Identifying macroeconomic parameters with forecasts, asset prices, and structure*

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Preliminary and incomplete

Abstract

Identification problems arise naturally in forward-looking models when agents observe more than econometricians. One approach is additional information, including economic forecasts and asset prices. Another is tighter economic structure: what is sometimes called cross-equation restrictions. We show how each aids in the identification of structural parameters, including the inflation parameter of a Taylor rule. As a rule, asset prices help to identify the state, and cross-equation restrictions help to identify structural parameters. All of this is done with variations on a single example.

JEL Classification Codes: E43, E52, G12.

Keywords: forward-looking models; information sets; forecasts; forward rates; monetary policy.

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*Preliminary and incomplete: no guarantees of accuracy or sense. We welcome comments, including references to related papers we inadvertently overlooked. This started with our reading of John Cochrane's paper on the same subject and subsequent emails and conversations with him and Mark Gertler. We thank them both. We also owe an obvious debt to Lars Hansen, Thomas Sargent, and Christopher Sims.

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1 Introduction

We explore the possibility of using forecasts, asset prices, and cross-equation restrictions to identify structural parameters of macroeconomic models. In stationary Markov models, macroeconomic quantities and asset prices are functions of the state, so we might consider using a combination of the two to reverse the operation and infer the state from the values. In parsimonious models, we might also expect the additional information from asset prices to ....

... This unified field theory of macro-finance is far from complete, but we argue that identification is not one of its problems.

We work through some examples that illustrate the central issues governing identification in forward-looking models, including the potential role played by forward rates. We wouldn't call the example simple, but it's the simplest we've been able to come up with. Even here some subtle issues arise. The central issue is what agents and econometricians observe. When econometricians (namely, us) observe less than agents, there are inevitably difficulties with identification. The logic follows Hansen and Sargent (1980, 1991) and related work by them and others. Forward rates can be helpful here, since they provide the econometrician with additional information. Whether this information is enough to identify the model's parameters depends on the model.

Model: typically expressed as a forward-looking difference equation
Solution: state follows a stationary process, and prices and quantities are functions of the state
Identification concerns the estimation of parameters

Punchline...

2 Forward-looking models 1: moving average version

We'll start with this one-dimensional example of a forward-looking rational expectations model:

$$y_t \;=\; \lambda E_t y_{t+1} + x_t. \tag{1}$$

Here y is an endogenous forward-looking variable, x is an exogenous shock or forcing process, and λ is a parameter whose absolute value is less than one. The shock x is an arbitrary scalar moving average,

$$x_t \;=\; \sum_{j=0}^{\infty} \alpha_j w_{t-j} \;=\; \alpha(L) w_t,$$

where x is stationary (square summable) and $\{w_t\} \sim \text{NID}(0, 1)$. Under these conditions, y has a unique stationary solution. In the scalar moving average case, the solution has the form $y_t = \beta(L)w_t$, with

$$\beta_j = \sum_{i=0}^{\infty} \lambda^i \alpha_{j+i}$$

for $j \geq 0$. See Appendix A.

Our mission is to estimate the parameters, particularly λ. Identification depends in large part on what we observe. We assume throughout that the agents in this economy observe x and y. Indeed, without this information, our solution of (1) wouldn't make sense. The question is what econometricians observe. We assume we always observe y and consider the impact of observing or not observing x. The question is whether we, the econometricians, can estimate the parameters anyway. We illustrate the argument with a first-order moving average or MA(1) for x:

$$x_t = \alpha_0 w_t + \alpha_1 w_{t-1} = \alpha_0(w_t + \theta w_{t-1}).$$

Then y is also MA(1), with parameters $\beta_0 = \alpha_0 + \lambda\alpha_1$ and $\beta_1 = \alpha_1$.

In this setting, identification is basically a matter of matching up parameters with properties of observables. We assume we know the autocovariance function $\gamma_y(k)$ of y, where $\gamma_y(k) \equiv E(y_t y_{t-k})$. If x is observed, we also know the autocovariance function $\gamma_x(k)$ for x.

