their shock ℓ_t as latent, consistent with their interpretation of ℓ_t as a policy shock. At their model estimates they find that the fitted $\hat{\ell}_t$ has a correlation of 94% with the short (one-quarter) rate and 98% with the five-year rate, indicating that $\hat{\ell}_t$ is highly correlated with the level factor *PC1*. Thus, through the lens of our canonical form it is seen that their normalization scheme and estimation algorithm were effectively selecting out a portfolio matrix W that set ℓ_t to the first *PC* of bond yields.

Motivated by their experience, we proceed to estimate the canonical version of $GM_{p,3}(g, \pi)$ under three equivalent models: \mathcal{P}_t^1 is rotated to one of *PC1*, *PC2*, or *PC3*. Proceeding in this manner is without loss of generality because, with all yields filtered, the fitted yields will be exactly the same regardless of which rotation of \mathcal{P}_t^1 to a theoretical *PC* is chosen. This is true even though the macro factors (g_t, π_t) may not (in general will not) be spanned to equal degrees by the individual low-order *PCs* of yields, because the spanning constraint links M_t to the entire vector \mathcal{C}_t^N . We estimate the model with \mathcal{P}_t^1 rotated to *PC1*; and then rotate the estimated model to obtain the cases in which \mathcal{P}_t^1 is rotated to *PC2* or *PC3*.

Equipped with the Kalman filter/*ML* estimates of these models, we report the loadings on g and π as well as the standard deviation of the “residual” $\rho_{\hat{\mathcal{P}}}^W \hat{\mathcal{P}}_t^1$ implied by these choices of W . The version of the “Taylor rule” we examine is what is often referred to as the *benchmark* model that expresses r_t as a contemporaneous affine function of the risk factors. We set r_t to the actual three-month Treasury bill rate for the purpose of computing residual “policy” shocks $e_t \equiv \rho_{\mathcal{P}^1}^W \mathcal{P}_t^1$, as this is the shortest maturity we used in constructing our *PCs* and in estimation of the models. Comparable results were obtained with r_t replaced by the fitted one- or three-month yields from the *MTSMs*. The *AIC* and *BIC* model selection criteria pointed to different lag lengths, and we chose the longer $p = 4$ in light of the use of lags out to a year in previous studies.

The model-implied loadings for the short-rate are displayed in Table 3. To facilitate comparisons of the estimated loadings to results in the literature, we scale INFPC (REALPC) to have the same standard deviation as the annualized CPI inflation rate (annualized growth rate of industrial production) over our sample period. In all three rotations $\rho_\pi^W > \rho_g^W$, which is consistent with *MTSM*-based estimates in Ang, Dong, and Piazzesi (2007) and the

Rotation	ρ_0^W	$\rho_{\mathcal{P}^1}^W$	ρ_π^W	ρ_g^W	$std(e)$	$corr(e, \pi)$	$corr(e, g)$
$\mathcal{P}^1 = PC1$	-0.0111	0.2930	0.1697	0.0233	0.0271	0.4275	-0.0483
$\mathcal{P}^1 = PC2$	-0.1920	-6.299	2.1836	0.4318	0.0631	-0.7737	-0.5184
$\mathcal{P}^1 = PC3$	0.7575	-100.880	1.5159	-1.2429	0.0982	-0.2671	0.9247

Table 3: Coefficients and moments of the variables in the short-rate equation under different rotations of the latent shock L_t implied by the $GM_{4,3}(g, \pi)$ model. The residual “policy” shock $e_t \equiv \rho_{\mathcal{P}^1}^W \mathcal{P}^{1W}$.

Rotation	ρ_0^W	$\rho_{\mathcal{P}^2}^W$	ρ_π^W	ρ_g^W	$std(e)$	$corr(e, \pi)$	$corr(e, g)$
$\mathcal{P}^2 = (PC1, PC2)$	0.0036	0.3184	0.5229	-0.0055	-0.0117	0.0297	0.5648
$\mathcal{P}^2 = (PC1, PC3)$	-1.207	0.8459	153.960	-2.8458	2.1920	0.1897	0.5133
$\mathcal{P}^2 = (PC2, PC3)$	0.7348	0.8385	-92.937	1.7075	-1.3401	0.1064	-0.2995

Table 4: Coefficients and moments of the variables in the short-rate equation under different rotations of the latent shock L_t implied by the $GM_{3,4}(g, \pi)$ model. The residual “policy” shock $e_t \equiv \rho_{\mathcal{P}^2}^W \cdot \mathcal{P}^{2W}$.

instrumental variables estimates reported in [Clarida, Gali, and Gertler \(2000\)](#) for their post-Volker disinflation periods. Despite differences in sample periods and estimation methods, the loadings reported by [Ang, Dong, and Piazzesi \(2007\)](#) ($\rho_\pi^W = 0.322$ (s.e. = 0.143) and $\rho_g^W = 0.091$ (s.e. = 0.064)) are quite close to our estimates for the rotation with $\mathcal{P}^1 = PC1$, particularly so in light of their reported standard errors.

