Identifying Taylor rules in macro-finance models[*]

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Abstract

Identification problems arise naturally in forward-looking models when agents observe more than economists. We illustrate the problem in several macro-finance models with Taylor rules. When the shock to the rule is observed by agents but not economists, identification of Taylor rule parameters requires restrictions on the form of the shock.

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1 Introduction

The macro-finance literature has the potential to give us deeper insights into macroeconomics and macroeconomic policy by using information from both aggregate quantities and asset prices. The link between bond-pricing and monetary policy seems particularly promising if central banks implement monetary policy through short-term interest rates, as they do in models with Taylor rules.

If the combination of macroeconomics and finance holds promise, it also raises challenges. We address one of them here: the challenge of identifying monetary policy parameters in a modern macroeconomic model. Identification problems arise in many economic models, but they play a particularly important role in assessments of monetary policy. If we see that the short-term interest rate rises with inflation, does that reflect the policy of the central bank or something else? How would we know? Since identification is a feature of models, the question is what we need in a model to be able to identify the parameters governing monetary policy.

We illustrate the problem and point to its resolution in a series of examples similar to those in the New Keynesian literature. The source of identification problems here is that we do not observe the shock to monetary policy. As a result, it's difficult, and perhaps impossible, to disentangle the systematic aspects of monetary policy from shocks to it. Without more information about the shock, we may not be able to identify the parameter tying, say, interest rate policy to the inflation rate. This issue about the difference between what agents and economists observe goes back to work by Hansen and Sargent (1980, 1991), who considered differences in the information sets of agents and economists in dynamic rational expectations models. In our examples, the issue is the Taylor rule shock. If agents observe it but economists do not, then we need restrictions on the shock to identify the Taylor rule's parameters. The identification problem in these models is different from the classical Cowles Commission work on simultaneous equation systems. The central issue is observability of the shock rather than simultaneity. If we observe the shock, identification follows immediately.

We show how this works in a series of examples that illustrate how observability interacts with dynamics in forward-looking rational expectations models. The first example is adapted from Cochrane (2011). Later examples introduce exponential-affine pricing kernels and New Keynesian Phillips curves. In all of them, we need restrictions on hidden shocks to identify structural parameters, including those of the Taylor rule. This logic is clearest when the state variables of the model are themselves observable. We consider a number of ways of observing the state, including state-space models and latent factors in bond-pricing models.

2 Two examples

Two examples illustrate the nature of identification problems in macro-finance models with Taylor rules. The first is adapted from Cochrane (2011). The second is an exponential-affine bond-pricing model. The critical ingredient in both cases is what we observe. We assume that economic agents observe everything, but we economists do not. In particular, we do not observe the shock to the Taylor rule. The question is how this affects our ability to infer the Taylor rule's parameters. We provide answers for these two examples and discuss some of the questions they raise about identification in such settings.

2.1 Cochrane's example

Cochrane's example consists of two equations, an asset pricing relation (here the Fisher equation) and a Taylor rule:

$$i_t = E_t \pi_{t+1} \tag{1}$$

$$i_t = \tau \pi_t + s_t. \tag{2}$$

Here i_t is the (one-period) nominal interest rate, π_t is the inflation rate, and s_t is a monetary policy shock. The Taylor rule parameter $\tau > 1$ describes how aggressively the central bank responds to inflation. There's not much economic content here, but it suffices to illustrate the identification problem.

Let us say, to be specific, that the shock is a linear function of a state vector x_t, $s_t = d^\top x_t$, and that x_t is autoregressive,

$$x_{t+1} = Ax_t + Cw_{t+1}, \tag{3}$$

with disturbances $\{w_t\} \sim \text{NID}(0, I)$ and A is stable. Let us denote the covariance matrix of the state by V_x, which solves $V_x = AV_xA^\top + CC^\top$. This structure allows easy comparison to models ranging from exponential-affine to vector autoregressions.

We solve the model by standard methods. We assume agents know the model and observe all of its variables. Equations (**??**) and (**??**) imply the forward-looking difference equation or rational expectations model

$$E_t \pi_{t+1} \quad = \quad \tau \pi_t + s_t.$$

The solution for inflation has the form $\pi_t = b^\top x_t$ for some coefficient vector b to be determined. Then $E_t \pi_{t+1} = b^\top E_t x_{t+1} = b^\top A x_t$. Lining up terms, we see that b satisfies

$$b^\top A \quad = \quad \tau b^\top + d^\top \quad \Rightarrow \quad b^\top \quad = \quad -d^\top (\tau I - A)^{-1}. \tag{4}$$

See Appendix **??**. This is the unique stationary solution if A is stable (eigenvalues less than one in absolute value) and $\tau > 1$ (the so-called Taylor principle).

