Investors' and Central Bank's Uncertainty Measures Embedded in Index Options[*]

Alexander David

University of Calgary

Pietro Veronesi

University of Chicago

NBER and CEPR

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Abstract

Index options' implied volatility (ATMIV) and out-of-the-money put-call implied volatility ratios (P/C) lead key macroeconomic variables such as industrial capacity utilization and short term interest rates by up to eight quarters. We show that this interaction between options, real activity, and policy variables arises in an equilibrium model where investors learn about the trend-growth regimes of economic data, and the central bank uses a learning-based Taylor rule. The model endogenously generates several time series properties of option prices including the counter (pro) cyclicality of ATMIV (P/C), the V-shape (inverse V-shape) relation between ATMIV (P/C) and monetary policy variables, the positive relation between the level and absolute changes in ATMIV, and an economically significant amount of time variation in the volatility premium.

Since the classic work of Breeden and Litzenberger (1978) it has been clear that option prices contain valuable information on investors' forward looking state price density function. It has been less clear, however, if this information is in any way linked to macroeconomic fundamentals that should affect agents' state prices. While it is intuitive that corporate stock and options prices react to corporate news, there is also substantive evidence that they react to monetary policy shocks.[1] This empirical evidence though spurs fundamental questions such as: What is the relation between option prices, corporate fundamentals, and the action of the central bank? Can option prices help us identify investors' and central bank's beliefs about future economic growth and inflation? In this paper we provide an equilibrium model that links option prices to both fundamentals and monetary policy and provide a dynamic and time consistent methodology to extract investors' beliefs on the regime of the macroeconomy and central bank policy.

Before describing the model and its implications, it is useful to present some empirical relations between option prices, the state of the economy, and monetary policy. Our analysis focuses on two popularly quoted measures constructed from options prices. The first is the implied volatility of at-the-money (ATMIV) options, which was originally created by Whaley (1993), and it has been trading on the Chicago Board Options Exchange under the ticker VIX since 1993.[2] The CBOE describes this index eloquently as:

> One of the most interesting features of VIX, and the reason it has been called the "investor fear guage," is that, historically, VIX hits its highest levels during times of financial turmoil and investor fear. [CBOE Bulletin on VIX, 2003].

The second measure is the ratio of implied volatilities of out-of-the-money put to call options (P/C), which is a direct market assessment of downside relative to upside risk. This measure has also been studied extensively since the work of Bates (1991) to guage investors' worries about a market decline.

Quarterly time series plots of these variables for options with three months to maturity are shown in Figure 1 for the 23-year period 1986 - 2008. Comparing the two panels in Figure 1, we see that, surprisingly perhaps, ATMIV and the P/C are negatively related, with the ATMIV (P/C) being generally counter (pro) cyclical. While it is intuitive that implied volatility ATMIV is high during downturns, it

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[1]For example, Bernanke and Kuttner (2005) report that monetary policy surprises affect the stock market, while Rigobon and Sack (2003) show that the monetary policy responds to stock returns with a greater reaction during times of higher volatility, and more recently Bekaert, Hoerova, and Duca (2010) find a significant reaction of options prices to lead and lag measures of monetary policy.

[2]In September 2003, the CBOE changed the computation method for the VIX index using a new methodology that does not involve the Black and Scholes formula. The two measures are highly correlated and we will compare their two time series in an online appendix, which is to be made available to readers.

is less obvious why the downside-risk index P/C is high during booms and low during recessions. Our model explains why by tying this variation to monetary policy dynamics.

Indeed, it is revealing to see how monetary policy relates to the two options' indices. By way of motivation, we estimate pairwise Vector Auto Regressions (VAR) with the options' indices and monetary policy variables. The left panels of Figure 2 report resulting impulse responses for the historical series over the options subsample of 1986:Q2 – 2008. The results are striking and all in one direction: shocks to both ATMIV and P/C lead to sustained impacts on future monetary policy. In contrast, we do not find that monetary policy measures have any sustained impacts on the two options measures (results not shown). Moreover, and more interestingly, the first panel shows that the 3-month Treasury rate decreases for up to eight quarters in response to a shock to ATMIV.[3] Even more interestingly, a shock to the downside-risk index P/C induces the 3-month Treasury rate to increase for up to eight quarters in the future. Under the interpretation of Bates (1991), the latter result implies that when investors become more worried about a stock market decline, future short-term rates increase. What is the economic mechanism generating this empirical observation? [4]

To further our understanding and link the two options' indices to a policy-relevant fundamental variable, the bottom two panels of Figure 2 show that indeed we obtain the same directional impulses for ATMIV and P/C shocks on capacity utilization (CU), an important determinant in the Federal Reserve monetary policy rule. That is, positive shocks to ATMIV (P/C) lead to a decrease (increase) in CU. In contrast, the impulse responses of CU on the options' indices are statistically insignificant (results not shown). Taken together, these results suggest intriguing dynamics between the information captured by option prices, and the actions of the central bank. Our model aims at capturing these dynamics.

We provide a dynamic equilibrium model of learning that links options to investors' and central banks' uncertainty about fundamentals. In order to have a model amenable to the empirical investigation, we follow the recent macro-finance term-structure literature (e.g. Ang and Piazzesi (2003) and Ang, Piazzesi, and Wei (2006)) and posit a structural econometric model for the equilibrium dynamics of fundamental variables along with the specification of a forward looking Taylor rule, which directly

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[3]In our empirical analysis, we use the 3-month T-bill rate as our short-term rate, rather than the Federal Funds rate, as the latter is affected by banks' default premium, which is absent in our model. The 3-month T-bill rate and the Fed Funds rate are very highly correlated.

[4]Interestingly, we find that the put-to-call ratio P/C strongly predicts future interest rates, while we do not find such relation with other measures of crash risk, such as the difference in implied volatility of out-of-the-money puts versus at-the-money puts. Our fitted model is consistent also with this evidence.

links macroeconomic variables to target interest rates set by the Federal Reserve.[5] We ensure that no-arbitrage restrictions hold in our economy by also positing the equilibrium dynamics for the state price density, which we use to price all traded assets in the economy – namely, stocks, Treasury bonds, and options – from the fundamental variables, endogenous investors' beliefs about the economy, and the Taylor-rule-based riskless rate.

We generalize the Taylor rule by introducing three key features, which we provide evidence for: First, we specify an unobserved regime switching model with composite regimes of macroeconomic and policy fundamental variables. The composite regime formulation explicitly recognizes that the econometrician observes fundamental transitions only in the presence of policy intervention. Second, we specify a learning-based Taylor rule, in which neither the central bank nor investors observe the true trend growths of nominal as well as real variables. Agents in the economy (investors and the central bank) are econometricians in the sense of Hansen (2007), that is, they attempt to learn about the drift regimes of fundamentals from the observation of past and current fundamentals. Their Bayesian learning dynamics about the regime of the economy are the key driver of our results, as explained below. Finally, to extend the current understanding of the effects of monetary aggregates on the stock market we follow the suggestions in Lucas (2007) to allow real money growth to affects transitions between fundamental drift regimes.[6] The observations of money growth affect investors' beliefs about future fundamental regimes, which the econometrician can extract from fundamentals and price data.

Our model sheds light on the compelling dynamic one-way relation between options' ATMIV and P/C and monetary policy, discussed earlier in Figure 2. The strong evidence that these option-based

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[5]The New Keynesian Economics approach shows the optimality of such rules in settings where price stickiness implies deviations from short run full employment and capacity utilization [see, e.g. Woodford (2003)]. Gallmeyer, Hollifield, and Zin (2005), Gallmeyer, Hollifield, Palomino, and Zin (2007), and Bekaert, Cho, and Moreno (2010) build term-structure models using policy variables.

[6]Lucas (2007) complains about the lack of use of monetary aggregates in recent models of monetary policy and recommends their use in information extraction:

> One source of this concern is the increasing reliance of central bank research on New-Keynesian modeling. New-Keynesian models define monetary policy in terms of a choice of money market rate and so make direct contact with central banking practice. Money supply measures play no role in their estimation, testing or policy simulation. A role for money in the long run is sometimes verbally acknowledged, but the models themselves are formulated in terms of deviation from trends that are themselves somewhere off stage. It seems likely that these models could be reformulated to give a unified account of trends, including trends in monetary aggregates, and deviations about trend but so far they have not been. This remains an unresolved issue on the frontier of monetary theory. Until it is resolved, monetary information should continue to be used as a kind of add-on or cross-check, just as it is in the ECB policy formulation today.

Coenen, Levin, and Wieland (2005) and Beck and Wieland (2008) show that money growth can help predict real activity when the real output and real money are economically linked but the central bank, which partially controls money growth, receives noisy information on the former.

indices lead policy variables, but not the reverse, is not driven by differences in information between investors and the central bank, as our model assumes they observe the same data and have the same information. Instead, our analysis points to a compelling *real effect* of uncertainty. In fact, our model shows that a higher ATMIV occurs when uncertainty about the current regime is high, because from Bayes' formula high uncertainty leads to faster revision of beliefs to news and thus higher return volatility. Consistent with the real options literature, high uncertainty predicts declines in future capacity utilization as firms delay the costly abandonment of plants and factories but instead operate below full capacity. Facing the same fundamental uncertainty and hence predicting lower CU, the central bank reacts by lowering the cost of capital.

Similarly, our model shows why increases in the P/C ratio predict future increases in short rates. A direct implication of Bayesian learning is that investors downwardly revise their beliefs in response to bad news by a larger amount in good times than in bad times. Therefore, in good times investors' perceive greater downside risk in stocks, or a more conditionally strongly negatively skewed return distribution. The negatively skewed distribution raises the price of put options relative to call options (P/C). However, the strong fundamental growth in good times also encourages firms to run their production processes at full capacity, raising inflation fears. Once again, the central bank with the same information of investors, responds to forecasts of tightening capacity utilization (CU) by raising the risk-free rate. These effects also explain why ATMIV and the P/C are negatively correlated (see Figure 1) since in periods of strong growth with stable policy variables, investors' overall belief volatility is generally low, so that the ATMIV is low.

Our dynamic model highlights the non-linear nature of the relation between macro-economic fundamentals, monetary policy variables, and option indices. Indeed, consistent with the existing empirical evidence, we find that linear regressions of ATMIV and P/C on a set of relevant macroeconomic variables lead to a small R^2, indicating a limited ability of macro-variables to explain the variation of option indices. In contrast, our model which endogenously generates strong non-linearities between fundamentals and asset prices through the learning mechanism, explains about 50 percent of the variation in the two options measures and explains their negative relation.

How can our model help understand these non-linearities? Essentially in our model, inflation is a key signal of future real activity. Corporate earnings growth is stable at moderate levels of inflation but it is expected to decline when inflation expectations are either too high or too low. This "Goldilocks" relationship between expected inflation and growth implies that the sign of the conditional reaction of the stock market to CPI fluctuations can vary over time. This time variation is a key mechanism for

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understanding several of the time-varying phenomena we see in the options market, and their relation to monetary policy. Indeed, we show that the central bank's efforts to stabilize growth and inflation implies that ATMIV (P/C) have a V-shape (inverse V-shape) relation with respect to the key policy variables, industrial capacity utilization and money growth. In particular, high ATMIV occurs both when capacity utilization is very low or very high. Indeed, high ATMIV correlates with lower capacity utilization as firms slow down their activity in the face of high uncertainty, which lowers inflationary pressures in the economy below regular levels. However, high ATMIV also occurs when capacity utilization is very high, as investors now fear a future slowdown of the economy. Similarly, ATMIV (P/C) has also a V-shape (inverse V-shape) relation with respect to money supply growth. In particular, ATMIV is high in those recessionary periods characterized by a strong money growth, as the Federal Reserve attempts to stimulate the economy. However, ATMIV is also high in a high-inflation recessionary period, when the Federal Reserve tightens money growth to rein in inflation. Because P/C is negatively related to ATMIV, it follows that the relations between the P/C and the policy variables are inverse V-shaped. We find support for these nonmonotonic relations between policy variables and the options' indices ATMIV and P/C in the data, although it is useful to note that for the post-1986 subsample of our data for which options prices are available, there were few periods of very high capacity utilization.

We fit the parameters of our structural model with an overidentified Simulated Method of Moments (SMM) procedure, which uses the likelihood of observing the fundamentals as well as stock, Treasury bond, and options prices to extract investor's beliefs. It is important to note that our estimation methodology ensures that the extracted beliefs are time-consistent and respect Bayes formula over the whole sample period. This implies that the estimated dynamics of uncertainty is also time consistent and is an outcome of the realization of fundamentals. This distinguishes our work from related work on options with learning that resets the model uncertainty in each period to some proxy of uncertainty in the data and focuses on conditional reactions in options prices.[7]

While our estimation method explicitly uses ATMIV and P/C as overidentifying moments, the uncertainty process of the model has additional implications for other options- related statistics. Because such statistics were not used in the estimation, their comparison with their model-implied counterparts

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[7] For example Guidolin and Timmermann (2005) and Buraschi and Jiltsov (2006) study option prices and volume in models with learning about fundamentals. Dubinsky and Johannes (2006) study the reaction of options prices on individual stocks to news about earnings. Benzoni, Collin-Dufresne, and Goldstein (2011) show that the increase in investors' perception about the average jump size of stock prices led to a steepening of the implied volatility smirk after the stock market crash of 1987, but do not study its time variation in subsequent years. In a paper related to ours, Shaliastovich (2009) models investors' non-Bayesian (behavioral) learning about the long run drift of consumption to generate the smile, but does not study its time series fit to data series.

offer additional evidence in support of the model. The first statistic is the strong positive relation between ATMIV and absolute changes in ATMIV, which is incompatible with the Heston (1993) stochastic volatility model, and it is believed that it requires an explosive volatility process (one which violates certain regularity conditions, see Jones (2003)). Our model volatility process is instead nonexplosive but thanks to Bayes' law, displays the positive correlation: in periods of greater uncertainty (and hence volatility), investors also update their beliefs faster, creating the positive relation between volatility and its absolute changes.

The second statistic that we take our model implications to is the implied volatility premium, which is the difference between implied volatility and the expectation of volatility under the objective measure. The volatility premium is currently one of the most actively researched statistics in empirical option pricing, and we show that our model volatility premium explains economically significant amounts of variation in the historical volatility premium. We use our fitted model to show that the risk premium component of the implied volatility premium is very highly correlated with the volatility of volatility, which is consistent with volatility being a systematic factor with a negative beta [see e.g. Buraschi and Jackwerth (2001) and Bakshi and Kapadia (2003)].

Related Literature

Besides the literature on option prices with learning in Footnote 7, this paper contributes to a small set of papers that provides economic explanations of the implied volatility curve for options.[8] Bollen and Whaley (2004) and Garleanu, Pederson, and Poteshman (2008) find that net buying pressure affects the prices of options for several days as market makers fail to provide options at no-arbitrage prices, but charge for the residual risk due to the limits to arbitrage. In addition to focus on lower frequency data and explaining the entire time series of options prices, we do not depart from the no-arbitrage framework. Among theoretical explanations for smirks, Liu, Pan, and Wang (2005) study the implications for ambiguity about rare event risk that raise the prices of puts relative to calls. Drechsler (2008) and Du (2010) provide calibrated models with time-varying ambiguity and with jumps with habit formation

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[8] There is also a large literature that explains the volatility smile by assuming exogenous processes for stock prices, volatilities, and jumps. Indeed, since the classic work of Black and Scholes (1977) the major innovations have been the addition of stochastic volatility [see, e.g., Hull and White (1987) and Heston (1993)], jumps in prices [see e.g. Bates (1996) and Bates (2000), and Pan (2002)], and jumps in volatility [see, e.g. Eraker, Johannes, and Polson (2003)]. A tremendous amount of empirical work has been done on these extensions of the BS formula that has enriched our understanding of stock price dynamics, and of options returns. Bakshi, Cao, and Chen (1997) provides a specification analysis of some of these models. Among more recent innovations, Christoffersen, Jacobs, Ornthanalai, and Wang (2008) build multi-factor stochastic volatility models, and somewhat related to our paper, Polson, Johannes, and Stroud (2008) price options when exogenously specified volatility follows an unobserved process that investors learn about. Constantinides, Jackwerth, and Perrakis (2008) find that several exogenously specified volatility models, such as GARCH, can be rejected as possible data generating processes for S&P 500 index options.

preferences, respectively, to generate the left skewed implied volatility smile, but neither paper studies the time series properties of the smile, nor their interaction with monetary policy.

Our work also complements the papers constructing structural models of options prices to understand the volatility premium. Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2010) and Eraker (2008) construct equilibrium models with "long run risks" in the consumption process to understand the size of the IVP and some of its unconditional moments. In work related to ours, Bekaert and Engstrom (2010) model the time variation in higher order moments of fundamentals to generate a volatility premium. This paper uses habit preferences to generate time variations in the price of risk. Unlike these papers, we shut off both the fundamental heteroskedasticity and time varying risk aversion channels in generating the volatility risk premium, and instead explicitly incorporate a role for monetary policy. As we will show, most of the time variation in the volatility premium in our model arises from variations in two fundamental uncertainties, which affect the speed of revision of beliefs and the volatility of stock market volatility. In particular, the volatility premium and hence the richness of option prices, is driven most strongly by investors' uncertainty about firms' earnings, and the uncertainty about money growth, which is an important signal for the stability of economic growth.

The layout of the paper is as follows. In section 1, we provide the structure of the model and derive some key pricing results. In section 2 we estimate the parameters of our model using an overidentified simulated method of moments procedure. In section 3, we study the ability of our model to explain the two fear indices, and in section 4, we study its ability to understand the volatility of stock market volatility and the volatility premium. Section 5 concludes. Two technical appendices provides proofs of technical results and the estimation methodology, respectively. In addition, an online appendix to be made available to readers contains a 10-year out-of-sample forecasting exercise, some additional robustness issues, and a more detailed analysis of some time series properties of the model.

1 Structure of the Model

Our main assumption throughout the paper is the drift rates of the fundamental processes are driven by an $N-$regime, continuous time, hidden Markov chain process. It is useful to describe this process first. We denote by s_t the regime at time t, where $s_t \in \{s^1, .., s^N\}$, and we let Λ denote the Markov chain infinitesimal generator matrix. That is, over the infinitesimal time interval of length dt

$$\lambda_{ij}dt \; = \; \text{prob}\left(s_{t+dt} = s^j | s_t = s^i\right), \quad \text{for} \quad i \neq j, \quad \lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}.$$

We assume that all agents in our economy, both investors and the central bank, do not observe the realizations of s_t but learn about it from the observation of numerous signals, including realized fundamental variables. Given an information filtration $\{\mathcal{F}_t\}$ generated by such signals, we denote the agents' common beliefs at time t about regime s^i as

$$\pi_{it} = \text{prob}(s_t = s^i | \mathcal{F}_t), \quad i = 1, ..., N \tag{1}$$

Lemma 1 below characterizes the dynamics of the vector $\pi_t = \{\pi_{1t}, ..., \pi_{Nt}\}$, but before we introduce the learning result, we need to introduce the rest of the model.

There is a single homogeneous good in the economy whose price, Q_t, follows:

$$\frac{dQ_t}{Q_t} = \beta(s_t)\, dt + \sigma_Q\, dW_t, \tag{2}$$

where $W_t = (W_{1t}, W_{2t}, W_{3t}, W_{4t}, W_{5t})'$ is a 5-dimensional vector of independent Weiner processes, inflation volatilities are summarized in 1×5 constant vector σ_Q, and the drift rate $\beta(s_t)$ depends on the realization of the (hidden) regime s_t.

The main real corporate fundamental in the economy is the process of real earnings, E_t, which follows the jump-diffusion process

$$\frac{dE_t}{E_t} = (\theta(s_t) - \kappa\, \xi_1)\, dt + \sigma_E\, dW_t, + (e^{Y_{1t}} - 1)\, dL_t \tag{3}$$

where fundamental volatilities, σ_E, are constant over time, the drift rate $\theta(s_t)$ depends on the realization of the regime s_t, L_t is the counter of a Poisson process with constant intensity κ, i.e. $\text{Prob}(dL_t = 1) = \kappa\, dt$, the jump size Y_{1t} is i.i.d. normal with mean μ_1 and volatility σ_1, and $\xi_1 = e^{\mu_1 + 0.5\sigma_1^2} - 1$. The regime process, s_t, the Brownian Motions, W_t, and the jump process L_t are all independent of each other. Under the assumption of continuous observation of fundamentals, and hence their quadratic variation processes, investors can perfectly observe jumps. In our model, jumps to earnings play three important roles: First, their inclusion permits a better estimation of the earnings process, which we will see has some large negative outcomes in our sample. Second, negative mean jumps will be shown to increase the average put-call implied volatility ratio (P/C), and, third, they increase the average volatility premium priced in options in our sample. It is important to note however that we model i.i.d. jump sizes and constant jump intensity so that the modeled jumps in themselves are unable to explain the time series variation in either the P/C or the volatility premium, which is the subject of our paper.

The next important fundamental in the economy is de-meaned industrial capacity utilization (CU), K_t which follows the process

$$dK_t = \rho(s_t)\,dt + \sigma_K\,dW_t, \tag{4}$$

where the volatilities σ_K are constant and assumed known by investors and the drift $\rho(s_t)$ depends on the realization of of the regime s_t. Note that unlike the other state variables, CU is stated in levels, and hence can become negative. The use of CU improves the term structure fit of our model.

The final state variable in our model is aggregate real money in the economy, H_t, which follows

$$\frac{dH_t}{H_t} = \omega(s_t)\,dt + \sigma_H\,dW_t, \tag{5}$$

where the volatilities σ_H are once again constant and the drift $\omega(s_t)$ depends on the regime s_t. We emphasize that H_t is the equilibrium quantity of real money in the economy determined both by its demand and supply. It is also useful to note that while ours is not a full structural model in which the quantity of money is endogenously determined, the statistical properties of dH_t/H_t affect agents' beliefs' dynamics, and thus equilibrium prices.

1.1 The Central Bank Policy Rule

We assume that all agents, investors and central bank, observe the same data and thus have the same information about the regime of the economy. Thus, the regime probabilities π_{it} defined in (1) are common across all agents. We assume that the central bank sets the real rate of the economy $\bar{\phi}_t$ by using a forward looking Taylor rule, namely

$$\bar{\phi}_t = \alpha_0 + \alpha_\beta\,\mathbb{E}\left[\beta_t|\mathcal{F}_t\right] + \alpha_\rho\,\mathbb{E}\left[\rho_t|\mathcal{F}_t\right]. \tag{6}$$

where the expectations are taken with respect to all of the information available at time t, $\mathcal{F}_t$.[9] The second and third terms of the real rate capture the essential elements of the Taylor rule, which posits that the central bank increases rates in response to increases in expected inflation and the expected real slack in the economy [see Taylor (1993)]. Our policy rule is hence 'forward-looking' in the sense of Clarida, Gali, and Gertler (2000), who suggested replacing current and/or lagged values of inflation and the output gap by their forward-looking conditional expectations. A significant contribution of

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[9]We allowed for a generalization the Taylor rule to let interest rates directly be impacted by money growth but did not estimate a significant effect.

our analysis is to jointly estimate the expectations from corporate earnings as well as regular macroeconomic variables, so that there is interaction between uncertainty in the corporate sector and central bank policy. In addition, following the assumption in Rudebusch and Wu (2008) we use the industrial capacity utilization series obtained from the Federal Reserve Board rather than the output gap, in the original Taylor rule. This choice is in part based on the observation in Gordon (1989) that capacity utilization is more closely related to inflation than the output gap.

We finally note that in standard Taylor rules, the central bank sets the nominal interest rate.[10] In our model, we will show that the inflation risk premium is constant, so the policy rule above can equivalently be written as setting of the nominal rate by adding expected inflation and the inflation risk premium on both sides of equation (6).

1.2 No Arbitrage Pricing

To build the policy rule of the central bank into a no-arbitrage framework, we follow Ang and Piazzesi (2003) and Piazzesi (2005) in specifying a state price density to price all cash flows in our model. Let M_t be the state price density at date t. As in the modern classic asset pricing theory (see, e.g. Cochrane (2001)), a generic random real cash flow $\{D_t\}$ is priced as

$$M_t P_t \;=\; \mathbb{E}\left[\int_t^\infty M_s\, D_s\, ds | \mathcal{F}_t\right]. \tag{7}$$

It is convenient to first write the process of the state price density in terms of the original hidden Markov process s_t and Brownian motions W_t. We specify M_t taking the form

$$\frac{dM_t}{M_t} \;=\; (-\phi(s_t) - \kappa\,\xi_2)dt \;-\; \sigma_M\, dW_t \;+\; (e^{Y_{2t}} - 1)\, dL_t, \tag{8}$$

where $\phi(s_t)$ denotes the real rate *conditional* on observing the regime (see discussion below), σ_M is a 1×5 constant vector of the market prices of risk, L_t is the same Poisson counter as in the earnings process in (3), Y_{2t} has an i.i.d. normal distribution with mean μ_2 and volatility σ_2 and perfectly correlated with Y_{1t}, and $\xi_2 = e^{\mu_2 + 0.5\sigma_2^2} - 1$. Note that the jumps in earnings are systematic since they are correlated with the marginal utility of the representative investor in the economy. We note that constant prices of risk also arise in a simple Lucas (1978) economy with no government where the representative agent has constant relative risk aversion, and where the fundamental volatility of consumption

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[10]The original Taylor rule [see Taylor (1993)] is $i_t = \pi_t + r_t^* + a_\pi(\pi_t - \pi_t^*) + a_y(y_t - \bar{y}_t)$, where i_t is the target nominal rate, π_t is the realized rate of inflation, r_t^* is the assumed equilibrium real rate of interest, π_t^* is the desired inflation rate, y_t is the log of GDP, and $\bar{y}_t$ is the log of potential GDP.

(dividends) is constant. This assumption along with the homoskedasticity of fundamentals ensure that all fluctuations in volatilities and premiums in our model arise endogenously due to learning and not from either time variation in risk aversion or built-in fundamental heteroskedasticity.