Clearly the loadings on (g, π) change substantially with the choice of weight matrix W . This in turn has a large effect on the associated standard deviations of the shock e_t . To put the values of $std(e)$ in perspective, note that the sample standard deviation of the three-month Treasury bill rate is 2.96%. Thus, under the rotation $\mathcal{P}^1 = PC1$, e_t and r_t have comparable volatilities. This is supported by the negative correlation between e_t and g_t . When W is chosen to select higher-order PC s for \mathcal{P}^1 , the volatility of e_t increases substantially. The correlations between e_t and (g_t, π_t) — the within-month responses of a monetary authority to an increase in inflation or output growth— also change in magnitude and sign.

We stress that none of these representations of r_t can be viewed as the most “economically plausible” since, by construction, all three are observationally equivalent within the family of $MTSM$ s being studied. The results for the rotation with $\mathcal{P}^1 = PC1$ appear to correspond most closely (in qualitative terms) with the estimates of Taylor rules in the literature, and it has the within-month response of the U.S. monetary authority focusing mostly on inflation (r_t rises with inflation). Yet absent additional identifying restrictions that pin down a unique W (and hence \mathcal{P}^1 as the policy shock), there is no basis for selecting this rotation over the infinity of other rotations with similarly plausible features.

Up to this point we have focused on model $GM_{4,3}(g, \pi)$ with $p=4$ lags under \mathbb{P} . A shortcoming of this model, evident from [Table 1](#), is its poor fit to individual yields: root-

mean-squared pricing errors are as large as 60 basis points. To investigate whether adding an additional risk factor— equivalently, under a Taylor rule interpretation, allowing for a two-dimensional policy shock— impacts estimates of the loadings (ρ_g, ρ_π) , we undertake a similar exercise using the four factor model $GM_{3,4}(g, \pi)$. This model is similar in structure to the *MTSM* examined in [Bikbov and Chernov \(2010\)](#).

In [Table 4](#), we report the model-implied loadings for the short rate under three different rotations of the (now two) yield portfolios \mathcal{P}_t^2 : to $(PC1, PC2)$, $(PC1, PC3)$, or $(PC2, PC3)$. Under none of these rotations do we obtain estimates of (ρ_g, ρ_π) consistent with standard priors about interest rate rules of the Federal Reserve. For instance, with the rotation $\mathcal{P}^{2'} = (PC1, PC2)$, which matches the standard deviation of e_t in the previous case of three factors and $\mathcal{P}^1 = PC1$, neither of the reaction coefficients are positive! Their magnitudes are also notably smaller than in the first row of [Table 3](#). The reason for these, possibly counter-intuitive loadings is that the shock e_t is linearly dependent on $PC2$ and this yield portfolio is positively correlated with output growth (note that $\text{corr}(e_t, g_t)$ has changed signs relative to the first row of [Table 3](#)). Therefore, the roles of both g_t and π_t in the short-rate equation are much attenuated and the loadings flip signs relative to priors for a Taylor rule. This is to some extent mechanical: $(PC1, PC2)$ explain most of the variation in bond yields, with relatively weak explanatory power attributed to M_t . Yet it underscores the weak foundations of interpretations of the loadings (ρ_g, ρ_π) as policy reaction coefficients. Importantly, the primary change in moving from model $GM_{4,3}(g, \pi)$ to model $GM_{3,4}(g, \pi)$ was the introduction of a more flexible, two-dimensional policy shock.

5 Macro-Yield Curve Dynamics In *MTSMs* with Lags

Perhaps the most notable empirical weakness of *MTSMs* that include macro factors M_t in the set of pricing factors Z_t , with or without lags, is that they typically imply that M_t is spanned by the information in the current yield curve. Enforcement of the spanning condition (4) is what underlies the poor pricing performance of the $\mathcal{N} = 3$ model $GM_{p,3}(g, \pi)$ documented in [JLS](#) and [Section 3.4](#). Expanding to $\mathcal{N} = 4$ factors, holding M fixed, improves the pricing performance of *MTSMs*, but the counterfactual spanning restriction remains.