Now consider estimation. Do we have enough information to estimate the Taylor rule parameter τ? We might try to estimate equation (**??**) by running a regression of i_t on π_t, with s_t as the residual. That won't work because s_t drives both i_t and π_t and we need to distinguish its direct effect on i_t from its indirect effect through π_t. Indeed, OLS would deliver a coefficient equal to $(b^\top V_x b)^{-1} b^\top A V_x b$ instead of τ.

Can we estimate τ even if the OLS coefficient is biased? The critical issue is whether we observe the shock s_t. Let us say that we observe the state x_t, but may or may not observe the shock s_t or the coefficient vector d that connects it to the state. Because we observe x_t, we can estimate A and V_x. We can also estimate the parameter vector b connecting inflation to the state. If we observe the shock s_t, then we can estimate the parameter vector d. We now have all the ingredients of the OLS coefficient but τ, which we can now find. Evidently the Taylor rule parameter is identified.

However, if we don't observe s_t, and therefore do not know d, we're in trouble. In economic terms, we can't distinguish the effects on the interest rate of inflation (the parameter τ) and the shock (d). In this model, there's not much we can do about that. Cochrane (2011, page 606) says simply: "the crucial Taylor rule parameter is not identified." Sims and Zha (2006, page 57) elaborate on the same issue in the context of a vector autoregression: "The ... problem ... is that the Fisher relation is always lurking in the background. The Fisher relation connects current nominal rates to expected future inflation rates and to real interest rates, which are in turn plausibly determined by expected output growth rates. So one might easily find an equation that had the form of the forward-looking Taylor rule, satisfied the identifying restrictions, but was something other than a policy reaction function."

2.2 An exponential-affine example

In finance, it's common to model interest rates with exponential-affine models, in which bond yields are linear functions of the state. In the macro-finance branch of this literature, the state includes macroeconomic variables like inflation and output growth. Examples include Ang and Piazzesi (2003), Chernov and Mueller (2012), Jardet, Monfort, and Pegoraro (2012), Joslin, Le, and Singleton (2013), Moench (2008), Rudebusch and Wu (2008), and Smith and Taylor (2009). In these models the short rate depends on, among other things, inflation.

An illustrative example follows from the log pricing kernel,

$$m_{t+1}^{\$} \quad = \quad -\lambda^\top \lambda/2 - \delta^\top x_t + \lambda^\top w_{t+1}, \tag{5}$$

and the linear law of motion (**??**). Here the nominal (log) pricing kernel $m_t^{\$}$ is connected to the real (log) pricing kernel m_t by $m_t^{\$} = m_t - \pi_t$. The one-period nominal interest rate is then

$$i_t \quad = \quad -\log E_t \exp(m_{t+1} - \pi_{t+1}) \quad = \quad -\log E_t \exp(m_{t+1}^{\$}) \quad = \quad \delta^\top x_t, \tag{6}$$

a linear function of the state. Since we observe the state x_t, we can estimate δ by projecting the interest rate on it.

If the first element of x_t is the inflation rate, then it's tempting to interpret equation (**??**) as a Taylor rule, with the first element of δ the inflation coefficient τ. But is it? The logic of equation (**??**) is closer to the asset-pricing relation, equation (**??**), than to the Taylor rule, equation (**??**). Without more structure, we can't say whether it's one, the other, or something else altogether.

More formally, consider an interpretation of (**??**) as a Taylor rule (**??**). Since we observe inflation π_t and the state x_t, we can estimate the parameter vector b connecting the two: $\pi_t = b^\top x_t$. Then the Taylor rule implies

$$i_t \quad = \quad \tau \pi_t + s_t \quad = \quad \tau b^\top x_t + d^\top x_t$$

Equating our two interest rate relations gives us $\delta^\top = \tau b^\top + d^\top$. It's clear, now, that we have the same difficulty we had in the previous example: If we do not know the shock parameter d, we cannot infer τ from estimates of δ. If x_t has dimension n, we have n equations to solve for $n+1$ unknowns (d and τ).

2.3 Discussion

These examples illustrate the challenge we face in identifying the parameters of the Taylor rule, but they also suggest follow-up questions that could lead to a solution.

One such question is whether we can put shocks in other places and use them for identification. Gertler (private communication) suggests putting a shock in Cochrane's first equation, so that the example becomes

$$\begin{aligned} i_t &= E_t \pi_{t+1} + s_{1t} \\ i_t &= \tau \pi_t + s_{2t}. \end{aligned}$$

Can the additional shock identify the Taylor rule?

