Econometrica, Vol. 49, No. 1 (January, 1981)

RATIONAL EXPECTATIONS IN STATIONARY LINEAR MODELS

BY CARL A. FUTIA[1]

Linear time series models have come to dominate the macroeconomic literature on rational expectations and equilibrium business cycle theory. But the explicit solution of such models has generally required strong restrictions upon the exogenous process of stochastic shocks (e.g., temporal independence) as well as upon the values of various demand and supply elasticities. This paper exhibits a solution technique, the method of z-transforms, which does not require one to impose such restrictions. The value of this method is illustrated by applying it to completely characterize the symmetric, stationary, rational expectations equilibria of a naive linear model of land speculation. This approach also permits systematic study of the informationally asymmetric equilibria of the model.

THIS PAPER develops a method for analyzing rational expectations (RE) equilibria in linear economic models. The methods I shall discuss usually enable one to determine whether or not a given model has a RE equilibrium and, if one exists, to exhibit an explicit expression for the stochastic process of equilibrium prices. The techniques apply to linear models driven by stationary processes of exogenous random shocks.

In a linear model the concept of a "rational" expectation is typically identified with that of a conditionally unbiased point forecast. This is a restrictive assumption; it forecloses the possibility of modeling the responses optimizing risk-averse agents will make to changes in the riskiness of their economic environment.

Nonetheless linear models have dominated the macroeconomic literature on rational expectations (see, for example, [**11**, **15**, and **17**]). The linearity restriction (plus some additional assumptions) permits the derivation of explicit expressions for the stochastic process of equilibrium prices. Such formulae are valuable for two reasons.

First, they permit comparisons of the economic properties of alternative equilibria (which may, for example, arise from alternative stabilization policies). These comparisons are very difficult to make in a stochastic economic model unless one has an explicit formula for its equilibrium; the calculus is not helpful as an analytical tool in these contexts.

Secondly, explicit equilibrium formulae facilitate the econometric implementation of RE models. For they enable one to express the cross equation restrictions that the hypothesis of rational expectations imposes [**5**].

But to date this macroeconomic literature has dealt almost exclusively with linear models disturbed by random shocks assumed to be independent over time. This independence assumption typically plays a key role in the solution of the model. Unfortunately it can also severely limit one's ability to analyze the effects of alternative stabilization policies. For such a policy can often be represented as a

[1] I would like to thank B. C. N. Greenwald, R. Radner, R. Rosenthal, A. Postlewaite, J. Stiglitz, S. Shavell, and an anonymous referee for many helpful comments on an earlier version of this paper. My special thanks go to H. J. Landau who guided me through the mathematical thickets I encountered in this work. They are all of course entitled to the usual exemption from responsibility.

stationary process of (non-independent) random shocks which has been superimposed upon the original shock process. Even if the original shocks were temporally independent, the policy modified process need not have this property. In order to evaluate the effects of such policies one must therefore be able to solve these linear models for a fairly general stationary process of exogenous random shocks. The techniques developed in this paper permit one to achieve this goal in a wide variety of circumstances.

This paper's key technical idea is to reduce the problem of finding a RE equilibrium to that of solving a system of linear equations in an appropriate Hilbert space. This reduction depends upon the fact that in Gaussian probability models unbiased predictors can be obtained via the linear operation of orthogonal projection. The relevant linear equations can usually be solved quite explicitly through the use of the Z-transform (see Section 4 for definitions).

The mathematics employed in this paper are well known [**7, 8, 12**]. What is new is the application of these mathematical techniques to the analysis of rational expectations equilibria in linear models. Similar techniques have been employed in a macroeconomic context by Saracoglu and Sargent [**14**]. Their results can be sharpened, strengthened, and, I think, simplified using the methods found in the following pages. Hansen and Sargent [**5**] have recently applied the same circle of mathematical ideas to the econometric problems which arise in linear RE models.

Although the principal motivation for developing these techniques lies in their macroeconomic applications, I shall not discuss any macroeconomic models in this paper. (But see [**3**] for an analysis of Lucas' business cycle model [**11**] which uses these techniques.) Instead I shall exposit this paper's principal ideas in the context of a naive and extremely simplified linear model of a speculative market for land (Section 1). The results which one can derive for this microeconomic model are of some independent interest. They enable one to exhibit robust specifications of the model which have no RE equilibria (Section 5). They also lead to robust examples which have only informationally asymmetric equilibria (Section 6).

Furthermore, they enable one to construct examples of markets which have equilibria with the property that each trader is fully informed after observing equilibrium prices but in which prices alone reveal only part of this information (Section 5). Thus each trader continues to have incentive to collect his private information in equilibrium for it enhances the value of the information which can be extracted from prices. Such equilibria are likely to be economically more viable than those in which prices alone reveal all the relevant information known to traders in the aggregate [**1, 4, 13**]. Equilibria in which prices alone convey all relevant information eliminate the incentive for individual traders to collect private information; how then could this information be incorporated into equilibrium prices?

We shall find therefore that linear RE models exhibit the same nonexistence pathologies noticed in the microeconomic literature on rational expectations [**6, 9, 10**]. They also make it easy to construct examples of new phenomena (markets having only asymmetric equilibria and markets having symmetric equilibria where prices alone are not fully revealing). Finally, linear RE models can be studied using techniques which permit explicit solutions in very general circumstances. This should open the door to a deeper analysis of linear, macroeconomic models than has been possible to date.

1. LAND SPECULATION IN HILBERT SPACE

In this section I shall outline the basic structure of the model of land speculation studied in this paper.

To begin let us first consider the supply side of the market. This is easy enough. I shall assume there is a fixed quantity of land, each parcel of which must be assigned some owner at each date t. (I use the symbol t to index time periods and allow t to assume all positive and negative integer values. The negative direction along the time axis points towards the past and the positive direction towards the future.)

The demand for land at each date is assumed to arise from two sources. The first source I shall term nonspeculative demand (e.g., demand for commercial or residential use). The nonspeculative demand for land at date t is assumed to be a random variable which is completely independent of all past, current, and future values of the price of land. (This extreme assumption of a zero price elasticity is quite inessential but greatly simplifies the exposition.) I shall assume that the nonspeculative demand for land never exceeds the total supply of land. The difference between total supply and nonspeculative demand at date t shall be denoted by s_t. I shall term the random variable s_t the supply of land available to speculators (or the *speculative supply*) at date t.

The second source of demand in this market comes from land speculators. Let p_t denote the logarithm of the price of land at date t. Let $\hat{p}_t^i(i = 1, \ldots n)$ denote the point forecast of p_{t+1} made by speculator i at date t. The way in which $\hat{p}_t^i$ is determined by each speculator will be carefully discussed in the following section.

I now assume that the demand for land at date t by speculator i is given by the equation $q_t^i = \hat{p}_t^i - \alpha p_t$ for some constant $\alpha > 1$ which is assumed to be the same for all n speculators. Speculators may well make different forecasts of p_{t+1} at date t, but two speculators who make the same forecast express the same demand for land at any current price.

At this point, I would like to assure the reader that the arbitrary distinction I have drawn between speculative and nonspeculative demand is an unnecessary one. It is adopted for expository convenience only.

The price of land at data t is assumed to be completely determined by the forecasts $\hat{p}_t^i$ and by the equality of speculative supply with speculative demand at date t, i.e., by the equation $s_t = \Sigma_{i=1}^n \hat{p}_t^i - n\alpha p_t$.

Since s_t is a random variable p_t and the forecasts $\hat{p}_t^i$ will in general be random variables, I shall be interested only in "stochastically stable" equilibria for this market. My attention shall therefore be restricted to the case where the sequences of random variables (s_t), $(\hat{p}_t^i)$, and (p_t), $t = 0, \pm 1, \pm 2, \ldots,$ are weakly stationary.