Which aspects of the joint distribution of macro risks and the yield curve does the spanning restriction distort? As a first look at this issue, [Figures 3 and 4](#) display the spanned values of *INFPC* and *REALPC* implied by models $GM_{4,3}^S(g, \pi)$ and $GM_{3,4}^S(g, \pi)$ examined in [Section 4](#) (now with a superscript “S” to highlight its spanning property) against their historical values. Enforcement of the macro-spanning restriction leads to substantially distorted representations of the historical macro series. The spanned series, which are linear combinations of the *PCs* of zero-coupon bond yields, show much more volatility and exhibit much wider swings at turning points in the macro series.

Next, we examine the extent to which enforcement of the spanning restriction also distorts the *IRs* of the macro factors to yield-curve shocks and vice versa. Specifically, we compare the *IRs* implied by two three-factor *MTSMs*, one that enforces the spanning restrictions and one that does not. The model with spanned macro risks is the model $GM_{4,3}^S(g, \pi)$

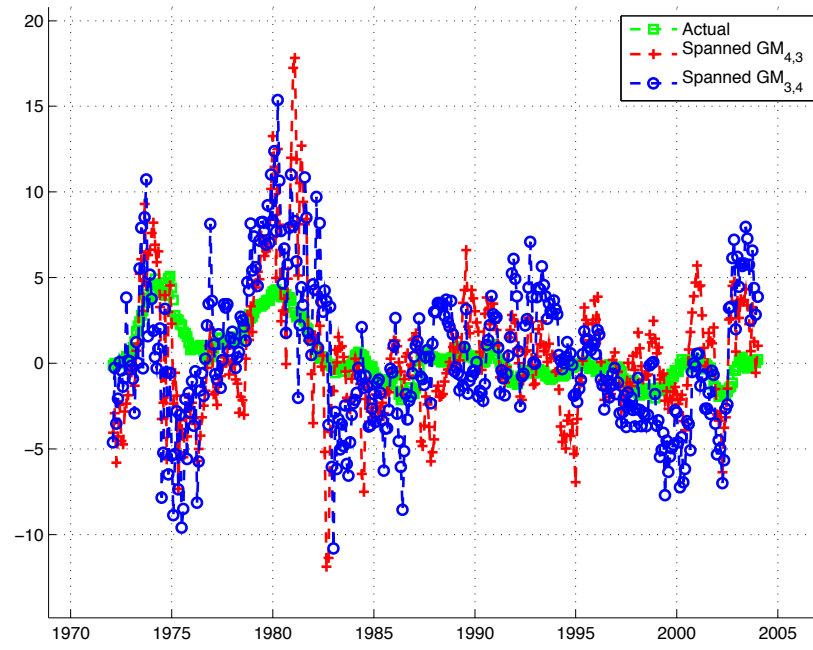


Figure 3: Actual and spanned INFPC.

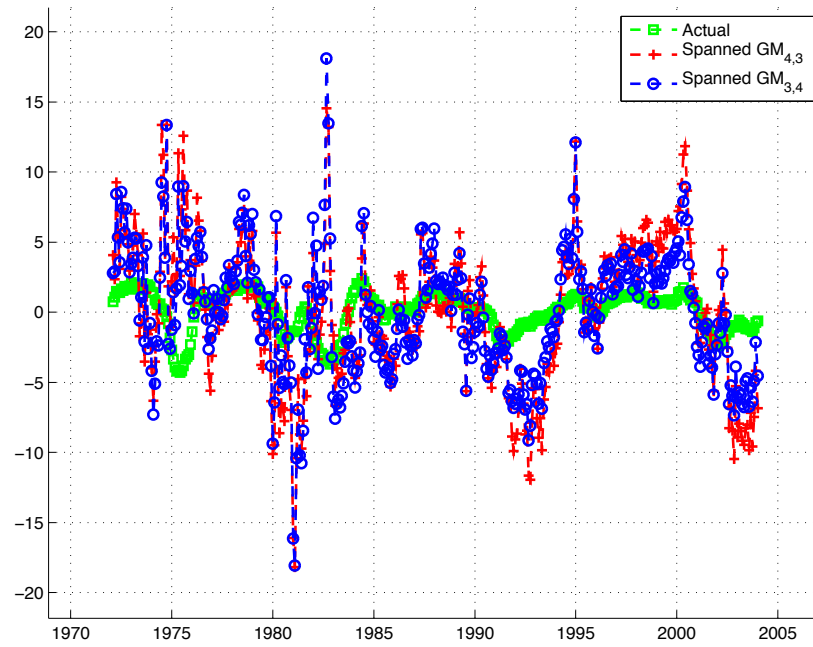


Figure 4: Actual and spanned REALPC.