Suppose, as Gertler suggests, that s_{1t} and s_{2t} are independent. Then, if it is observed, we can use s_{1t} as an instrument for π_t to estimate the Taylor rule equation, which gives us an estimate of τ. Given τ, we can then back out the shock s_{2t}. We give a different rationale for this result in the next section, but there are two conclusions here of more general interest. One is that identification requires some restriction on the shocks. Here the restriction is independence, but in later examples other restrictions serve the same purpose. The other is that identifying τ and backing out the unobserved shock are complementary activities. Generally if we can do one, we can do the other as well.

Another question is whether we can use long-term interest rates to help with identification. The answer is no if the idea is to use long rates to observe the state. In exponential-affine models, the state spans bond yields of all maturities. In many cases of interest, we can invert the mapping and express the state as a linear function of a subset of yields. In this sense, we can imagine using a vector of bond yields to observe the state. We have seen, though, that observing the state is not enough. We observe the state in both examples, yet cannot identify the Taylor rule. We explore the issue of state observability further in Section **??**.

3 Macro-finance models with Taylor rules

Macro-finance models, which combine elements of macroeconomic and asset-pricing models, bring evidence from both macroeconomic and financial variables to bear on our understanding of monetary policy. It's not easy to reconcile the two, but if we do, we gain perspective that's missing from either approach on its own.

We show how Gertler's insight can be developed to identify the Taylor rule in such models. We use two examples, one based on a representative agent, the other on an exponential-affine model. We explore identification in these models when we observe the state, the short rate, and inflation, but not the shock to the Taylor rule. In both cases, we need one restriction on the shock to identify the (one) policy parameter.

3.1 A representative-agent model

One line of macro-finance research combines representative-agent asset pricing with a rule governing monetary policy. Gallmeyer, Hollifield, and Zin (2005) is a prominent example. We simplify their model, using power utility instead of recursive preferences and a simpler law of motion for the state.

The model consists of

$$\begin{aligned} i_t &= -\log E_t \exp(m_{t+1} - \pi_{t+1}) && (7) \\ m_t &= -\rho - \alpha g_t \\ g_t &= g + s_{1t} \\ i_t &= r + \tau\pi_t + s_{2t}. && (8) \end{aligned}$$

Equations (**??**) and (**??**) mirror the two equations of Cochrane's example. The former is a variant of equation (**??**), a more complex version of equation (**??**) that represents the finance component of the model. The latter is a Taylor rule, representing monetary policy, the sole macroeconomic component. The second and third equations characterize the real pricing kernel. The second is the logarithm of the marginal rate of substitution of a power utility agent with discount rate ρ, curvature parameter α, and log consumption growth g_t. The third connects fluctuations in log consumption growth to a shock s_{1t}. As before, the state x_t follows the law of motion (**??**) and shocks are linear functions of it, $s_{it} = d_i^\top x_t$ for $i = 1, 2$.

The solution now combines asset pricing with a forward-looking difference equation. We posit a solution of the form $\pi_t = b^\top x_t$. Solving (**??**) then gives us

$$i_t = \rho + \alpha g - V_m/2 + a^\top x_t, \quad (9)$$

with

$$\begin{aligned} a^\top &= (\alpha d_1^\top + b^\top)A \\ V_m &= a^\top C C^\top a. \end{aligned}$$

Note that the short rate equation (**??**) now has a shock, as Gertler suggests. Equating (**??**) and (**??**) gives us

$$\rho + \alpha g - V_m/2 + (\alpha d_1^\top + b^\top)Ax_t \quad = \quad r + (\tau b^\top + d_2^\top)x_t.$$

Lining up similar terms, we have $r = \rho + \alpha g - V_m/2$ and

$$(\alpha d_1^\top + b^\top)A \quad = \quad \tau b^\top + d_2^\top \quad \Rightarrow \quad b^\top \quad = \quad (\alpha d_1^\top - d_2^\top)(\tau I - A)^{-1}.$$

As before, this gives us a unique stationary solution under the stated conditions: A stable and $\tau > 1$.

Now consider identification. Suppose we observe the state x_t, the interest rate i_t, the inflation rate π_t, and log consumption growth g_t, but not the shock s_{2t} to the Taylor rule. From observations of the state, we can estimate the autoregressive matrix A, and from observations of consumption growth we can estimate the shock coefficients d_1. We can also estimate a and b by projecting i_t and π_t on the state. With $a^\top = (\alpha d_1^\top + b^\top)A$ known, that leaves us to solve

$$a^\top \quad = \quad \tau b^\top + d_2^\top \tag{10}$$

for the Taylor rule's inflation parameter τ and shock coefficients d_2: n equations in the $n+1$ unknowns (τ, d_2). The problem is the same as before: Without further restrictions, the Taylor rule is not identified. This is Cochrane's conclusion in somewhat more general form.