To ensure no-arbitrage, the expected drift rate of the state price density must equal the real rate $\bar{\phi}_t$ in (6), so that we impose

$$\mathbb{E}\left[\frac{dM_t}{M_t}|\mathcal{F}_t\right] = -\bar{\phi}_t dt$$

Since investors and the central bank have the same information, this no arbitrage restriction is naturally obtained by requiring that regime by regime:[11]

$$\phi(s_t) = \alpha_0 + \alpha_\beta\,\beta(s_t) + \alpha_\rho\,\rho(s_t).$$

1.3 Learning Dynamics

For notational convenience, we stack the fundamental processes (2), (3), (4), and (5) that are observed by the econometrician as signals in a vector $dY_t = \left(\frac{dQ_t}{Q_t}, \frac{dE_t}{E_t}, dK_t, \frac{dH_t}{H_t}\right)'$, so that

$$dY_t = \varrho(s_t)\,dt + \Sigma_4\,dW_t + J_{4t}\,dL_t, \tag{9}$$

where the drift vector process is $\varrho(s_t) = (\beta(s_t), \theta(s_t) - \kappa\xi_1, \rho(s_t), \omega(s_t))'$, the volatility matrix is $\Sigma_4 = (\sigma'_Q, \sigma'_E, \sigma'_K, \sigma'_H)'$, and the vector of jump sizes is $J_{4t} = (0, e^{Y_{1t}} - 1, 0, 0,)$. In particular, note that we assume the econometrician does not observe investors' state price density M_t. Agents in the economy, instead, observe both signals dY_t and dM_t and we denote the full set of signals as $dZ_t = \left(dY'_t, \frac{dM_t}{M_t}\right)'$, which has the drift vector $\nu(s_t) = (\varrho(s_t)', -\phi(s_t) - \kappa\,\xi_2)'$, volatility matrix $\Sigma = (\Sigma'_4, \sigma'_M)'$, and jump size of $J_t = (J'_{4t}, e^{Y_{2t}} - 1)'$.

The following Lemma characterizes the dynamics of beliefs $\pi_{it} = \text{prob}(s_t = s^i|\mathcal{F}_t)$. For notational convenience, we denote the drift of the signal vector dZ_t in regime i by $\nu^i = \nu(s^i)$.

Lemma 1. *Given an initial condition $\pi_0 = \hat{\pi}$ with $\sum_{i=1}^N \hat{\pi}_i = 1$ and $0 \leq \hat{\pi}_i \leq 1$ for all i, the probabilities π_{it} satisfy the N-dimensional system of stochastic differential equations:*

$$d\pi_{it} = \mu_i(\pi_t)dt + \sigma_i(\pi_t)d\tilde{W}_t, \tag{10}$$

$$\textit{in which} \quad \mu_i(\pi_t) = [\pi_t\Lambda]_i, \quad \sigma_i(\pi_t) = \pi_{it}\left[\nu^i - \bar{\nu}(\pi_t)\right]'\Sigma'^{-1}, \tag{11}$$

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[11]Indeed, from (8): $-\mathbb{E}\left[\frac{dM_t}{M_t}|\mathcal{F}_t\right] = \mathbb{E}\left[\phi(s_t)|\mathcal{F}_t\right] = \alpha_0 + \alpha_\beta\,\mathbb{E}\left[\beta(s_t)|\mathcal{F}_t\right] + \alpha_\rho\,\mathbb{E}\left[\rho(s_t)|\mathcal{F}_t\right]$, which yields (6).

$$\bar{\nu}(\pi_t) = \sum_{i=1}^{N} \pi_{it}\,\nu^i = \mathbb{E}_t\,(dZ_t|\mathcal{F}_t), \quad and$$

$$d\tilde{W}_t = \Sigma^{-1}[dZ_t - J_t dL_t - \bar{\nu}(\pi_t)] = \Sigma^{-1}\,(\nu_t - \bar{\nu}(\pi_t))dt \; + \; dW_t. \tag{12}$$

Moreover, for every $t > 0$, $\sum_{i=1}^{N} \pi_t = 1$.

This filtering result is a straightforward extension of the Wonham filter (see Wonham (1964)), which characterizes the Bayesian learning about the hidden drift with Brownian noise.[12] In the setup here, the observed fundamental vector process has observable jumps in some elements, which do not affect investors' beliefs about the hidden drift. In particular, the high frequency variation in investors' beliefs is driven by investors' inferred shocks, $d\tilde{W}$, in equation (12) as opposed to the true shocks, dW, which affect fundamentals. It is also possible to write the fundamental process vector $dZ_t = \nu_t dt + \Sigma dW + J_t dL_t = \bar{\nu}(\pi_t)dt + \Sigma d\tilde{W} + J_t dL_t$. The right hand side of (12) also reveals that the inferred shocks process $d\tilde{W}$, does not depend on the jump parameters, since investors are able to observe jumps which thus do not affect their inference about s_t.

1.4 Stock Prices and the Term Structure of Interest Rates

The following proposition provides expressions for the price-earnings (henceforth P/E) ratio and the nominal bond price:

Proposition 1.

   (a) The P/E ratio at time t is

$$\frac{P_t}{E_t}(\pi_t) \;=\; \sum_{j=1}^{N} C_j\,\pi_{jt} \;\equiv\; C \cdot \pi_t, \tag{13}$$

where the vector $C = (C_1,..,C_N)$ *satisfies* $C \;=\; A^{-1} \cdot \mathbf{1}_N$,

$$A = Diag(\phi^1 - \theta^1 + \sigma_M\,\sigma_E' - \kappa(\xi_3 - \xi_1 - \xi_2), \cdots, \phi^N - \theta^N + \sigma_M\,\sigma_E' - \kappa(\xi_3 - \xi_1 - \xi_2)) - \Lambda. \tag{14}$$

---

[12]The first application of the Wonham filter in financial economics, as well as several properties of the filtering process, are derived in David (1997). We find it useful to recall that a main advantage of this modeling strategy as opposed to the more commonly used Kalman filter is that investors uncertainty (conditional variance of expectations about the drift terms) fluctuates forever, while in the Kalman filter, this uncertainty converges to a constant. The fluctuating confidence (inverse of the conditional variance) is the driver of the options' indices that we seek to explain in this paper.

(b) The price of a nominal zero-coupon bond at time t with maturity τ is

$$B_t(\pi_t, \tau) = \sum_{i=1}^{N} \pi_{it} B_i(\tau), \tag{15}$$

where the $N \times 1$ vector valued function $B(\tau)$ with element $B_i(\tau) = E\left(\frac{M_{t+\tau}}{M_t} \cdot \frac{Q_t}{Q_{t+\tau}} | \nu_t = \nu^i\right)$ is given by

$$B(\tau) = \Omega\, e^{\omega\,\tau}\, \Omega^{-1}\, 1_N. \tag{16}$$

In (16), Ω and ω denote the matrix of eigenvectors and the vector of eigenvalues, respectively, of the matrix $\hat{\Lambda} = \Lambda - Diag(r^1, r^2, \cdots, r^n)$, where each $r^i = k^i + \beta^i - \sigma_M \sigma_Q' - \sigma_Q \sigma_Q'$, is the nominal rate that would obtain in the ith regime, were the regimes observable. In addition, $e^{\omega\tau}$ denotes the diagonal matrix with $e^{\omega_i \tau}$ in its (i, i) position.

The proof for stocks is in the appendix. The proof for bonds follows from a simple extension of the proof in a similar setting in David and Veronesi (2009). The stock price formula has a similar form to that developed in the pure diffusion setup of David and Veronesi (2009) and further intuition on the formula is provided there. The major difference here is the jump risk in earnings and kernel, which is priced, and adds to the equity risk premium. The constant C_i is the P/E as in the Gordon growth model. In contrast to stocks, bond prices do not jump since the belief processes are continuous and the main bond fundamental, inflation, is continuous. It is useful to note that the actual dynamics of stock and bond prices here are quite different from those in David and Veronesi (2009) since they are in part determined by policy variables, not in their paper, and in addition, stock prices can jump.

Let $P_t^n = P_t \cdot Q_t$ be the nominal value of stock, where P_t is the real value of stocks in Proposition 1. Using the dynamics of the inflation and earnings processes under the observed filtration, we now formulate the nominal return processes for stocks and bonds.

Proposition 2.

(a) The nominal stock return process under the investor's filtration is given by

$$\frac{dP_t^n}{P_t^n}(\pi_t) = (\mu^n(\pi_t) - \delta(\pi_t))\, dt + \sigma^n(\pi_t)\, d\tilde{W}_t + (e^{Y_{1t}} - 1)\, dL_t,$$

where $\delta(\pi) = 1/(C \cdot \pi_t)$ is the earnings yield, $\mu^n(\pi) = r_t^n + \sigma^n(\pi)(\sigma_M + \sigma_Q)$ is the nominal expected return, and the nominal stock price volatility is

$$\sigma^n(\pi_t) = \sigma_E + \sigma_Q + \frac{\sum_{i=1}^{N} C_i \pi_{it} (\nu_i - \overline{\nu}(\pi_t))'(\Sigma')^{-1}}{\sum_{i=1}^{N} C_i \pi_{it}}. \tag{17}$$

The proof follows from a simple adaptation of the proof in Veronesi (2000) and an application of Ito's formula for jump-diffusions. Asset volatilities have exogenous as well as learning-based components, which depends on the volatility of each regime probability π_i. We will discuss these further in the empirical sections of this paper.

1.5 Return Volatility and its Dynamic Properties

A key variable for understanding a number of features of options prices is the volatility of stock variance. We develop its properties here. We start by introducing the following notation. Let

$$\pi_i^\circ = \frac{\pi_i C_i}{\Sigma_{j=1}^{N} \pi_j C_j} \tag{18}$$

As in Veronesi (2000), we call $\pi^\circ = (\pi_1^\circ, ..., \pi_n^\circ)$ the *value-weighted probabilities* (notice that $\pi_i^\circ \geq 0$ for each i and $\sum_{i=1}^{N} \pi_i^\circ = 1$,). From now on, a "$\circ$" denotes a quantity computed with respect to the distribution π°. For example, $\overline{\theta}^\circ$ denotes the mean of the drift vector θ computed using the distribution π° (whereas e.g. $\overline{\theta}$ denotes the mean drift vector computed using the original distribution π_t), and

$$\sigma_{\theta\beta} = \sum_{i=1}^{N} \pi_i(\theta_i - \overline{\theta})(\beta_i - \overline{\beta}); \text{and} \quad \sigma_{\theta\beta}^\circ = \sum_{i=1}^{N} \pi_i^\circ(\theta_i - \overline{\theta}^\circ)(\beta_i - \overline{\beta}^\circ) \tag{19}$$

are the covariances of the drift vectors θ and β computed using π and π°, respectively. In addition we denote $\sigma_{\theta\nu}$ and $\sigma_{\theta\nu}^\circ$ to be the vectors of covariances of θ with each element of the vector ν using the two sets of probabilities respectively. We then have:

Proposition 3 *(a) Stock return variance is given by*

$$V = \sigma^n(\pi_t)\sigma^{n'}(\pi_t) = (\sigma_E + \sigma_Q)(\sigma_E + \sigma_Q)' + (\overline{\nu}^\circ - \overline{\nu})'(\Sigma\Sigma')^{-1}(\overline{\nu}^\circ - \overline{\nu}) + 2[(\overline{\theta}^\circ - \overline{\theta}) + (\overline{\beta}^\circ - \overline{\beta})] \tag{20}$$

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(b) Return variance V follows the process $dV = \mu_V dt + \sigma_V d\tilde{W}$, where $\sigma_V =$

$$2\left[\sum_i \left([\pi_i^\circ (\nu_i - \bar{\nu}^\circ) - \pi_i \nu_i]'(\Sigma\Sigma')^{-1}(\bar{\nu}^\circ - \bar{\nu})(\nu_i - \bar{\nu})'\right) + (\sigma_{\theta_\nu}^\circ - \sigma_{\theta_\nu})' + (\sigma_{\beta_\nu}^\circ - \sigma_{\beta_\nu})'\right]\Sigma'^{-1} \tag{21}$$

(c) The volatility of stock volatility is

$$\sigma_\sigma = 0.5 \frac{\sigma_V}{\sqrt{V}} \tag{22}$$

The proposition implies that return variance is stochastic and so is the covariance between return and variance, given by

$$\mathrm{Cov}\left(dV, \frac{dS}{S}\right) \equiv \sigma_V \sigma_S' \, dt, \tag{23}$$

where stock volatility is in (17) and the volatility of variance in (22). We will see below in Section 3 that for our estimated model this covariance can change sign and magnitude leading to changes in the slope of the implied volatility curve for options prices.

We finally show that the stock price process in our model satisfies important regularity conditions, which guarantee the solutions to the option pricing partial differential equation as well as estimation of the likelihood function. These conditions will be useful to compare the properties of our model with standard option pricing models in Section 4.

Proposition 4 *The stock price process, P_t^n, in Proposition 2 satisfies global Lipschitz and growth conditions.*

1.6 Option Prices

To compute option prices we need to find the process for the stock index under the risk-neutral measure, obtained next:

Proposition 5 *The stock price under the risk-neutral measure follows:*

$$\frac{dP_t^{n*}}{P_t^{n*}}(\pi_t^*) = (\mu^n(\pi_t^*) - \delta(\pi_t^*)) \, dt + \sigma^n(\pi_t^*) \, d\tilde{W}_t^* + (e^{Y_{1t}^*} - 1) \, dL_t^*,$$

$$d\pi_t^* = (\mu(\pi_t^*) - \vartheta(\pi_t^*)) \, dt + \sigma(\pi_t^*) \, d\tilde{W}_t^*,$$

where $d\tilde{W}_t^* = d\tilde{W}_t + (\sigma_M + \sigma_Q)dt$, L_t^* is the counter of a Poisson process with intensity $\kappa^* = \kappa \cdot e^{\mu_2 + .5\sigma_2^2}$, and Y_{1t}^* is distributed $N(\mu_1 + \sigma_1\sigma_2, \sigma_1^2)$. Finally the market price of risk of the belief of

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regime i, which is the covariance of π_i with the nominal pricing kernel is given by

$$\vartheta_i(\pi_t^*) = \pi_{it}^* \left((\beta_i - \bar{\beta}(\pi_t^*)) - (\phi_i - \bar{\phi}(\pi_t^*)) \right). \tag{24}$$

The proof is in Appendix 1.

We appeal to the Feynman-Kac formula to use Monte-Carlo simulations to evaluate the expectation

$$f(t, \pi_t, P_t^n) = \mathbb{E}^Q \left[\exp \left(- \int_{s=t}^{T} r(\pi_s) ds \right) g(P_T^n, \pi_T) \right]. \tag{25}$$

We use some variance reduction techniques for efficiency. The advantage of the simulation method is that it does not suffer from the curse of dimensionality, which would be the case if we directly attempted to solve the fundamental PDE for derivatives prices. Details of the simulation procedure are provided in Appendix 2.

2 Estimation

Ours is a regime switching model in which the regime s_t affects the drift rates of four different fundamental series. This feature of our model is important as it introduces an important low-frequency comovement of fundamental variables in addition to the high frequency Brownian shocks. For example, we will see that earnings growth is more stable in period of moderately low inflation and is unstable when the inflation drift is either too high or too low. In addition, the persistence and transition between regimes is partly determined by the central bank's efforts to stabilize the economy, a feature captured in our model by the joint specification of macroeconomic and policy variables regimes. An important feature of our methodology is that asset prices in the model are functions of both macro and policy variables and these are used by the econometrician to back out investors' beliefs about these regimes.

2.1 Estimation Methodology

It is important for the goal of this paper to extract investors' beliefs in a dynamic, forward-looking, and time-consistent manner. This is accomplished by using an overidentified Simulated Method of Moments (SMM) of the learning model, in which the structural parameters are constant over time, but the arrival of new information leads investors to update their beliefs about the composite regimes of macro and policy variables. We (the econometricians) estimate the model by using information in fundamentals (macro and policy), and market prices (stock, bond, and options). Fundamentals are included

since investors' information sets clearly contain the history of all fundamental data. However, since investors' information sets are likely to be considerably richer than the history of fundamentals, we attempt to extract their forward-looking beliefs embedded in asset prices at discrete (quarterly) points of time. It is important to note that the SMM likelihood function of observing the fundamentals is exactly identified by all the structural parameters. It follows that all asset prices used in the procedure are overidentifying restrictions on the model, and lead to an omnibus test of the model. In fact, we use the objective function to guide us on the number and specification of the composite regimes of fundamentals and essentially use a stopping rule when the model is no longer rejected. It is also worth noting that we used the SMM objective function to restrict several of the elements of the variance covariance matrix of fundamentals to be zero. The only non-diagonal element that made any improvement in the objective function is the covariance between inflation and capacity utilization, and we will comment on this further in Section 2.5 below. Details of the SMM procedure are available in Appendix 2.

2.2   Data Description

Our data sample runs from 1967 to 2008. The definitions of the fundamental series are as follows. Aggregate quarterly earnings for the economy are approximated as the operating earnings of S&P 500 firms, and these data are obtained from Standard and Poor's. Dividends for these firms, also obtained from Standard and Poor's, are used with the prices to compute returns. The other three fundamentals, the Consumer Price Index (CPI), Industrial CU and money (M1) are obtained from the Federal Reserve Board. Of the monetary aggregates, we find the growth in M1 real balances (which we refer to as money growth) to be most highly related with the other macroeconomic variables. A discussion of some other variables is in Section 4 of the online appendix.

Stock prices are obtained from S&P and P/E ratio is estimated as the equity value of these firms divided by their operating earnings. The time series of zero-coupon yields and returns on Treasury bonds of different maturities are obtained from the Fama-Bliss data set available at the University of Chicago. Options data are obtained from two sources. We obtain transactions data on S&P 500 index options from 1986:Q2 to 1996:Q1 from the CBOE. These data are no longer available from 1996:Q1, and therefore, we use data on these same options from Option Metrics from 1996:Q2 to 2008:Q3. It is important to note that Option Metrics provide the average of bid and ask prices at the end of each trading day, and not prices based on actual transactions. Prices at the beginning of each quarter are fitted with fundamental data available at the end of the previous quarter.

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2.3 Estimation Results for the Regime Switching Model

In this subsection, we briefly describe the results of the estimation of our model. The procedure in the Appendix 2 finally settles on N = 8 regimes. Although eight regimes seems like a large number, there are two important qualifications to be made: First, we should recall that we have four fundamental variables (inflation, real earnings, capacity utilization, and money growth). We estimate from the individual quarterly time series of 42 years that each fundamental variable has four distinct regimes. Had we assumed that these regimes were independent across variables, we would have ended up with 256 composite regimes. Our eight *composite* regimes accomplish our goal of inducing important low-frequency comovement across these four variables, without sacrificing much the fit for each individual series. Second, we use the entire time series of asset prices, namely the P/E ratio, short-term and long term bonds, and option prices, to estimate these regimes and the transition matrix. This is important as asset prices contain information on agents' beliefs about regimes that *could* happen, but haven't in the sample. As a simple example, a long deflationary period *did* happen in the 1930s, but its possibility only emerged again in the last decade. Asset prices in the interim period crucially depend on these unrealized regimes – a classic Peso Problem issue – and hence are informative of the number of regimes.

The fundamental composite regimes that we estimate are provided below, and investors' conditional probabilities of these regimes are in Figure 3.

Regime s^1: $(\beta = 1.5\%, \theta = 6.1\%, \rho = 1\%, \omega = 1.2\%)$. This is the "regular boom" regime of the economy. Inflation is low, earnings growth is strong, and the policy variables are just above their average levels, which are all conditions for stability in future fundamentals. Investors believed on *average* that this regime is the most likely to be in the sample period, and in particular during non-recessionary periods as classified by the NBER.

Regime s^2: $(\beta = 6.5\%, \theta = -5.2\%, \rho = 1\%, \omega = 5.3\%)$. This is "regular recession" regime. Inflation is at a medium level, earnings are shrinking, CU is above average, but money growth is very strong. The strong money growth is consistent with stimulative efforts by the central bank, but also with high demand for money that is pushing up goods prices. The filtered probability of this regime was at its maximum in the 1982 recession, at about 50 percent. In the past three recessions, the probability of this regime has been small.

Regime s^3: $(\beta = 6.5\%, \theta = 6.1\%, \rho = 8.7\%, \omega = 1.2\%)$. This is the "over-heating" regime. In this regime, earnings growth is still strong, while the other fundamentals warn of impending trouble. Inflation

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hits a medium level, CU is unusually tight, although money growth remains mild, likely as the central bank has not decided to intervene yet. The filtered probabilities suggest that this has been the second most likely regime in the sample, and fears of it have sporadically increased in most boom periods in the sample.

Regime s^4: ($\beta = 9.1\%, \theta = -5.2\%, \rho = -2.5\%, \omega = -5.7\%$). This is the "stagflation" regime. In this regime, fundamentals are at about their worst shape, with high inflation, low profit growth, low CU and very low money growth. The low money growth is consistent with attempts by the central bank to rein in inflation. Investors' filtered probability of this regime peaked around the 1981 recession, and did not fully subside until the end of the following recession in 1983. Notably, the belief of this regime increased to nearly 10 percent right before the 2008 financial crisis.

Regime s^5: ($\beta = 1.5\%, \theta = 7.7\%, \rho = 1\%, \omega = -3.1\%$). This is the "new economy" regime. Earnings growth is at its most rapid, inflation and capacity utilization are low, but money has tightened likely a reflection of the central bank's efforts to moderate growth. Investors' probability of this regime peaked at about 50% in the late 1990s, but crashed during the 2001 recession as investors' hopes of the new economy tanked. There was a mild increase of this probability in the boom period in the 2000s, which again tanked in the 2008 recession.

Regime s^6: ($\beta = -0.2\%, \theta = -5.7\%, \rho = -6.6\%, \omega = 5.3\%$). This is the "deflation" regime of the economy, in which earnings shrink at their most rapid rate in the cycle. CU is 6.6% below its historical average, and money growth is very rapid as the central bank attempts to stimulate growth. Investors' deflation expectations have spiked in the recessions of the current millennium, but were also high after the 1982 recession after the Fed's efforts to tame strong inflation expectations.

Regime s^7: ($\beta = 6.5\%, \theta = -5.2\%, \rho = -6.6\%, \omega = -3.1\%$). This is a "deep recession" regime, in which inflation and earnings growth are similar to those in the mild recession (regime 2), but CU is very low and money shrinks, which likely results as monetary policy is no longer effective in stimulating the economy. Investors' filtered probability of this regime was at its highest after the oil price induced recession in 1973, but has also been as high as 30% in the current recession. Combined with their high deflation probability in this period, investors' inflation uncertainty has been very high in this recession.

Regime s^8: ($\beta = 1.5\%, \theta = 6.1\%, \rho = -6.6\%, \omega = 5.3\%$). We call this the "low capacity boom" regime of the economy in which inflation and earnings growth are as in the regular boom regime (regime

1), however, the growth seems shaky since CU is very low and money growth is very strong likely as a result of very proactive stimulative efforts by the central bank. Investors' probability of this regime hit close to 30 percent in the recovery periods following recessions in the early 1970s and mid 1980s, and was even higher in 2000s prior to the most recent recession. The high money growth in this period is consistent with the easy credit regime that is oft cited as the cause of the increase in stock and house prices in this decade.

The uncertainty of investors is generated in large part by their estimations of transitions between regimes, which we show in the top and middle panels of Table 2. In the top panels we see how inflation interacts with earnings stability. In the boom regimes with low inflation (regimes 1,5 and 8), investors' estimate only a 1.2 percent chance of a recession in the next year, while in regime 3, when earnings are still booming, but inflation heats up to a medium level, the transition to a recession regime in the following quarter rises to about 8.5 percent. The role of low CU in affecting transitions can be seen by comparing the annual transitions in the regular (regime 2) and deep (regime 7) recession regimes to the deflation regime. Indeed, the risk of entering deflation rises from about 0.5% from the regular recession, to about 7.7 percent from a deep recession, when CU is extremely weak, which explains the spike in deflation fears in the two recessions of the current millennium (see Figure 3). The middle panel of the table, shows the 5-year transitions between regimes, which show the medium-term risks to fundamentals. Notable among these transitions, is the large persistence of the new-economy growth regime, which suggests that even after five years, investors expect to remain in that regime with a probability exceeding 92 percent. It is also relevant to point out that the estimated persistence of the deflation regime is the lowest among all regimes. This estimate likely arises from the fairly rapid recovery of industrial CU from its troughs, which we have seen in our sample that began in 1967.

2.4 Estimation Results for the Taylor Rule and State Price Density

We next turn to the parameter estimates determining the pricing kernel. As shown in Table 1, the interest rate rule parameters suggest that the real rate in the economy depends positively on both expected inflation and expected change in CU, $\alpha_\beta = 0.362$ and $\alpha_\rho = 0.257$. These estimates are similar to estimates of the Taylor rule in many other papers and indeed Taylor's own work suggested values of each parameter of 0.5. It is useful to remember that in our model we use industrial CU rather than the output gap used by Taylor, and the rates depend on the expected drifts of the variables rather than the variable realizations themselves.

The next line in Table 1 shows the prices of risk. Most notably, the prices or risk of the earnings shock, the CU shock, and kernel shock itself are all positive and large (around 0.3), however, the prices of risk of the inflation and money growth shocks are small. As seen in our regime specification, neither of these latter variables is consistently pro or countercyclical and the implied parameters suggest that investors do not consistently associate these shocks with periods of high marginal utility.