To make this idea precise and to aid the formal analysis of this market, I shall now explain the underlying probability structure of the model.

I shall assume that we are given an infinite sequence of m-dimensional random vectors (ε_t), $t = 0, \pm 1, \pm 2, \ldots$. Each $\varepsilon_t = (\varepsilon_t^1, \ldots, \varepsilon_t^m)$ is an m-tuple of real valued random variables. I assume that the following condition holds:

CONDITION P: The random variables (ε_t^i) are independent, normal variables with mean zero and unit variance.

The mathematical techniques used in this paper do not depend upon the normality assumption. Normality simply assures that point forecasts which minimize the expected value of the square forecast error will also be conditionally unbiased ones.

If $a = (a^1, \ldots, a^m)$ and $b = (b^1, \ldots, b^m)$ are m-tuples of real numbers, define their dot product $ab \equiv \sum_{i=1}^m a^i b^i$ in the usual way. Define the real valued random variable $a\varepsilon_t$ by the formula $a\varepsilon_t = \sum_{i=1}^m a^i \varepsilon_t^i$.

I shall say that an arbitrary random variable x is *admissible* if:

(a) $$x = \sum_{t=-\infty}^{\infty} a_t \varepsilon_t,$$

(b) $$\sum_{t=-\infty}^{\infty} a_t a_t < \infty.$$

Such a random variable is nothing more than a linear combination of the variables (ε_t^i) with square summable coefficients. An admissible random variable $x = \sum_{t=-\infty}^{\infty} a_t \varepsilon_t$ has mean zero and variance given by the formula $\sum_{t=-\infty}^{\infty} a_t a_t$.

Now let H denote the set of admissible random variables. The usual application of Minkowski's inequality shows that H is a vector space in the obvious way. In fact, H is actually a Hilbert space with an inner product defined as follows. Let $x = \sum_{t=-\infty}^{\infty} a_t \varepsilon_t$ and $y = \sum_{t=-\infty}^{\infty} b_t \varepsilon_t$ be two admissible variables. Their inner product (x, y) is defined by the formula $(x, y) \equiv \sum_{t=-\infty}^{\infty} a_t b_t$. Note that the inner product of two random variables in H is simply their covariance. If $(x, y) = 0$ the variables x, y are said to be orthogonal. Two variables are orthogonal if and only if they are uncorrelated.

The Hilbert space H has an obvious and very useful orthonormal basis, namely the set of random variables (ε_t^i).

Now suppose that (x_t), $t = 0, \pm 1, \pm 2$, is a sequence of random variables in H. The process (x_t) is defined to be *covariance stationary* if and only if the covariance $(x_t, \varepsilon_{t+k}^j)$ depends only upon the integers j and k and not upon t.

Define the backward shift (or lag) operator B on H to be the continuous linear map defined on basis elements as $B\varepsilon_t^i = \varepsilon_{t-1}^i$ for all i, t. Then a sequence of random variables (x_t) is covariance stationary if and only if $Bx_t = x_{t-1}$ for all t.

My description of the probability structure underlying this model of land speculation is now complete. The sequences (s_t), (p_t), and $(\hat{p}_t^i)$ will be assumed to be covariance stationary sequences of variables in H. (The usual trick of expressing a random variable in terms of deviations from its mean shows that the zero expected value restriction is no loss of generality in a linear model.) The Hilbert space H will play an important conceptual and technical role in the development to follow.

2. INFORMATION

In the preceeding section we assumed that the price of land at date t is determined by two factors: the quantity of land supplied to speculators at that date, s_t, and speculators' period t forecasts $(\hat{p}_t^i)$ of the logarithm of the price of land at date $t+1$. In this section, I shall describe how these forecasts depend upon the information available to speculators at date t.

The assumption I wish to make is that $\hat{p}_t^i$ is the mathematical expectation of p_{t+1} conditional upon the information speculator i has at date t. This information will be represented by a collection I of random variables from H.

Because any finite collection of random variables in H is jointly normal (see [**2**, Section 4]) we know that $\hat{p}_t^i$ is a random variable which is a linear function of the speculator's information; specifically, it is contained in the smallest closed subspace Y of H which contains I. Since $\hat{p}_t^i$ is a conditional expectation we conclude that it is the unique element of Y which predicts p_{t+1} with smallest mean square forecast error. In other words, $\hat{p}_t^i$ is the orthogonal projection of p_{t+1} upon the subspace Y generated by the speculator's information. I shall denote this relationship by $\hat{p}_t^i = \Pi(Y)p_{t+1}$ where $\Pi(Y)$ denotes the linear operator on H defined by orthogonal projection onto the subspace Y.

In order to obtain covariance stationary processes as equilibria of the model it is necessary to impose some stationarity restrictions upon the nature of each speculator's information.

DEFINITION 2.1: A *stationary information process* (W_t), $t = 0, \pm 1, \pm 2, \ldots,$ is a sequence of closed subspaces of H with the following two properties:

(a) $\quad W_t \subseteq W_{t+1}, \quad$ all t,

(b) $\quad BW_{t+1} = W_t, \quad$ all t.

I shall assume that each speculator's information can be represented by some stationary information process. The subspace W_t represents the information available to the speculator at date t. This subspace will, in general, include both information exogenous to the market as well as information endogenously generated by the market process (e.g., past prices). The assumption that $W_t \subseteq W_{t+1}$ says simply that a speculator never forgets information. The assumption that $BW_{t+1} = W_t$ expresses the requirement that the new information acquired between dates t and $t+1$ be generated by a stationary stochastic process.

Here are some examples of stationary information processes. I shall adopt the following notation. If $x^i = (x_t^i)$, $i = 1, \ldots, k$, are stationary processes in H, then $V_t(x^1, \ldots, x^k)$ is defined to be the smallest closed subspace of H containing the variables $x_t^1, \ldots, x_t^\kappa, x_{t-1}^1, \ldots, x_{t-1}^k, x_{t-2}^1, \ldots, x_{t-2}^k, \ldots .$

EXAMPLE 2.2: $W_t = 0$ all t. This process provides no information to a speculator at any date.

EXAMPLE 2.3: $W_t = H$ all t. This process provides complete information about all past and future events to a speculator at any date.

EXAMPLE 2.4: $W_t = V_{t+k}(s)$. At date t, this process provides a speculator with the knowledge of speculative supplies up through date $t+k$. If $k>0$, the speculator knows the values of speculative supplies k periods into the future. If $k<0$, the speculator learns the values of speculative supplies with a k period lag.

EXAMPLE 2.5: $W_t = V_t(s, p)$. At date t, this process provides a speculator with the knowledge of the history of market prices and speculative supplies.

3. EQUILIBRIUM

In this section, I shall define the concept of rational expectations (or RE) equilibrium appropriate for the model of land speculation under discussion. The principal result of this section is Theorem 3.15 which completely characterizes those markets having symmetric RE equilibria in terms of the parameters defining the market. A symmetric RE equilibrium is one in which all speculators have identical relevant information once they have observed the history of equilibrium prices. Theorem 3.15 together with the systematic use of the Z-transform will enable us to construct robust parametrizations of markets having no RE equilibrium (Section 5).

But it is the proof of 3.15 rather than the result itself which has direct applications to other linear, rational expectations models. The basic idea is to deduce that symmetric RE equilibrium prices must convey a predetermined amount of information to each speculator. This implies that if such a price process exists then it must be identical to what is termed the full communication price process. These are market clearing prices which arise when each speculator makes forecasts based upon the exogenous information which symmetric RE prices must reveal (when they exist). It is easy, using the Z-transform, to derive explicit expressions for the full communication prices. Using these expressions one can determine whether full communication prices reveal enough information to qualify as the symmetric RE equilibrium prices. If they do, they constitute the unique symmetric RE equilibrium. If they do not, there is no symmetric equilibrium.