We can, however, identify the monetary policy rule if we place one or more restrictions on the shock coefficients d_2. One example was mentioned earlier: choose d_1 and d_2 so that the shocks s_{1t} and s_{2t} are independent. We'll return to this shortly. Another example is a zero in the vector d_2 — what is traditionally termed an exclusion restriction. Suppose the ith element of d_2 is zero. Then the ith element of (**??**) is

$$a_i \quad = \quad \tau b_i,$$

which we can solve for τ. Given τ, and our estimates of a and b, we can now solve (**??**) for the remaining components of d_2. We can do the same thing with restrictions based on linear combinations. Suppose $d^\top f = 0$ for some known vector f. Then we find τ from $a^\top f = \tau b^\top f$. Any such linear restriction on the shock coefficient d_2 allows us to identify the Taylor rule.

Cochrane's example is a special case with shocks to consumption growth turned off: $d_1 = 0$. As a result, all the variation in inflation and the short rate comes from monetary policy shocks s_{2t}. Special case or not, the conclusion is the same: we need one restriction on d_2 to identify the (one) Taylor rule parameter τ.

3.2 An exponential-affine model

We take a similar approach to an exponential-affine model, adding a Taylor rule to an otherwise standard bond-pricing model. The model consists of a real pricing kernel, a Taylor rule, and the linear law of motion (**??**). The first two are

$$\begin{aligned} m_{t+1} &= -\rho - \delta^\top x_t + \lambda^\top w_{t+1} \\ i_t &= r + \tau\pi_t + s_t. \end{aligned}$$

We complete the model by specifying the Taylor rule shock as a linear function of the state: $s_t = d^\top x_t$.

We solve the model by the usual method. Given a guess $\pi_t = b^\top x_t$ for inflation, the nominal pricing kernel is

$$m^{\$}_{t+1} \;=\; m_{t+1} - \pi_{t+1} \;=\; -\rho - (\delta^\top + b^\top A)x_t + (\lambda^\top - b^\top C)w_{t+1}.$$

The short rate follows from (**??**):

$$i_t \;=\; \rho - V_m/2 + a^\top x_t,$$

where $V_m = (\lambda^\top - b^\top C)(\lambda - C^\top b)$ and $a^\top = \delta^\top A + b^\top$. Equating this to the Taylor rule gives us $r = \rho - V_m/2$ and

$$\delta^\top + b^\top A \;=\; \tau b^\top + d^\top \quad\Rightarrow\quad b^\top \;=\; (\delta^\top - d^\top)(\tau I - A)^{-1}.$$

This is the unique stationary solution for b under the usual conditions.

Identification follows the same logic as the preceding example. Let us say, again, that we observe the state x_t, the short rate i_t, and inflation π_t. From the latter we estimate a and b. That gives us n equations in the $n+1$ unknowns (τ, d). The model is identified only when we impose one or more restrictions on the monetary policy shock. If, for example, the ith element of d is zero, then τ follows from $a_i = \tau b_i$.

This model is a generalization of the previous one in which we've given the real pricing kernel a more flexible structure. Evidently the structure of the pricing kernel has little bearing on identification. We need instead more structure on the shock to the Taylor rule to compensate for not observing it.

3.3 Discussion

We have seen that one restriction suffices to identify the Taylor rule in these examples. One obvious question why this works. Is it similar to the use of exclusion restrictions in simultaneous equations models? Most econometrics textbooks illustrate exclusions of this kind with supply and demand. There we need a variable in one equation that's missing — excluded — from the other. To identify the demand equation, we need a variable in the supply equation that's excluded from demand. That's not the case here. We can identify the Taylor rule even when there are no shocks in the other equation if we have a restriction on the Taylor rule shock. In this respect, our identification issues are different. In our example, the central issue is not whether we have the right configuration of shocks across equations, but whether we observe them. When we don't observe the shock to the Taylor rule, we need additional structure in the same equation to deduce its parameters.

The same logic applies to Gertler's example in Section **??**, where we used independence of the two shocks to identify the Taylor rule. Doesn't that involve shocks in the second equation? Well, yes, but the critical feature of independence here is the restriction it places on the Taylor rule shock. The shocks are uncorrelated, hence independent, if $d_1^\top V_x d_2 = 0$. But that's a linear restriction on the coefficient vector d_2 of the Taylor rule shock.

A similar issue arises with restrictions on interest rate coefficients. Suppose we know that a linear combination of interest rate coefficients is zero: $a^\top f = 0$ for some known f. Then (**??**) gives us a restriction connecting the Taylor rule shock and coefficient: $\tau b^\top f + d^\top f = 0$. One interpretation is that we've used a restriction from another part of the model to identify the model. We would say instead that any such restriction on interest rate behavior implies a restriction on the Taylor rule, which identifies the policy rule for the usual reasons.

Another difference from traditional simultaneous equation methods is that single-equation estimation methods generally won't work. We need information about the whole model to deduce the Taylor rule. This reflects, in part, what Hansen and Sargent (1980, p 37) call the hallmark of rational expectations models: that cross-equation restrictions connect the parameters in one equation to those in the others.