with state vector (the factors that forecast excess returns) $Z_t^{S'} = (g_t, \pi_t, PC1_t)$. The model with unspanned macro risks, denoted $GM_{2,3}^U(g, \pi)$, maintains the assumption that there are three priced risks, but it presumes that risk premiums depend on the expanded state $Z_t^{U'} = (g_t, \pi_t, PC1_t, PC2_t, PC3_t)$. The order-selection criteria call for a higher order VAR representation for Z_t^S ($p = 4$) than for Z_t^U ($p = 2$), which is not surprising given the higher dimension of the latter state vector.

For an economy with unspanned macro risks, the focus on the subvector of states $(g_t, \pi_t, PC1_t)$ in model $GM_{4,3}^S(g, \pi)$ will in general lead to misspecified risk premiums since relevant conditioning information is omitted. Similarly, IR s among the included states will in general be inaccurate. Finally, the model with spanned macro risks is, by construction, completely silent about the impact of unspanned macro risks on the yield curve and vice versa. The remainder of this section illustrates the quantitative importance of these points.

To set up this analysis we briefly review the structure of $MTSM$ s with unspanned M developed in JPS. Their canonical form replaces (15) and (16) with

$$r_t = \rho_{0P} + \rho_{1P} \cdot \mathcal{P}_t^{\mathcal{N}}, \quad (28)$$

$$\mathcal{P}_t^{\mathcal{N}} = K_{0P}^{\mathbb{Q}} + K_{PP}^{\mathbb{Q}} \mathcal{P}_{t-1}^{\mathcal{N}} + \sqrt{\Sigma_{PP}} \epsilon_{Pt}^{\mathbb{Q}}. \quad (29)$$

Since, as we have seen, an \mathcal{N} -factor $MTSM$ that enforces spanning can always be rotated so that the \mathcal{N} pricing factors are $\mathcal{P}^{\mathcal{N}}$, this premise leads to identical priced risks as in the \mathcal{N} -factor models encompassed by [Theorem 1](#).

Where $MTSM$ s with unspanned risks differ from those with spanned risks is in their specifications of the distribution of the state under \mathbb{P} . Instead of imposing the spanning condition (4), the pricing factors $\mathcal{P}_t^{\mathcal{N}}$ and macro variables M_t are assumed to jointly follow an unconstrained $\text{VAR}^{UP}(p)$:

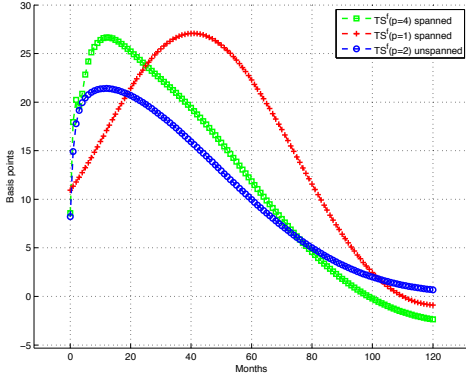
$$\begin{bmatrix} \mathcal{P}_t^{\mathcal{N}} \\ M_t \end{bmatrix} = \begin{bmatrix} K_{0P}^{\mathbb{P}} \\ K_{0M}^{\mathbb{P}} \end{bmatrix} + \begin{bmatrix} K_{PP}^{\mathbb{P}} & K_{PM}^{\mathbb{P}} \\ K_{PX}^{\mathbb{P}} & K_{MM}^{\mathbb{P}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{t-1}^{\mathcal{N}} \\ \mathcal{M}_{t-1}^p \end{bmatrix} + \sqrt{\Sigma_Z} \epsilon_{Zt}^{\mathbb{P}}, \quad (30)$$

where now $Z_t^{U'} = (\mathcal{P}_t^{\mathcal{N}'}, M_t')$, $\epsilon_{Zt}^{\mathbb{P}} \sim N(0, I_{\mathcal{N}+\mathcal{M}})$, the $(\mathcal{N} + \mathcal{M}) \times (\mathcal{N} + \mathcal{M})$ matrix Σ_Z is nonsingular, and Σ_{PP} is the upper $\mathcal{N} \times \mathcal{N}$ block of Σ_Z . Accordingly, M_t is not deterministically spanned by \mathcal{P}_t and forecasts of \mathcal{P} are conditioned on the full set of $\mathcal{N} + \mathcal{M}$ variables Z_t^U . JPS consider the special case of $p = 1$ but, using the logic of [Theorem 1](#), their canonical form is easily extended to the case of lags under \mathbb{P} .