The reader should note that the RE equilibria studied in most macroeconomic models are symmetric ones.

To begin the discussion of symmetric equilibria, I shall introduce some useful terminology.

DEFINITION 3.1: An *exogenous information structure* $U = (U^1, \ldots, U^n)$ is an n-tuple of stationary information processes (Definition 2.1), one for each speculator. The exogenous information structure is *symmetric* if $U^i = U^j$ all j.

If $U = (U^1, \ldots, U^n)$ is an exogenous information structure, then the information process U^i represents speculator i's exogenous information. The subspace U_t^i represents the exogenous information speculator i has at date t. This information is exogenous in the sense that it is unaffected by the market equilibration process and the market activities of any and all speculators.

The structure of a market for land is completely determined by the specification of an exogenous information structure U and of a stationary speculative supply process (s_t).

DEFINITION 3.2: A *rational expectations equilibrium* for the land market defined by U and (s_t) is a covariance stationary price process $p^* = (p_t^*)$ with the following two properties:

$$\text{RE1:} \qquad p_t^* = \frac{1}{n\alpha}\left(\sum_{i=1}^{n} \Pi(U_t^i + V_t(p^*))p_{t+1}^* - s_t\right),$$

$$\text{RE2:} \qquad p_t^* \in \sum_{i=1}^{n} U_t^i + V_t(s).$$

Before explaining this definition, let me first clarify the notation. If W_1, W_2 are two closed subspaces of H, their sum $W_1 + W_2$ is defined to be the smallest closed subspace of H containing both W_1 and W_2. Recall that $\Pi(U_t^i + V_t(p^*))$ denotes the orthogonal projection of H onto the closed subspace $U_t^i + V_t(p^*)$ and that $V_t(p^*)$ denotes the smallest enclosed subspace of H containing $p_t^*, p_{t-1}^*, p_{t-2}^*, \ldots$.

Condition RE1 expresses the requirement that an equilibrium price process clear the market for land at each date when each speculator makes a conditionally unbiased forecast of the relevant future equilibrium price. This forecast is based upon the information available to speculator i at date t, viz. the exogenous information U_t^i together with the history of equilibrium prices $V_t(p^*)$. The forecast by i of p_{t+1}^* is then simply the orthogonal projection of p_{t+1}^* onto the subspace $U_t^i + V_t(p^*)$.

The interpretation of condition RE2 is more subtle. This condition asserts that any information a speculator obtains by observing equilibrium prices could have been obtained from direct knowledge of the exogenous information possessed by all speculators together with the history of speculative supplies. By imposing RE2, I prevent equilibrium prices at date t from conveying any more exogenous information than that which could in principle be available to speculators at date t. I like to think of RE2 as the axiom of "no divine revelation." Notice, however, that RE2 does not prevent equilibrium prices from depending upon irrelevant information (e.g., the outcomes of coin tossing experiments). For such information could well be included in the information structure U.

The reader should note that RE2 is actually equivalent to the requirement that p_t^* be measurable with respect to the sigma field $\boldsymbol{A}$ generated by the random variables in the set $\Sigma_{i=1}^m U_t^i + V_t(s)$. For if x is $\boldsymbol{A}$ measurable, then $x = E(x|\boldsymbol{A})$. But since all the variables in question are jointly normal $E(x|A)$ is actually an element in $\Sigma_{i=1}^m U_t^d + V_t(s)$.

DEFINITION 3.3: Let (p_t^*) be an RE equilibrium of the market defined by U and (s_t). This RE equilibrium is called *symmetric* if, for all i and j,

$$\Pi(U_t^i + V_t(p^*))p_{t+1}^* = \Pi(U_t^j + V_t(p^*))p_{t+1}^*.$$

Thus, a symmetric RE equilibrium is one in which all speculators make identical forecasts of equilibrium prices. Note that if the exogenous information structure U is a symmetric one, then all RE equilibria for the market defined by U and (s_t) are symmetric. However, symmetric RE equilibria can easily arise in markets defined by asymmetric exogenous information structures. In such cases, symmetric equilibrium prices convey enough information to remove the initial informational asymmetries.

The study of symmetric RE equilibria is facilitated by the introduction of another equilibrium concept first proposed by Radner [**13**].

DEFINITION 3.4: The *full communication* (FC) *equilibrium* for the market defined by (s_t) and U is a stationary price process (p_t^*) satisfying:

$$\text{FC1:} \qquad p_t^* = \frac{1}{n\alpha}\left[\sum_{i=1}^{n} \Pi\left(\sum_{j=1}^{n} U_t^j + V_t(s)\right) p_{t+1}^* - s_t\right].$$

A full communication equilibrium arises from the following thought experiment. Before time begins traders get together and agree to exchange their exogenous information U_t^i at each date. In addition, they agree to exchange information with nonspeculative demanders and thus will be informed of speculative supplies at each date. At date t, each speculator makes an unbiased forecast of p_{t+1}^* based upon the information he has at date t, $\Sigma_{j=1}^n U_t^j + V_t(s)$. Naturally at a given date all speculators make identical forecasts. A sequence of market clearing prices determined by forecasts made in this way is a FC equilibrium.

One indication that a FC equilibrium is a useful benchmark in any given market is the following result.

THEOREM 3.5: *A FC equilibrium for the market defined by* (s_t) *and* U *exists and is unique.*

PROOF: Define $W_t = \Sigma_{j=1}^n U_t^j + V_t(s)$. The theorem asserts that there is a unique stationary sequence of random variables (p_t^*) which solve the equations

$$(3.6) \qquad p_t^* = \frac{1}{n\alpha}\left[n\ \Pi(W_t)p_{t+1}^* - s_t\right].$$

If (p_t^*) is a stationary solution to these equations then $Bp_{t+1}^* = p_t^*$ for all t. Clearly B is an invertible operator in H and B^{-1} is defined on basis elements by the formula $B^{-1}\varepsilon_t^i = \varepsilon_{t+1}^i$. Thus, $B^{-1}p_t^* = p_{t+1}^*$. Equation (3.6) then takes the form

$$(3.7) \qquad p_t^* = \frac{1}{n\alpha}\left[n\ \Pi(W_t)B^{-1}p_t^* - s_t\right].$$

This equation has a unique solution for any given s_t if and only if the linear operator $[I-(1/\alpha)\Pi(W_t)B^{-1}]$ is invertible. To see that this operator is in fact invertible simply note that the norm of the operator $(1/\alpha)\Pi(W_t)B^{-1}$ is strictly less than one. For $\alpha > 1$, the norm of $\Pi(W_t)$ is one (because $\Pi(W_t)$ is a nontrivial orthogonal projection), and the norm of B is one. It follows that

$$\left[I-\frac{1}{\alpha}\Pi(W_t)B^{-1}\right]^{-1}=\sum_{j=0}^{\infty}\left[\frac{1}{\alpha}\Pi(W_t)B^{-1}\right]^{j}.$$

Therefore, the unique solution to (3.7) is given by the formula

$$p_t^* = -\frac{1}{n\alpha}\sum_{j=0}^{\infty}\left[\frac{1}{\alpha}\Pi(W_t)B^{-1}\right]^{j} s_t. \tag{3.8}$$

It remains to show that the variables p_t^* defined by (3.8) form a stationary sequence, i.e., that $Bp_t^* = p_{t-1}^*$, all t. To do this, simply note that the characterization of orthogonal projections [**16**, pp. 313–314] together with the observation that $BW_t = W_{t-1}$ implies the formula

$$B\Pi(W_t)B^{-1}=\Pi(W_{t-1})$$

or

$$B\Pi(W_t)=\Pi(W_{t-1})B.$$

Using this last relationship after applying B to both sides of 3.8 shows that $Bp_t^* = p_{t-1}^*$ and thus completes the proof of Theorem 3.5. *Q.E.D.*

Formula (3.8) is important for the following reason. It shows how to calculate the FC prices in any market using only the parameters defining the market, viz. U and (s_t). The main result of this section (Theorem 3.15) shows how to determine whether or not a symmetric RE equilibrium exists in a given market via a calculation involving only the FC prices, U and (s_t). This last calculation can, therefore, in principle be done starting only with the parameters U and (s_t).