4 A model with a Phillips curve

We apply the same logic to an example closer to a New Keynesian model. We add a Phillips curve to the model and an output gap to the Taylor rule. Similar models are described by Canova and Sala (2009), Clarida, Gali, and Gertler (1999), Cochrane (2001), Gali (2008, ch 3), King (2000), Shapiro (2008), and Woodford (2003), among many others. Our contribution is a modest one: to add an asset pricing relation, which we think of as an illustration of how asset prices might be introduced into the analysis.

Despite the added economic structure, the logic for identification is the same: we need restrictions on the shock coefficients to identify the Taylor rule. What changes is that we need two restrictions, one for each of the two parameters of the rule. We face similar issues in identifying the Phillips curve. If its shock isn't observed, we need restrictions to identify its parameters.

Our model consists of

$$\begin{aligned} i_t &= -\log E_t \exp(m_{t+1} - \pi_{t+1}) \\ m_t &= -\rho - \alpha g_t \\ \pi_t &= \beta E_t \pi_{t+1} + \kappa g_t + s_{1t} \\ i_t &= r + \tau_1 \pi_t + \tau_2 g_t + s_{2t}. \end{aligned}$$

The first equation is the usual pricing relation, the second is the real pricing kernel, the third is a New Keynesian Phillips curve, and the fourth is a Taylor rule (now with an extra term). In addition, we have the law of motion (**??**) for the state, and the shocks $s_{it} = d_i^\top x_t$ for $i = 1, 2$.

We now have a two-dimensional rational expectations model in the forward-looking variables π_t and g_t. We guess a solution that includes $\pi_t = b^\top x_t$ and $g_t = c^\top x_t$. Then the pricing relation gives us

$$i_t = \rho - V_m/2 + a^\top x_t$$

with $a^\top = (\alpha c^\top + b^\top)A$ and $V_m = a^\top CC^\top a$. If we equate this to the Taylor rule and collect terms, we have $r = \rho - V_m/2$ and

$$a^\top = \tau_1 b^\top + \tau_2 c^\top + d_2^\top. \tag{11}$$

The Phillips curve implies

$$b^\top = \beta A b^\top + \kappa c^\top + d_1^\top. \tag{12}$$

As many others have noted, the conditions for this to be a unique stationary solution are more stringent than before. We'll assume that they're satisfied.

Let's turn to identification. Suppose we observe the state x_t, the interest rate i_t, the inflation rate π_t, and log consumption growth g_t. From them, we can estimate the autoregressive matrix A and the coefficient vectors (a, b, c). In the Taylor rule, the unknowns are the policy parameters (τ_1, τ_2) and the coefficients d_1 of the shock. If we observe the shock s_{2t}, equation (**??**) gives us n equations to solve for τ_1 and τ_2. As long as the dimension n of the state is at least two, the Taylor rule parameters are identified. If we do not observe the shock, then we need two restrictions on its coefficient vector d_2. The conclusion is the same as before, but with two parameters to identify we need two restrictions on the vector of shock coefficients.

The same logic applies to identifying the parameters of the Phillips curve. If we observe the shock s_{1t}, equation (**??**) gives us n equations to solve for the parameters β and κ. If we do not observe the shock, then two restrictions are needed to identify the two parameters. The identification problem for the Phillips curve has the same structure as that for the Taylor rule, although in practice they've been treated separately. For more on the subject, see the extensive discussions in Canova and Sala (2009), Gali and Gertler (1999), Nason and Smith (2008), and Shapiro (2008).

In fact, standard theoretical implementations of the New Keynesian models assume, in addition to independence between shocks, that s_{it} is affected by one disturbance (see, e.g., Gali, 2008 for many examples). Such assumption amounts to $n-1$ zero restrictions on d_i. As we have seen, these theoretical restrictions are much more generous than what's needed to identify such parameters of interest as τ_1 or κ. For all k such that $d_{2k} = 0$, equation (**??**) simplifies to $a_k = \tau_1 b_k + \tau_2 c_k$. So, our only worry about identification of τ_1 should be to check that there are at least two k's such that $(a_k, b_k, c_k) \neq (0, 0, 0)$. A similar logic applies to equation (**??**) while identifying κ.

5 Observing the state

Our logic so far is predicated on observing the state. What happens if we observe the state indirectly? Or a noisy signal of the state? Our current claim, subject to change, is that neither changes our understanding of the identification problem.

5.1 Identifying the state

In many applications, the state variable is latent: we don't observe it directly, but we may be able to infer something about it. Examples include a variety of dynamic factor models and exponential-affine bond-pricing models.