2.5 Model Fit to the Data

Using the time series of investors' regime probabilities in Figure 3 and the estimated parameters, we generate time series of model-implied expected fundamental growth and stock and Treasury bond prices in Figures 4 and 5. The fits of the model are reported in Table 3 (see Table 8 in the online appendix for similar out-of-sample results). The model expected growth rate explains 62, 16, 75, and 37 percent of the variation in the realized fundamental for inflation, earnings growth, capacity utilization, and real money growth, respectively. We note that our SMM procedure, which maximizes the likelihood of investors observing the historical fundamental processes, does not have an explicit prediction on the fitted *actual* fundamentals in each period, but instead characterizes expected fundamental growth. Therefore, these fits reflect not simply the accuracy of our model, but in addition, investors' estimates on the fraction of variation in fundamental growth that is related to shifts in trend growth rates as opposed to purely idiosyncratic variation. So, for instance, earnings growth is the most volatile fundamental, and the model explains the lowest amount of its variation, while CU is the smoothest, and the model explains most of its variation. Also note that the regression coefficients in the expectations regressions are between 1.3 and 2.2 so that actual fundamentals are more volatile than their expectations.

Figure 4 shows some interesting comovements of real money growth with the business cycle, which our model captures correctly. In the 1970s, money growth was tightened during recessions, while in the recessions of the 2000s, money growth was very rapid. Therefore, a simplified view on the pro or countercyclicality of money would be misleading, and is perhaps a source of some of the disillusionment with monetary aggregates. The different policy response in these recessions is likely in part determined by the different trends in CU in these recessions. In particular CU has the opposite trends as money growth, very tight in the 1970s and weak in the 2000s. It is important to note though that the model is not able to explain all the volatility in historical money growth in the 1979-1982 period when the Federal Reserve experimented with targeting money growth.

The ability of the model to replicate the fluctuations in stock and Treasury bond prices is displayed in Figure 5 and the fit statistics are reported in Table 3. The model explains most fluctuations in the

S&P500 P/E ratio, in particular the single digit P/Es in the late 1970s and early 1980s, the return to high teens levels in the 1980s, the rapid rise to over 25 in the late 1990s. and the decline in the 2000s again, overall explaining about 60 percent of the historical variation. The middle and bottom panel shows that the model is also quite successful in explaining most of the variation in the short rate and the slope of the term structure of Treasuries. The use of a Taylor-type rule is mostly instrumental in explaining the sharp dips in short rates following most of the recessions, although notably, the model short rate recovered more rapidly than the historical series in the early 1990s and the 2000s. In addition, the historical slope was higher in these two episodes than our model can explain. Overall, our model explains more than 50 percent of the variation in the short rate and slope. It is worth noting that our model does not rely on 'unobservable' factors that are used in the exponential affine term structure literature [see e.g. Dai and Singleton (2002)] to explain the fluctuations in these variables, but only on beliefs π_{it} which are fully tied down by the dynamics of fundamentals.

The final components of our SMM error term are the moments based on option prices that we discuss separately in Section 3. Using the scores of the likelihood function and the errors of the price and volatility variables, we evaluate the SMM objective function, which serves as an omnibus test statistic. The overall SMM objective function value, which has a chi-squared distribution with five degrees of freedom, is 10.47, implying a p-value larger than 5%, so we fail to reject our model.

3 Options' Indices and Monetary Policy

This section contains our main results: Before discussing the relation between options and monetary policy, we first show that the model fits well the ATMIV and P/C indices, and elaborate on their relation with several macro-economic control variables that have been used in the literature. We then turn to the model implications for options and monetary variables. We extend our analysis by providing out-of-sample results in the online appendix.

3.1 Explaining Time-Variation in ATM Implied Volatility

To guage the impact of fundamentals on ATMIV, the bottom panel of Table 2 computes the model's implied ATMIV when investors are 80 percent sure of being in each of the eight regimes respectively.[13]

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[13]An important caveat to note is that the model ATMIV is not a linear function of beliefs but instead is more directly associated with the uncertainty about the regimes. Therefore, the implications for intermediate beliefs should not be approximated by interpolating the ATMIV at the regime beliefs in the table.

It is immediately evident from the table that the model implies that implied volatility is generally coun-
tercyclical, being higher in regimes with negative earnings growth (regimes 2,4, 6, and 7). Its highest
levels occur during stagflation periods (regime 4) when investors' beliefs are the most reactive to in-
flation news. However, implied volatility can be high in strong regimes of the economy as well. In
particular in the new economy regime (regime 5), ATMIV is close to that in some recession regimes. In
this regime, strong economic growth raises investors' future earnings uncertainty causing high volatil-
ity. This observation is made in David and Veronesi (2009) for explaining the conditional positive
relation between P/E ratios and realized volatility in the late 1990s.

The historical and model-fitted ATMIV series are shown in the top panel of Figure 1 and some
regressions examining the fits are in Table 4. Our historical time series spans a long period of nearly
23 years that covers the recessions of 1991, 2001, and 2008, as well as unusual events such as the stock
market crash of 1987, the collapse of LTCM in 1998, and the bursting of the technology bubble in
2000. As seen in the figure, during each of these events implied volatility increased above 30 percent,
while its average over the sample is 18.5 percent. Our model implied volatility, which only builds in the
impact of macroeconomic uncertainty, follows closely the increase in data implied volatility during the
three recessions, and remains high in the post technology bubble period. Although the model is unable
to explain the surge in volatility during the 1987 crash or the LTCM episode, media commentary at or
around these episodes confirms that macro events were not the cause of these crises.[14]

It is also noteworthy that the model captures well the post-recession decline in implied volatility
from 1991 to 1996 and from 2001 to 2006 despite the fairly different macroeconomic conditions in
these recessions. While earnings growth rebounded after each recession, the 1991 recession had tight
CU and weak money growth, while the opposite conditions prevailed in the 2001 recession. In the
former recession, rising inflation was a concern, while in the latter, investors were concerned about
deflation. The unwinding of these conditions was therefore quite different in the two periods, but at the
end of these cathartic periods, investors' beliefs of regime 1 (the regular boom) increased to over 60
percent (Figure 3) and implied volatility hit lows in the 10-13 percent range. The model is also quite
successful in explaining the spike in implied volatility in the current recession, which started with the
fear of an increase in inflation to a medium level and an increase investors' probability of the economy
overheating (regime 3) and deep recession (regime 7), followed by the collapse of inflation and increase
in the fear of deflation in the second half of 2008. The model ATMIV hit about 50 percent at the end

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[14]Microstructure issues have been attributed to each of these two crises. Trading problems arose due to the breakdown in
market mechanisms by the large trades of portfolio insurers in 1987, while the shutdown of several markets simultane-
ously led to the severe liquidity problems in 1998.

of 2008, its highest level in the 23 year sample, although it was lower than the nearly 70 percent in the data.[15]

Table 4 contains formal statistics of the fit of our model for the data ATMIV series. Line 1 shows that the $\bar{R}^2$ of the simple regression of the data ATMIV on the model ATMIV for the full sample is 53 percent, and the beta coefficient is 0.91, which is very close to 1. Excluding the fourth quarter of 1987 from the regression increases the $\bar{R}^2$ by another 5 percentage points (not reported). As well known in the GARCH literature, most measures of volatility are persistent. Line 2 shows that the regression $\bar{R}^2$ of 32 percent of the data ATMIV series on its own lag. In line 3, we include both the model and the lag, and find an $\bar{R}^2$ of 58 percent, or an increase of about 5 percentage points over our model.

We next provide regression results for the five macroeconomic variables that we found significant individually in lines 4 through 8. These results show that ATMIV is a procyclical variable but none of them is able to explain much of the variation of ATMIV. Indeed, line 9 shows that together all these controls explain 36 percent of the variation in the ATMIV, significantly below that of our model. However, our model suggests, that while these variables are important determinants of implied volatility, they affect it jointly and in a nonlinear way, which our model captures through the combination of the expected discounted value of cash flows in our asset pricing formulae and the use of Bayes' rule. In addition, some variables, like inflation and money growth, are not significant in a linear regression, but their effect is embedded in our model ATMIV, through the joint regime specification with real fundamentals.

We finally consider the effects of combining the lags and the controls with our model ATMIV (line 10) and add the lagged data ATMIV (line 11). The $\bar{R}^2$ increase to 61% and 66% in the two cases, respectively. The 8% increase in explanatory power in line 10 over line 1 (model only) is likely the result of the highly parsimonious nature of our model, which nonetheless does explain most of the variation in the joint specification. The further 5 percentage point increase over line 10 in line 11 tells us that there are persistent economic forces that explain incremental amounts of variation in ATMIV, which are not in our model or controls. The information in the lag likely includes trading disruptions, which as mentioned above were particularly important at the time of the stock market crash and the LTCM failure. It remains a challenge to include such information into a model that already builds in the macroeconomic effects as in our model specification, which as noted explain about 80% of the predictable variation.

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[15]Undoubtedly, the selling pressure from the collapse of Lehman Brothers and troubles at other financial institutions worldwide had an impact of volatility in this period more than can be accounted for by our model. However, these failures were endogenous, and our model does suggest that there were greater fundamental stresses in this period than any other period in our sample.

3.2 Explaining Time-Variation in the Put-Call Ratio

In this section we study the model's ability to explain the put-call ratio (P/C), which we defined to be the ratio of implied volatilities of 5 percent out-of-the money put and call options.[16] To understand what forces affect the P/C ratio, it is useful to first look at Figure 6 that shows the densities of stock returns when investors are 80 percent certain of being in each of the eight regimes (the remaining regimes each have equal probability). The top (bottom) panel shows the densities for the boom (recession) regimes, which are regimes with expected positive (negative) earnings growth. As seen, the densities are negatively (positively) skewed for the boom (recession) regimes. These shapes naturally imply that the P/C is greater (smaller) than one for the boom (recession) regimes, the first step in understanding the puzzling pro-cyclical nature of P/C.[17]

The conditional values of the put call ratio and higher moments of the stock return distribution under the risk-neutral measure are given in the bottom panel of Table 2. The table also shows that the densities at these eight beliefs are all highly non-Gaussian with skewness coefficients of between -3.3 and 1.4, and kurtosis coefficient of between 4.3 and 21.5. The non-Gaussianity partly stems from the jumps in returns due to jumps in earnings, and partly from the continuous shifting of the instantaneous moments of the return distribution. Indeed, because the jump intensity in earnings is constant, *all* the time variation in the densities arises from the shifting moments. Its is also interesting to note that the sign of the skewness in each of the regimes can be calculated quite easily by looking at the sign of covariance between stock returns and stock variance in each regime from its closed-form expression in (23). As seen in Table 2, the sign of the conditional correlation mostly matches the sign of conditional skewness in each regime.

The intuition is as follows: By Bayes' Law, in the boom regimes, investors revise their beliefs more rapidly when they receive negative shocks to earnings than positive shocks, leading to more volatility with negative outcomes, which is a negative correlation between returns and variance. The higher volatility with bad news gives a negative sloped implied volatility curve or a put call ratio larger than one. The opposite holds true for the low earnings growth rate regimes.

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[16]To ensure intertemporal consistency of put-call ratios we follow Bates (1991) and set $K_{\text{put}} = S_t e^{(r-\delta)\tau}/1.05$ and $K_{\text{call}} = S_t e^{(r-\delta)\tau} \times 1.05$. In the online appendix, we compare this measure to a time series of the 10% OTM put-call ratio.

[17]Notice from the filtered probabilities, that investors were never 80 percent certain of any of the regimes in our sample, so that the model P/Cs were never as extreme as reported for these regimes. In particular the model P/C was almost always positive in our sample. It is also important to note, that the P/C is not linear in the beliefs, and values for intermediate beliefs will not be well approximated by interpolation. In particular we find that that the P/C at intermediate regimes is outside of the range in Table 2.

The bottom panel of Figure 1 shows the data and model P/Cs. As can be seen, both series are almost always greater than one (each is less than one only once in the sample) and the model P/C tracks the data P/C quite closely. The figure shows that the P/C is procyclical, falling in each NBER-dated recession. Comparing to the top panel, gives the surprising stylized fact that the ATMIV and the P/C are negatively related. In particular, in 1995 when the ATMIV hovered around lows of near 10 percent, the P/C hovered above 1.6. Similarly, around 2006, when the ATMIV was again around 10 percent, the P/C ratio was again above 1.4. In and after the three recessions in the sample, when the ATMIV rose above 30 percent, the P/C fell below 1.2. We return to the issue of negative correlation between ATMIV and P/C below.

Line 1 of Table 5 provides the simple regressions of the data P/C on the model P/C. As seen, the fit is very solid with a statistically insignificant alpha coefficient, a beta of 0.72 (not different statistically from 1) and $\bar{R}^2$ of 45.4 percent. Line 2 reports that the lagged P/C explains a similar $\bar{R}^2$ of 45.8 percent, but of course does not provide us intuition on the underlying economic forces driving the P/C. When both model and lag are included, each of the variables is statistically significant, and the explanatory power increases by 8 percentage points, implying that the lag has some information over and above that of our model.

Line 4 to 8 provide regression results for the five the five macroeconomic variables that we found significant for the ATMIV, and are inputs to our model. In addition, we also report in Line 9 and 10 results for two market sentiment measures advanced by Han (2008): The first, a"trader sentiment," is the net long position of large speculators on S&P 500 index futures obtained from the *Commodity Futures Trading Commission's Commitment of Traders Report*. The second, an "investor sentiment" is the bull-bear spread (proportion of traders bullish less bearish) in *Investor's Intelligence's* survey of investment newsletter writers.[18]

Line 10 include all the macroeconomic controls and the sentiment variables in a joint regression in line 11, and find an adjusted R^2 of only about 21.9 percent, far below our model. This reinforces the view that the macroeconomic variables affect the P/C nonlinearly. Finally, using all the variables along with the model and lagged P/C, leads to a very small increase in explanatory power over using just the model and lag.

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[18]Both variables are measured within a week prior to the options trades. These measures are alternative measures of fear in the market and are thus compelling control variables for our measures of downside risk obtained from asset prices. While Han (2008) suggests that the significance of these measures supports a behavioral view of asset prices, we note that they could be consistent with a rational model of heterogeneous learning about the regimes of fundamentals such as in David (2008a).

3.3 Understanding the Relation between Options and Monetary Policy

As noted in the introduction, and shown in the left panels of Figure 2, shocks to both indices ATMIV and P/C lead to a sustained impact on future monetary policy, while monetary policy measures do not impact on the two indices. More specifically, a shock to ATMIV leads to a decrease in future interest rates while a shock to the P/C leads to an increase in future interest rates. Is the model able to replicate such results?

The right panels of Figure 2 show the analogous impulse responses when the same vector autoregression is computed on the model fitted ATIMIV and P/C indices and policy variables. Comparing to the panels on the left, we find that our fitted model has exactly the same relationships as in the data: Both fitted ATMIV and P/C lead to sustained effects for up to eight quarters on policy variables, while the reverse impulses are statistically insignificant. Having a fully dynamic model that can replicate the historical impulses is useful since it can rule out certain channels for the effects. In particular, in our model the central bank and investors have exactly the same information. So, the reason why policy follows variation in option prices is not due to differences in information of the two groups of agents.

What economic channel then explains the one-directional impulses? The bottom panels of Figure 2 suggest that the two option indices have a sustained effect on industrial CU. In particular, in our model an increase in ATMIV occurs because of increases in the uncertainty about the current economic regime. The impact of higher uncertainty on future industrial CU then easily follows from the real options literature, whereby firms delay the abandonment of plants and factories in the face of higher uncertainty, instead choosing to operate them at less than full capacity [see e.g. Bloom (2009)]. Because the central bank has the same information as the investors, the learning-based Taylor rule then implies that shocks to uncertainty (ATMIV) have a sustained future impact on interest rates.

Our model also uncovers another basic economic mechanism that explains why a shock to the downside-risk index P/C is correlated with a future increase in CU. The reason is that in our model, a positive shock to P/C occurs when investors increase their beliefs to be in an expansionary phase of the economy (see Figure 6). Because of symmetric information, such beliefs also affect the central bank's learning-based Taylor rule. Thus, the increase in P/C is correlated with an increase in expected future CU, which then leads to a tightening of monetary policy through higher interest rates.

In Figure 2 we only report impulse responses options' indices to interest rates and capacity utilization. What about the relation between these indices and money growth? We did compute the impulse responses for the effects of the two options' indices on money growth and did not find significant

responses in either direction. This (negative) result is interesting as well. The next subsection further develops the relation between option's indices and monetary policy variables, and explains these findings.

Before concluding this section we add two caveats to these results. First, the impulse response functions are estimated for the options subsample, when the relationship between CU and these options' indices happen to be monotonic. As we show next in Section 3.4, over the full sample, the relationship is nonlinear and impulse responses may well have the opposite signs for the first part of our sample, when tight CU lead to monetary policy tightening. Second, the impulses studied here are at a quarterly frequency, and we do not rule out that the central bank can have significant impulses at shorter horizons in higher frequency data.

3.4 The Nonlinear Relations between Options' Indices and Macro-Policy Variables

The results in Section 3.3 calls for further investigation of the relation between option prices and policy variables. In this subsection, we document strong nonlinear relationships between macro-policy variables and option prices. In addition, we show that the money growth affects both ATMIV and P/C more noticeably in periods when policy makers stimulate the economy.

The top panels of Figure 7 show the relationship between the ATMIV (left panel) and P/C (right panel) and expected real money growth. The x-axis in these plots has the expected money growth of investors which we have shown in Figure 4.[19] As can be seen, the ATMIV has an V-shape, with its minimum at close to 1.5 percent real money growth. The fitted curve shows a higher ATMIV when money growth is tight (around 24 percent) as opposed to when its accommodative (around 20 percent), and dips to around 16, when money growth is neutral. By comparing the estimated ATMIV and P/C values in the different regimes in Table 2 we can see how our model generates the V-shape. Consider regimes 2 and 4 for example, which have the highest ATMIV. Real money growth is strongly positive in regime 2, and is strongly negative in regime 4. In regime 2, money growth is rapid as the central bank attempts to stimulate the economy in the regular recession, and the conditional relationship between money growth and ATMIV is negative. In regime 4, volatility is high even as the central bank attempts to rein in high inflation even though real growth is weak and the conditional relationship is positive. Similarly, in the new economy growth rate regime, the ATMIV is high as discussed, and money growth is tight, as the central bank attempts to rein in lofty expectations of real growth. On the low end, in

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[19]We plot this relationship rather than the one between the ATMIV and realized money growth, since the latter is fairly noisy and provides a less precise relationship with a similar shape.

the stable regime 1 with low inflation and high real growth, real money growth is one percent and the ATMIV is the lowest.

The bottom panels of Figure 7 show the relationship between ATMIV (left panel) and P/C (right panel) calculated from our model using the filtered beliefs in Figure 3 and realized CU (which is a fairly smooth series). As seen, the ATMIV again has a V-shape relationship, while the P/C has an inverse V-shape. We plot the relationship for the full sample for our model since de-meaned CU never rose above 3 percent in the 1986-2008 period, when we have options data. In this period, the relationship between the data ATMIV and CU is negative, while that with the P/C is positive. In the period of the options data, higher CU was taken as good news for fundamentals, which lowered uncertainty and the ATMIV and raised the P/C. For the full sample, which includes periods with very high CU, the relationships are non-monotonic as described above, since during periods of high CU, an increase in CU increased uncertainty about future fundamentals and had the opposite effects on ATMIV and P/C. The estimated model-implied ATMIV and P/C in the alternative regimes in Table 2 also show this relationship, as high ATMIV can result in recessions with high CU (regime 3) or low CU (regime 7). Once again, fundamentals' uncertainty is negatively related to P/C so that it has an inverse V-shape relation with CU (bottom right panel of Figure 7).

We end this discussion on the nonlinear relation between macro-finance variables and options indices by studying their conditional relation during stimulative periods. We define stimulative periods as those where the 3-month Treasury Bill Yield is below the annualized inflation (CPI) rate. In the sub-sample where we have options data (1986:2 – 2008) there are 20 quarters that we characterize as stimulative. These are periods of extreme stress in the market and it is of interest to study the response of the stock markets to money growth in these times. The left panel of Figure 8 shows that in this period ATMIV and money growth were negatively correlated. In the right panel we plot our model ATMIV and expected money growth and find a negative correlation which is somewhat stronger. The negative relation tell us that the actions of the central bank to boost money growth in such periods lowered the uncertainty and volatility in stock market, and our regime-switching model captures well this conditional relation. This role of monetary policy in reducing market uncertainty in stressful times is obviously not evident in simple linear regressions and could be the responsible for the general disillusionment with monetary aggregates in the literature (see e.g. Volcker (1977) and the more recent New-Keynesian literature cited in the introduction). Our results also provide direct evidence from the options markets on the effectiveness of monetary policy during "stimulative" periods.

4 Additional Properties of ATMIV

In this section we discuss features of observed option prices that are not directly fitted by our empirical methodology. The ability of our model to replicate these additional features provides further support for the economic mechanism that determines option prices in our model. As we will see the key variable in the model that enables it to explain these additional facts is the volatility of stock volatility, and we will end the section by providing its determinants.

4.1 The Volatility of ATM Volatility

In the previous section we saw that our model ATMIV was able to explain about 53 percent of the variation in the data ATMIV. In the model, the implied volatility is to a large part determined by the endogenous volatility of stock prices, which increases during periods of greater investor and central bank uncertainty. Looking again at the top panel of Figure 1, we see that during episodes of high volatility around the three NBER dated recessions in the options subsample, ATMIV also fluctuated by large amounts. The positive relation between volatility and the volatility of volatility is noted in Jones (2003) who further notes that it cannot arise in the Heston (1993) stochastic volatility model, which has been the workhorse of the option pricing literature. To obtain the level dependence of volatility, Jones (2003) proposes a generalization of the Constant Elasticity of Variance (CEV) model of Chan, Karolyi, Longstaff, and Sanders (1992). One drawback of the volatility processes in such models is that they do not satisfy global growth and Lipschitz conditions, which are commonly used sufficient statistics for a number of important results. In contrast, our model, which satisfies these two regularity conditions (see Proposition 4), is able to provide an economic explanation of the positive relation between volatility and the volatility of volatility.

Indeed, the top panels of Figure 9 show the scatter plots of implied volatility and absolute changes in implied volatility for the data and model series. Both show a positive association of similar magnitude between these variables with correlations of 41 percent and 49 percent, respectively, and these correlations are statistically significant. The economic explanation offered from the model can be readily seen in (10). In particular, the Bayesian learning mechanism that drives volatility in our model implies that investors revise their beliefs faster during periods of high uncertainty as they have low confidence in their estimates of the current regime of the fundamentals.

If the mechanism implied by the model is correct, we should see a similar positive association between implied volatility and absolute changes in the volatility of stock returns. We construct a time

series of the model's volatility-of-volatility (VV) using expression (22) and evaluate it at each date using the filtered beliefs in Figure 3. The scatter plot of absolute changes in implied volatility (data and model) with this series are shown in the bottom panels. As seen, the model volatility of volatility is highly correlated with both the data and model absolute changes in implied volatility with correlations of 34 and 61 percent respectively. Note that the model series measures the ex-ante volatility of volatility at each date and is compared to the ex-post realized absolute changes in ATMIV and our model predicts a positive but not one-to-one association between these variables. This is highlighted by the fact that the correlation between these variables is only about 61 percent correlation even when both variables are generated by our model.

4.2 The Implied Volatility Premium

The volatility premium is an ex-ante measure of the stock market volatility forecast of investors' priced into options relative to a volatility forecast under the objective (P) measure, and is currently one of the most actively researched statistics in empirical option pricing. If volatility is systematically positively related to investors' pricing kernel (marginal utility of consumption), then as a priced factor it carries a negative risk premium, which leads to a higher forecast of volatility under the Q measure, or a positive volatility premium. The strong evidence that volatility is countercyclical, which we have already discussed in Section 3, suggests that the volatility premium should be positive.

The empirical finance literature now has more than one operational definition of this quantity. The first, which we call the *implied volatility premium* (IVP) is defined as the difference between at-the-money implied volatility and a forecast of future volatility to the maturity of the option under the objective (P) measure. The forecast under the P measure is constructed for specific volatility models. A second definition, which we call the *forecast volatility risk premium* (FVRP), simply takes the difference in forecasts of future volatility with the *same* structural model but under the two measures.[20]

We now study the ability of our model to explains a significant amount of time variation in the IVP. To construct a data based ex-ante IVP series we need forecasts of realized volatility, which we discuss first. We construct two forecasts using well established results in the volatility forecasting literature, which we discussed in Section 3.1. The specifications we use are similar to those in Drechsler and Yaron (2010). The first specification for our sample from 1986:Q2 to 2008:Q4 is a regression of

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[20]In addition, Bollerslev, Tauchen, and Zhou (2009) use a measure of an ex-post volatility premium that takes the difference between implied volatility and realized volatility to predict future stock returns. In early work on this issue, Canina and Figlewski (1993) and Christensen and Prabhala (1998) first reported that implied volatility is a useful albeit biased forecaster of future realized volatility.

realized volatility on its one-quarter lag, the lagged P/E ratio, and lagged returns on the S&P 500 index in periods when they are negative, which we call Projection 1. The results of this regression are:

$$\text{Vol}(t+1) \quad = \quad 3.291 + 0.548 \,\text{Vol}(t) + 0.212 \,\text{P/E}(t) \quad - \quad 0.470 \,\text{Ret}^-(t); \ \bar{R}^2 = 0.238 \quad (26)$$
$$[1.170] \quad [2.122]^* \qquad [1.237] \qquad\qquad [-2.004]^*$$

where $\text{Vol}(t+1)$ is the volatility realized in quarter $t+1$, which we define as the square root of the sum of squared S&P 500 index returns in the quarter, $P/E(t)$ is the S&P 500 price-to-operating earnings ratio, and $\text{Ret}^-(t)$ is the return on the S&P 500 index in periods when it is negative. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The regression $\bar{R}^2$ improves to 47 percent for the post-crash subsample starting in 1988:Q2.[21] The second forecast is similar to that constructed in Drechsler and Yaron (2010), which used the lagged implied volatility to forecast realized volatility and adjusts for the forecast bias, which we call Projection 2. The results of this regression are:

$$\text{Vol}(t+1) \quad = \quad 2.334 \quad + 0.019 \,\text{Vol}(t) + 0.709 \,\text{I. Vol}(t); \qquad \bar{R}^2 = 0.393 \qquad (27)$$
$$[1.659]^* \quad [0.165] \qquad\quad [4.986]^*$$

The $\bar{R}^2$ of this regression improves to over 63 percent if the stock market crash is excluded. It is important to note that for each projection we use non-overlapping data by constructing one-quarter ahead volatility forecasts at the quarterly frequency so that the t-statistics are more reliable.

