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Notes on New Keynesian Models*

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The idea is to work through some simple models to see how they work. The emphasis is on their dynamic structure, not their economic foundations.

Cochrane's example

Model. This is pretty basic, but illustrates how the model is solved. We adapted it from Cochrane's "identification" paper. Not really a New Keynesian Model, more like a Cagan-style model of hyperinflation. The model has two equations,

$$\begin{aligned} i_t &= r + E_t p_{t+1} + e_1^\top x_t \\ i_t &= r + \tau p_t + e_2^\top x_t, \end{aligned}$$

plus a stationary process for the shocks,

$$x_{t+1} = A x_t + B w_{t+1},$$

where $\{w_t\} \sim \text{NID}(0, I)$. The variables are the nominal interest rate i and inflation p. Think of the first equation as a simple version of an Euler equation (EE) and the second as a Taylor rule (TR). We set $r = 0$ to keep things simple.

Solution. The model has two endogenous variables, one static (i) and one dynamic (p), terminology that should be clear in a minute. If we substitute for i (the static endogeneous variable) we're left with the expectational difference equation

$$E_t p_{t+1} = \tau p_t + (e_2 - e_1)^\top x_t.$$

Solution methods are summarized in the Appendix. The simplest one is to guess a solution of the form $p_t = a^\top x_t$. Then $E_t p_{t+1} = a^\top A x_t$. The equation becomes

$$a^\top A = \tau a^\top x_t + (e_2 - e_1) x_t.$$

Collecting coefficients of x_t, we have the solution,

$$a^\top = (e_2 - e_1)^\top (\tau I - A)^{-1}.$$

If we expand the implied geometric series, we see this as a linear combination of the discounted sum of expected future x.

*Working notes, no guarantee of accuracy or sense.

New Keynesian model

Model. Here's another one, a streamlined version of Clarida-Gali-Gertler. [Sorry, I couldn't resist playing with the notation.]

$$
\begin{aligned}
i_t &= \alpha E_t y_{t+1} + E_t p_{t+1} \\
p_t &= \psi_g g_t + \psi_p E_t p_{t+1} + e_p^\top x_t \\
i_t &= \tau_g g_t + \tau_p p_t + e_m^\top x_t \\
y_t &= a_t + g_t \;=\; e_a^\top x_t + g_t.
\end{aligned}
$$

The variables are the nominal short rate i_t, consumption/output growth y_t, inflation p_t, growth of the output gap (deviation from optimum) g_t, and growth of "full-employment" output a_t. The equations are, in order: an Euler equation, a Phillips curve, a Taylor rule, and the definition of output (=consumption).

Solution. Endogenous variables: y, p, i, g. Of these y and p are dynamic forward-looking variables, the others are static and can be substituted out. The system looks like

$$
\begin{aligned}
\alpha E_t y_{t+1} + E_t p_{t+1} &= \tau_g (y_t - e_a^\top x_t) + \tau_p p_t + e_m^\top x_t \\
p_t &= \psi_g (y_t - e_a^\top x_t) + \psi_p E_t p_{t+1} + e_p^\top x_t
\end{aligned}
$$

or

$$
\begin{bmatrix} \alpha & 1 \\ 0 & \psi_p \end{bmatrix} \begin{bmatrix} E_t y_{t+1} \\ E_t p_{t+1} \end{bmatrix} = \begin{bmatrix} \tau_g & \tau_p \\ -\psi_g & 1 \end{bmatrix} \begin{bmatrix} y_t \\ p_t \end{bmatrix} + \begin{bmatrix} e_m^\top - \tau_g e_a^\top \\ \psi_g e_a^\top - e_p^\top \end{bmatrix} [x_t] \,.
$$

Appendix: Hansen-Sargent formulas

Univariate version. Here's a useful result from Hansen and Sargent (JEDC, 1980, p 14) and Sargent (*Macroeconomic Theory, 2e*, 1987, pp 303-304). An expectational difference equation with stationary forcing variable x generates a "geometric distributed lead":

$$\begin{aligned} y_t &= \lambda E_t y_{t+1} + x_t \\ &= \lambda E_t(\lambda E_{t+1} y_{t+2} + x_{t+1}) + x_t \\ &= \sum_{j=0}^{\infty} \lambda^j E_t x_{t+j}. \end{aligned}$$

If $x_t = \sum_{j=0}^{\infty} \chi_j w_{t-j} = \chi(L) w_t$, with w white noise, then what is y_t? A unique stationary solution $y_t = \psi(L) w_t$ exists if x is stationary and $|\lambda| < 1$, but what is $\psi(L)$?

Note how the distributed lead works. Conditional expectations of x have the form

$$E_t x_{t+j} \;=\; [\chi(L)/L^j]_+ w_t \;=\; \sum_{i=0}^{\infty} \chi_{j+i} w_{t-i}$$

(The subscript "+" means ignore negative powers of L.) Therefore the coefficient of w_{t-i} in the distributed lead is

$$\psi_i \;=\; \sum_{j=0}^{\infty} \lambda^j \chi_{i+j}.$$

This tells us, for example, that if x is MA(q), then so is y: if $\chi_j = 0$ for $j > q$, then $\psi_j = 0$ above the same limit.

There's a "lag notation" version that expresses the result in compact form. We're not sure whether it's all that useful for our purposes, but here it is. We're looking for a solution $y_t = \psi(L) w_t$ satisfying the expectational difference equation:

$$\psi(L) w_t \;=\; [\psi(L)/L]_+ w_t + \chi(L) w_t.$$

... [flesh this out]

See also Hansen and Sargent ("A note on Wiener-Kolmogorov prediction," ms, 1981).

Vector version. Here's a related result adapted from Ljungqvist and Sargent (*Recursive Macroeconomic Theory, 2e*, 2005, section 2.4). It extends the previous result to higher dimensional forcing processes that can be expressed as stationary vector autoregressions. Consider the system

$$\begin{aligned} y_t &= \lambda E_t y_{t+1} + u^\top x_t \\ x_{t+1} &= A x_t + B w_{t+1}, \end{aligned}$$

where u is an arbitrary vector and w is iid with mean zero and variance I. The solution in this case is

$$y_t \;=\; \sum_{j=0}^{\infty} \lambda^j u^\top E_t x_{t+j} \;=\; u^\top \sum_{j=0}^{\infty} \lambda^j A^j x_t \;=\; u^\top (I - \lambda A)^{-1} x_t.$$

[The last step follows from the matrix geometric series.]

There's a method of undetermined coefficients version of this. Guess $y_t = a^\top x_t$ for some vector a (we know the solution has this form from what we just did). Then the difference equation tells us

$$a^\top x_t \;=\; a^\top \lambda A x_t + u^\top x_t.$$

Collecting terms in x_t gives us $a^\top = u^\top (I - \lambda A)^{-1}$, as stated. What's missing from this approach is an indication that λA must be stable.

Here's a vector version. Let y be a vector with

$$y_t \;=\; L E_t y_{t+1} + G x_t.$$

Use the usual law of motion for x_t. If we guess $y_t = F x_t$, substitution seems to give us

$$F \;=\; LFA + G.$$

How do we solve this for F? Is there a formula or are we stuck with numerical methods?