It will turn out that a symmetric RE equilibrium exists if and only if the FC equilibrium has certain information revealing properties. To express these properties, I introduce the following terminology.

DEFINITION 3.9: Let $x=(x_t)$ be a stationary sequence of random variables. Then x is said to: (a) be *strictly informative* if for all i

$$U_t^i + V_t(x) = \sum_{j=1}^{n} U_t^j + V_t(s);$$

(b) be *informative* if for all i

$$\Pi(U_t^i + V_t(x))x_{t+1} = \Pi\left(\sum_{j=1}^{n} U_t^j + V_t(a)\right)x_{t+1};$$

(c) *reveal supplies* if

$$\sum_{j=1}^{n} U_t^j + V_t(x) = \sum_{j=1}^{n} U_t^j + V_t(s).$$

Notice that if x is strictly informative then x is informative and reveals supplies. The reverse implications need not hold.

I now note that every RE equilibrium price process reveals supplies.

PROPOSITION 3.10: *Let (p_t^*) be any RE equilibrium for the market defined by U and (s_t). Then*

$$\sum_{j=1}^{n} U_t^j + V_t(p^*) = \sum_{j=1}^{n} U_t^j + V_t(s).$$

PROOF: By hypothesis (p_t^*) satisfies RE1. Thus

$$s_t = -n\alpha p_t^* + \sum_{i=1}^{n} \Pi(U_t^i + V_t(p^*))p_{t+1}^*.$$

We conclude that

$$V_t(s) \subseteq \sum_{j=1}^{n} U_t^j + V_t(p^*)$$

and therefore that

$$\sum_{j=1}^{n} U_t^j + V_t(s) \subseteq \sum_{j=1}^{n} U_t^j + V_t(p^*).$$

On the other hand (p_t^*) also satisfies RE2. Thus

$$V_t(p^*) \subseteq \sum_{j=1}^{n} U_t^j + V_t(s)$$

and therefore

$$\sum_{j=1}^{n} U_t^j + V_t(p^*) \subseteq \sum_{j=1}^{n} U_t^j + V_t(s).$$

These inclusions, when combined, show that

$$\sum_{j=1}^{n} U_t^j + V_t(p^*) = \sum_{j=1}^{n} U_t^j + V_t(s). \qquad Q.E.D.$$

Proposition 3.10 shows that RE prices reveal supplies. The following result shows that symmetric RE equilibrium prices reveal even more information.

PROPOSITION 3.11: *Let (p_t^*) be a RE equilibrium for the market defined by U and (s_t). Then (p_t^*) is a symmetric RE equilibrium if and only if (p_t^*) is informative.*

PROOF: That an informative RE equilibrium is symmetric is obvious. Suppose that the equilibrium is symmetric. We must show that for all i

$$\Pi(U_t^i + V_t(p^*))p_{t+1}^* = \Pi\Big(\sum_{j=1}^n U_t^j + V_t(s)\Big)p_{t+1}^*.$$

From Proposition 3.10, we know that it is equivalent to show that for all i

$$\Pi(U_t^i + V_t(p^*))p_{t+1}^* = \Pi\Big(\sum_{j=1}^n U_t^j + V_t(p^*)\Big)p_{t+1}^*.$$

Notice that both the right and left hand sides of the last equation are elements of the subspace $\Sigma_{j=1}^n U_t^j + V_t(p^*)$. To show that they are equal, it will suffice to show that they have the same orthogonal projection onto each of the n subspaces $U_t^k + V_t(p^*)$, $k = 1, \ldots, n$. In other words, we wish to establish the following equality for each i and k:

$$\begin{aligned}(3.12)\qquad &\Pi(U_t^k + V_t(p^*))\Pi(U_t^i + V_t(p^*))p_{t+1}^* \\ &= \Pi(U_t^k + V_t(p^*))\Pi\Big(\sum_{j=1}^n U_t^j + V_t(p^*)\Big)p_{t+1}^*.\end{aligned}$$

To do this, first observe that because the equilibrium is symmetric, we have the equality

$$\Pi(U_t^k + V_t(p^*))p_{t+1}^* = \Pi(U_t^i + V_t(p^*))p_{t+1}^*$$

for all i, k. Since the square of a projection operator is the operator itself, this last equality shows that the left hand side of (3.12) equals

$$(3.13)\qquad \Pi(U_t^k + V_t(p^*))p_{t+1}^*.$$

Since

$$U_t^k + V_t(p^*) \subseteq \sum_{j=1}^n U_t^j + V_t(p^*),$$

the right hand side of (3.12) is also equal to (3.13). This completes the proof.

Q.E.D.

COROLLARY 3.14: *A symmetric RE equilibrium for the market defined by U and* (s_t) *is identical to the FC equilibrium for that market.*

PROOF: By Proposition 3.11 a symmetric RE equilibrium is informative. Thus, we have the equality

$$\Pi(U_t^i + V_t(p^*))p_{t+1}^* = \Pi\Big(\sum_{j=1}^n U_t^j + V_t(s)\Big)p_{t+1}^*$$

for all i. By hypothesis (p_t^*) satisfies RE1. Substituting the last equality into RE1 yields FC1 (Definition 3.4) and shows that (p_t^*) is also the FC equilibrium for the market defined by U and (s_t). *Q.E.D.*

We can now establish the main result of this section.

THEOREM 3.15: *A symmetric RE equilibrium exists if and only if the FC equilibrium is informative. When a symmetric RE equilibrium does exist, it is identical with the FC equilibrium. Consequently, a given market has at most one symmetric RE equilibrium (by Theorem 3.5).*

PROOF: From Proposition 3.11, we conclude that a symmetric RE equilibrium is informative. From Corollary 3.14, we conclude that such an equilibrium is identical with the FC equilibrium. Thus, the FC equilibrium is informative.

Conversely, suppose (p_t^*) is an informative FC equilibrium. It then has the property that

$$\Pi(U_t^i + V_t(p^*))p_{t+1}^* = \Pi\left(\sum_{j=1}^n U_t^j + V_t(s)\right)p_{t+1}^*$$

for all i. Substituting the left hand side into FC1 yields RE1. Since FC1 implies RE2, we conclude that informative FC prices satisfy both RE1 and RE2 and therefore constitute a symmetric RE equilibrium. *Q.E.D.*

The following corollary of Theorem 3.15 is useful in constructing examples.

COROLLARY 3.16: *Let (p_t^*) be the FC equilibrium. Assume that for all i and j*

$$U_t^i + V_t(p^*) = U_t^j + V_t(p^*).$$

Then a symmetric RE equilibrium exists if and only if the FC equilibrium is strictly informative.

PROOF: By Corollary 3.14, a symmetric RE equilibrium is an FC equilibrium which, by Proposition 3.10, must therefore reveal supplies. Thus, the FC equilibrium satisfies the equality

$$\sum_{j=1}^n U_t^j + V_t(p^*) = \sum_{j=1}^n U_t^j + V_t(s).$$

By hypothesis the left hand side of this equality is $U_t^i + V_t(p^*)$. Thus, if a symmetric RE equilibrium exists, the FC equilibrium is strictly informative. On the other hand, a strictly informative FC equilibrium is informative. By 3.15, it is therefore a symmetric RE equilibrium. *Q.E.D.*

The reader should note that the hypothesis of Corollary 3.16 is always satisfied if the exogenous information structure is a symmetric one.