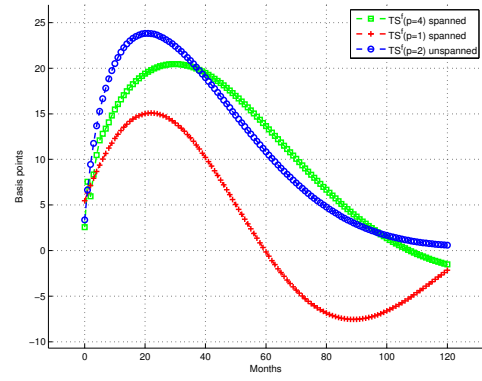
Analogously to (22), the conditional density of (Z_t^U, \mathcal{P}_t^e) is given by

$$f(Z_t^U, \mathcal{P}_t^e | \mathcal{Z}_{t-1}^{Up}; \Theta) = f(\mathcal{P}_t^e | \mathcal{P}_t^{\mathcal{N}}; r_{\infty}^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_P, \Sigma_e) \times f(Z_t^U | \mathcal{Z}_{t-1}^{Up}; K_{1Z}^{\mathbb{P}}, K_{0Z}^{\mathbb{P}}, \Sigma_Z). \quad (31)$$

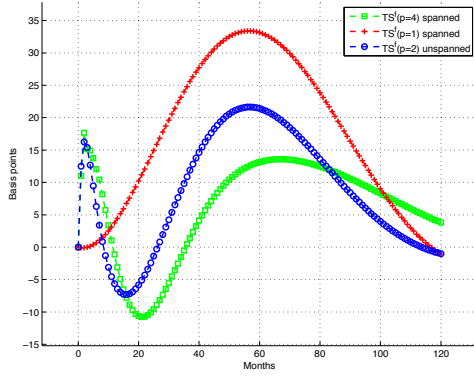
Just as in the models with spanning, the associated likelihood function factors into a piece dependent on the risk-neutral parameters $(r_{\infty}^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_P)$ and a second piece involving the parameters of the conditional \mathbb{P} -mean of Z_t^U , with the only overlap being the parameters governing Σ_Z . Therefore, the no-arbitrage structure of a $MTSM$ with unspanned risk is also likely to be (nearly) irrelevant for studying the joint distribution Z_t^U under \mathbb{P} . This will be