2.1 What do we observe?

2.2 Example: streamlined New Keynesian model

The simplest example in Cochrane (2007) consists of two equations:

$$i_t = r + E_t p_{t+1} + x_{1t} \tag{2}$$
$$i_t = r + \tau p_t + x_{2t}, \tag{3}$$

The variables are the nominal interest rate i and inflation p. Think of the first equation as a simple version of an Euler equation (EE) and the second as a Taylor rule (TR). Let the shocks x_{it} be independent moving averages $\alpha_i(L)w_{it}$. We'll set the real interest rate $r = 0$ for now.

[Should we switch to vector version, which imposes invertibility on the overall system but not on subsets?]

Solution. Equations (2) and (3) give us the stochastic difference equation

$$E_t p_{t+1} = \tau p_t - x_{1t} + x_{2t}. \tag{4}$$

We're looking for a stationary solution for p. If $|\tau| > 1$, there's a unique stationary solution of the form $p_t = \pi_1(L)w_{1t} + \pi_2(L)w_{2t}$. Thus: given parameters $\{\tau, \alpha_{ij}\}$ we can derive the coefficients $\{\pi_{ij}\}$ of the inflation process.

Identification. Suppose we observe inflation (p) and the interest rate (i), but not the shocks (x_1 and x_2). Can we identify the Taylor rule parameter (τ)? The difficulty is that the behavior of p and i combines the behavior of the shocks and the TR parameter, and it's not clear we can disentangle them. Here are some examples.

- Special case 1 ($x_2 = 0$). Identification follows directly from estimating the Taylor rule (3), which is exact in this case. Both the lhs and rhs are observable.

  A little algebra shows how this works. Since the model has a univariate shock, we can estimate $\pi_1(L)$ from a long enough time series for p. From this process, we can infer the coefficients for expected inflation ($[\pi_1(L)/L]_+$ in Hansen-Sargent notation). The interest rate and the Euler equation then tell us x_1 and its coefficients $\alpha_1(L)$. The coefficients of the lhs ($[\pi_1(L)/L]_+ + \alpha_1(L)$) are τ times those of the rhs ($\pi_1(L)$):

  $$w_{t-j}: \qquad \pi_{1j+1} + \alpha_{1j} \;=\; \tau^j \sum_{i=j+1}^{\infty} \tau^{-i}\alpha_{1i} + \alpha_{1j} \;=\; \tau\pi_{1j}.$$

  [This holds by construction; verification here simply tells us we did the calculations right.]

- Special case 2 ($x_1 = 0$). This is the example Cochrane looks at. Again, we observe p and i. The former gives us the inflation coefficients π_2. The latter gives us no new information, since with $x_1 = 0$ expected inflation is implied by the inflation process. The result is that we can't isolate the roles of p_t and x_{2t} in the Taylor rule (3), hence can't estimate τ.

  Here's an example. Suppose x_2 is MA(1). Then p is MA(1), too. The inflation coefficients are

  $$\begin{aligned} \pi_{20} &= -\tau^{-1}(\alpha_{20} + \tau^{-1}\alpha_{21}) \\ \pi_{21} &= -\tau^{-1}\alpha_{21}. \end{aligned}$$

  We can estimate the π's from inflation data, but we can't disentangle the impact of the shock (the α's) from the policy parameter (τ). That's true even if x_2 is white noise ($\alpha_{2j} = 0$ for $j \geq 1$): the inflation coefficient is $\pi_{20} = -\tau^{-1}\alpha_{20}$ and we can't separate the two components. Alternatively, let x_2 be AR(1), so that $\alpha_{2j} = \varphi^j\alpha_{20}$. In this case, $\pi_{2j} = -\varphi^j\alpha_{20}/(\tau - \varphi)$, so inflation is AR(1) with the same autoregressive parameter. Inflation and the shock are perfectly correlated, so there's no way we can disentangle their effects in the Taylor rule. An inflation process can be reconciled with any choice of τ we like by adjusting α_{20}.