Using the difference between the implied volatility at the beginning of the quarter t and the expectation of quarter t realized volatility based on data available at the end of quarter $t - 1$, we form the ex-ante volatility premium series. Using the two alternative forecasts of realized volatility, we have two measures of the ex-ante volatility premium, which we display in the top and middle panels of Figure 10. The two series have a correlation of 88 percent, and their means are very similar at 2.7 and 2.9 percent respectively.

We similarly construct a model based IVP series by taking the difference between the model implied volatility analyzed in Section 3.1 and the model forecast of volatility under the P-measure using simulation methods as described in equation (47) in Appendix 2. The model IVP, IVP^M, is also displayed in Figure 10. The mean of the IVP^M is 4.4 percent, which is not statistically significantly different from

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[21]Most papers on volatility premium studies exclude the stock market crash from their samples. We find that the ability of our model in explaining the time variation in the volatility premium is not sensitive to the exclusion of the crash.

the empirical IPVs since their sample standard deviations are about 6 percent. However, our main goal is to understand the time-variation in the volatility premium, which we discuss next.

By regressing the volatility premium from projections 1 and 2, we get the following fits:

$$\text{IVP}^1(t) \;=\; 1.091 \;+\; 0.691\,\text{IVP}^M(t); \quad \bar{R}^2 = 0.163. \tag{28}$$
$$[0.759] \quad [3.228]^*$$

$$\text{IVP}^2(t) \;=\; 1.158 \;+\; 0.571\,\text{IVP}^M(t); \quad \bar{R}^2 = 0.122. \tag{29}$$
$$[1.151] \quad [2.958]^*$$

As can be seen, the intercept terms are small and statistically insignificant, and the betas of the regression are instead strongly significant, and the model explains about 16 percent of the variation in the data IVP from Projection 1, which is economically significant. As seen in the plot, both data and model volatility premiums are countercyclical, and are higher in periods of higher volatility. We emphasize that the fact that the model-IVP explains the measured IVP in the data is not hardwired in our estimation procedure, in which filtered beliefs dynamics are determined only by fundamentals and policy variables, and the model parameters were estimated to fit ATMIV and P/C. Thus, it is indeed comforting to see that our model generates the proper dynamics of the IVP in this effectively out-of-sample test of the model. The fit for the IVP from Projection 2 is similar, although the explanatory power is lower at 12 percent. Looking at the middle panel, the qualitative feature of the model fit is very similar though.

As is well known, the implied volatility premium is a measure of the overall value of the option, which includes the risk premium for volatility fluctuations as well as other non-Gaussian aspects of the stock return distribution. Our model can shed light on the source of variation of IVP. In particular, first, we decompose the model-IVP into a premium for the forecasted future volatility variability (FVRP) and a residual that is due to other non-Gaussian components. The FVRP is simply the difference in the forecasts under the Q and the P measure, using the model.[22] Constructing the series conditional on beliefs for our full sample we find an average FVRP of 1.75 percent, so that the FVRP comprises about 40 percent of the total IVP. The model IVP and FVRP are plotted in the third panel of Figure 10, and are highly correlated (correlation coefficient of 93.8 percent).

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[22]The Q forecast of our model is constructed using the same methodology as the P forecast and is shown in (46) in Appendix 2.

Second, we show in Section 5 of the online appendix that the model-IVP is largely explained by the model's volatility-of-volatility (VV). Therefore, in times when investors' assess that volatility will fluctuate more, the premium for volatility fluctuations is higher. We show further that the VV explains nearly 50 percent of the variation in the FVRP and a much smaller amount of variation in the jump component of the IVP.

4.3   What Drives the Volatility of Stock Volatility?

In Section 4.1 we show that VV explains the positive comovement of ATMIV and its absolute changes. The online appendix also shows that VV explains the dynamics of the model IVP. We now relate the VV to the fundamental uncertainties in our model.

The VV is endogenously generated by the learning process in our model and is a result of the the nonlinear updating inherent in Bayes' law. In periods where investors are more uncertain about fundamentals, they put less weight on their current beliefs and more weight on incoming news so that revisions to beliefs and hence stock market volatility are higher. This implies that the VV should be directly related to measures of investors' uncertainty.

To see the relationship explicitly, we define earnings uncertainty (and analogous definitions for the other fundamentals) as

$$\text{EU}\left(t\right) = \sqrt{\sum_{i=1}^{N} \pi_i\left(t\right) \left(\theta_i - \overline{\theta}\left(t\right)\right)^2}. \tag{30}$$

We plot the time series of VV and the four fundamental uncertainties in Figure 11, and Table 6 provides results on the simple OLS regressions of the VV on these variables for options subsample (1986:Q2 – 2008). While all four uncertainties are strongly countercyclical, we find that earnings uncertainty has been the single most important driver of the VV explaining more than 70 percent of its variation. Money growth uncertainty also explains a significant amount of variation in the VV, while the other two uncertainties have been of minor importance. As noted in Section 3.4, CU itself has an impact on ATMIV and P/C, but as seen here, uncertainty about CU is not a significant driver of VV. This result arises because CU is a fairly smooth process so that uncertainty about it does not drive major changes in investors uncertainty. Money growth uncertainty turns out to be an important driver as investors likely perceive changes in money as an important signal of the view of the central bank about the regime of the economy.

Taken together the four fundamental uncertainties explain more than 80 percent of the variation in VV. Relatedly, Beber and Brandt (2009) show that volatility in stock and bond markets declines faster

following periods of high macroeconomic uncertainty extracted from the economic derivative markets over a shorter sample from 2002-2005.

5 Conclusion

Option prices provide key forward looking information on investors' expectations, and market attention is often focused on two uncertainty measures from options, the at-the-money implied volatility (ATMIV) and the ratio of implied volatilities of out-of-the-money puts and calls (P/C). The former is measure of market turbulence, while the latter is a measure of downside risk. We show that both measures are empirically relevant for monetary policy, but in opposite direction: a positive shock to ATMIV leads to a decline in future rates, while the opposite is true for a positive shock to P/C.

Standard option pricing models use exogenous stock prices and their volatilities that are unrelated to fundamentals, and are hence unable to identify specific economic factors that can explain the variation of options' uncertainty measures and their impact on monetary policy. We instead provide a model in which stock, bond, and option prices, are functions of investors' beliefs of the joint states of macroeconomic and policy fundamentals through a forward-looking Taylor rule. The model is able to shed light on the counter (pro) cyclicality of the ATMIV (P/C), is able to explain about half their time series variation, and their compelling nonlinear relations with policy variables. In particular, the AT-MIV (P/C) has a (inverse) V-shape with expected money growth and capacity utilization. The model's ability to explain the time-series properties of these options' indices is based on its inherent Bayesian learning framework in which volatility is high during periods of greater uncertainty, and bad news leads to sharper downward revisions of beliefs in good times.

Our analysis also shows that investors' uncertainty in the options market has real economic consequences, which is tempered by the efforts of the central bank to smooth fluctuations. In particular, these specific options' measures are able to predict future movements in interest rates. The model also explains that the relationship between the options' indices and money growth is specially strong in periods of extreme stress when the central bank follows a stimulative policy by keeping the short rate below the inflation rate. Thus our reduced form model for equilibrium fundamental processes suggests some further support for the Taylor type rules, but also some additional factors to be worked on in future macro research such as the direct impact of uncertainty on interest rates and the role of money in monetary policy, which has been conspicuously absent in recent modeling.

The fitted dynamic updating of investors' beliefs have implications for the volatility of stock market variance and additional properties of options prices. We use these additional predictions of the model, which were not used in its estimation, to provide further support for our model. In particular, we show our fitted model is able to explain the positive correlation between ATMIV and absolute changes in ATMIV (a feature that is not consistent with standard option pricing models) and additionally is able to explain an economically significant amount of variation in the implied volatility premium. The model's implied volatility premium is driven to a large extent by the risk premium for volatility shocks and to a lesser extent by the fatness of tails created by the continuous shifting of moments of the return distribution from the Bayesian updating.

An important caveat is that the model, which structurally estimates the impact of investors uncertainty about the macroeconomy on options, is unable to explain some important surges in these indices at times when microstructure liquidity issues have roiled markets,such as the crash of 1987 and the collapse of LTCM in 1998. It is relevant to note that our empirical methodology, which uses information in fundamentals as well as prices, does not provide false alarms about macroeconomic problems in these episodes that were clearly not macro related. It also mitigates concerns about overfitting all observed variations in asset prices. Still, it remains a challenge to include microstructure information into a model that already builds in the macroeconomic effects as in our model specification.

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Appendix 1

For proving Proposition 1 we will need the following lemma.

Lemma 2 *Given the process of earnings in (3) and the SPD in (8), over a small interval of time Δ we have*

$$\mathbb{E}\left[\frac{M_{t+\Delta}E_{t+\Delta}}{M_t E_t}|\nu_t = \nu_i\right] = e^{[\theta_i - \phi_i - \sigma_M \sigma'_E + \kappa(\xi_3 - \xi_1 - \xi_2)]\Delta} + o(\Delta),$$

where $\xi_3 = e^{\mu_1 + \mu_2 + 0.5\,(\sigma_1 + \sigma_2)^2} - 1$.

Proof. From (3) and (8) we have

$$\frac{E_s}{E_t} = \exp\left(\int_t^s [\theta_u - \kappa\xi_1 - 0.5\,\sigma_E\sigma'_E]\,du + \sigma_E(W_s - W_t) + \sum_{j=L_{t+1}}^{L_s} Y_{1j}\right)$$

$$\frac{M_s}{M_t} = \exp\left(\int_t^s [-\phi_u - \kappa\xi_2 - 0.5\,\sigma_M\sigma'_M]\,du - \sigma_M(W_s - W_t) + \sum_{j=L_{t+1}}^{L_s} Y_{2j}\right).$$

Multiplying the two equations we have

$$\frac{E_s M_s}{E_t M_t} = \exp\left(\int_t^s [\theta_u - \phi_s - \sigma_E\sigma'_M - \kappa(\xi_1 + \xi_2) - 0.5\,(\sigma_E\sigma'_E + \sigma_M\sigma'_M)]du + (\sigma_E - \sigma_M)(W_s - W_t)\right)$$

$$\times \exp\left(\sum_{j=L_{t+1}}^{L_s} Y_{1j} + Y_{2j}\right).$$

Now for a small interval of time Δ and the fact that jumps in the drift processes and L_t are independent of each other and each occurs with probability of order $O(\Delta)$, we have

$$\begin{aligned}
\mathbb{E}[\frac{E_{t+\Delta}\,M_{t+\Delta}}{E_t M_t}|\nu_t = \nu_i] &= e^{[\theta_i - \phi_i - \sigma_E\sigma'_M - \kappa(\xi_1+\xi_2)]\Delta} \cdot \mathbb{E}\left[e^{\sum_{j=L_{t+1}}^{L_{t+\Delta}} Y_{1j}+Y_{2j}}\right] \\
&= [1 + (\theta_i - \phi_i - \sigma_E\sigma'_M - \kappa(\xi_1 + \xi_2))\Delta][1 - \kappa\Delta + \kappa\Delta(1+\xi_3)] + o(\Delta) \\
&= 1 + [\theta_i - \phi_i - \sigma_E\sigma'_M + \kappa(\xi_3 - (\xi_1 + \xi_2))]\Delta + o(\Delta) \\
&= e^{[\theta_i - \phi_i - \sigma_M\sigma'_E + \kappa(\xi_3 - \xi_1 - \xi_2)]\Delta} + o(\Delta),
\end{aligned}$$

as claimed. Note in the first equality above we have used the independence property of the drift process and the jump process, while in the second we have used the definition of $e^x = 1 + x + x^2/2!\cdots$. ∎

Proof of Proposition 1: The P/E ratio at time t satisfies

$$\begin{aligned}
\frac{P_t}{E_t} &= \mathbb{E}\left[\int_t^\infty \frac{M_s\,E_s}{M_t\,E_t}\,ds|\mathcal{F}_t\right] \\
&= \sum_{i=1}^N \pi_{it}\mathbb{E}\left[\int_t^\infty \frac{M_s\,E_s}{M_t\,E_t}\,ds|\nu_t = \nu_i\right] \equiv \sum_{i=1}^N \pi_{it}V_{it}.
\end{aligned}$$

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Let $\hat{\theta}_i = \theta_i - \phi_i - \sigma_M \sigma'_E + \kappa(\xi_3 - \xi_1 - \xi_2)$. Using Lemma 2 to evaluate the expectations over a time interval Δ, we have

$$
\begin{aligned}
V_{i,t} &= \mathbb{E}\left[\int_t^{t+\Delta} \frac{M_s E_s}{M_t E_t}\, ds\,|\nu_t = \nu_i\right] + \mathbb{E}\left[\frac{M_{t+\Delta} E_{t+\Delta}}{M_t E_t} \int_{t+\Delta}^{\infty} \frac{M_s E_s}{M_{t+\Delta} E_{t+\Delta}}\, ds\,|\nu_t = \nu_i\right] \\
&= \int_t^{t+\Delta} e^{\hat{\theta}_i ds} + e^{\hat{\theta}_i \Delta}\mathbb{E}\left[\int_{t+\Delta}^{\infty} \frac{M_s E_s}{M_{t+\Delta} E_{t+\Delta}}\, ds\,|\nu_t = \nu_i\right] \\
&= \frac{e^{\hat{\theta}_i \Delta} - 1}{\hat{\theta}_i} + e^{\hat{\theta}_i \Delta}\left[(1 + \lambda_{ii}\Delta)V_{i,t+\Delta} + \sum_{j \neq i} \lambda_{ij}\Delta V_{j,t+\Delta}\right].
\end{aligned}
$$

Since $V_{i,t}$ is time homogeneous, we have $V_{i,t} = V_{i,t+\Delta} = C_i$. Now collecting terms and taking the limit as $\Delta \to 0$, we get

$$
\begin{aligned}
C_i \frac{1 - e^{\hat{\theta}_i \Delta}}{\Delta} &= \frac{e^{\hat{\theta}_i \Delta - 1}}{\hat{\theta}_i \Delta} + e^{\hat{\theta}_i \Delta}\left[\lambda_{ii} C_i + \sum_{j \neq i} \lambda_{ij} C_j\right] \\
-\hat{\theta}_i C_i &= 1 + \left[\lambda_{ii} C_i + \sum_{j \neq i} \lambda_{ij} C_j\right].
\end{aligned}
$$

In vector form we can write this equality as

$$
\left(\text{Diag}(-\hat{\theta}) - \Lambda\right) C = \mathbf{1}_N,
$$

whose solution is $C = A^{-1} \cdot \mathbf{1}_N$, as in the statement of the proposition. ∎

For proving Proposition 3 we will use the algebraic result stated in the following lemma.

Lemma 3

$$
\frac{\partial \overline{\theta}^{\circ}}{\partial \pi_i} = \frac{C_i\left(\theta_i - \overline{\theta}^{\circ}\right)}{\left(\sum_j \pi_j C_j\right)}.
$$

Proof of Lemma 3:

$$
\begin{aligned}
\frac{\partial \overline{\theta}^{\circ}}{\partial \pi_i} &= \frac{\partial\left(\frac{\sum_j \pi_j C_j \theta_j}{\sum_j \pi_j C_j}\right)}{\partial \pi_i} = \frac{C_i \theta_i \left(\sum_j \pi_j C_j\right) - C_i \left(\sum_j \pi_j C_j \theta_j\right)}{\left(\sum_j \pi_j C_j\right)^2} \\
&= \frac{C_i \theta_i}{\left(\sum_j \pi_j C_j\right)} - \frac{C_i \left(\sum_j \pi_j C_j \theta_j\right)}{\left(\sum_j \pi_j C_j\right)^2} = \frac{C_i \theta_i}{\left(\sum_j \pi_j C_j\right)} - \frac{C_i \overline{\theta}^{\circ}}{\left(\sum_j \pi_j C_j\right)} \\
&= \frac{C_i\left(\theta_i - \overline{\theta}^{\circ}\right)}{\left(\sum_j \pi_j C_j\right)},
\end{aligned}
$$

which completes the proof. ∎

Proof of Proposition 3: Let the second term in the variance equation be $V_2 = (\bar{\nu}^\circ - \bar{\nu})'(\Sigma\Sigma')^{-1}(\bar{\nu}^\circ - \bar{\nu})$. Then, using Lemma 3 on each element of the drift vector ν we have

$$\frac{\partial V_2}{\partial \pi_i} = 2\left[\frac{C_i(\nu_i - \overline{\nu}^\circ)}{\sum_j \pi_j C_j} - \nu_i\right]'(\Sigma\Sigma')^{-1}(\bar{\nu}^\circ - \bar{\nu}).$$

Then, using the volatilities of the beliefs process in equation (11), we have $dV_2 = \mu_{V,2}dt + \sigma_{V,2}$, where

$$
\begin{aligned}
\sigma_{V,2} &= \sum_i \frac{\partial V_2}{\partial \pi_i}\sigma_i \\
&= 2\sum_i \pi_i\left[\frac{C_i(\nu_i - \overline{\nu}^\circ)}{\sum_j \pi_j C_j} - \nu_i\right]'(\Sigma\Sigma')^{-1}(\bar{\nu}^\circ - \bar{\nu})(\nu_i - \bar{\nu})'\Sigma^{'-1} \\
&= 2\sum_i [\pi_i^\circ(\nu_i - \overline{\nu}^\circ) - \pi_i\nu_i]'(\Sigma\Sigma')^{-1}(\bar{\nu}^\circ - \bar{\nu})(\nu_i - \bar{\nu})'\Sigma^{'-1}.
\end{aligned}
$$

Similarly, let the third term in the variance equation be $V_3 = 2\left[(\overline{\theta}^\circ - \overline{\theta}) + (\overline{\beta}^\circ - \overline{\beta})\right]$. Then we have $dV_3 = \mu_{V,3}dt + \sigma_{V,3}$, where

$$
\begin{aligned}
\sigma_{V,3} &= \sum_i \frac{\partial V_3}{\partial \pi_i}\sigma_i = 2\sum_i \frac{\partial[(\overline{\theta}^\circ - \overline{\theta}) + (\overline{\beta}^\circ - \overline{\beta})]}{\partial \pi_i}\sigma_i \\
&= 2\sum_i \pi_i\left(\left[\frac{C_i(\theta_i - \overline{\theta}^\circ)}{\sum_j \pi_j C_j} - \theta_i\right] + \left[\frac{C_i(\beta_i - \overline{\beta}^\circ)}{\sum_j \pi_j C_j} - \beta_i\right]\right)(\nu_i - \bar{\nu})'\Sigma^{'-1} \\
&= 2\left[(\sigma_{\theta_\nu}^\circ - \sigma_{\theta_\nu})' + (\sigma_{\beta_\nu}^\circ - \sigma_{\beta_\nu})'\right]\Sigma^{'-1},
\end{aligned}
$$

where the second equality follows from Lemma 3, the third the definition of π_i°, and the fourth from the fact that

$$\sum_j \pi_j^\circ(\theta_j - \overline{\theta}^\circ)(\beta_j - \overline{\beta}) = \sum_j \pi_j^\circ \theta_j \beta_j - \overline{\theta}^\circ\overline{\beta}^\circ = \sigma_{\theta\beta}^\circ,$$

and analogous terms for the other elements of ν. Now summing $\sigma_{V,2}$ and $\sigma_{V,3}$ provides the statement of (b).

Proof of Proposition 4 Since $||\sigma^n(\pi)||$ in (17) is a continuous function of π on the N dimensional simplex, which is a compact set, it has a maximum and minimum, which we denote by $||\bar{\sigma}^n||$ and $||\underline{\sigma}^n||$. Therefore, $||S_1\sigma^n(\pi_1) - S_2\sigma^n(\pi_2)|| \le (||\bar{\sigma}^n|| - ||\underline{\sigma}^n||)\cdot|S_1 - S_2|$ so that the Lipschitz condition is satisfied for the stock price. Similarly, $||S\,\sigma^n(\pi)||^2 \le ||\bar{\sigma}^n||^2\,S^2 < (1 + ||\bar{\sigma}^n||^2\,S^2)$, so that the growth condition holds as well. Similarly the norm of the volatility of beliefs in (11) is bounded by $||\bar{\sigma}_i||$ and $||\underline{\sigma}_i||$ and both conditions hold for the beliefs processes, which completes the proof. ∎

Proof of Proposition 5: The change of measure with respect to the Brownian motions in the context of the filtering setup has been derived in David (2008b). For brevity, we only provide the proof of the change of measure for the jump component.

Lets show the change of measure for the jump.

$$
\begin{aligned}
\kappa\mathbb{E}\left[\frac{M^+ - M}{M}\frac{S^+ - S}{S}\Big|\mathcal{F}_t\right] &= \kappa\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{Y_1}e^{Y_2}f(Y_1, Y_2)dY_2dY_1 \\
&= \kappa\int_{-\infty}^{\infty}e^{Y_1}f(Y_1)\int_{-\infty}^{\infty}e^{Y_2}f(Y_2|Y_1)dY_2dY_1
\end{aligned}
$$

$$
\begin{aligned}
&= \kappa \int_{-\infty}^{\infty} e^{Y_1} e^{\mu_2 + \frac{\sigma_2}{\sigma_1}(Y_1 - \mu_1)} f(Y_1) dY_1 \\
&= \kappa e^{\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1} e^{\mu_1(1 + \frac{\sigma_2}{\sigma_1}) + 0.5(1 + \frac{\sigma_2}{\sigma_1})^2 \sigma_1^2} \\
&= \kappa e^{\mu_2 + .5\sigma_2^2} e^{\mu_1 + \sigma_1 \sigma_2 + 0.5\sigma_1^2} \\
&= \kappa^* \mathbb{E}^*[e^{Y_1}].
\end{aligned}
$$

In the above, the second equality arises from the definition of a conditional expectation, the third because the two jump processes are perfectly correlated, and the fourth from the moment generating function of a normal distribution.

Appendix 2

1. SMM Estimation of the Regime Switching Jump-Diffusion Model

We start by providing here the details of the SMM estimation procedure, which is used to estimate the model. The procedure uses the SML methodology of Brandt and Santa-Clara (2002), which has already been extended to to learning framework in the pure diffusion setting in David (2008b). We provide here the extension to the case of observable jumps in the fundamental processes. Piazzesi (2005) has extended the procedure to a setting with jump-diffusions.

Using the definition of the inferred shocks (12) we can write the variables observed by the econometrician in (9) as perceived by the investors as $\frac{dY_t}{Y_t} = \varrho(\pi_t)dt + \Sigma_4 dW_t + J_{4t}dL_t$. Similarly the pricing kernel in (8) under investors' filtration can be written as $dM_t/M_t = (-\bar{\phi}(\pi_t) - \kappa\xi_2)dt - \sigma_M d\tilde{W}_t - (e^{Y_{2t}} - 1)dL_t$, where the real rate in the economy, $\bar{\phi}(\pi_t)$, is the expected value of ϕ_t in (6) conditional on investors' filtration. Since fundamentals are stationary in growth rates, we start by defining logs of variables: $y_t = \log(Y_t)$, and $m_t = \log(M_t)$. Using these characterizations we can write

$$
dy_t = (\bar{\varrho}(\pi_t) - \frac{1}{2}\operatorname{diag}(\Sigma_4 \Sigma_4'))dt + \Sigma_4 d\tilde{W}_t + J_{4t}dL_t, \tag{31}
$$

$$
dm_t = (-\bar{\phi}(\pi_t) - \kappa\xi_2 - \frac{1}{2}\sigma_M \sigma_M')dt - \sigma_M d\tilde{W}_t - (e^{Y_{2t}} - 1)dL_t, \tag{32}
$$

where $\operatorname{diag}(x)$ is a column vector composed of the diagonal elements of a square matrix x. It is immediate that investors' beliefs π_t completely capture the state of the system (y_t, m_t) for forecasting future growth rates. The specification of the system is completed with the belief dynamics in (10).

The econometrician has data series $\{y_{t_1}, y_{t_2}, \cdots, y_{t_K}\}$. Let Ψ be the set of parameters of the model. Let

$$
\mathcal{L}(\Psi) \equiv p(y_{t_1}, \cdots, y_{t_K}; \Psi) = p(\pi_{t_0}; \Psi) \prod_{k=1}^{K} p(y_{t_{k+1}} - y_{t_k}, t_{k+1}|\pi_{t_k}, t_k; \Psi),
$$

where $p(y_{t_{k+1}} - y_{t_k}, t_{k+1}|\pi_{t_k}, t_k; \Psi)$ is the marginal density of fundamentals at time t_{k+1} conditional on investors' beliefs at time t_k. Since $\{\pi_{t_k}\}$ for $k = 1, \cdots, K$ is not observed by the econometrician, we maximize

$$
E[\mathcal{L}(\Psi)] = \int \cdots \int \mathcal{L}(\Psi) f(\pi_{t_1}, \pi_{t_2}, \cdots, \pi_{t_K}) d\pi_{t_1}, d\pi_{t_2}, \cdots, d\pi_{t_K}, \tag{33}
$$

where the expectation is over all sample paths for the fundamentals, $\tilde{y}_t$, such that $\tilde{y}_{t_k} = y_{t_k}$, $k = 1, \cdots, K$. In general, along each path, the sequence of beliefs $\{\pi_{t_k}\}$ will be different.