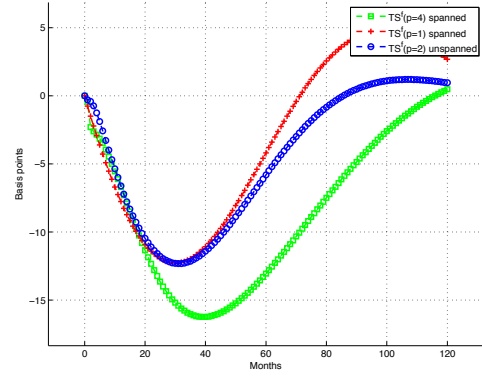
(a) Response of $PC1$ to $REALPC$



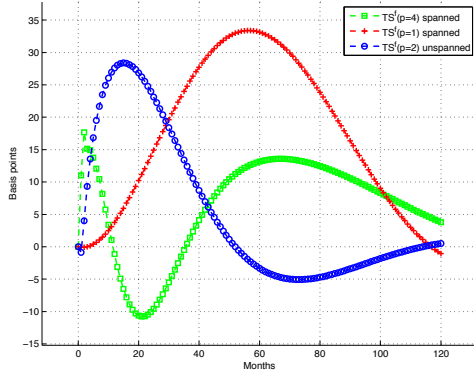
(b) Response of $PC1$ to $INFPC$



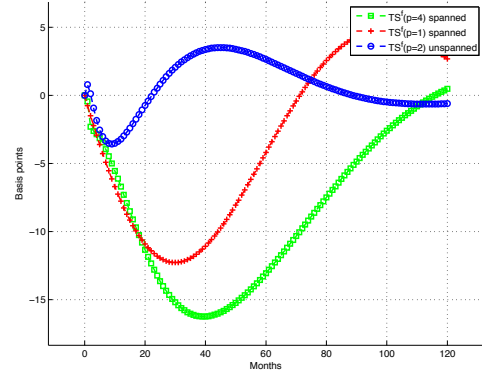
(c) Response of $REALPC$ to $PC1$



(d) Response of $INFPC$ to $PC1$



(e) Response of $REALPC$ to $PC2$



(f) Response of $INFPC$ to $PC2$

Figure 5: The states are ordered as (INFPC, REALPC, PC1, PC2, PC3) for the unspanned models and (INFPC, REALPC, PC1) for the spanned models. To draw IRs from and to PC2 implied by the spanned models, we first rotate the model to one in which the states are (INFPC, REALPC, PC2). The models with spanned risks are $GM_{4,3}^S(g, \pi)$ ($p = 4$) and $GM_{1,3}^S(g, \pi)$ ($p = 1$), and the model with unspanned risks is $GM_{2,3}^U(g, \pi)$.

the case even when we allow all yields to be priced with errors, so long as the average pricing errors are small relative to their volatilities in the sense made precise in JLS.

Panels (a) and (b) of [Figure 5](#) display the *IRs* of *PC1*— the level of the Treasury yield curve— to one standard deviation shocks in (g_t, π_t) (measured by *REALPC* and *INFPC*). Panels (c) and (d) display the reverse response of the macro factors to a shock to *PC1*.¹³ For comparison we have also included the *IR* for model $GM_{1,3}^S(g, \pi)$, since many *MTSMs* adopt first-order Markov representations of Z_t under both \mathbb{P} and \mathbb{Q} .

An immediately striking feature of the *IRs* from models $GM_{4,3}^S(g, \pi)$ and $GM_{1,3}^S(g, \pi)$ is how different they are as a consequence of changing the order of the $\text{VAR}^{\mathbb{P}}(p)$ from $p = 1$ to $p = 4$. Particularly in Panel (c) the responses within the first two years are in opposite directions, and differences persist for nearly eight years. One might be inclined to attribute these differences to over-fitting in the case of $p = 4$, but we reiterate that model selection criteria called for $p > 1$ and, moreover, the response patterns for model $GM_{4,3}^S(g, \pi)$ are qualitatively similar to those for model $GM_{2,3}^S(g, \pi)$ (not displayed). Inference about the joint distribution of macro risks and bond yields can be very sensitive to the order of the $\text{VAR}^{\mathbb{P}}(p)$ representation of the risk factors in *MTSMs* that enforce the macro-spanning restriction.

Comparing Panels (a) and (b) to their counterparts in [Figure 1](#) it is seen that models $GM_{4,3}^S(g, \pi)$, $GM_{3,4}^S(g, \pi)$, and $GM_{2,3}^U(g, \pi)$ all imply qualitatively similar responses of *PC1* to the macro shocks. The timings and magnitudes of the peak responses differ somewhat across models, but the broad patterns are similar. Evidently the joint dynamics of $(g_t, \pi_t, PC1_t)$ implied by VARs (with model-specific lag lengths) with and without the inclusion of $(PC2, PC3)$ are roughly comparable, owing to the inclusion in all three models of $(g_t, \pi_t, PC1_t)$ in the state vector Z . Similar observations apply to the responses of the macro factors to a shock in *PC1* as displayed in [Figure 5](#) Panels (c) and (d).

At the same time, along other dimensions, the model-implied joint distributions of (M_t, y_t) are strikingly different. Consider, for instance, the joint distribution of *PC2* and M_t . The slope of the yield curve *PC2* is not included as an independent source of risk in the models $GM_{p,\mathcal{N}}^S(g, \pi)$ with spanned macro risks. Equivalently, the theoretical risk factors are the first \mathcal{N} *PCs* $\mathcal{C}_t^{\mathcal{N}}$ and (g_t, π_t) are linearly spanned by $\mathcal{C}_t^{\mathcal{N}}$ according to (4). In contrast, *PC2* enters both as a risk factor and as an element of the Z_t^U in the model with unspanned macro risks so it is a distinct source of risk relative to $(M_t, PC1_t)$.

Panels (e) and (f) of [Figure 5](#) compare the responses of $(REALPC, INFPC)$ to a *PC2* shock for our illustrative *MTSMs* with lags. Focusing first on output growth, after a shared initial positive response, the response patterns are almost mirror images of each other (highly negatively correlated). The unconstrained VAR with unspanned macro risks has *REALPC* rising for about eighteen months, and then gradually declining towards zero over a span of about three years. In contrast, the positive impact under the macro-spanning constraint dies out within a year, turns negative over the following year, and gradually cycles to a positive effect that persists for an additional seven years or so.