In Section 5, I shall apply Theorem 3.15 and Corollary 3.16 to exhibit robust examples of markets having no RE equilibria at all. But in order to check the hypotheses of 3.15 or 3.16 in specific examples, one needs some additional tools. These are discussed in the following section.

4. THE Z-TRANSFORM

To apply Theorem 3.15 and Corollary 3.16 one must have a practical method for determining whether two closed subspaces of the Hilbert space H are equal. In addition, it is often necessary to analyze equations similar to, but more complicated than, (3.6) which cannot be solved by explicit inversion of a linear operator as was done in the proof of Theorem 3.5. Both types of problems can often be easily solved through the application of the Z-transform.

Let x be an element of H which is contained in $V_0(\varepsilon)$. (Recall that $V_t(\varepsilon)$ is the subspace of H generated by the random variables ε_k^i, $k \leq t$, $i = 1, \ldots, m$.) The random variable x can therefore be expressed by the formula $x = \Sigma_{j=1}^m f_j(B)\varepsilon_0^j$ where $f_j(B) = \Sigma_{k=0}^\infty a_{jk}B^k$ and $\Sigma_{k=0}^\infty a_{jk}^2 < \infty$ for $j = 1, \ldots, m$.

DEFINITION 4.1: The *Z-transform* of x is the m-tuple of power series in the complex variable z given by the expression $(f_1(z), \ldots, f_m(z))$.

If (x_t) is a stationary stochastic process with $V_t(\varepsilon)$ containing x_t, then the Z-transform of the process (x_t) will be defined to be the Z-transform of x_0. Note that each of the power series $f_j(z)$ defines an analytic function on the open disk of complex numbers $|z| < 1$. In fact it is easy to see that taking Z-transforms establishes an isomorphism between the Hilbert space $V_0(\varepsilon)$ and the Hilbert space of Z-transforms. *Thus two random variables* x, y in $V_0(\varepsilon)$ *are equal if and only if their Z-transforms are equal as vectors of analytic functions on the open disk.* This last fact is the key to obtaining explicit solutions to linear RE models and it will be relied upon heavily in the following sections.

I shall next present a result which often enables one to determine whether two closed subspaces of H are in fact equal. Suppose $2m$ stationary processes in H are given: $(x_t^1), (x_t^2), \ldots, (x_t^m), (y_t^1), \ldots, (y_t^m)$. Recall that $V_0(x^1, \ldots, x^m)$ is the closed subspace generated by $x_0^j, x_{-1}^j, x_{-2}^j, \ldots$ for $j = 1, \ldots, m$. We wish to determine whether or not $V_0(x^1, \ldots, x^m) = V_0(y^1, \ldots, y^m)$. Assume that both of these subspaces are contained in $V_0(\varepsilon^1, \ldots, \varepsilon^m)$. Hence

$$x_0^j = \sum_{k=1}^m f_{jk}(B)\varepsilon_0^k,$$

$$y_0^j = \sum_{k=1}^m g_{jk}(B)\varepsilon_0^k \qquad (j = 1, \ldots, m).$$

Let $f(B), g(B)$ denote, respectively, the determinants of the matrices $(f_{jk}(B)), (g_{jk}(B))$. Assume that at least one of these functions is not identically zero. Finally, assume that $f(z)$, $g(z)$ are analytic functions for $|z| \leq 1$ (as would be the case for example if the f_{jk}, g_{jk} were all polynomials of finite degree).

PROPOSITION 4.2: $V_0(x^1, \ldots, x^m) = V_0(y^1, \ldots, y^m)$ *if and only if* $f(z), g(z)$ *have the same zeroes of modulus strictly less than one when each zero is counted according to its multiplicity.*

Proposition 4.2 can be proven as an immediate consequence of statements 7.2 on p. 67 and 8.33 on p. 86 together with the discussion on p. 87 in [**7**].

5. SOLVING FOR SYMMETRIC EQUILIBRIA

In this section I shall use Theorem 3.15 together with the Z-transform to determine whether or not certain parametrizations of the land market model have symmetric equilibria. The key step is the derivation of the Z-transform of the full communication equilibrium. Although this can be read off from formula (3.8) it is instructive to derive it directly from equation (3.6). The technique which enables one to solve (3.6) using the Z-transform generalizes to other linear RE models while the algebraic method of solution which led to (3.8) does not.

To economize on notation I shall assume throughout this section that $m = 1$ so that H is generated by the random variables (ε_t^1); in what follows the superscript "1" will be dropped. Suppose that the covariance stationary process of speculative supplies can be defined by the equation $s_t = g(B)\varepsilon_t$. Assume also that no speculator comes to the market with any exogenous information; thus $U_t^i = 0$ for all t and i. The exogenous information structure thus defined is a symmetric one. All RE equilibria must then be symmetric and by Theorem 3.15 there can be at most one such equilibrium. Because the exogenous information structure is symmetric, Corollary 3.16 shows that the equality $V_t(p^*) = V_t(s)$ is necessary and sufficient for the existence of a RE equilibrium where (p_t^*) is the full communication equilibrium price process.

To keep things simple I shall next assume that the Z-transform $g(z)$ of s_0 is an analytic function for $|z| \leq 1$ and that it has no zeroes of modulus strictly less than one. It then follows trivially from Proposition 4.2 that

$$V_0(s) = V_0(\varepsilon). \tag{5.1}$$

The necessary and sufficient condition for existence thus reduces to the equality

$$V_0(p^*) = V_0(\varepsilon). \tag{5.2}$$

To determine whether or not (5.2) holds we must first compute the full communication prices (p_t^*) which define its left-hand side.

From Definition 3.4, from (5.1) and the assumption that $U_t^i = 0$, we know that p_0^* solves the equation

$$p_0^* = \frac{1}{\alpha}\Pi(V_0(\varepsilon))p_1^* - \frac{1}{n\alpha}s_0 \tag{5.3}$$

where $Bp_1^* = p_0^*$. Without loss of generality we can write

$$p_0^* = \gamma\varepsilon_0 + Bf(B)\varepsilon_0. \tag{5.4}$$

Then (5.3) takes the form

$$[\gamma + Bf(B)]\varepsilon_0 = \frac{1}{\alpha}\Pi(V_0(\varepsilon))[B^{-1}\gamma + f(B)]\varepsilon_0 - \frac{1}{n\alpha}g(B)\varepsilon_0. \tag{5.5}$$

Noting that $\Pi(V_0(\varepsilon))B^{-1}\varepsilon_0 = 0$ and taking the Z-transform of both sides of (5.5) we obtain

$$(5.6) \qquad \gamma + zf(z) = \frac{1}{\alpha} f(z) - \frac{1}{n\alpha} g(z).$$

Equation (5.3) has a solution if and only if there is a real number γ and a function $f(z) = \Sigma_{k=0}^{\infty} b_k z^k$ with $\Sigma\, b_k^2 < \infty$ solving (5.6) for all values $|z| < 1$. If there is a solution then $\gamma + zf(z)$ is the Z-transform of p_0^*.