As a first step, we need to calculate $p(y_{t_{k+1}} - y_{t_k}, t_{k+1}|\pi_{t_k}, t_k; \Psi)$. Following Brandt and Santa-Clara (2002), we simulate paths of the state variables over smaller discrete units of time using the Euler discretization scheme

(see also Kloeden and Platen 1992):

$$\tilde{y}_{t+h} - \tilde{y}_t \;=\; \left(\bar{\varrho}(\pi_t) - \frac{1}{2}(\sigma_Q\sigma_Q', \sigma_E\sigma_E')'\right)h + \Sigma_2\,\sqrt{h}\tilde{\epsilon}_{2t} + \mathbf{1}_{\tilde{u}_t<\kappa h}\tilde{\epsilon}_{2t}, \tag{34}$$

$$m_{t+h} - m_t \;=\; \left(-\bar{\phi}(\pi_t) - \kappa\zeta_2 - \frac{1}{2}\sigma_M\,\sigma_M'\right)h - \sigma_M\,\sqrt{h}\tilde{\epsilon}_{1t} + \mathbf{1}_{\tilde{u}<\kappa h}\tilde{\epsilon}_{2t}, \tag{35}$$

$$\pi_{t+h} - \pi_t \;=\; \mu(\pi_t)\,h + \sigma(\pi_t)\sqrt{h}\tilde{\epsilon}_{1t}, \tag{36}$$

where $\tilde{\epsilon}_{1t}$ and $\tilde{\epsilon}_{2t}$ are 5- and 1- dimensional standard normal variables, respectively, $\tilde{u}_t$ is uniformly distributed, and $h = 1/M$ is the discretization interval. The Euler scheme implies that the marginal conditional density of the 4×1 fundamental growth vector y_t over h is 4-dimensional normal.

We approximate $p(\cdot|\cdot)$ with the density $p_M(\cdot|\cdot)$, which obtains when the state variables are discretized over M subintervals. Since the drift and volatility coefficients of the state variables in (10), and (31) to (32) are infinitely differentiable, and $\Sigma\Sigma'$ is positive definite, Lemma 1 in Brandt and Santa-Clara (2002) implies that $p_M(\cdot|\cdot) \to p(\cdot|\cdot)$ as $M \to \infty$. The Chapman-Kolmogorov equation implies that the density over the interval (t_k, t_{k+1}) with M subintervals satisfies $p_M(y_{t_{k+1}} - y_{t_k}, t_{k+1}|\pi_{t_k}, t_k; \Psi) =$

$$\int\int \phi\left(y_{t_{k+1}} - y;\,;\Psi\right) \times p_M\left(y - y_{t_k}, \pi, m, t_k + (M-1)h|\pi_{t_k}, t_k\right) d\pi\, dy, \tag{37}$$

where $\phi(y;\psi)$, denotes the mixture-of-normals density given as:

$$\phi(y;\psi) \;=\; N(\varrho(\pi_t)h, \Sigma_4\Sigma_4'h) \quad \text{with probability} \quad \kappa\,h, \tag{38}$$
$$=\; N(\varrho(\pi_t)h + (0,1,0,0)'\mu_1, \Sigma_4\Sigma_4'h + i_2\sigma_1^2) \quad \text{with probability} \quad 1 - \kappa\,h, \tag{39}$$

where i_2 is the 4×4 square matrix with zero in all elements except the $(2,2)$ element, which is 1. Now $p_M(\cdot|\cdot)$ can be approximated by simulating L paths of the state variables in the interval $(t_k, t_k + (M-1)h)$ and computing the average

$$\hat{p}_M\left(y_{t_{k+1}} - y_{t_k}, t_{k+1}|\pi_{t_k}, t_k; \Psi\right) \;=\; \frac{1}{L}\sum_{l=1}^{L}\phi\left(y_{t_{k+1}} - y^{(l)}; \Psi\right). \tag{40}$$

The Strong Law of Large Numbers (SLLN) implies that $\hat{p}_M \to p_M$ as $L \to \infty$.

To compute the expectation in (33), we simulate S paths of the system (34) to (36) "through" the full time series of fundamentals. Each path is started with an initial belief, $\pi_{t_0} = \pi^*$, the stationary beliefs implied by the generator matrix Λ. In each time interval (t_k, t_{k+1}) we simulate $(M-1)$ successive values of $\tilde{y}_t^{(s)}$ using the discrete scheme in (34), and set $\tilde{y}_{t_k}^{(s)} = y_{t_k}$. The results in the paper use $M = 90$ for quarterly data, so that shocks are approximated at roughly a daily frequency. The pricing kernel and beliefs along the entire path of the sth simulation are obtained by iterating on (35) and (36). We approximate the expected likelihood as

$$\hat{\mathcal{L}}^{(S)}(\Psi) = \frac{1}{S}\sum_{s=1}^{S}\prod_{k=0}^{K-1}\hat{p}_M(y_{t_{k+1}}^{(s)} - y_{t_k}^{(s)}, t_{k+1}|\pi_{t_k}^{(s)}, t_k; \Psi), \tag{41}$$

where $\hat{p}_M(\cdot|\cdot)$ is the density approximated in (40). The SLLN implies that $\hat{\mathcal{L}}^{(S)}(\Psi) \to E[\mathcal{L}(\Psi)]$ as $S \to \infty$. We often report $\bar{\pi}_{t_k} = 1/S \sum_{s=1}^{S}\pi_{t_k}^{(s)}$, which is the econometrician's expectation of investors' belief at t_k.

To extract investors' beliefs from data on price levels and volatilities in addition to fundamentals we add overidentifying moments to the SML method above. From Proposition 1, we can compute the time series of model-implied price-earning ratios and bond yields at the discrete data points t_k, $k = 1, \cdots, K$ as

$$\widehat{\text{P/E}}_{t_k} \;=\; C \cdot \bar{\pi}_{t_k}, \qquad\qquad \hat{i}_{t_k}(\tau) = -\frac{1}{\tau}\log\left(B\left(\tau\right)\cdot\bar{\pi}_{t_k}\right).$$

44

We note that the constants Cs and the functions $B(\tau)$ both depend on the parameters of the fundamental processes, Ψ. Hence, we let the pricing errors be denoted

$$e_{t_k}^P = \left(\widehat{\text{P/E}}_{t_k} - \text{P/E}_{t_k}, \widehat{i}_{t_k}(0.25) - i_{t_k}(0.25), \left(\widehat{i}_{t_k}(5) - \widehat{i}_{t_k}(1) \right) - \left(i_{t_k}(5) - i_{t_k}(1) \right) \right).$$

We similarly formulate the errors from options prices as

$$e_{t_k}^O = \left(\widehat{V}_{t_k} - V_{t_k}, \widehat{(P/C)}_{t_k} - (P/C)_{t_k} \right),$$

where V is the ATMIV, and P/C is the put-call ratio as discussed. The model-implied options prices are calculated using Monte-Carlo simulations as described below.

To estimate Ψ from data on fundamentals as well as financial variables, we form the overidentified SMM objective function

$$c = \left(\frac{1}{T} \sum_{t=1}^T \epsilon_{\mathbf{t}} \right)' \cdot \Omega^{-1} \cdot \left(\frac{1}{T} \sum_{t=1}^T \epsilon_{\mathbf{t}} \right). \tag{42}$$

The moments used are the scores of the log likelihood function from fundamentals, and the pricing errors from stock, Treasury bond, and options prices. Since the number of scores in $\frac{\partial \log(\hat{\mathfrak{L}})}{\partial \Psi}(t_k)$ equals the number of parameters driving the fundamental processes in Ψ, and the number of pricing errors is five, the statistic c in (42) has a chi-squared distribution with five degrees of freedom. We correct the variance covariance matrix for autocorrelation and heteroskedasticity using the Newey-West method [see, for example, Hamilton (1994) equation 14.1.19] using a lag length of $q = 12$. A long lag length is chosen since interest rates and P/E ratios used in the error terms are highly persistent processes.

We end the description of our estimation methodology with two important details. First, for determining the number of specified regimes we do not use likelihood ratio tests, which are computationally extremely demanding and beyond the scope of this paper (see Garcia (1998)). Instead, we follow the simpler and more practical methodology of using the overidentified SMM objective to determine a stopping rule on the number of regimes used in a number of papers modeling regimes shifts (e.g. Gray (1996), Bansal and Zhou (2002)). Second, to reduce the number of parameters, we follow a two-step procedure. First we estimate an unrestricted generator matrix and rank all its elements into 7 bins. All elements in the smallest bin (whose values we estimated to be below 0.001) were set to zero. Elements in each of the remaining bins were constrained to be equal in our second step estimation.

2. Options Prices

As for the likelihood function we formulate options prices as expected discounted values of their terminal payoffs under the risk-neutral measure. Expectations are approximated using Monte Carlo simulation while discretizing the dynamics of the state variables of our system along the sth sample path under the risk-neutral measure as:

$$\pi_{t+h}^{*(s)} - \pi_t^{*(s)} = \left(\mu(\pi_t^{*(s)}) - \rho(\pi_t^{*(s)}) \right) h + \sigma(\pi_t^{*(s)}) \sqrt{h} \tilde{\epsilon}_t^{*(s)}, \tag{43}$$

$$P_{t+h}^{n*(s)} = P_t^{n*(s)} \exp[\left(r(\pi_t^{*(s)}) - \delta(\pi_t^{*(s)}) \right) h + \sigma^n(\pi_t^{*(s)}) \sqrt{h} \tilde{\epsilon}_{1t}^{(s)*} + \mathbf{1}_{\tilde{u}_t < \kappa^* h} \tilde{\epsilon}_{2t}^{(s)*}], \tag{44}$$

$$B_{t+h}^{*(s)} = B_t^{*(s)} \exp[-r(\pi_t^{*(s)}) h], \tag{45}$$

where $\tilde{\epsilon}_{1t}^*$ and $\tilde{\epsilon}_{2t}^*$ are 5- and 1- dimensional standard normal variables, respectively, $\tilde{u}_t$ is uniformly distributed, and $h = 1/M$ is the discretization interval. On each sample the process for the state variables is simulated starting with $\pi_t^{*(s)} = \pi_t$, the assumed beliefs of investors at time t. Then the value of a European call option at

time t when investors have beliefs π_t that matures at $t + T$ is given by

$$C^{M*}(t, T, \pi_t) = \frac{1}{S} \sum_{s=1}^{S} B_{t+T}^{*(s)} \max\left[P_{t+T}^{n*(s)} - K, 0 \right].$$

We report option prices for $M = 90$. To reduce the time of computations we use three variance reduction techniques: the first two, antithetic and control variate (with Black-Scholes prices), are well known. In addition, we use the expected martingale simulation technique of Duan et. al. The volatility forecast under the Q-measure is approximated from the path of forecasted beliefs under this measure as

$$\sigma^{M*}(t, T, \pi_t) = \sqrt{\frac{1}{S} \sum_{j=1}^{(T-t)M} \sigma^n(\pi_{t+jh}^{*(s)}) \sigma^n(\pi_{t+jh}^{*(s)})' h}. \tag{46}$$

Similarly, using the discretized beliefs processes as in (36), volatility forecasts under the objective measure are analogously constructed as

$$\sigma^n(t, T, \pi_t) = \sqrt{\frac{1}{S} \sum_{j=1}^{(T-t)M} \sigma^n(\pi_{t+jh}^{*(s)}) \sigma^n(\pi_{t+jh}^{*(s)})' h}. \tag{47}$$

Table 1: Parameter Estimates of Regime Switching Model From SMM Procedure

Fundamental Drifts	β_1	β_2	β_3	β_4	θ_1	θ_2	θ_3	θ_4
	-0.002	0.015	0.065	0.091	-0.057	-0.052	0.061	0.077
	(0.003)	(0.008)	(0.021)	(.005)	(0.002)	(0.007)	(.007)	(0.012)
	ρ_1	ρ_2	ρ_3	ρ_4	ω_1	ω_2	ω_3	ω_4
	- 0.066	-0.025	0.01	0.086	-0.057	-0.031	0.012	0.052
	(0.010)	(0.008)	(0.002)	(0.007)	(0.001)	(0.015)	(0.007)	(.005)

Fundamental Volatilities:	$\sigma_{Q,1}$	$\sigma_{E,2}$	$\sigma_{K,1}$	$\sigma_{K,4}$	$\sigma_{H,5}$
	0.025	0.084	0.047	0.056	0.049
	(0.012)	(0.036)	(0.017)	(0.016)	(0.021)

Interest Rate Rule:	α_0	α_β	α_ρ
	0.017	0.362	0.257
	(0.008)	(0.017)	(0.003)

Prices of Risk:	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\sigma_{M,4}$	$\sigma_{M,5}$
	0.045	0.421	0.300	0.3381	-0.045
	(0.129)	(0.201)	(0.111)	(0.149)	(0.078)

Jump Parameters:	κ	μ_1	σ_1	μ_2	σ_2
	0.432	-0.052	0.036	0.293	-0.790
	(0.195)	(0.027)	(0.016)	(0.133)	(0.364)

Generator Elements:	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
	0.005	0.018	0.043	0.051	0.097	0.199
	(0.002)	(0.011)	(0.007)	(0.003)	(0.023)	(0.007)

SMM Error Value ($\chi^2(7)$):   10.470   P-Value: 0.063

The table reports SMM estimates of the following model for CPI, Q_t, real earnings, E_t, the real pricing kernel, M_t, de-meaned capacity utilization, and real money growth:

$$\frac{dQ_t}{Q_t} = \beta_t \, dt + \sigma_Q \, dW_t,$$

$$\frac{dE_t}{E_t} = (\theta_t - \kappa \, \xi_1) \, dt + \sigma_E \, dW_t, + (e^{Y_{1t}} - 1) \, dL_t,$$

$$\frac{dM_t}{M_t} = (-\phi_t - \kappa \, \xi_2) dt - \sigma_M \, dW_t + (e^{Y_{2t}} - 1) \, dL_t,$$

$$dK_t = \rho_t \, dt + \sigma_K \, dW_t,$$

$$\frac{dH_t}{H_t} = \omega_t \, dt + \sigma_H \, dW_t.$$

W_t is a 5×1 vector of Standard Brownian Motions, L_t is the counter of a Poisson process with constant intensity κ, and Y_{it} i.i.d. $N(\mu_i, \sigma_i)$, $i = 1, 2$. The drift of the stacked state vector $\nu_t = (\beta_t, \theta_t - \kappa \, \xi_1, -\phi_t - \kappa \, \xi_2, \rho_t, \omega_t)'$, follows an eight-state unobserved regime switching model over the composite states listed in the bottom panel of Table 2 with the following generator matrix:

$$\Lambda = \begin{pmatrix}
-\sum_j \lambda_{1j} & 0 & \lambda_6 & 0 & \lambda_2 & \lambda_1 & 0 & 0 \\
0 & -\sum_j \lambda_{2j} & \lambda_5 & \lambda_6 & 0 & \lambda_1 & \lambda_4 & 0 \\
0 & \lambda_3 & -\sum_j \lambda_{3j} & 0 & 00 & \lambda_1 & \lambda_4 & \lambda_5 \\
0 & \lambda_6 & 0 & -\sum_j \lambda_{4j} & 0 & 0 & \lambda_4 & 0 \\
\lambda_1 & 0 & \lambda_1 & 0 & -\sum_j \lambda_{5j} & \lambda_1 & 0 & 0 \\
\lambda_5 & \lambda_3 & 0 & \lambda_3 & 0 & -\sum_j \lambda_{6j} & 0 & \lambda_5 \\
\lambda_5 & 0 & 0 & 0 & 0 & \lambda_5 & -\sum_j \lambda_{7j} & 0 \\
\lambda_3 & 0 & \lambda_5 & 0 & 0 & 0 & 0 & -\sum_j \lambda_{8j}
\end{pmatrix}.$$

The pricing kernel, M_t is observed by investors but not by the econometrician. Investors beliefs about the underlying drift states follow the filtering equation in (10). Estimates are obtained from data on the fundamentals as well five price series listed in Table 3 using the SMM methodology described in Appendix 2. Standard errors are in parentheses.

Table 2: Model Implied Transition Probabilities, Stationary Probabilities, Stock and Bond Price Valuations, Fear Indices, and Higher Moments in the Eight States

Implied Annual Quarterly Transition Probability (Percent) Matrix

	1	2	3	4	5	6	7	8
1	80.1	0.3	16.2	0.0	1.6	0.5	0.4	0.8
2	0.2	71.9	7.5	14.9	0.0	0.6	4.5	0.4
3	0.4	3.3	82.8	0.3	0.0	0.6	4.3	8.2
4	0.2	14.9	0.8	79.4	0.0	0.3	4.5	0.0
5	0.5	0.0	0.5	0.0	98.4	0.4	0.0	0.0
6	7.7	3.5	1.3	3.6	0.0	75.7	0.2	7.9
7	8.2	0.2	0.8	0.2	0.1	7.7	82.4	0.4
8	3.6	0.2	8.6	0.0	0.0	0.0	0.2	87.4

Implied 5-Year Transition Probability (Percent) Matrix

	1	2	3	4	5	6	7	8
1	34.7	3.6	37.8	1.3	5.2	2.0	5.1	10.3
2	3.7	30.1	16.8	27.0	0.1	3.5	13.6	5.0
3	5.2	7.3	46.1	3.5	0.2	3.3	11.4	23.0
4	3.4	27.0	8.3	42.2	0.1	3.1	14.0	1.9
5	2.0	0.33	2.6	0.22	92.3	1.5	0.3	0.9
6	16.4	8.9	14.9	9.6	1.0	25.6	3.5	20.0
7	21.4	2.6	11.4	2.3	1.2	15.5	39.3	6.2
8	9.7	2.2	26.7	0.7	0.6	0.8	3.2	56.1

Implied Stationary Probabilities, P/E Ratios, the Term Structure, Option Prices, and Higher Order Moments

#	$\beta(\%)$	$\theta\ (\%)$	$\rho(\%)$	$\omega(\%)$	$\bar{\pi}(\%)$	P/E	$i_{0.25}(\%)$	$S(\%)$	ATM(%)	P/C	Skw	Kur	$\rho_{SV}(\%)$
1	1.5	6.1	1.0	1.2	11.0	17.3	4.3	1.5	15.2	1.66	−1.9	9.4	−71.1
2	6.5	−5.2	1.0	5.3	7.6	8.9	10.9	−0.1	33.0	0.81	1.0	4.7	96.4
3	6.5	6.1	8.7	1.2	25.8	14.5	12.6	−1.9	16.3	1.19	−1.1	6.6	−78.3
4	9.1	−5.2	−2.5	−5.7	6.9	7.9	13.4	−1.1	41.6	0.76	1.4	5.1	99.1
5	1.5	7.7	1.0	−3.1	12.1	32.0	4.1	0.1	22.8	1.39	−2.3	10.8	−88.0
6	−0.2	−5.7	−6.6	5.3	4.8	12.9	0.1	2.3	21.5	0.90	−0.3	5.9	83.5
7	6.5	−5.2	−6.6	−3.1	10.5	11.4	8.7	−1.3	24.2	0.71	0.8	4.3	95.7
8	1.5	6.1	−6.6	5.3	21.2	19.7	2.3	1.3	17.5	1.65	−3.3	21.5	−92.2

The **top** and **middle panels** report the quarterly and 5-year implied transition probability matrix between the eight states implied from the generator matrix elements displayed in Table 1. Rows may not sum to one due to rounding. The **bottom panel** report the implied stationary probabilities and implied prices of the variables used in the SMM estimation procedure in the eight states. $\bar{\pi}$ is the stationary probability of each state; P/E the price-earnings ratio, and, $i_{0.25}$, the 3-month Treasury yield, and S the 5-year less 1-year Treasury yield; ATM is the at-the-money implied volatility (ATMIV); P/C is the ratio of 5% OTM put-to-call implied volatilities; Skw and Kur are the skewness and kurtosis of the 3-month risk-neutral return distribution, respectively; ρ_{SV} is the correlation between stock returns and stock variance calculated using (23). The P/E ratio and bond yields are computed as shown in Proposition 1. Implied Volatility, Put-Call Ratio and the higher order moments are for options with three months to maturity as calculated using Monte Carlo simulations as shown in Appendix 2.

Table 3: Model Fits for Expected Fundamentals, Stocks, Bonds, and Options Prices from SMM Procedure

Variable	α	β	R^2
Inflation	-0.009	1.638	0.623
	[-1.538]	[8.071]*	
Real Earnings Growth	-0.045	2.099	0.159
	[-1.501]	[3.617]*	
De-Meaned Capacity Utilization	0.002	1.353	0.751
	[0.068]	[13.540]*	
Real Money Growth	-0.005	2.15	0.365
	[-1.785]	[5.957]*	
P/E Ratio	-3.028	1.249	0.606
	[-1.915]	[19.976]*	
3-Month Yield	-0.002	0.970	0.535
	[-0.246]	[5.991]*	
5-Year Minus 1-Year Treasury Yield	0.003	0.857	0.502
	[2.456]*	[8.071]*	
ATM	0.004	0.913	0.502
	[1.522]	[5.539]*	
P/C	0.19	0.710	0.451
	[1.645]*	[8.212]*	

We display the fits of the variables used in our SMM procedure: the fundamentals, and the five pricing variables, which are used to overidentify the model. For the four fundamentals we provide the regression results for the equation $x(t) = \alpha + \beta \, \mathbb{E}[x|\mathcal{F}_{t-1}] + \epsilon(t)$, where $x(t)$ is the realized growth and $\mathbb{E}[x|\mathcal{F}_t]$ is investors' conditional expected growth of the fundamental under consideration. The conditional expected growth is obtained from the filtered probabilities $\pi(t)$ displayed in Figure 3, and for earnings, for example, is given by $\sum_i^N \theta_i \, \pi_i(t)$. For the price series, we present the regression results for the equation $p(t) = \alpha + \beta \, p(\pi(t)) + \epsilon(t)$, where $p(t)$ and $p(\pi(t))$ are the realized and model price conditional on investors' beliefs at t, respectively. T-statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation.

Table 4: Explaining At-The-Money Implied Volatility of 3-Month S&P 500 Options (1986:Q2 – 2008)

No.	Constant	ATMIV^M	Lag	P/E	CU	Earn	NBER	R^-_{t-1}	$\bar{R}^2$
1	1.769	0.91							0.528
	[0.454]	[3.977]*							
2	10.211		0.415						0.323
	[4.336]*		[3.346]*						
3	−1.847	0.374	0.742						0.576
	[−0.526]	[3.444]*	[2.781]*						
4	10.776			0.405					0.041
	[3.308]*			[2.989]*					
5	18.107				−0.655				0.045
	[15.277]*				[−1.719]				
6	18.888					−0.616			0.114
	[17.108]*					[−2.552]*			
7	27.409						−0.102		0.168
	[7.198]*						[−2.547]*		
8	16.593							−2.562	0.269
	[17.312]*							[−4.462]*	
9	13.781			0.343	−0.035	−0.275	−0.035	−2.128	0.361
	[2.618]*			[2.159]*	[−0.151]	[−1.486]	[−0.882]	[−7.376]*	
10	−2.002	0.990		0.089	0.760	0.098	0.000	−1.238	0.609
	[−0.518]	[3.276]*		[0.508]	[2.754]*	[0.4972]	[0.020]	[−3.065]*	
11	4.921	0.88	0.381	−0.058	0.742	0.1	0.01	−1.206	0.663
	[1.458]	[4.285]*	[2.660]*	[−0.373]	[2.755]*	[0.580]	[0.550]	[−3.169]*	

The table reports the quarterly time series regressions

$$\text{ATMIV}(t) = \beta_0 + \beta_1 \text{ATMIV}^M(t-1) + \beta_2 \text{ATMIV}(t-1) + +\beta_3 \text{P/E}(t-1) + \beta_4 \text{C.U.}(t-1)$$
$$+ \beta_5 \text{Earn}(t-1) + \beta_6 \text{NBER}(t-1) + \beta_7 R^-_{t-1} + \epsilon(t).$$

In different lines some of the β_i are set to zero. ATM(t) is the at-the-money Black-Scholes implied volatility on S&P 500 index options traded on the CBOE with approximately three months to maturity and trading at the beginning of the quarter. ATM^M is the at-the-money implied volatility implied by our model and calculated as shown in Appendix 2. The historical and model implied series are shown in the top panel of Figure 1. The latter are calculated conditional on investors' beliefs of fundamental drift states that are extracted and displayed in Figure 3. P/E is the price to operating income ratio of S&P 500 firms, CU is the demeaned industrial capacity utilization in the United States obtained from the Federal Reserve Board, Earn stands for the real operating earnings growth of S&P 500 firms, NBER is 100 times the quarterly expansion indicator created by the NBER, and R^- is percentage one quarter lagged returns in periods when it is negative on the S&P 500 index. Besides options prices, all other variables are measured at the end of the previous quarter. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The symbol * indicates statistical significance at the 5% level.