In some of these cases, the state is not completely determined: a transformation of the state is observationally equivalent. Consider a model based on the linear law of motion (**??**). If the state x_t is not observed directly, then it is indistinguishable econometrically from a model with state $x'_t = Qx_t$, where Q is a square matrix of full rank. In terms of this new state variable, the law of motion becomes

$$x'_{t+1} \quad = \quad QAQ^{-1}x'_t + QCw_{t+1} \quad = \quad A'x'_t + C'w_{t+1}.$$

where $A' = QAQ^{-1}$ and $C' = QC$.

We can transform the rest of the model the same way. Consider the representative agent model described in Section **??**. For any choice of state x'_t, we can estimate the associated parameter vectors for the observables. The short rate is then related to the state by $i_t = r + a'^{\top}x'_t$ where $a'^{\top} = a^{\top}Q^{-1}$. Similarly, inflation is related by $\pi_t = b'^{\top}x'_t$, where $b'^{\top} = b^{\top}Q^{-1}$. The Taylor rule then implies

$$a'^{\top} \quad = \quad \tau b'^{\top} + d'^{\top},$$

the analog of equation (**??**) for the transformed state. Here $d'^{\top} = d^{\top}Q^{-1}$.

Identification is the same as before: we need one restriction on $d'^{\top}$ to identify the single Taylor rule parameter τ. The only question is whether the restrictions we placed on d translate to d'. Consider a general linear restriction $d^{\top}r = 0$, where at least one element of r is non-zero. This restriction can be re-written in terms of d' as $d'^{\top}Qr = 0$. Thus, the restricting vector is $r' = Qr$.

So, we have to pick a Q to identify the state. There are many choices, but $Q = C^{-1}$ is used often. It leads to unit variance of disturbances, variances of all observable variables are controlled by loadings on states (a', b', and so on), and all the interdependencies of the variables are controlled by A'. Variants of this approach are used in the VAR literature (Leeper, Sims, and Zha, 1996), factor analysis (Bernanke, Boivin, and Eliasz, 2005), and term structure modelling (Joslin, Singleton, and Zhu, 2011).

5.2 Filtering the state

The next step is to infer the state from observed variables using the Kalman filter. The canonical state-space model starts with the usual linear law of motion (**??**). If we denote the endogenous variables of the model by z_t, the solution has the form $z_t = Bx_t$, where each row of B represents one of the model's endogenous variables. In our examples, these variables include the one-period interest rate i_t, the inflation rate π_t, and so on.

A state-space model adds a measurement equation connecting observed variables y_t to the state:

$$y_t \quad = \quad Gx_t + Hv_t. \tag{13}$$

The observables are likely to include at least some of the endogenous variables, including those mentioned above, and possibly some of the state variables, too. In modern "data-rich" applications, the dimension of y_t is large relative to that of x_t. We posit measurement errors $v_t \sim \text{NID}(0, I)$ that are independent of w_t, but could easily incorporate arbitrary correlation between them.

Our question then becomes how well observations of y_t substitute for observations of the state x_t. The question is a classical one and has a standard answer based on the Kalman filter: use y_t to estimate the state x_t and proceed as before using the estimate in place of the state. The Kalman filter is widely used in macroeconomics, so our treatment will be brief. See Appendix **??** for more.

Two conditions guarantee that the state is, for our purposes, observed. First, we need the law of motion (**??**) to generate a state of dimension n, not a lower dimensional subspace. It's sufficient to assume that C has rank n, but when (**??**) is in companion form that's not realistic. In formal terms, we need (A, C) to be controllable, so that we can attain any x_t in $\mathbb{R}^n$ with some realization of disturbances w_t. Second, and more critically, we need (A, G) to be observable; for the matrix

$$\mathcal{O} \quad = \quad \begin{bmatrix} G \\ GA \\ \vdots \\ GA^{n-1} \end{bmatrix}$$

to have rank n. Roughly speaking: that the state generates enough variation in the observed variables that we can (approximately) reverse engineer the state. With these two conditions, we can estimate the state from observations of the infinite history $y^t = (y_t, y_{t-1}, \ldots)$.

Here the state x_t isn't delivered exactly, but we generate a recursive estimate of the state based on the history to date of the observables:

$$\hat{x}_t \quad = \quad E\left(x_t | y^t\right).$$

The state is then $x_t = \hat{x}_t + e_t$ and the error e_t is orthogonal to the history y^t. The short rate is therefore

$$i_t \;=\; a^\top x_t \;=\; a^\top \hat{x}_t + a^\top e_t.$$

The last term is noise, orthogonal to the history y^t and error e_t, but the first one gives us a. This is just a projection on a smaller information set than x_t. We can nevertheless line up coefficients as before, using $\hat{x}_t$ in place of x_t.