The positive response of *REALPC* in model $GM_{2,3}^U(g, \pi)$ is consistent with the earlier

¹³The variables *REALPC* and *INFPC* were normalized to have the same mean and standard deviation as the growth in industrial production and annualized inflation, respectively, for reporting these *IRs*.

findings of [Estrella and Hardouvelis \(1991\)](#). [Stock and Watson \(2003\)](#) survey more recent evidence of a positive relationship, and also review the wide use of the yield-curve slope as a leading indicator of real economic activity. These positive responses do not match the response in the *MTSM* examined in [Rudebusch and Wu \(2008\)](#) and, indeed, their pattern more closely resembles the negative response in model $GM_{4,3}^S(g, \pi)$ (absent the initial positive response). Their model, which has a more structural foundation than our reduced-form *MTSMs*, implicitly enforces macro spanning just as in model $GM_{4,3}^S(g, \pi)$.

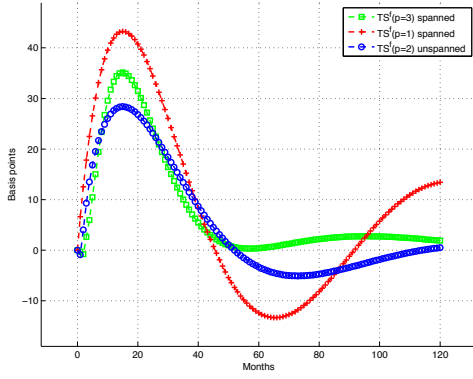
Turning to the inflation responses in [Figure 5](#), when accommodating unspanned inflation risk there is at most a small response of *INFPC* to a shock to the slope of the yield curve. This is consistent with the descriptive time-series evidence discussed in [Stock and Watson \(2003\)](#). However a very different response pattern emerges from model $GM_{4,3}^S(g, \pi)$ with spanned macro risks. In this case inflation responds negatively with a trough occurring over three years after the shock.

Why do *MTSMs* with spanned and unspanned macro risks provide such different evidence on nature of the comovement among the slope of the yield curve, inflation, and output growth? For the reasons discussed previously (and elaborated on in JLS), the *IRs* implied by model $GM_{2,3}^U(g, \pi)$ are essentially those implied by the unconstrained $\text{VAR}^{\mathbb{P}}(2)$ model for Z_t^U – the no-arbitrage restrictions are (nearly) irrelevant for these calculations. That is, this *MTSM* is essentially just replicating what is in the data for our choice of macro factors, bond yields, and sample period.

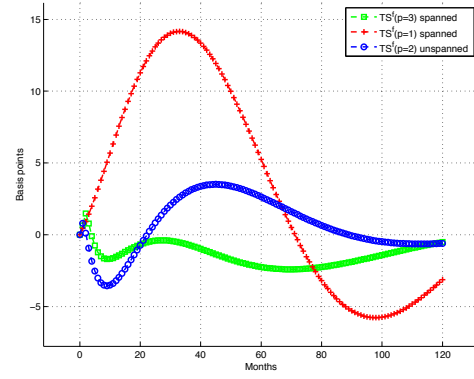
On the other hand, *MTSMs* that enforce macro-spanning fundamentally change the connection between the information in the yield curve and the macroeconomy. In our illustrative model $GM_{4,3}^S(g, \pi)$, *PC2* is effectively omitted from the set of pricing factors ($Z_t^{S'} = (g_t, \pi_t, PC1_t)$). Therefore, this model has to synthesize *PC2* as an affine function of Z_t^S through the macro-spanning constraint. The filtered values $PC2_t^f$ of the theoretical construct *PC2* in this model typically bear little resemblance to the historical time series $PC2^o$. Consequently, assessments of the responses of macro factors to the slope of the yield curve under spanning are highly inaccurate in this model.

This same reasoning suggests that a model with macro-spanning and an expanded state vector $(g_t, \pi_t, PC1_t, PC2_t)$ would give similar responses among the $(PC2_t, M_t)$. To confirm this we display in [Figure 6](#) the *IRs* of *REALPC* and *INFPC* to *PC2* shocks from the models $GM_{3,4}^S(g, \pi)$, $GM_{1,4}^S(g, \pi)$, and $GM_{2,3}^U(g, \pi)$. The patterns of responses of *REALPC* are similar across these three models, with the *IRs* for models $GM_{3,4}^S(g, \pi)$ and $GM_{2,3}^U(g, \pi)$ being especially close. The responses of *INFPC* are also similar across the latter two models.