Table 5: Explaining the Ratio of Implied Volatilities of 5% Out-of-the-Money Puts to Calls for 3-Month S&P 500 Options (1986:Q2 – 2008)

No.	Constant	P/C^M	Lag	P/E	CU	Earn	NBER	R_{t-1}^-	COT	II	$\bar{R}^2$
1	0.385 [3.603]*	0.717 [8.338]*									0.454
2	0.394 [4.886]*		0.693 [10.912]*								0.458
3	0.237 [2.918]*	0.42 [3.454]*	0.406 [3.570]*								0.538
4	1.466 [9.536]*			−0.01 [−1.344]							0.041
5	1.297 [37.324]*				0.015 [1.780]						0.048
6	1.279 [41.974]*					0.013 [2.521]*					0.092
7	1.176 [60.65]*						0.001 [3.170]*				0.039
8	1.31 [36.721]*							0.029 [4.050]*			0.057
9	1.267 [27.522]*								−0.901 [−0.760]		0.008
10	53.99 [30.724]*									−2.262 [−1.488]	0.044
11	1.499 [10.53]			−0.006 [−1.145]	−0.004 [−0.453]	0.01 [1.315]	0 [0.047]	0.041 [4.846]*	0.357 [0.399]	−0.004 [−2.530]*	0.219
12	0.441 [2.128]*	0.682 [6.086]*		−0.001 [−0.129]	−0.014 [−2.383]*	0.002 [0.539]	0.001 [1.089]	0.022 [2.311]*	0.654 [0.897]	−0.003 [−1.973]*	0.470
13	0.264 [1.783]	0.412 [3.468]*	0.392 [3.288]*	−0.001 [−0.351]	−0.012 [−2.420]*	0.005 [0.126]	0.001 [1.447]	0.014 [1.952]	0.681 [1.325]	−0.002 [−2.031]*	0.551

The table reports the quarterly time series regressions

$$
\begin{aligned}
\text{P/C}(t) \quad = \quad & \beta_0 + \beta_1\,\text{P/C}^M(t-1) + \beta_2\,\text{P/C}(t-1) + \beta_3\,\text{P/E}(t-1) + \beta_4\,C.U.(t-1) \\
+ \quad & \beta_5\,\text{Earn}(t-1) + \beta_6\,\text{NBER}(t-1) + \beta_7\,R_{t-1}^- + \beta_8\,\text{COT}(t) \; + \; +\beta_8\,\text{II}(t)\epsilon(t).
\end{aligned}
$$

In different lines some of the β_i are set to zero. P/C(t) is the ratio of Black-Scholes implied volatilities of 5% out-of-the-money puts to calls for S&P 500 options with about three months to maturity measured at the beginning of the quarter. P/E^M is the analogous put-call ratio implied by our model and calculated as shown in Appendix 2. The historical and model implied series are shown in the top panel of Figure 1. The latter are calculated conditional on investors' beliefs of fundamental drift states that are extracted and displayed in Figure 3. P/E is the price to operating income ratio of S&P 500 500 firms; CU is the demeaned industrial capacity utilization in the United States obtained from the Federal Reserve Board; Earn stands for the real operating earnings growth of S&P firms; NBER is 100 times the quarterly expansion indicator created by the NBER; R_{t-1}^- is the one quarter lagged returns in periods when it is negative on the S&P 500 index; COT stands for the sentiment of traders measured as the net long position of large speculators on S&P 500 index futures obtained from the Commodity Futures Trading Commission's Commitment of Traders Report; II stands for investor sentiment (bullish less bearish proportion) measured in Investor's Intelligence's survey of investment newsletter writers. Besides options prices and sentiment variables, all other variables are measured at the end of the previous quarter. COT is measured on the day of the options trade, and the II on the Wednesday before the options trade. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The symbol * indicates statistical significance at the 5% level.

Table 6: Model Volatility of Stock Volatility and Fundamental Uncertainties (1986:Q2 – 2008)

No.	Constant	Inf Unc	Earn Unc	CU Unc	MG Unc	$\bar{R}^2$
1.	2.046	0.646				0.054
	(1.535)	(1.817)				
2.	0.933		0.346			0.725
	(1.554)		(8.347)*			
3.	1.767			0.677		0.037
	(1.023)			(1.729)		
4.	-0.226				1.952	0.234
	(-0.143)				(2.553)*	
5.	0.989	0.0313	0.353	-1.116	1.751	0.891
	(1.032)	(0.182)	(14.295)*	(-3.444)*	(5.910)*	

The table reports the quarterly time series regressions

$$\text{VV}(t) = \beta_0 + \beta_1 \, \text{IU}(t) + \beta_2 \, \text{EU}(t) + \beta_3 \, \text{CU}(t) + \beta_4 \, \text{MG}(t) + \epsilon(t). \qquad (48)$$

In different lines some of the β_i are set to zero. Inf Unc stands for inflation uncertainty, Earn Unc for earnings uncertainty, CU Unc for Capacity Utilization uncertainty, and MG Unc for money growth uncertainty. Uncertainty for each fundamental variable is measured using equation (30) and the models volatility of stock volatility is computed using (22). Time series of all variables are evaluated at the filtered belief series in Figure 3. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The symbol * indicates statistical significance at the 5% level.

Figure 1: Data and Model Fitted At-the-Money Implied Volatility (ATMIV) and 5% Out-of-the-Money Put-Call Implied Volatility Ratio (P/C) from (1986:Q2-2008)
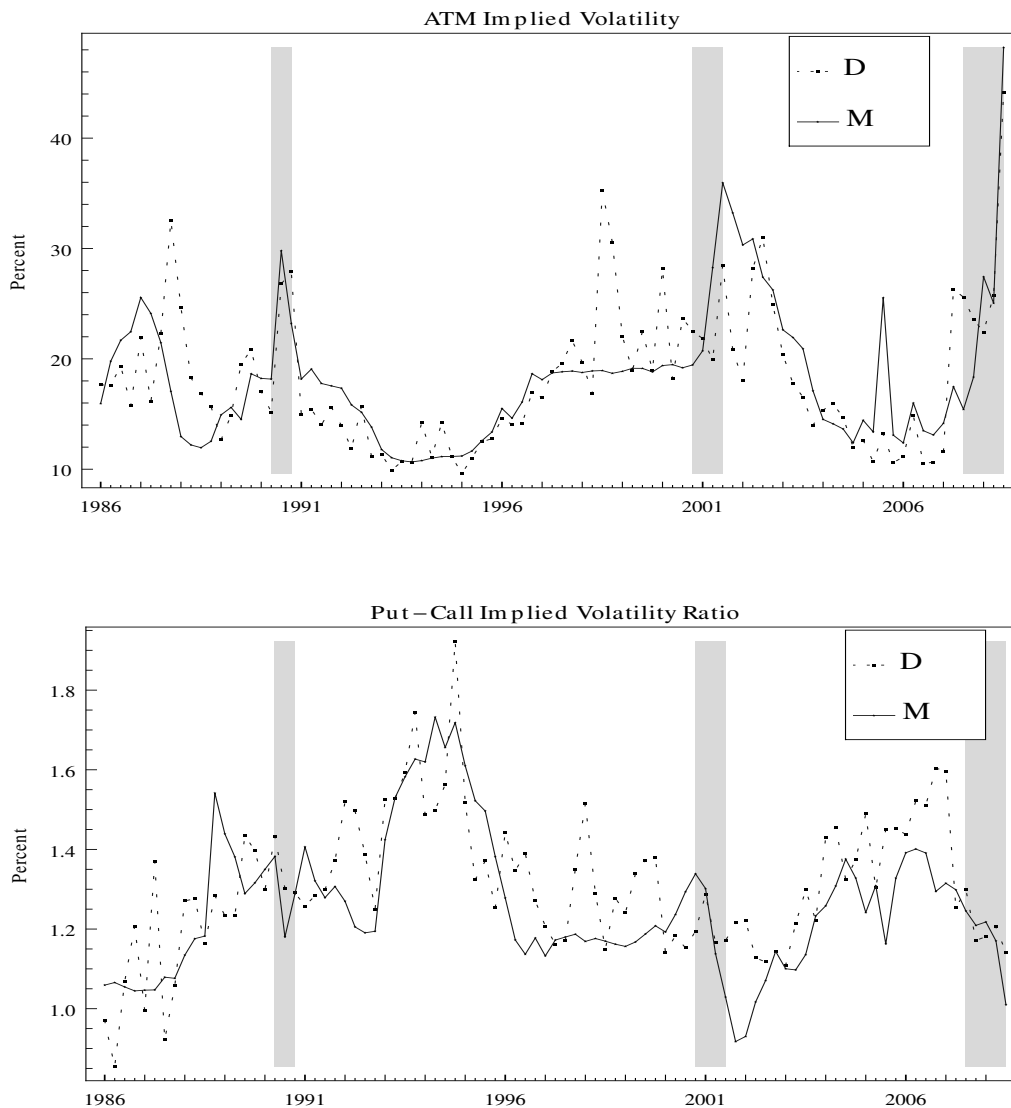

The at-the-money implied volatility (ATMIV) and ratio of 5% OTM put-to-call implied volatilities (P/C) at about three months to maturity are constructed at a quarterly frequency from S&P 500 index options prices as discussed in Section 2.2. The legend "D" denotes the historical data series, while "M" denotes those from our model. The model series are calculated using Monte Carlo simulations as shown in Appendix 2. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 2: Impulse Responses of Short Rates and Industrial Capacity Utilization to Shocks to Fear Measures (1986:Q2-2008)
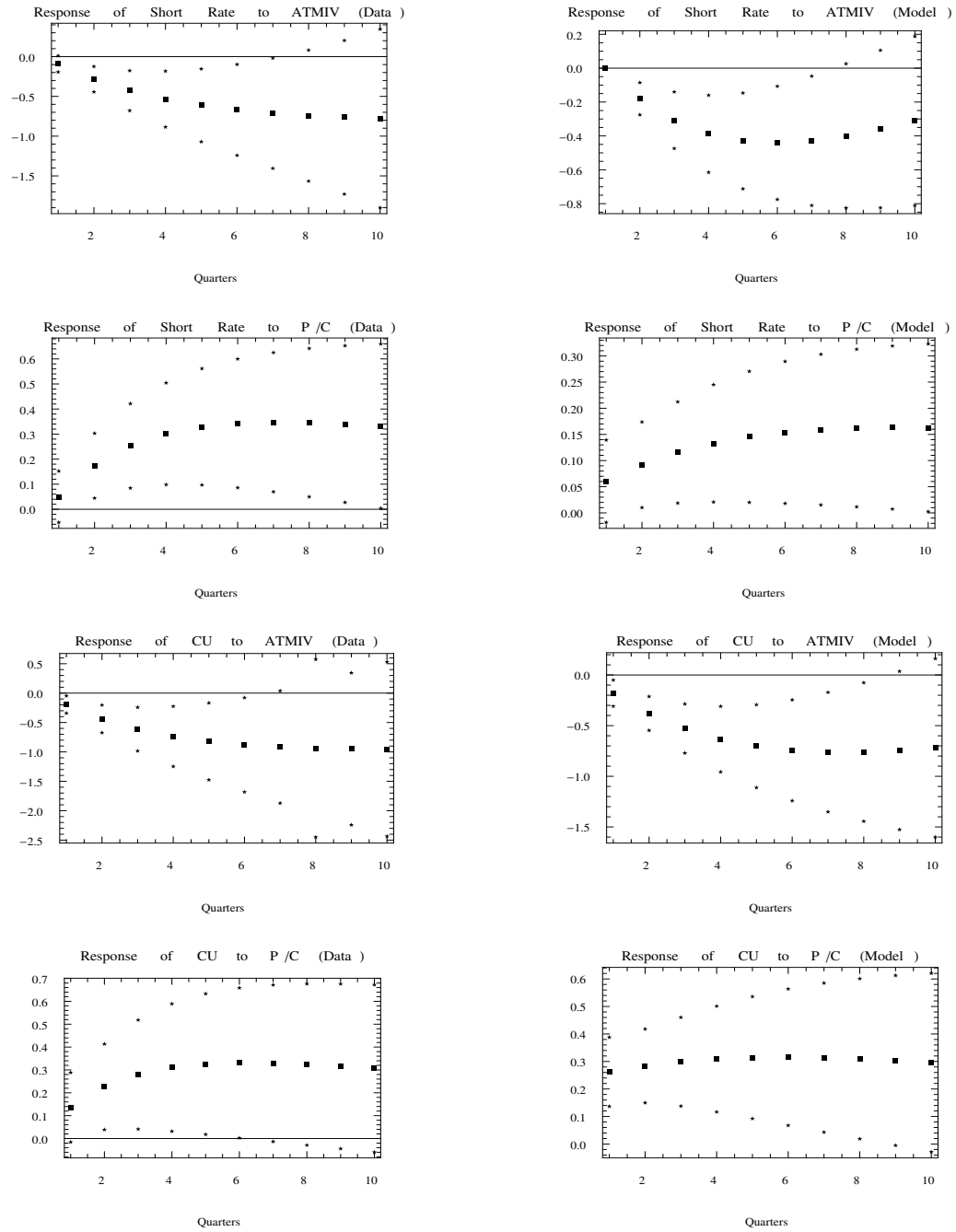
We report the generalized impulse response function (IRF) of Pesaran and Shin (1998) for the 1st order VAR system with the two variables in each panel. The IRF using this definition is independent of the order of variables in the VAR. Two standard error bands using bootstrap with 5000 repetitions are also displayed.

Figure 3: Conditional Probabilities of Eight Regimes (1967-2008)
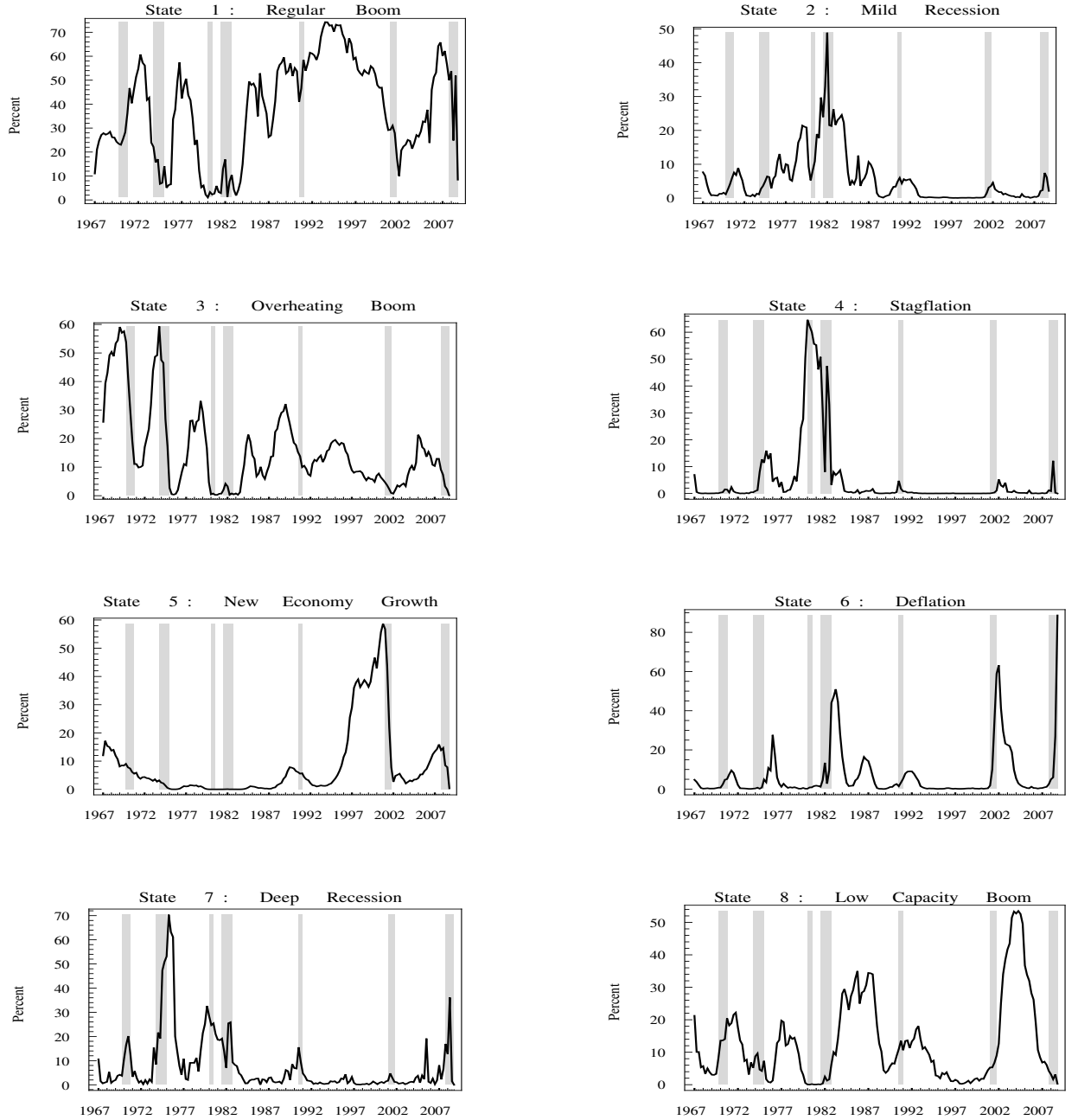
The regime definitions are in Table 2. β, θ, ρ, and ω are the drifts of inflation, earnings growth, de-meaned capacity utilization, and real money growth, respectively. The filtered beliefs are obtained from the SMM procedure in Appendix 2. The calibrated values of the parameters are shown in Table 1. Shaded areas represent NBER-dated recessions.

Figure 4: Fundamentals: Empirical and Model Fitted (1960-2008)
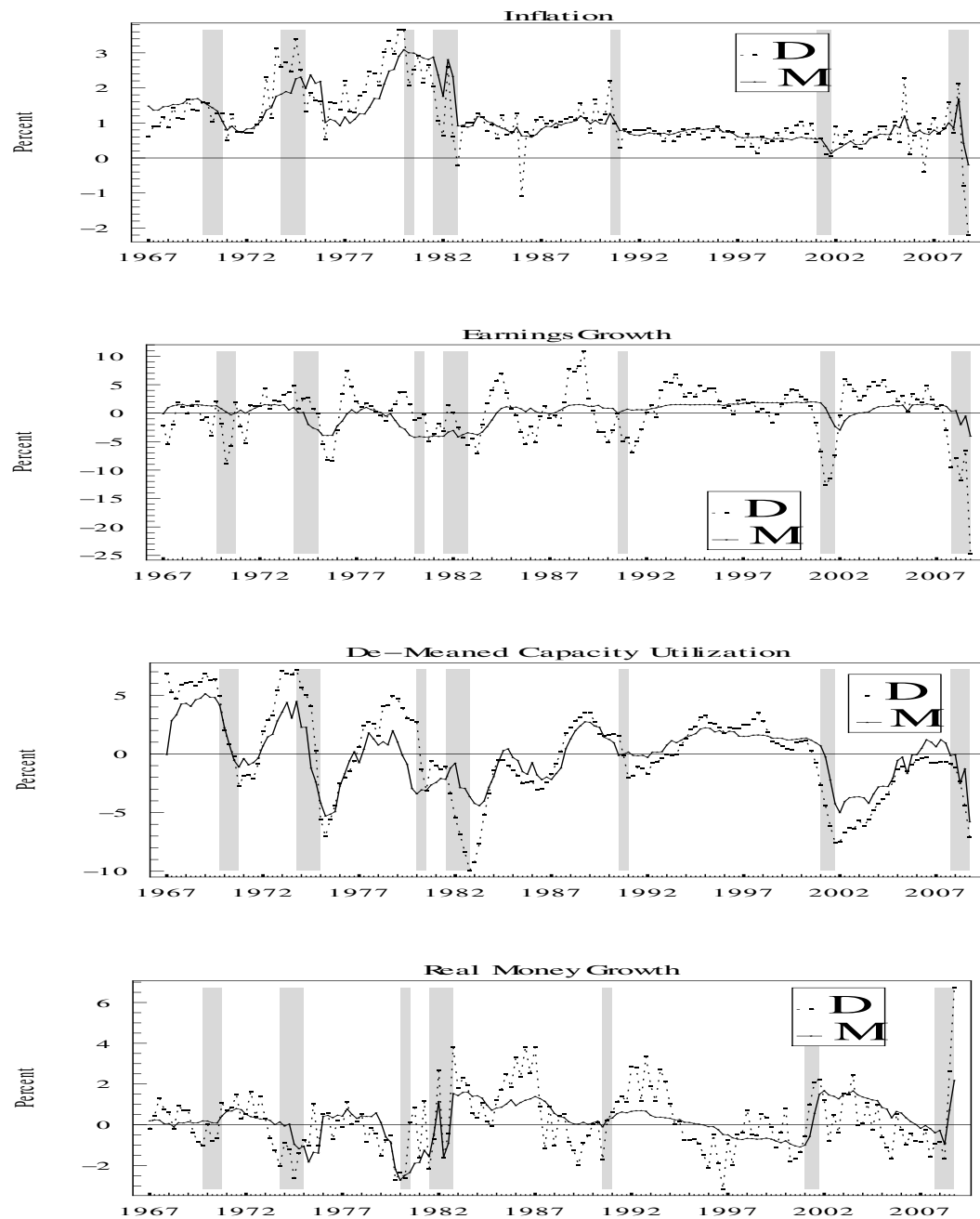
Historical values of financial and fundamental variables series (D) are in solid lines and their fitted values (M) from the SMM estimation procedure in Appendix B are in dashed lines. The calibrated values of the parameters are shown in Table 1. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 5: Stock and Bond Prices: Empirical and Model Fitted (1960-2008)
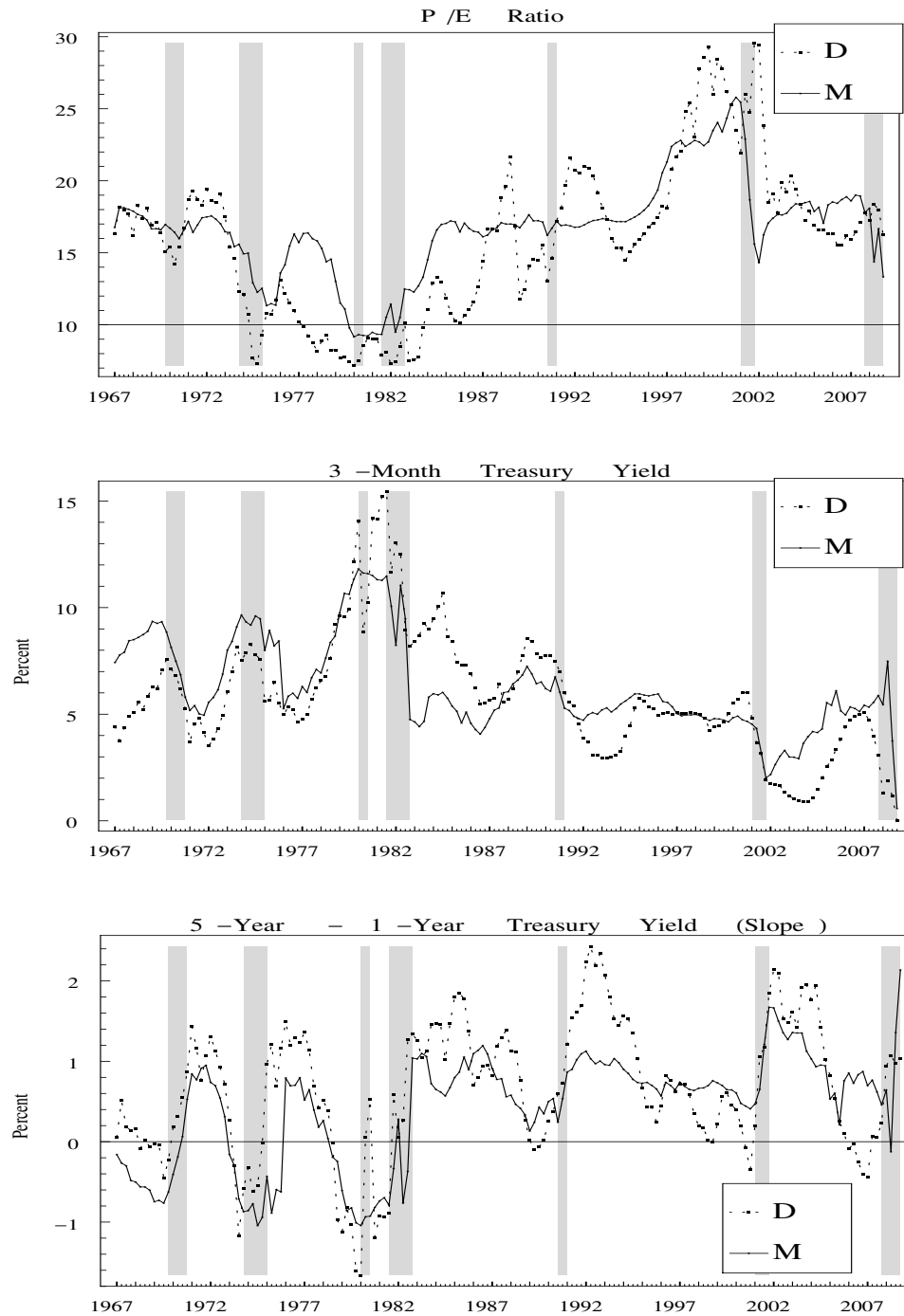
Historical values of financial and fundamental variables series (D) are in solid lines and their fitted values (M) from the SMM estimation procedure in the Appendix are in dashed lines. The estimated values of the parameters are shown in Table 1 and the implied asset price valuations are in Table 2 The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 6: Strike-Adjusted 3-Month Densities of Stock Returns Under the Risk-Neutral Measures in the Eight Regimes
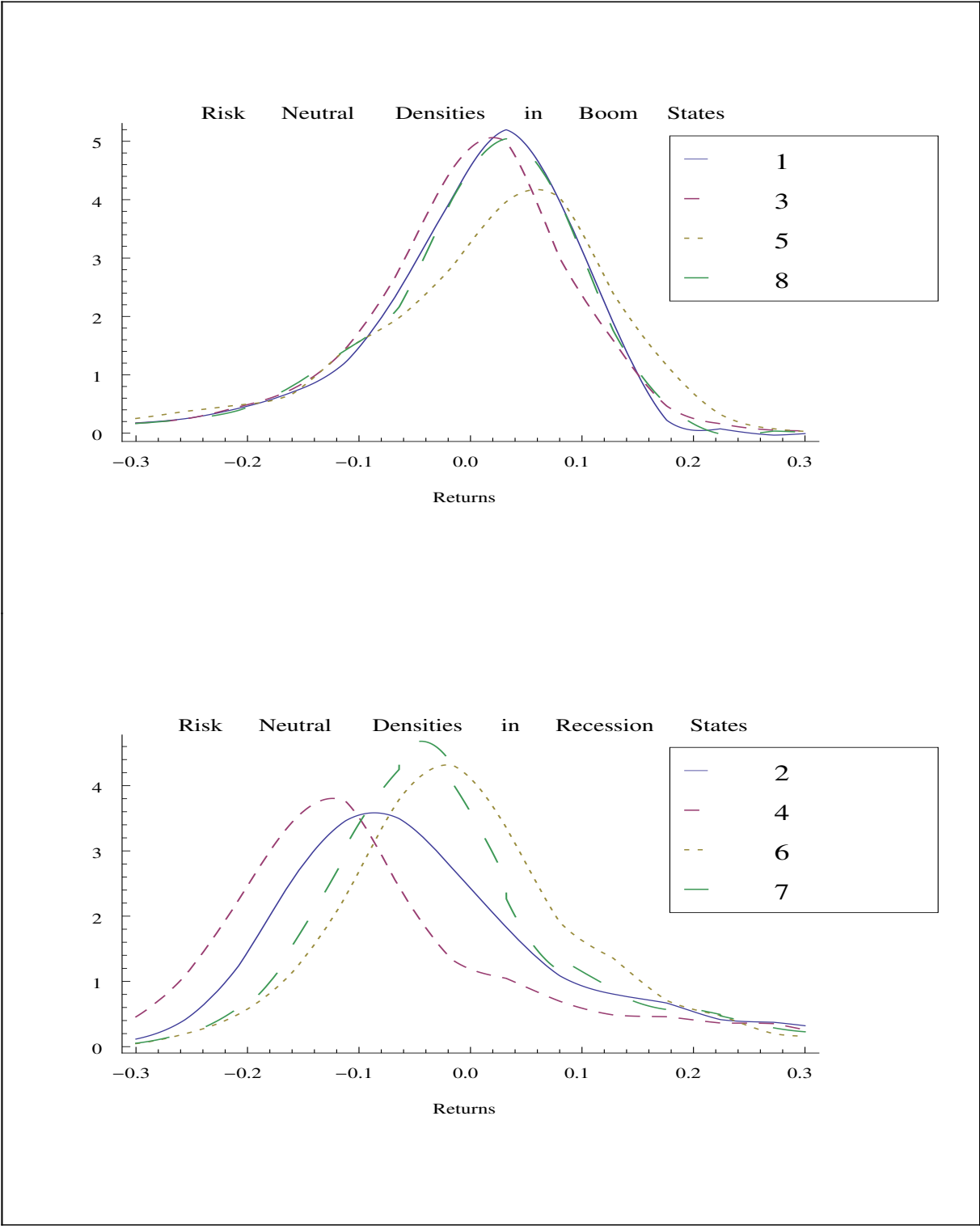
Stock return risk-neutral densities conditional on investors having 80 percent probability of each of the eight regimes (and equal probabilities for the other regimes) are calculated using our estimated model parameters in Table 1 using Monte Carlo simulations as shown in Appendix 2.