If the observability condition does not hold, then we can estimate the full state, that is, we can obtain $\hat{x}_{1:m,t}$ with $m < n$. The key is that by construction of the Kalman filter the unspanned part of the state $x_{m+1:n,t}$ is orthogonal to the filtered component of the state. Therefore, we can still estimate a subset of coefficients $a_{1:m}$ without a bias. As a result, we will obtain m equations with $m+1$ unknowns (in the examples of Section **??**). The number of needed exclusion restrictions have not changed. The same logic can be extended to the Phillips curve example.

5.3 Examples

How would we estimate the states in practice? The idea is to use observations of variables that are sensitive to the state under the null of the model. Intuitively, the more data one uses the more likely is (A, G) to be observable (although size of the observation vector is not sufficient) and the more precise the estimate of x_t is going to be. We briefly mention two examples of such data sources: prices of financial assets and survey forecasts.

A financial application offers one straightforward way to estimate the state. Exponential-affine models have a structure that is similar to what we've discussed and whose form is most evident with forward rates. If q_t^h is the price at date t if a claim to one dollar at $t+h$, then continuously-compounded forward rates are defined by $f_t^h = \log(q_t^h / q_t^{h+1})$. The short rate is $i_t = f_t^0$. For the model of Section **??**, the short rate is (ignoring the intercept) $i_t = a^\top x_t$ and the jth forward rate has the form $f_t^j = a^\top A^j x_t$. The vector f_t of the first h forward rates has the form

$$f_t \;=\; \begin{bmatrix} f_t^0 \\ f_t^1 \\ \vdots \\ f_t^{h-1} \end{bmatrix} \;=\; \begin{bmatrix} a^\top \\ a^\top A \\ \vdots \\ a^\top A^{h-1} \end{bmatrix} [x_t] \;=\; B x_t.$$

If B is invertible, we can use the vector of forward rates as the state. The same holds for yields, defined by $i_t^h = h^{-1} \sum_{j=1}^{h} f_t^{j-1}$.

If the dimension of state n is the same as size the vector of forward rates, we can literally solve for x_t by inverting the equation above: $x_t = B^{-1} f_t$. In typical applications, x_t is low-dimensional and the vector of observables is large. In these cases, researchers attach measurement errors to all forward rates as in (**??**) and use the Kalman filter to estimate the state.

Another source of observations are survey forecasts of macro variables. These forecasts fit perfectly into our framework as they deliver (noisy) expectations of inflation, output, nominal interest rate, and so on at horizon h. For example,

$$\begin{aligned} f_t(i_{t+h}) &= E_t(i_{t+h}) + v_t = a^\top A^h x_t + v_t \\ f_t(\pi_{t+h}) &= E_t(\pi_{t+h}) + v_t = b^\top A^h x_t + v_t \\ f_t(g_{t+h}) &= E_t(g_{t+h}) + v_t = c^\top A^h x_t + v_t. \end{aligned}$$

Here, for simplicity, we assumed identical forecasting noise v_t, but its structure can be as elaborate as one wishes. Surveys provide forecasts for a range of h, so again one can use the Kalman filter to estimate the state. Chernov and Mueller (2012), Chun (2011), and Kim and Orphanides (2012) are examples of using survey forecasts in a state-space framework.

6 Conclusions

Identification is always an issue in applied economic work, perhaps nowhere more so than in the study of monetary policy. That's still true. We have shown, however, that (i) the problem of identifying the systematic component of monetary policy (the Taylor rule parameters) in macro-finance models stems from our inability to observe the nonsystematic component (the shock to the rule) and (ii) the solution is to impose restrictions on the form of the shock. We are left where we often are in matters of identification: trying to decide which restrictions are plausible, and which are not.

A Solution of rational expectations models

We show how to solve linear rational expectations models. We consider models with a single forward-looking variable first, then move to multi-dimensional systems.

Consider the class of forward-looking linear rational expectations models,

$$\begin{aligned} z_t &= \Lambda E_t z_{t+1} + D x_t \\ x_{t+1} &= A x_t + C w_{t+1}. \end{aligned}$$

Here x_t is the state, Λ is stable (eigenvalues less than one in absolute value), A is also stable, and $w_t \sim \mathrm{NID}(0, I)$. The goal is to solve the model and (thereby) link z_t to the state x_t.

One-dimensional case. If z_t is a scalar and the shock is $s_t = d^\top x_t$, we have

$$z_t = \lambda E_t z_{t+1} + d^\top x_t. \tag{14}$$

Substituting repeatedly gives us

$$z_t = \sum_{j=0}^{\infty} \lambda^j d^\top E_t x_{t+j} = d^\top \sum_{j=0}^{\infty} \lambda^j A^j x_t = d^\top (I - \lambda A)^{-1} x_t.$$

The last step follows from the matrix geometric series if A is stable and $|\lambda| < 1$. Under these conditions, this is the unique stationary solution.