- Shocks in both equations. One of the lessons here is getting extra information out of the interest rate (special case 1). If $x_1 = 0$ we can't do that. Another is that the TR shock x_2 shows up in p, which makes it difficult to separate their effects in the TR (special case 2). The question is how far we can go if we have shocks in both equations.

  Gertler's example shows how we might handle two shocks. Let the two state variables be AR(1):

$$x_{it} = \varphi_i x_{it-1} + w_{it}.$$

  with $\varphi_2 = 0$ (we can generalize this later on). In either case, the state space is essentially (x_1, x_2), so we drop the infinite MA notation. The inflation process has the form $p_t = \pi_1 x_{1t} + \pi_2 x_{2t}$. Substituting into (4) and collecting terms gives us $\pi_1 = 1/(\tau - \varphi_1)$ and $\pi_2 = -1/\tau$. Observables are therefore

$$\begin{aligned} p_t &= [1/(\tau - \varphi_1)]x_{1t} - (1/\tau)x_{2t} \\ i_t &= [\tau/(\tau - \varphi_1)]x_{1t}. \end{aligned}$$

  We can estimate φ_1 from the autocorrelation of i and τ from the ratio of the interest rate to expected inflation.

3 Forward-looking models 2: state space version

Here's a more formal approach to the same problem. The idea is to write the dynamics of the observables in terms of the dynamics of the shocks. Here if the shocks are a VAR, then so are the observables. The VAR for the observables can be estimated, so the question is whether we can use its estimated coefficients to uncover the TR parameter τ. Suppose the shocks follow

$$x_{t+1} = Ax_t + Bw_{t+1},$$

with $x = (x_1, x_2)$ and w vector white noise. In our example, $A = [\varphi_1, 0; 0, 0]$. Then the observables $y = (p, i)$ are a linear transformation of x: $y = Cx$. In our example,

$$C = \begin{bmatrix} 1/(\tau - \varphi_1) & -1/\tau \\ \tau/(\tau - \varphi_1) & 0 \end{bmatrix}.$$

Let's assume that C is invertible. Then in terms of observables, we also have a VAR, namely

$$y_{t+1} = CAC^{-1}y_t + CBw_{t+1} = A^* y_t + B^* w_{t+1}$$

Now to identification. We can estimate the autoregressive parameters A^* of the observables; the question is whether we can deduce τ from them. In the example,

$$A^* = \begin{bmatrix} 0 & \varphi_1/\tau \\ 0 & \varphi_1 \end{bmatrix},$$

so we can compute $\tau = a^*_{22}/a^*_{12}$. [This is tedious; we did it with Matlab's Maple toolbox.]

This line of thought can easily be generalized, although the expressions can get complicated. Let the expectational difference equation be

$$E_t p_{t+1} \;=\; \tau p_t - u_1^\top x_t + u_2^\top x_t,$$

where u_1 and u_2 are known vectors. (In our example, u_i picks out the ith element of x.) We guess $p_t = a^\top x_t$ and derive

$$a^\top \;=\; (u_1 - u_2)^\top (\tau I - A)^{-1}. \qquad (5)$$

The observables are then

$$y_t \;=\; \begin{bmatrix} p_t \\ i_t \end{bmatrix} \;=\; \begin{bmatrix} a^\top \\ a^\top A + u_1^\top \end{bmatrix} x_t \;=\; C x_t.$$

We estimate A^* and then ask whether we can recover τ. By counting, you might guess that this won't work without some restrictions on A: we estimate 4 elements of A^*, which isn't enough to nail down the 4 elements of A plus τ. Clearly it works, as described, if all the elements of A but the first one are zero. Can we go beyond that? Suppose A is diagonal. This is horribly nonlinear, but the diagonal elements of A are the eigenvalues of A^*. [Remember: A and A^* are similar.] From there, we can find τ. For example [this courtesy of Matlab] the upper right element is

$$a^*_{12} \;=\; \frac{a_{11} - a_{22}}{\tau - a_{22}}.$$

As long as the eigenvalues aren't equal, we can find τ. [Problem: we don't know which eigenvalue is which element of A, so we get two possible estimates of τ.]