Figure 7: Relationship Between Fear Indices and Policy Variables From Kernel Regressions
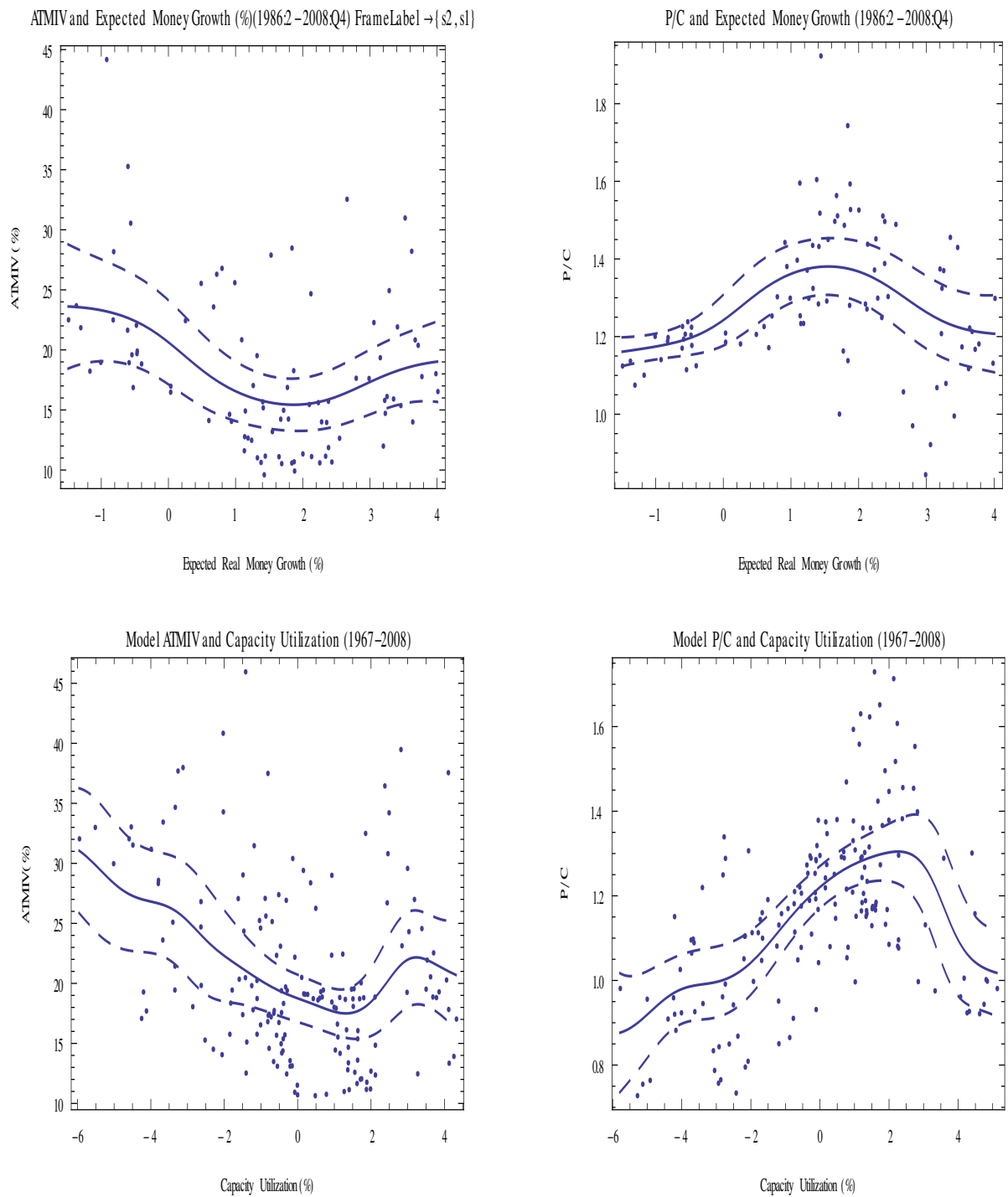
The fear indices (ATMIV and P/C) are shown in Figure 1. Data and model expected fundamentals are shown in Figure 4. The solid lines are the mean fitted values from the nonparametric regressions are estimated with a Gaussian kernel. 95% confidence bands are shown in dashed lines and are constructed as shown in Hardle (1990).

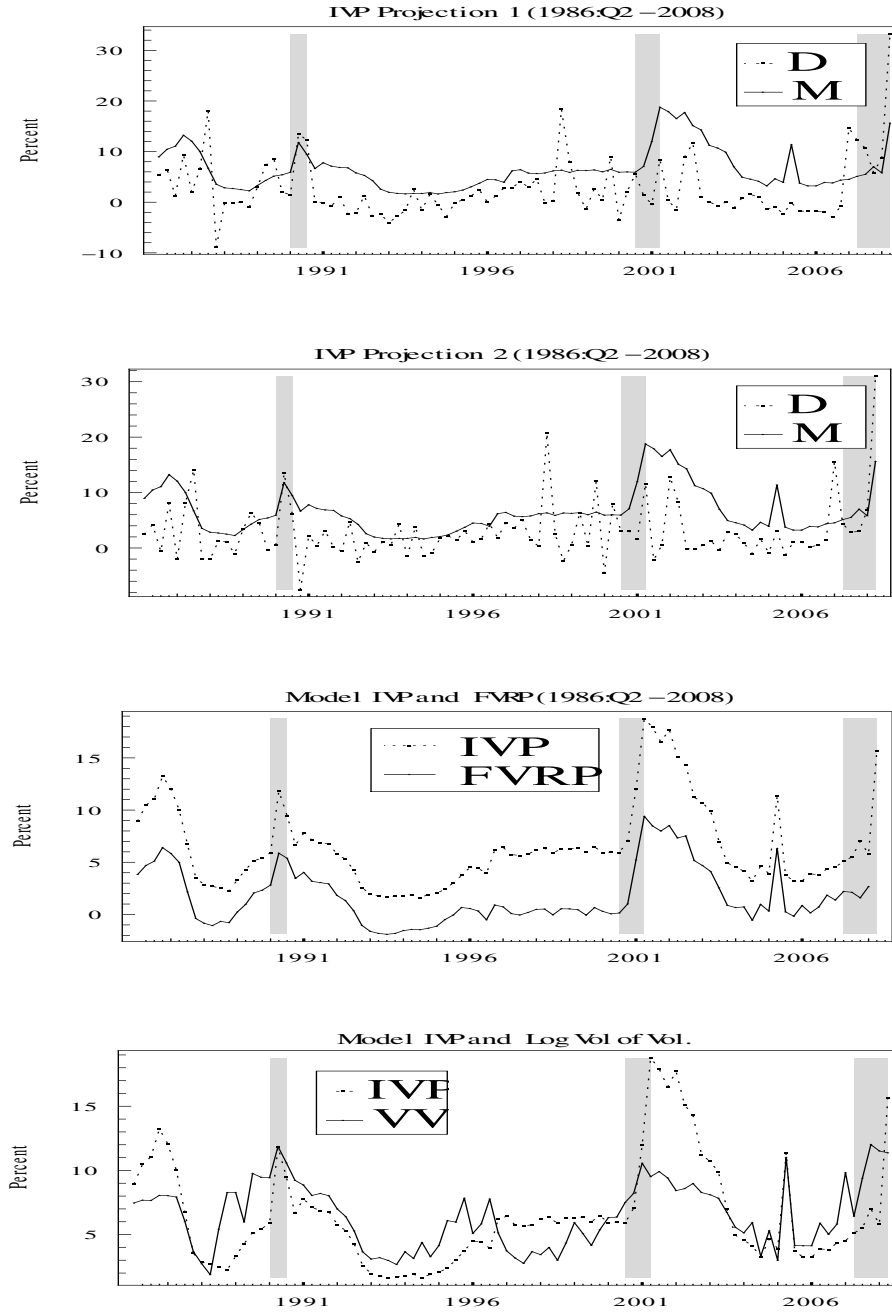Figure 8: Relationship Between ATMIV and Money Growth During Stimulative Periods
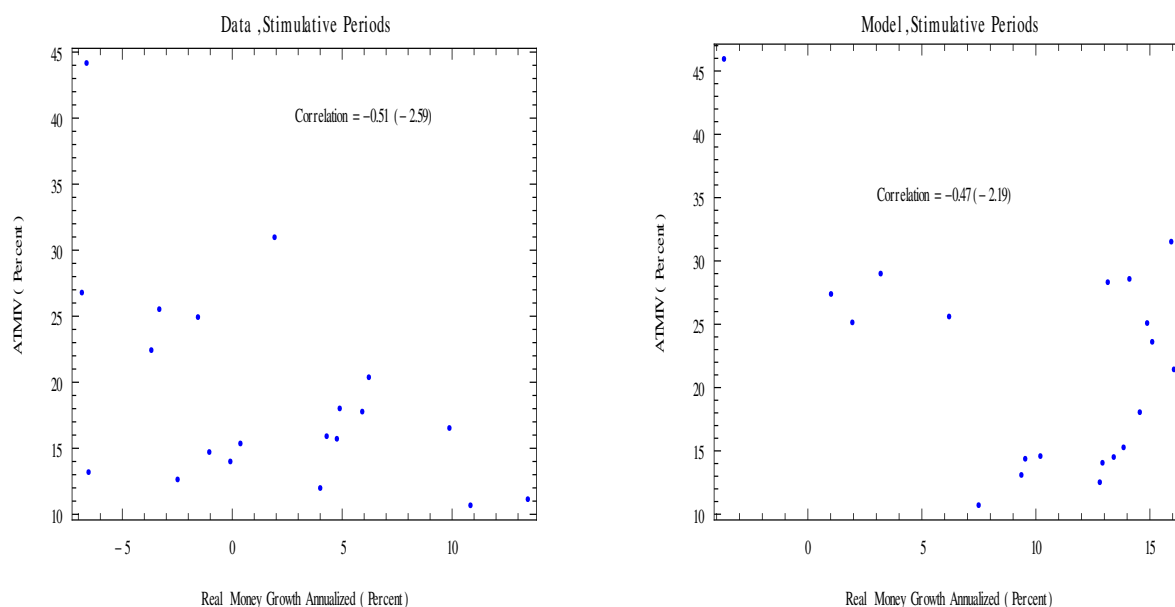
Data ,Stimulative Periods

Correlation = −0.51 (− 2.59)

ATMIV (Percent)

Real Money Growth Annualized (Percent)

Model ,Stimulative Periods

Correlation = −0.47 (− 2.19)

ATMIV (Percent)

Real Money Growth Annualized (Percent)

We define "Stimulative" periods as those where the 3-month Treasury Bill Yield is below the annualized inflation (CPI) rate. In the sub-sample of our date where we have options data (1986:2 − 2008) there are 20 quarters that we characterize as stimulative. The panels report the sample correlation and its t.statistic in parenthesis. The at-the-money implied volatility (ATMIV) at about three months to maturity is constructed at a quarterly frequency from S&P 500 index options prices as discussed in Section 2.2. The model series are calculated using Monte Carlo simulations as shown in Appendix 2. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3.

Figure 9: Relationship Between ATM Implied Volatility and Absolute Changes in ATM Implied Volatility, (1986:Q2-2008)
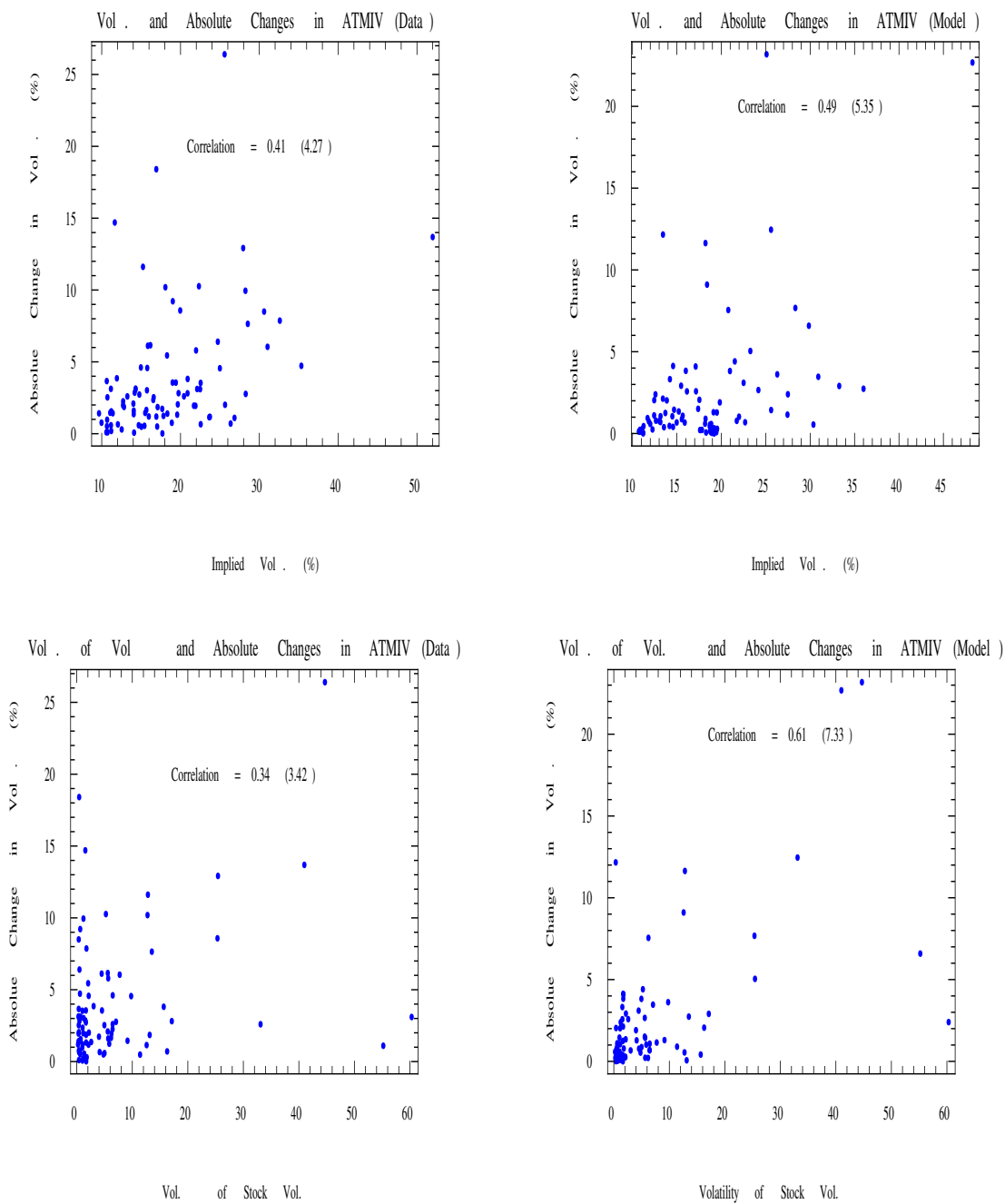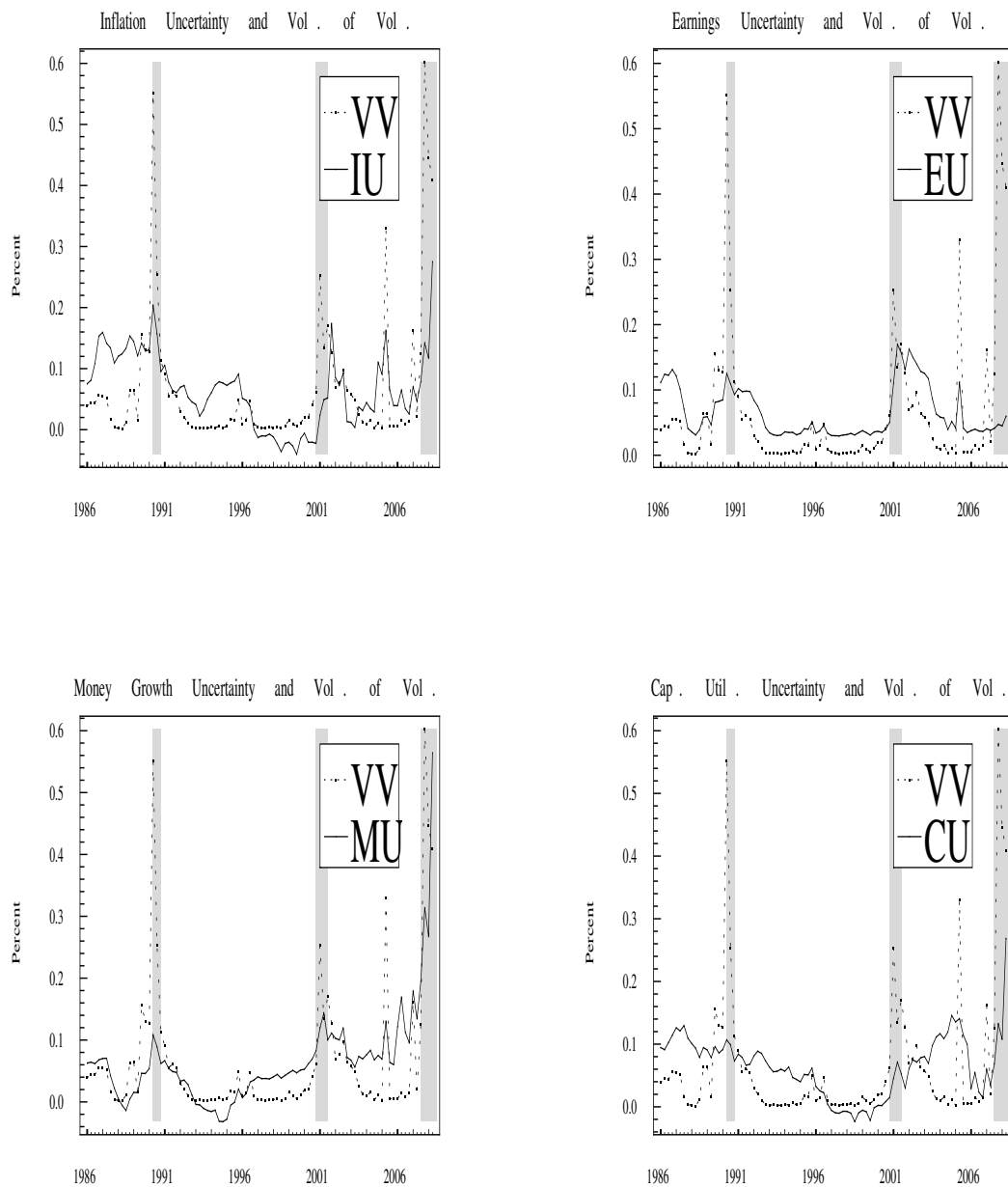

Data and model ATM implied volatility are shown in the top panel of Figure 1. The model volatility of volatility is computed using (22) and the filtered belief series in Figure 3.

Figure 10: Volatility Premium (1986:Q2-2008)

The **first** and **second** panels show the data implied volatility premiums (IVP), which are the difference between the ATMIV in Figure 1 and the P-measure forecast of realized volatility from from Projection 1 (equation (26)) and Projection 2 (equation (27)), respectively. The panels also show the analogous implied volatility premium from our model which is the difference between the model implied volatility in Figure 1 and the model forecast of volatility under the P-measure using simulation methods as described in equation (47) in Appendix 2. The **third** panel shows the IVP and the Forward Volatility Risk Premium (FVRP) from our model. The FVRP is the difference in volatility forecasts under the Q- and P-measures respectively. The **fourth** panel shows the model's IVP and volatility of volatility series as computed using (22). All model variables are computed using the filtered belief series in Figure 3.

Figure 11: Volatility of Stock Market Volatility and Fundamental Uncertainties (1986:Q2-2008)

The model volatility of volatility is computed using (22) and fundamental RMSE uncertainties are computed using (30). All variables are computed using the filtered belief series in Figure 3.

Online Appendix of "Investors' and Central Bank's Uncertainty Measures Embedded in Index Options" by Alexander David and Pietro Veronesi

In this online appendix we discuss the following five issues: Alternative measures of at-the-money implied volatility and the volatility Premium; examining the cyclical properties of deeper out-of-the-money put-call ratios; the out-of-sample performance of the model; alternative money growth measures; and, the relation between the model's volatility-of-volatility and implied volatility premium?

1. Alternative Measures of At-the-Money Implied Volatility and the Volatility Premium

In this subsection we compare two measures at-the-money implied volatility. The first measure is Black-Scholes at-the-money implied volatility (ATMIV) obtained by interpolation of nearly option prices, which we have used in the estimation of our model. The second is a model-free measure of expected forward volatility under the risk-neutral measure from options with about three months to maturity similar to the popularly traded VIX index, which uses 1 month options. We will refer to this latter measure as "3-Month VIX". In this approach the variance contract price estimated from out-of-the-money puts and calls is calculated as:

$$V(t,\tau) = \int_{S(t)}^{\infty} \frac{2(1 - \log[\frac{K}{S(t)}])}{K^2} C(t,\tau,K)dK + \int_{0}^{S(t)} \frac{2(1 + \log[\frac{S(t)}{K}])}{K^2} P(t,\tau,K)dK. \qquad (49)$$

For more details of this calculation see e.g. CBOE Bulletin on VIX, 2003, and among other authors by Britten-Jones and Neuberger (2000) and Bakshi, Kapadia, and Madan (2003).

One essential difference is that the ATMIV uses information in essentially close to the being at-the-money options, while the 3-Month VIX uses information in all traded options, albeit with declining weights for options further out-of-the-money. The time series of the two variables in Figure 12 shows that even though the two series are very highly correlated (correlation coefficient of 93%), the gap between the two series has increased in recent years. In the latter part of the sample, there is in increase in the trading of options that are further away from the money, so undoubtedly, in the 3-Month VIX such options have had a larger influence on the index. In particular, in the early part of our sample (1986-1996), obtained from the CBOE, there were very few options traded more that 5% out-of-the-money, and indeed, the two measures were very close. The reason we calibrated our model using ATMIV is that its model counterpart is faster to compute, using only the prices of at-the-money puts and calls at each date, while a model based 3-Month VIX will require computing at each date a different

set of options prices to match those in the data (for the latter part of the sample there are 30-50 strikes per quarter).

We next consider the implications for the volatility premium. Essentially, our main results are unaffected by replacing ATMIV by the model free approach. Figure 13 is analogous to Figure 10, with the only difference being that the ATMIV is replaced by the 3-Month VIX. As can be seen, the volatility premiums and related measures are very similar in the two figures and all the regression results are quite similar, except that the mean of the 3-month VIX based volatility premiums (using the alternative projections in Section 4.2) are 4.9 and 5.1 percent, and hence very close to our model IVP of 4.4 percent.

2. Cyclical Properties of Deeper Out-of-the-Money Put-Call Ratio

As discussed in the main body of our paper, the 5% OTM put-call ratio is generally pro-cyclical. Here we address the question as to whether the analogous ratio for deeper out-of-the-money options has similar cyclical properties, and in particular look at the 10% OTM options. One issue that we face is that in the first part of our sample, with data provided by the CBOE, there were many quarters where deeper OTM strikes were not traded, and extrapolating the traded prices led to a very volatile and likely unreliable time series. We therefore, limited our analysis to the second part of our sample, where the data are provide by Optionmetrics. The time series of the 5% and 10% OTM P/C ratios are plotted in Figure 14, and as evident from the figure, the two series are highly correlated, with a correlation coefficient 83%. In particular, in both recessions in this shorter sample, the P/C ratios have fallen from their recent levels, while the ratios were high in periods of stronger economic conditions (such as 1996 and 2006). Overall, considering deeper OTM options does not change our main finding that P/C ratios are generally pro-cyclical.