There's also a method of undetermined coefficients version. We guess $z_t = b^\top x_t$ for some vector b (we've just seen the solution has this form). Then the difference equation tells us

$$b^\top x_t = b^\top \lambda A x_t + d^\top x_t.$$

Collecting terms in x_t gives us $b^\top = d^\top (I - \lambda A)^{-1}$.

This model is close enough to the examples of Sections **??** and **??** that we can illustrate their identification issues in a more abstract setting. Suppose we observe the state x_t and the endogenous variable z_t, but not the shock s_t. Then we can estimate A and b. Equation (**??**) then gives us

$$b^\top = \lambda b^\top A + d^\top.$$

If x has dimension n, we have n equations in the $n+1$ unknowns (λ, d); we need one restriction on d to identify the parameter λ.

Multi-dimensional case. If z_t is a vector, as in Section **??**, we have

$$z_t = \Lambda E_t z_{t+1} + D^\top x_t.$$

Repeated substitution gives us

$$z_t = \sum_{j=0}^{\infty} \Lambda^j D A^j x_t.$$

That gives us the solution $z_t = Bx_t$ where

$$B = D + \Lambda B A = \sum_{j=0}^{\infty} \Lambda^j D A^j.$$

The solution is

$$\text{vec}(B) = (I - A^\top \otimes \Lambda)^{-1} \text{vec}(D).$$

See, for example, Anderson, Hansen, McGrattan, and Sargent (1996, Section 6) or Klein (2000, Appendix B). The same sources also explain how to solve rational expectations models with endogenous state variables.

B Elements of Kalman filtering

We outline the essential elements of Kalman filtering. Hansen and Sargent (2013) is a standard reference for economists. Anderson and Moore (1979) and Boyd (2009) are good technical references.

The starting point is the state-space system

$$\begin{aligned} x_{t+1} &= Ax_t + Cw_{t+1} \\ y_t &= Gx_t + Hv_t. \end{aligned}$$

Here $\{w_t\}$ and $\{v_t\}$ are vectors of independent standard normals — independent element by element, with each other, and across time. We refer to x_t as the state and y_t as the measurement. The state has dimension n, the measurement dimension p. In the technical literature, it's common to use w_t in place of w_{t+1} in the first equation, but since neither x_{t+1} nor w_{t+1} is ever observed, it's a convention without content.

Controllability and observability. We say (A, C) is controllable if

$$\mathcal{C} = \begin{bmatrix} C & AC & \cdots & A^{n-1}C \end{bmatrix}$$

has rank n. The word controllable is misleading in this context; the idea is simply that w_t generates variation across all n dimensions of x_t. We say (A, G) is observable if

$$\mathcal{O} = \begin{bmatrix} G \\ GA \\ \vdots \\ GA^{n-1} \end{bmatrix}$$

has rank n. The issue is similar to the previous one. The idea is that the condition guarantees us that observing the history of y_t is enough to come up with a full-rank estimate of x_t.

Controllability example. Here's one that fails:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \; C = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \;\Rightarrow\; \mathcal{C} = \begin{bmatrix} c_1 & a_{11}c_1 \\ 0 & 0 \end{bmatrix},$$

which has rank $1 < n = 2$. Here the innovation w_t never generates variation in x_{2t}, so we don't span the whole two-dimensional state. However, if a_{21} is nonzero we get controllability, because w_t affects x_{2t} with a one-period lag through its impact on x_{1t}. An example is an AR(2) in companion form.

Observability example. The logic is similar. Suppose x_t is n-dimensional and the nth column of G consists of zeros. There's no direct impact of the nth state variable on the observations y_t. In the bond-pricing literature, this might be a case in which one of the state variables doesn't appear in bond yields of any maturity. Nevertheless, the nth state variable might be (indirectly) observable if it feeds into other state variables: if a_{jn} is nonzero for some $j \neq n$. Here's an example:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \; G = \begin{bmatrix} g_1 & 0 \end{bmatrix} \;\Rightarrow\; \mathcal{O} = \begin{bmatrix} g_1 & 0 \\ a_{11}g_1 & a_{12}g_1 \end{bmatrix},$$

which has rank two. If $a_{12} = 0$, the condition fails.

Kalman filter. The idea is to estimate the state x_t given (infinite) measurement histories $y^s = (y_s, y_{s-1}, \ldots)$. One popular notation is to express conditional expectations of x_t by

$$\hat{x}_{t|s} = E(x_t | y^s).$$

We use a more compact notation that's also common,

$$\begin{aligned} \hat{x}_t &= E(x_t | y^t) \\ \hat{x}_t^- &= E(x_t | y^{t-1}). \end{aligned}$$

Etc.

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