Could we make A triangular? Not clear. Also not clear whether we could use information on covariances: eg, assume B diagonal.

Extensions: (i) VAR(2), (ii) bond yields (EH), (iii) larger vector of x's.

4 Term structure models

4.1 Essential Affine

Which model? Essential? Or skip this? Suggestion: start with model in which "P and Q" versions of A are different. What difficulties does that cause us?

4.2 A macro model

One of Stan's models...

5 Outstanding issues

Spanning ....

Errors in observations

6 Conclusions

A Solving expectational difference equations

Scalar moving average shock process. Here's a useful result from Hansen and Sargent (1980, section 2) and Sargent (1987, section XI.19). An expectational difference equation with stationary forcing variable x generates a "geometric distributed lead":

$$\begin{aligned} y_t &= \lambda E_t y_{t+1} + x_t \\ &= \lambda E_t(\lambda E_{t+1} y_{t+2} + x_{t+1}) + x_t \\ &= \sum_{j=0}^{\infty} \lambda^j E_t x_{t+j}. \end{aligned}$$

If $x_t = \sum_{j=0}^{\infty} \alpha_j w_{t-j} = \alpha(L) w_t$, with w white noise, then what is y_t? A unique stationary solution $y_t = \beta(L) w_t$ exists if x is stationary and $|\lambda| < 1$, but what is $\beta(L)$?

Conditional expectations of x have the form

$$E_t x_{t+j} = [\alpha(L)/L^j]_+ w_t = \sum_{i=0}^{\infty} \alpha_{j+i} w_{t-i}$$

(The subscript "+" means ignore negative powers of L.) Therefore the coefficient of w_{t-i} in the distributed lead is

$$\beta_i = \sum_{j=0}^{\infty} \lambda^j \alpha_{i+j}.$$

This tells us, for example, that if x is MA(q), then so is y: if $\alpha_j = 0$ for $j > q$, then $\beta_j = 0$ over the same range.

There's a "lag polynomial" version that expresses the result in compact form. We're looking for a solution $y_t = \beta(L) w_t$ satisfying the expectational difference equation:

$$\beta(L) w_t = [\beta(L)/L]_+ w_t + \alpha(L) w_t.$$

The solution is

$$\beta(L) = \frac{L\alpha(L) - \lambda\alpha(\lambda)}{L - \lambda}$$

See the references mentioned above.

Vector autoregressive shock process. Here's a related result adapted from Ljungqvist and Sargent (2005, section 2.4). [Earlier ref??] It extends the previous result to higher dimensional processes that can be expressed as stationary vector autoregressions. Consider the system

$$\begin{aligned} y_t &= \lambda E_t y_{t+1} + e^\top x_t \\ x_{t+1} &= A x_t + C w_{t+1}, \end{aligned}$$

where A is stable (eigenvalues less than one in absolute value), e is an arbitrary vector, and $w \sim \mathrm{NID}(0, I)$. The solution in this case is

$$y_t \;=\; \sum_{j=0}^{\infty} \lambda^j e^\top E_t x_{t+j} \;=\; e^\top \sum_{j=0}^{\infty} \lambda^j A^j x_t \;=\; e^\top (I - \lambda A)^{-1} x_t.$$

The last step follows from the matrix geometric series.

There's a method of undetermined coefficients version of this. Guess $y_t = \beta^\top x_t$ for some vector β (we know the solution has this form from what we just did). Then the difference equation tells us

$$\beta^\top x_t \;=\; \beta^\top \lambda A x_t + e^\top x_t.$$

Collecting terms in x_t gives us $\beta^\top = e^\top (I - \lambda A)^{-1}$, as stated. What this approach misses is the requirement that A be stable.

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