I shall now show that there is a unique γ and f solving (5.6). Since $1/\alpha < 1$, we know that if (5.6) has a solution the equality must hold for $z = 1/\alpha$. Thus

$$\gamma = -\frac{1}{n\alpha} g\left(\frac{1}{\alpha}\right)$$

and the constant γ has been determined as a function of the parameters defining the supply process. Substituting this value for γ into (5.6) one obtains

$$(5.7) \qquad \left(z - \frac{1}{\alpha}\right) f(z) = -\frac{1}{n\alpha}\left[g(z) - g\left(\frac{1}{\alpha}\right)\right].$$

Since both sides of (5.7) are analytic for $|z| < 1$ and vanish at $z = 1/\alpha$ we can divide both sides by $(z - (1/\alpha))$ and obtain

$$(5.8) \qquad f(z) = -\frac{1}{n\alpha} \frac{g(z) - g(1/\alpha)}{z - (1/\alpha)}.$$

Since g is analytic for $|z| \leqslant 1$, f is also analytic on the same domain. Therefore the Z-transform of p_0^* is the function analytic for $|z| \leqslant 1$ defined by

$$(5.9) \qquad \gamma + zf(z) = -\frac{1}{n\alpha}\left[g\left(\frac{1}{\alpha}\right) + z\frac{(g(z) - g(1/\alpha))}{z - (1/\alpha)}\right].$$

Let us now consider as an example the class of markets with supply process of the form

$$(5.10) \qquad s_t = a_0\varepsilon_t + a_1\varepsilon_{t-1}, \qquad |a_0| \geqslant |a_1|.$$

Then (5.9) simplifies to

$$(5.11) \qquad \gamma + zf(z) = -\frac{1}{n\alpha}\left(a_0 + \frac{1}{\alpha} a_1 + a_1 z\right).$$

This is the Z-transform of p_0^* when (5.10) holds. Proposition 4.2 then implies that (5.2) holds if and only if

$$(5.12) \qquad \left|a_0 + \frac{1}{\alpha} a_1\right| \geqslant |a_1|.$$

If (5.12) obtains, then (5.11) is the Z-transform of the unique RE equilibrium price process.

As an illustration suppose that $a_0 = 1$, $a_1 = -3/8$, and α is nearly equal to unity. Then (5.12) holds and there is a unique RE equilibrium. But if $a_0 = 1$ and $a_1 = -5/8$, condition (5.12) fails to hold and there is no RE equilibrium at all.

It is possible to interpret this existence failure in more familiar terms. Write the full communication prices in the form

$$(5.13) \qquad p_t^* = b_0\varepsilon_t + b_1\varepsilon_{t-1}.$$

Notice that for any integer J

$$(5.14) \qquad p_t^* = b_0\varepsilon_t + \sum_{j=1}^{J} (-1)^{j+1}\left(\frac{b_1}{b_0}\right)^j p_{t-j}^* + (-1)^{J+2} b_1 \left(\frac{b_1}{b_0}\right)^J \varepsilon_{t-J-1}.$$

Suppose we "started up" the market at time 0 by defining $0 = \varepsilon_{-1} = \varepsilon_{-2} = \varepsilon_{-3} = \ldots$ and by defining $0 = p_{-1} = p_{-2} = \ldots$. Given these initial conditions, let all speculators use the formula

$$(5.15) \qquad \hat{p}_t^i = \sum_{j=1}^{t+1} (-1)^{j+1}\left(\frac{b_1}{b_0}\right)^j p_{t-j+1}^*$$

to determine their point forecasts of p_{t+1}^*. By construction the market will clear at each date and the forecasts defined by (5.15) will be conditionally unbiased (recall that b_0, b_1 were defined by the full communication equilibrium condition). If $|b_0| > |b_1|$, i.e. if (5.12) holds, the resulting price process converges to the stationary process of symmetric RE prices. For these parameter values a symmetric RE equilibrium exists. But if $|b_0| < |b_1|$ the process is explosive. No (stationary) RE equilibrium exists. The important point to note is this. The price process one obtains when speculators forecast future values of full communication prices using only the past values of these prices is explosive precisely when these prices fail to reveal to speculators the information contained in the history of speculative supplies. At least in these examples one can interpret the nonexistence results as manifestations of inherent market instability. This connection between the dynamic stability of RE price processes and the amount of information they reveal seems to persist in other linear models and may be a quite general phenomenon.

Notice that the second example presented above is a robust one. For if δ is a sufficiently small positive number then any supply process

$$s_t' = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$$

satisfying

$$(a_0 - 1)^2 + (a_1 + 5/8)^2 + \sum_{k=2}^{\infty} a_k^2 < \delta$$

defines a market which also fails to have a RE equilibrium. In fact, one can also perturb the exogenous information structure slightly and preserve the nonexistence result. Two such structures $(U^1, \ldots, U^m)$, $(W^1, \ldots, W^m)$ are defined to be "close" if the norm of the linear operator $\Pi(U_0^i) - \Pi(W_0^i)$ is small for $i = 1, \ldots, n$. Note that there are no small perturbations of the zero exogenous information structure used in the examples above because the operator which projects onto the zero subspace is an isolated point in this metric. But if $m > 1$ one can easily modify the above example into one with a nonzero symmetric exogenous information structure and in which there is no RE equilibrium. Furthermore small symmetry preserving perturbations of the symmetric exogenous information structure define markets with the same property.

One can also draw an econometric moral from the above examples. Linear RE models are not nearly as well behaved as linear market models in which prices play no informational role. In estimating a linear RE model one must expect that the hypothesis that data are generated by a stationary RE equilibrium price process imposes a priori restrictions (like (5.12)) upon the parameters one wishes to estimate. In fact any stationarity hypothesis (e.g., that the nth differences are stationary) imposes analogous constraints.

The RE equilibria exhibited in this section have been symmetric equilibria. When $U_t^i = U_t^j$ for all i, j such equilibria must, by Corollary 3.16, convey information about the history of speculative supplies to each speculator. Specifically the equality

$$V_t(p^*) + U_t^i = V_t(s) + U_t^i$$

must hold for each i. I would now like to show that while symmetric RE prices are informative in this sense they need not be perfectly informative. In other words $V_t(p^*)$ may contain strictly less information than $V_t(p^*) + U_t^i$. Speculators may have incentive to collect private information even at a symmetric equilibrium.

To see this let us assume that, for each i, $U_t^i = V_t(y)$ where the process (y_t) is defined by the equation

$$y_t = \lambda \varepsilon_t - \varepsilon_{t-1}, \qquad |\lambda| < 1.$$

Assume that

$$s_t = a_0 \varepsilon_t + a_1 \varepsilon_{t-1}, \qquad |a_0| > |a_1|.$$

Because $V_t(s) = V_t(\varepsilon)$ we know that the full communication prices have a Z-transform given by (5.11). If we assume that

$$\text{(5.16)} \qquad \lambda \neq \frac{a_0 + (1/\alpha) a_1}{a_1},$$

then one can show (using the first corollary on page 101 of [**8**]) that

$$\text{(5.17)} \qquad V_t(p^*) + U_t^i = V_t(s) + U_t^i.$$

Hence by Corollary 3.16 there is a symmetric RE equilibrium. But notice that if

$$\left| a_0 + \frac{1}{\alpha} a_1 \right| < |a_1|,$$

then Proposition 4.2 implies that

$$\text{(5.18)} \qquad V_t(p^*) \neq V_t(\varepsilon) = V_t(s) + U_t^i.$$

In this situation each speculator makes more accurate forecasts using the history of (y_t) and (p_t) than is possible using the history of (p_t) alone. Prices do not reveal all private information even though after observing prices each speculator is completely informed. Thus the incentive to collect private information which will then be partially incorporated into prices can persist even in a RE equilibrium. But notice that if

$$\left|a_0+\frac{1}{\alpha}\right| \geqslant |a_1|$$

the situation is reversed. Now $V_t(p^*)=V_t(\varepsilon)$ and the information collected by speculators becomes useless.