These findings suggest that in some circumstances the joint dynamics of included macro and yield-based factors can be approximated within a model that enforces macro-spanning by adding additional lags of Z_t under \mathbb{P} . However, this improved fit may, as in the illustrative cases here, require the addition of extraneous pricing factors (more than are needed to fit the cross-section of bond yields) and extra lags (relative to the more parsimonious model with unspanned macro risks). This last point is illustrated by the very large differences between the responses of *INFPC* in models $GM_{3,4}^S(g, \pi)$ and $GM_{1,4}^S(g, \pi)$. Put differently, our findings suggest that the large values of p often adopted in the literature on *MTSMs* may well



(a) Response of *REALPC* to *PC2*



(b) Response of *INFPC* to *PC2*

Figure 6: Impulse responses of *REALPC* and *INFPC* given one standard deviation shock to *PC2*. The states are ordered as *INFPC*, *REALPC*, *PC1*, *PC2*, *PC3* for the unspanned models and *INFPC*, *REALPC*, *PC1*, *PC2* for the spanned models. The *INFPC*, *REALPC* variables are normalized to have the same mean and standard deviation as annualized *CPI*, and *IP* growth, respectively. The models with spanned risks are $GM_{3,4}^S(g, \pi)$ ($p = 3$) and $GM_{1,4}^S(g, \pi)$ ($p = 1$), and the model with unspanned risks is $GM_{2,3}^U(g, \pi)$.

have been necessary owing to the imposition of counterfactual macro-spanning restrictions. Furthermore, the spanning models $GM_{p,4}^S(g, \pi)$ omit *PC3* as a risk factor and, therefore, the *IRs* of *PC3* to macro shocks will likely be different from those in the model $GM_{2,3}^U(g, \pi)$.

6 Concluding Remarks

We have extended the family of *MTSMs* to accommodate higher-order (beyond first order) lags in the parameterization of the historical distribution of the risk factors, while preserving the parsimonious first-order Markov structure of these factors called for by the cross-sectional distribution of yields. Our framework encompasses both the models with spanned macro risks studied in JLS and the models with unspanned risks examined in JPS and Duffee (2011). Since the $\text{VAR}^{\mathbb{P}}(p)$ representation of the state can have p of any finite order, our canonical form nests most of the extant *MTSMs* with lags.

Our empirical analysis has addressed several distinct issues related to the specification of *MTSMs* with lags. First we argued, based on model-free evidence, that the macro and yield-curve data suggest a low-dimensional cross-sectional factor structure for bond yields. This structure is well described by a first-order Markov process under the pricing measure for the vector of risk factors. On the other hand, in several settings, statistical model selection criteria called for a higher order VAR representation of the state under the historical measure (for the data-generating process). Together, these observations motivated our asymmetric formulation of *MTSMs* based on a $\text{VAR}^{\mathbb{P}}(p)$ model with $p > 1$.

With the canonical version of this family of *MTSMs* in hand, we document through several examples that the no-arbitrage restrictions of a *MTSM* have very little impact on the properties of the joint distribution of the risk factors when the risk factors are rotated so that the non-macro factors are low-order *PCs* of bond yields. This shows that the prior results on the irrelevance of no-arbitrage restrictions in a canonical setting documented in JLS for the case of $p = 1$ extend to *MTSMs* with more lags under \mathbb{P} .

Next we examine the econometric identification of the reaction coefficients of a central bank policy rule— the Taylor rule— within *MTSMs* that enforce macro-spanning. We show that the policy parameters are in general not identified, absent the imposition of additional economic structure that is rarely imposed in the analysis of reduced-form *MTSMs*. The practical implication of this observation is that we are able to present several observationally equivalent (within an *MTSM*) representations of the short-term rate, many of which have loadings on output growth and inflation, and (what some have referred to as) policy shocks, that bear no resemblance to their counterparts in standard Taylor rules. Moreover, our empirical results illustrate that expanding the number of risk factors from three to four— thereby effectively allowing for a richer distribution of the monetary policy shock— can fundamentally change the signs and magnitudes of loadings for the short-term rate, and the covariances of the shock with output and inflation.

Finally, drawing upon the literature on macro risks that are not spanned by the current information in the yield curve (unspanned macro risks), we addressed the question of how enforcement of the macro-spanning constraint distorts the impulse response functions among the macro and yield-curve factors. Particularly for yield-curve constructs that are not included in the set of risk factors (e.g., the slope of the yield curve in the family of models $GM_{p,3}(g, \pi)$) the distortions were large. Models with macro spanning are also, by construction, completely silent about the impact of unspanned macro risks on risk premiums in bond markets.

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