6. ASYMMETRIC EQUILIBRIA

In this section I shall exhibit a set of conditions sufficient to ensure the existence of an asymmetric RE equilibrium. These are RE equilibria in which some speculators make different price forecasts than do others. The sufficient conditions are derived under the assumption that traders can be divided into two categories: the informed and the uninformed. When these conditions are satisfied one can derive an explicit expression for the Z-transform of the equilibrium price process. As an application of this result I shall exhibit examples of markets which have only asymmetric equilibria.

As in the previous section assume that $m=1$ so that H is generated by the real valued random variables (ε_t). Let speculators $i=1,\ldots,k<n$ be informed; this means by definition that

$$U_t^1=U_t^2=\ldots=U_t^k=V_t(\varepsilon).$$

Let the other $n-k$ speculators be uninformed so that

$$U_t^{k+1}=U_t^{k+2}=\ldots=U_t^n=0.$$

Assume that the process of speculative supplies is defined by the formula

$$s_t=a_0\varepsilon_0+a_1\varepsilon_{t-1}.$$

Finally, define functions $h_1(\lambda, z)$, $h_2(\lambda)$ by the formulae:

$$h_1(\lambda, z)=\frac{a_0+a_1\lambda}{k}+\frac{\lambda z}{1-\lambda z}\left(\frac{a_0+a_1\lambda}{k}+\frac{a_1}{n\alpha}\right),$$

$$h_2(\lambda)=\frac{a_0}{k}+\frac{a_1}{n\alpha}-\left[\frac{a_0}{n\alpha}-a_1\left(\frac{1}{k}-\frac{1}{n\alpha^2}\right)\right]\lambda-\frac{a_1}{n\alpha}\lambda^2.$$

THEOREM 6.1: *The market described above has an asymmetric RE equilibrium provided that* (i) *there is a real number* λ^* *such that* $h_2(\lambda^*)=0$ *and* $|\lambda^*|<1$ *and* (ii) *for the* λ^* *satisfying* (i) *the function* $h_1(\lambda^*, z)$ *does not vanish for any complex number* $|z|<1$. *When these conditions are satisfied the Z-transform of the asymmetric equilibrium price process is* $(z-\lambda^*)h_1(\lambda^*, z)$.

Before proving Theorem 6.1, let us apply it to construct some examples. Notice that

$$h_2(\lambda)\equiv\frac{1}{k}\left[a_0+\frac{k}{n\alpha}a_1-\left(\frac{k}{n\alpha}a_0-a_1\left(1-\frac{k}{n\alpha^2}\right)\right)\lambda-\frac{k}{n\alpha}a_1\lambda^2\right].$$

If $a_0+(k/n\alpha)a_1=0$, then $\lambda^*=0$ is a zero of $h_2(\lambda)$. Then

$$h_1(0, z)=\frac{1}{k}a_0\neq 0.$$

Hence we have a market with an asymmetric equilibrium. In fact it is obvious that as long as $a_0+(k/n\alpha)a_1$ is nearly equal to zero one again obtains a market having an asymmetric equilibrium with λ^* nearly equal to zero. Thus this family of examples is robust to slight perturbations of a_0, a_1, and k/n.

But notice also that if $a_0+(k/n\alpha)a_1=0$ then the market has no symmetric equilibrium. To see this notice that $\Sigma_{i=1}^m U_t^i+V_t(s)=V_t(\varepsilon)$ and therefore the full communication prices have a Z-transform defined by (5.9). Then (5.12) is necessary and sufficient for the existence of a symmetric equilibrium. But

$$\begin{aligned}\left|a_0+\frac{1}{\alpha}a_1\right|&=\left|a_0+\frac{k}{n\alpha}a_1+\frac{n-k}{n\alpha}a_1\right|\\&=\left|\frac{n-k}{n\alpha}a_1\right|\\&<|a_1|.\end{aligned}$$

Thus there is no symmetric equilibrium.

I shall now prove Theorem 6.1. To do so one must show that the sufficient conditions stated in the theorem permit the solution of the following equation which defines the asymmetric equilibrium price process:

$$(6.5)\qquad p_t^*=\frac{k}{n\alpha}\Pi(V_t(\varepsilon))p_{t+1}^*+\frac{n-k}{n\alpha}\Pi(V_t(p^*))p_{t+1}^*-\frac{1}{n\alpha}s_t.$$

Furthermore, the solution to (6.5) must also have the property that

$$(6.6)\qquad V_t(p^*)\subsetneqq V_t(\varepsilon).$$

Condition (6.6) ensures that informed traders will make more accurate forecasts than uninformed traders (provided $s_t\neq\varepsilon_t$).

To solve (6.5) subject to (6.6) I shall begin by assuming that there is a solution of the form

$$(6.7)\qquad p_t^*=(B-\lambda)(\rho+Bf(B))\varepsilon_t.$$

This solution will also be assumed to have the additional properties

(6.8) the function $f(z)$ is analytic for $|z|\leqslant 1$;

(6.9) $|\lambda|<1$;

(6.10) $\rho+zf(z)$ has no zeroes of modulus less than one.

The proof of Theorem 6.1 will then take the following form. Using (6.7)–(6.10) one can write the Z-transform of equation (6.5) in a simple form. Having done this one then uses the sufficient conditions in Theorem 6.1 to find specific values of ρ, λ, and f which not only solve (6.5) but which also have properties (6.7)–(6.10). Thus the assumptions which enabled one to write down the Z-transform of (6.5) in the first place are satisfied, the prices defined by (6.7) satisfy (6.5), and the proof is completed.

To carry out this program let us first use assumption (6.7) to rewrite (6.5) in the form

$$(6.11) \qquad (B-\lambda)(\rho+Bf(B))\varepsilon_0 = \frac{k}{n\alpha}[\rho+(B-\lambda)f(B)]\varepsilon_0 + \frac{n-k}{n\alpha}\Pi(V_0(p^*))[\rho+(B-\lambda)f(B)]\varepsilon_0 - \frac{1}{n\alpha}(a_0+a_1B)\varepsilon_0.$$

In order to exhibit the Z-transform of (6.11) we must first calculate the orthogonal projection of $[\rho+(B-\lambda)f(B)]\varepsilon_0$ onto $V_0(p^*)$. By assumptions (6.7)–(6.10) the Z-transform of p_0^* vanishes at only one point of modulus less than one, namely λ. By Theorem 3.14 in [**12**] one concludes that the co-dimension of $V_0(p^*)$ in $V_0(\varepsilon)$ is exactly equal to one. In fact, the orthogonal complement of $V_0(p^*)$ in $V_0(\varepsilon)$ consists of scalar multiples of the vector $(1-\lambda B)^{-1}\varepsilon_0$. Furthermore, 3.11 and 3.14 in [**12**] then imply that every vector in $V_0(p^*)$ can be written in the form $g(B)(B-\lambda)(1-\lambda B)^{-1}\varepsilon_0$ for some function g. Thus $[\rho+(B-\lambda)f(B)]\varepsilon_0$ can be written uniquely in the form

$$(6.12) \qquad [g(B)(B-\lambda)(1-\lambda B)^{-1}+(\text{constant})(1-\lambda B)^{-1}]\varepsilon_0.$$

Taking the Z-transform of this expression and equating it to the Z-transform of p_0^* at the point $z=\lambda$ we find that the unknown constant appearing in (6.12) equals $\rho(1-\lambda^2)$. We conclude then that

$$(6.13) \qquad \Pi(V_0(p^*))[\rho+(B-\lambda)f(B)]\varepsilon_0 = [\rho+(B-\lambda)f(B)-\rho(1-\lambda^2)(1-\lambda B)^{-1}]\varepsilon_0.$$

Substituting (6.13) into (6.11) and taking Z-transforms, one obtains the Z-transform of (6.5):

$$(6.14) \qquad (z-\lambda)(\rho+zf(z)) = \frac{k}{n\alpha}(\rho+(z-\lambda)f(z)) + \frac{n-k}{n\alpha}(\rho+(z-\lambda)f(z)) - \frac{n-k}{n\alpha}\frac{1-\lambda^2}{1-\lambda z}\rho - \frac{1}{n\alpha}(a_0+a_1z).$$

The proof of Theorem 6.1 can now be completed by finding real numbers ρ, λ and a function f which solves (6.14) and which also satisfy (6.8)–(6.10).

To do this first set $z=\lambda$ in (6.14). Then $\rho=(1/k)(a_0+a_1\lambda)$ and the value of ρ is determined in terms of a_0, a_1 and the yet unknown value of λ. This value of ρ causes the right hand side of (6.14) to vanish at $z=\lambda$. Therefore we can divide both sides of (6.14) by $(z-\lambda)$ and, after substituting this value of ρ and rearranging terms, we obtain

$$(6.15)\qquad \left(z-\frac{1}{\alpha}\right)f(z)=-\frac{(a_0+a_1\lambda)}{k}+\frac{1}{n\alpha}\frac{1}{z-\lambda}\times\left[\left(\frac{n}{k}-\frac{n-k}{k}\frac{1-\lambda^2}{1-\lambda z}\right)(a_0+a_1\lambda)-(a_0+a_1z)\right].$$

A short calculation shows that the second summand on the right hand side of (6.15) contains a factor $(z-\lambda)$ which then cancels with $1/(z-\lambda)$. Then 6.15 takes the form

$$(6.16)\qquad \left(z-\frac{1}{\alpha}\right)f(z)=-\frac{(a_0+a_1\lambda)}{k}-\frac{1}{n\alpha}\frac{1}{1-\lambda z}\left(\lambda a_0\frac{n-k}{k}+a_1+a_1\lambda^2\frac{n-k}{k}-a_1\lambda z\right).$$

Because the left hand side of (6.16) vanishes at $z=1/\alpha$, the right hand side must also vanish there. This implies that λ must be chosen to satisfy the equation

$$(6.17)\qquad -\left(\frac{a_0+a_1\lambda}{k}\right)-\frac{1}{n\alpha}\frac{1}{1-\lambda/\alpha}\left(\lambda a_0\frac{n-k}{k}+a_1+a_1\lambda^2\frac{n-k}{k}-a_1\frac{\lambda}{\alpha}\right)=0.$$

Assuming that such a λ can be found with $|\lambda|<1$ allows us to multiply both sides of (6.17) by the nonzero number $-(1-(\lambda/\alpha))$ to obtain the equation

$$(6.18)\qquad h_2(\lambda)\equiv\frac{a_0}{k}+\frac{a_1}{n\alpha}-\left(\frac{a_0}{n\alpha}-a_1\left(\frac{1}{k}-\frac{1}{n\alpha^2}\right)\right)\lambda-\frac{a_1}{n\alpha}\lambda^2=0.$$

Any real number λ^* such that $h_2(\lambda^*)=0$ has the property that the right hand side of (6.16) vanishes when $\lambda=\lambda^*$ and $z=1/\alpha$. If in addition $|\lambda^*|<1$, then the right hand side of (6.16) is analytic for $|z|\leqslant 1$. Both of the conditions are satisfied by hypothesis in Theorem 6.1. Therefore after setting $\lambda=\lambda^*$ one can divide both sides of (6.12) by $z-(1/\alpha)$ and obtain

$$(6.19)\qquad f(z)=-\frac{1}{1-\lambda^*z}\frac{1}{z-(1/\alpha)}\left[\frac{a_0+a_1\lambda^*}{k}(1-\lambda^*z)+\frac{1}{n\alpha}\left(\lambda^*a_0\frac{n-k}{k}+a_1+a_1\lambda^{*2}\frac{n-k}{k}-a_1\lambda^*z\right)\right].$$

The term in brackets in (6.19) vanishes at $z = 1/\alpha$ and hence contains a factor $(z-(1/\alpha))$. Removing this factor and cancelling it with $1/(z-(1/\alpha))$ gives us the expression

$$f(z)=\frac{\lambda^*}{1-\lambda^* z}\left(\frac{a_0+a_1\lambda^*}{k}+\frac{a_1}{n\alpha}\right). \tag{6.20}$$

Thus

$$\rho+zf(z)=\frac{a_0+a_1\lambda^*}{k}+\frac{\lambda^* z}{1-\lambda^* z}\left(\frac{a_0+a_1\lambda^*}{k}+\frac{a_1}{n\alpha}\right). \tag{6.21}$$

Notice that the right hand side of (6.21) equals $h_1(\lambda^*, z)$. Thus if the hypothesis of Theorem 6.1 is satisfied, $\rho+zf(z)$ has no zero of modulus less than one. Thus (6.8)–(6.10) are satisfied and the Z-transform of p_0^* is $(z-\lambda^*)h_1(\lambda^*, z)$. This completes the proof of Theorem 6.1.

Bell Laboratories, Murray Hill, New Jersey

Manuscript received February, 1979; final revision received December, 1979.

REFERENCES

[1] Allen, B.: "Generic Existence of Completely Revealing Equilibria," Dept. of Economics, University of California, Berkeley, 1978 (unpublished).
[2] Ansley, C. F., W. A. Spivey, and W. J. Wrobleski: "On the Structure of Moving Average Processes," *Journal of Econometrics*, 6 (1977), 121–134.
[3] Futia, C. A.: "Stochastic Business Cycles," mimeo, Bell Laboratories, 1979.
[4] Grossman, S.: "Further Results on the Informational Efficiency of Competitive Stock Markets," *Journal of Economic Theory*, 18 (1979), 81–101.
[5] Hansen, L. P., and T. J. Sargent: "Formulating and Estimating Dynamic Linear Rational Expectations Models: I," *Journal of Economic Dynamics and Control*, 2 (1980), 7–46.
[6] Hart, O. D.: "On the Optimality of Equilibrium When the Market Structure is Incomplete," *Journal of Economic Theory*, 11 (1975), 418–443.
[7] Helson, H.: *Lectures on Invariant Subspaces.* New York: Academic Press, 1964.
[8] Hoffman, K.: *Banach Spaces of Analytic Functions.* Englewood Cliffs: Prentice Hall, 1962.
[9] Jordan, J., and R. Radner: "The Non-existence of Expectations Equilibrium: A Robust Example," mimeo, Dept. of Economics, University of Minnesota, 1978.
[10] Kreps, D.: "A Note on Fulfilled Expectations Equilibria," *Journal of Economic Theory*, 14 (1977), 32–43.
[11] Lucas, R. E.: "An Equilibrium Model of the Business Cycle," *Journal of Political Economy*, 83 (1975), 1113–1144.
[12] Radjavi, H., and P. Rosenthal: *Invariant Subspaces.* New York: Springer Verlag, 1973.
[13] Radner, R.: "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices," *Econometrica*, 47 (1979), 655–677.
[14] Saracoglu, R., and T. J. Sargent: "Seasonality and Portfolio Balance Under Rational Expectations," *Journal of Monetary Economics*, 4 (1978), 435–458.
[15] Sargent, T. J., and N. Wallace: "Rational Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule," *Journal of Political Economy*, 83 (1975), 241–254.
[16] Schechter, M.: *Principles of Functional Analysis.* New York: Academic Press, 1971.
[17] Schiller, R. J.: "Rational Expectations and the Dynamic Structure of Macroeconomic Models," *Journal of Monetary Economics*, 4 (1978), 1–44.