3. 10-Year Out-of-Sample Performance of the Model

In this subsection, we consider the long term out-of-sample performance of the model. In particular, we estimate the parameters of our model on data until 1999, and then test its performance over the following decade. A few comments are useful about the goals of the exercise. First, to be clear, we are not forecasting the future value of fundamentals up to 10 years out. Instead, we are forecasting the reactions of changes in investors' beliefs to fundamental news over the 2000s, based on information that they would have until 1999. Essentially, our model is about about time variation in the relationship between macroeconomic and policy fundamentals in alternative macroeconomic regimes, and hence the test is of the stability of the estimated regimes. Second, asset prices are used in the estimation of

the model for data until 1999, since they provide useful forward-looking information about investors' assessment of regime changes. However, in the out-of-sample exercise, asset prices play no role in determining investors beliefs, which are completely determined by fundamental news. Asset prices are completely and endogenously determined by these beliefs, and we study the ability of the model to explain the realized prices in the 2000s. Third, in the out-of-sample exercise, which used data until 1999, the fundamentals and stock and bond prices have 33 years in common to the full sample estimation, or about three quarters of the full sample. However, the options prices used have only 54 quarters of the same observations as the full sample, closer to half the full sample. Therefore, we should expect the option prices to deviate the most among the all the variables in the out-of-sample exercise. Finally, for the out-of-sample study we have another year of options data available relative to the estimation in the body of the paper where we had options data until 2008.

The model parameters when estimated until 1999 are quite similar to those for the full sample and the differences in the estimations are subtle, but important in a few dimensions. In particular, the beliefs of investors in Figure 15, the fitted fundamentals in Figure 17, stock and bond prices in Figure 16, and implied transition matrices and option moments in Table 7 all look fairly similar to the analogous figures in the main body of the paper. The biggest changes can be seen in the fits of options prices in Figure 18, where comparing with Figure 1, we see larger option pricing errors. In particular in the burst of volatility after the 2001 recession the model ATMIV overshot the historical ATMIV much more substantially than in the full sample estimation. Similarly, the model P/C ratio in Figure 18 overshoots the historical P/C in 2005. Despite these errors, the overall trends in the two figures are similar, again displaying the pro (counter) cyclicality of ATMIV (P/C).

More formally, we compare the fits of the in-sample and out-of-sample exercise in Table 8. For the in-sample part, we find very similar R^2 for all the fundamental and pricing variables as in Table 3. However, in the out-of-sample part, there are some differences. In particular, the explanatory power of the stock and bond prices are lower, and we find statistically significant for α coefficients for 4 of the 9 variables. In the in-sample exercise, these coefficients are all statistically insignificant, as the model parameters were chosen to bring the model moments close to the data moments. Nonetheless, the strong R^2 in all cases show that the main implications of the model are intact even considering fundamental data 10 years out of sample.

4. Alternative Money Growth Measures

In this subsection we compare some alternative money growth measures. In our analysis of the body of the paper we use the real growth in M1, and we provide some comments for this choice.

3

The top panel of Figure 19 shows the growth in nominal and real M1. While the two series are highly correlated (correlation coefficient of nearly 70 percent), it is evident that the real growth series displays greater variation in the first half of our sample, when inflation was generally higher than in the second half of the sample. Indeed, average nominal M1 growth did not display the sharp declines in the first three recessions of our sample that were evident in real money growth. Overall, the fit of our regime model in the first half of the sample is not as good if we use the growth in nominal M1.

The bottom panel of Figure 19 shows the growth in real M1 and M2. It is useful to note that M2 is a broader measure of money supply mainly due to the inclusion of savings deposits, which are not part of M1. The two series are highly correlated (correlation coefficient of nearly 60 percent) however, there are some important differences in the 1990s. In particular, real M1 growth was strong following the 1991 recession, while real M2 growth was nearly flat. In the late 1990s period of very rapid earnings growth, real M1 growth was tight, but real M2 growth was quite stable. The use of real M2 in our estimation lowers makes investors' beliefs more smooth through the 1990s and lowers the ability of our model to fit asset prices. Overall, as a signal of real economic activity, it appears real M1 growth is a better measure. On a similar note, Buraschi and Jiltsov (2007) use M1 growth to fit the term structure of interest rates.

5. What Explains the Model's Implied Volatility Premium?

To understand the driving determinants, we investigate further some model generated time series. A natural candidate to consider is the volatility of volatility (VV).

$$\text{IVP}^M(t) = 5.216 + 1.663 \log(\text{VV}^M)(t); \quad \bar{R}^2 = 0.401. \tag{50}$$
$$[11.194]^* \quad [4.208]^*$$

In Figure 9 we showed that volatility fluctuations are higher in periods when the VV is higher. Equation (50) tells us that the options are priced more dearly when there is greater volatility in stock volatility, suggesting that volatility is itself a priced risk factor. We study this further by regressing the two components of the volatility premium on the VV:

$$\text{FVRP}^M(t) = 0.761 + 1.289 \log(\text{VV}^M)(t); \quad \bar{R}^2 = 0.530. \tag{51}$$
$$[3.145]^* \quad [5.243]^*$$
$$\text{IVP}^M(t) - \text{FVRP}^M(t) = 4.462 + 0.387 \log(\text{VV}^M)(t); \quad \bar{R}^2 = 0.103. \tag{52}$$
$$[16.084]^* \quad [1.991]^*$$

4

Equation (51) tells us that a large part of the fluctuation in the model IVP from the VV is captured by the fluctuation in the FVRP portion of the IVP. In fact the explanatory power of the VV for the FVRP is well over 50 percent. In contrast, the explanatory power for the the residual portion, IVP-FVRP, on the VV is only 11 percent. Therefore, only a small part of the residual premium, which essentially captures the non-Gaussian aspect of the return distribution under the risk-neutral measure is related to the VV, which directly impacts the fourth moment of returns. Finally, we note that returns in our model are non-normal also because of the jumps in the earnings and kernel processes. However, the jump parameters are all constant, so that any time-variation in higher moments in the model arises from the variation in beliefs about macroeconomic regimes, which affects the VV and as well as other higher order moments.

Figure 12: Black-Scholes and Model Free Implied Volatility (1986:Q2-2008)

ATM Implied Volatility
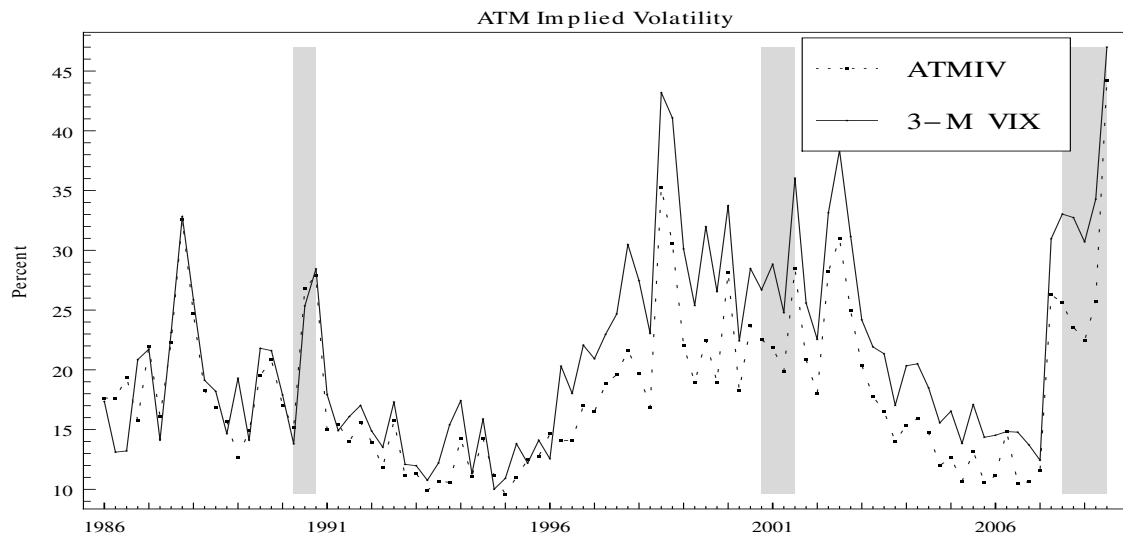
The line labeled as "ATMIV" is the Black-Scholes interpolated at-the-money implied volatility, which we have used in the estimation of our model. The line labeled as "3-M VIX" is a measure of expected forward volatility from options with about three months to maturity obtained using the formula in (49), similar to the construction of the VIX index. Shaded areas represent NBER-dated recessions.

Table 7: Model Implied Transition Probabilities, Stationary Probabilities, Stock and Bond Price Valuations, Fear Indices, and Higher Moments in the Eight States

Implied Annual Quarterly Transition Probability (Percent) Matrix

	1	2	3	4	5	6	7	8
1	80.0	0.4	16.	0.0	1.6	0.3	0.4	0.8
2	0.2	72.0	7.4	15.0	0.	00.5	4.2	0.4
3	0.4	3.4	83.0	0.4	0.0	0.5	4.0	8.1
4	0.2	15.0	0.75	80.0	0.0	00.2	4.2	0.0
5	0.3	0.0	0.35	0.0	990	0.3	0.0081	0.0
6	7.6	3.6	1.3	3.8	0.1	76.0	0.2	7.8
7	8.1	0.2	0.82	0.2	0.1	7.5	83.0	0.4
8	3.7	0.2	8.5	0.0	0.0	0.0	0.20	87.0

Implied 5-Year Transition Probability (Percent) Matrix

	1	2	3	4	5	6	7	8
1	34.8	3.7	38.4	1.3	5.2	1.5	4.8	10.1
2	3.43	31.1	16.9	27.7	0.1	3.0	12.9	4.8
3	4.9	7.7	47.1	3.7	0.2	2.8	10.9	22.8
4	3.2	27.7	8.3	42.9	0.1	2.7	13.2	1.9
5	1.3	0.2	1.7	0.2	94.9	0.9	0.2	0.6
6	16.2	9.4	14.9	10.0	1.03	25.5	3.3	19.7
7	21.2	2.70	11.4	2.4	1.2	15.2	39.8	6.0
8	9.9	2.2	26.8	0.7	0.6	0.7	3.0	56.0

Implied Stationary Probabilities, P/E Ratios, the Term Structure, Option Prices, and Higher Order Moments

| #	$\beta(\%)$	$\theta(\%)$	$\rho(\%)$	$\omega(\%)$	$\bar{\pi}(\%)$	P/E	$i_{0.25}(\%)$	$S(\%)$	ATM(%)	P/C	Skw	Kur	$\rho_{SV}(\%)$
1	1.7	6.0	0.9	1.6	10.1	14.9	4.4	1.7	15.5	1.7	−1.4	6.6	−67.9
2	6.9	−5.0	0.9	5.3	7.7	7.7	11.8	−0.2	33.8	0.8	1.8	9.5	98.0
3	6.9	6.0	9.3	1.6	24.7	12.2	14.0	−2.2	16.8	1.2	−0.4	5.5	−76.9
4	8.8	−5.0	−2.3	−5.7	7.0	7.0	13.5	−0.8	39.6	0.7	2.0	7.6	98.8
5	1.7	7.8	0.9	−2.7	16.9	29.2	4.1	0.0	23.2	1.4	−1.5	6.7	−85.8
6	−0.1	−5.6	−6.0	5.3	4.1	11.3	0.1	2.4	22.7	0.8	0.7	5.7	83.4
7	6.9	−5.0	−6.0	−5.7	9.7	10.0	9.4	−1.3	23.5	0.7	1.3	5.5	95.9
8	1.7	6.0	−6.0	1.6	19.6	17.0	2.3	1.4	16.3	1.6	−1.8	9.9	−84.8

The **top** and **middle panels** report the quarterly and 5-year implied transition probability matrix between the eight states implied from the generator matrix elements displayed in Table 1. Rows may not sum to one due to rounding. The **bottom panel** report the implied stationary probabilities and implied prices of the variables used in the SMM estimation procedure in the eight states. $\bar{\pi}$ is the stationary probability of each state; P/E the price-earnings ratio, and, $i_{0.25}$, the 3-month Treasury yield, and S the 5-year less 1-year Treasury yield; ATM is the at-the-money implied volatility (ATMIV); P/C is the ratio of 5% OTM put-to-call implied volatilities; Skw and Kur are the skewness and kurtosis of the 3-month risk-neutral return distribution, respectively; ρ_{SV} is the correlation between stock returns and stock variance calculated using (23). The P/E ratio and bond yields are computed as shown in Proposition 1. Implied Volatility, Put-Call Ratio and the higher order moments are for options with three months to maturity as calculated using Monte Carlo simulations as shown in Appendix 2.

Table 8: In-Sample (1967-1999) and Out-of-Sample (2000-2009) Model Fits for Expected Fundamentals, Stocks, Bonds, and Options Prices from SMM Procedure

| Variable | ————In-Sample———— | | | ————Out-of-Sample———— | | |
	α	β	R^2	α	β	R^2
Inflation	-0.052	1.477	0.613	-0.008	2.801	0.469
	[-0.387]	[7.182]*		[-2.076]*	[4.541]*	
Real Earnings Growth	-0.893	1.463	0.142	-0.053	4.092	0.210
	[-1.562]	[3.371]*		[-2.174]*	[2.998]*	
De-Meaned Capacity Utilization	-0.067	1.415	0.671	-.005	[1.473]	0.647
	[1.476]	[8.505]*		[-0.721]	[5.191]*	
Real Money Growth	-0.067	1.961	0.335	-0.005	1.474	0.336
	[-0.405]	[5.5887]*		[-1.352]	[3.202]*	
P/E Ratio	-2.878	1.182	0.728	4.751	0.622	0.215
	[-1.891]	[14.057]*		[3.694]*	[2.491]*	
3-Month Yield	0.001	0.793	0.483	[-0.009]	0.891	0.348
	[0.761]	[5.182]*		[-0.879]	[3.016]*	
5-Year Minus 1-Year Treasury Yield	0.002	0.917	0.627	-.002	1.093	0.398
	[0.757]*	[8.132]*		[-0.637]	[3.639]*	
ATM	0.028	0.922	0.337	0.081	0.578	0.275
	[1.086]	[5.5373]*		[2.706]*	[2.421]*	
P/C	0.15	0.654	0.492	0.431	0.482	0.315
	[1.777]	[6.662]*		[4.777]*	[4.548]*	

We display the fits of the variables used in our SMM procedure: the fundamentals, and the five pricing variables, which are used to overidentify the model. For the four fundamentals we provide the regression results for the equation $x(t) = \alpha + \beta \, \mathbb{E}[x|\mathcal{F}_{t-1}] + \epsilon(t)$, where $x(t)$ is the realized growth and $\mathbb{E}[x|\mathcal{F}_t]$ is investors' conditional expected growth of the fundamental under consideration. The conditional expected growth is obtained from the filtered probabilities $\pi(t)$ displayed in Figure 3, and for earnings, for example, is given by $\sum_i^N \theta_i \, \pi_i(t)$. For the price series, we present the regression results for the equation $p(t) = \alpha + \beta \, p(\pi(t)) + \epsilon(t)$, where $p(t)$ and $p(\pi(t))$ are the realized and model price conditional on investors' beliefs at t, respectively. T-statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation. The symbol * denotes statistical significant at the 5% level.

Figure 13: Volatility Premium Using the 3-Month VIX (1986:Q2-2008)
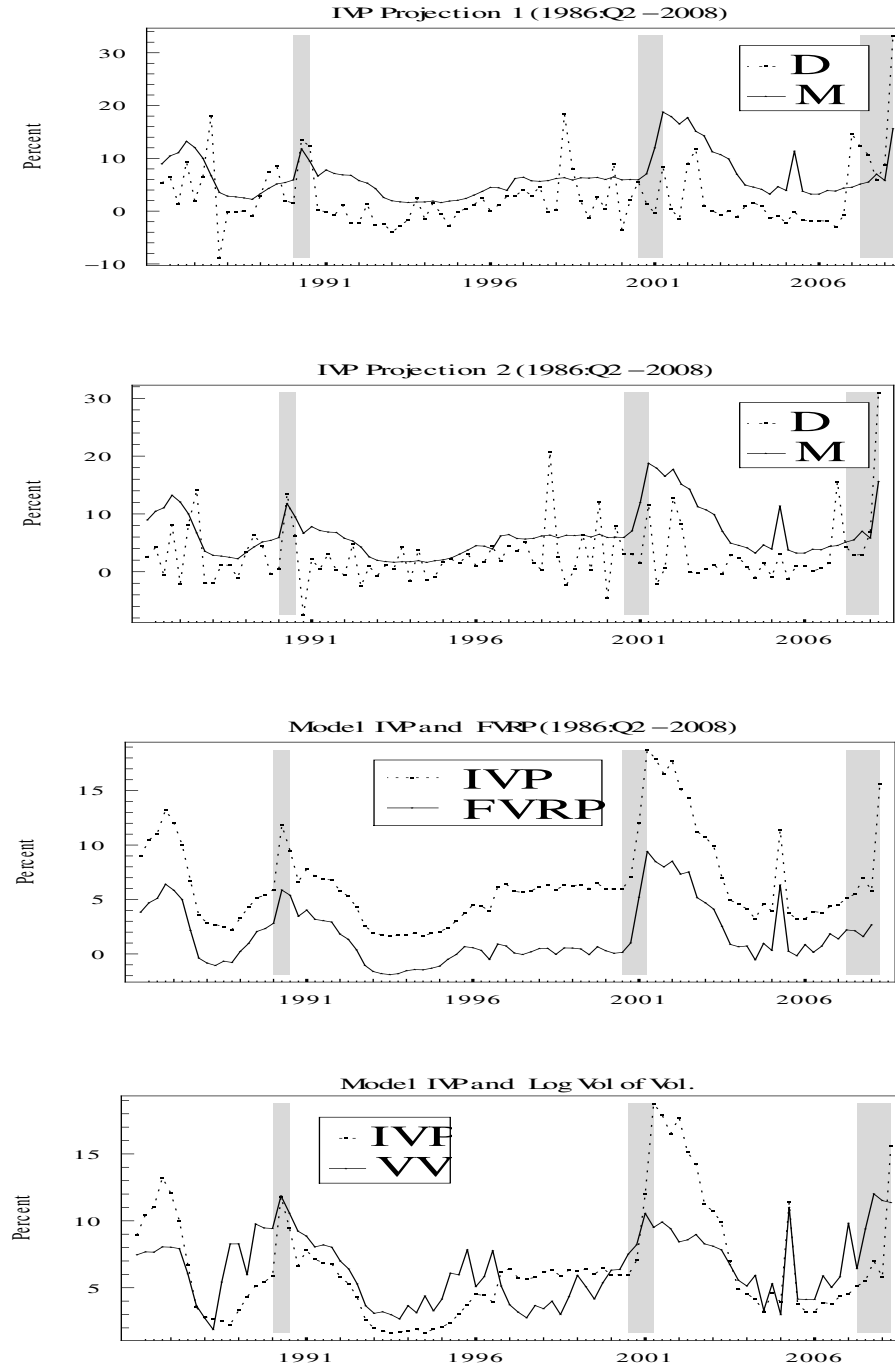
This panel is constructed similarly to that in Figure 10, with the only difference being that the Black-Scholes implied volatility (ATMIV) is replaced by the 3-Month VIX. These two alternative measures can be seen in Figure 12. The **first** and **second** panels show the data implied volatility premiums (IVP), which are the difference between the ATMIV in Figure 1 and the P-measure forecast of realized volatility from from Projection 1 (equation (26)) and Projection 2 (equation (27)), respectively. The panels also show the analogous implied volatility premium from our model which is the difference between the model implied volatility in Figure 1 and the model forecast of volatility under the P-measure using simulation methods as described in equation (47) in Appendix 2. The **third** panel shows the IVP and the Forward Volatility Risk Premium (FVRP) from our model. The FVRP is the difference in volatility forecasts under the Q- and P-measures respectively. The **fourth** panel shows the model's IVP and volatility of volatility series as computed using (22). All model variables are computed using the filtered belief series in Figure 3.

Figure 14: P/C Ratios for Alternative Levels of Moneyness (1996:Q3-2008)

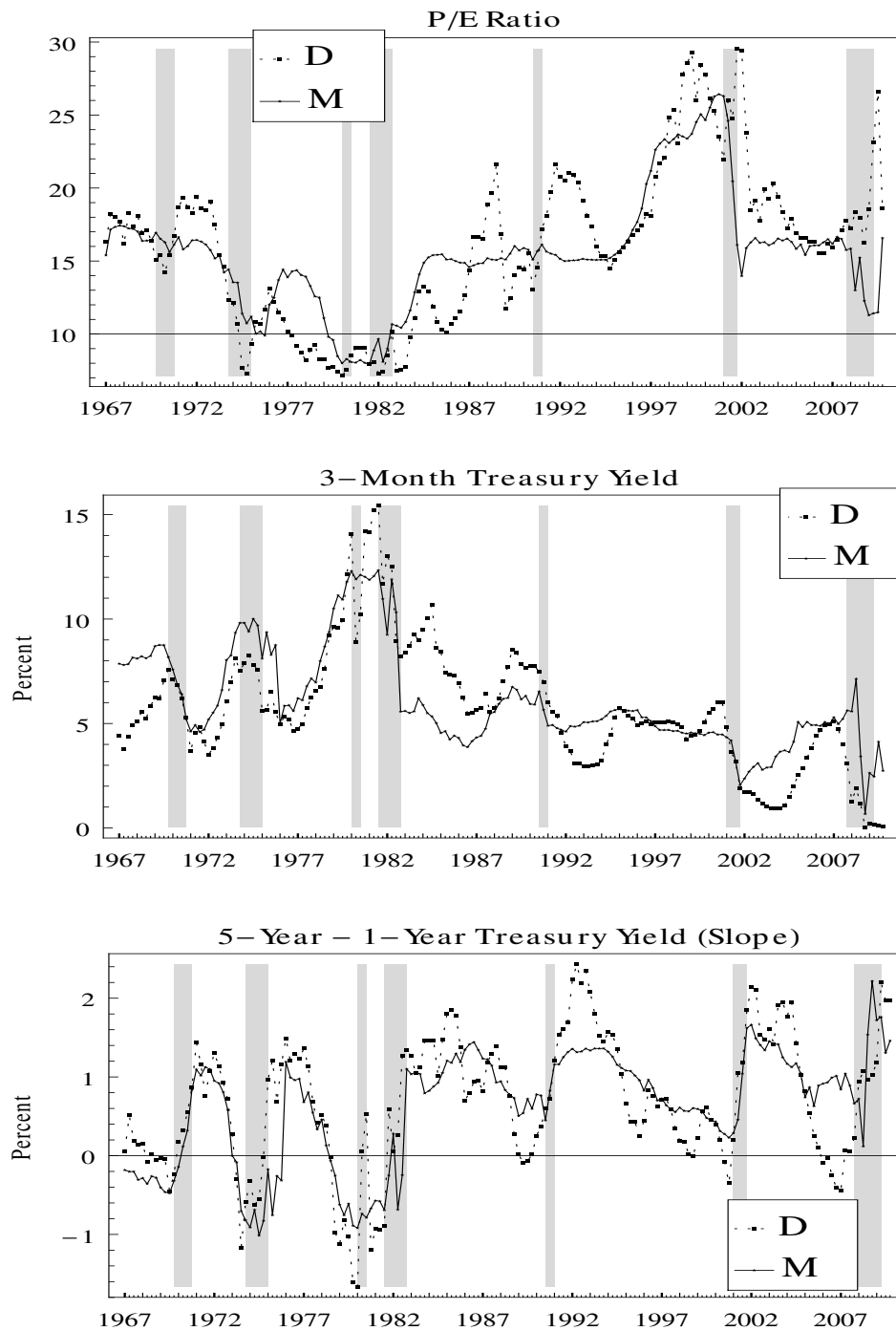5% and 10% Out−of−the−Money Implied Volatility Put−Call Ratios
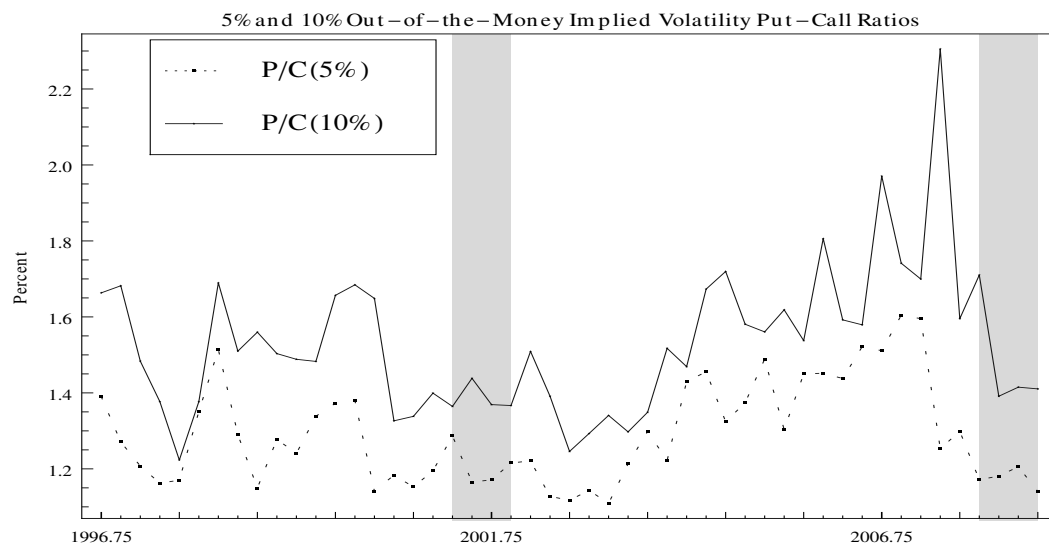
This panel is constructed similarly to the data P/C ratios in Figure 1, with alternative level of moneyness. Shaded areas represent NBER-dated recessions.

Figure 15: Conditional Probabilities of Eight Regimes (1967-2009) With Model Parameters Estimated on Data Until 1999
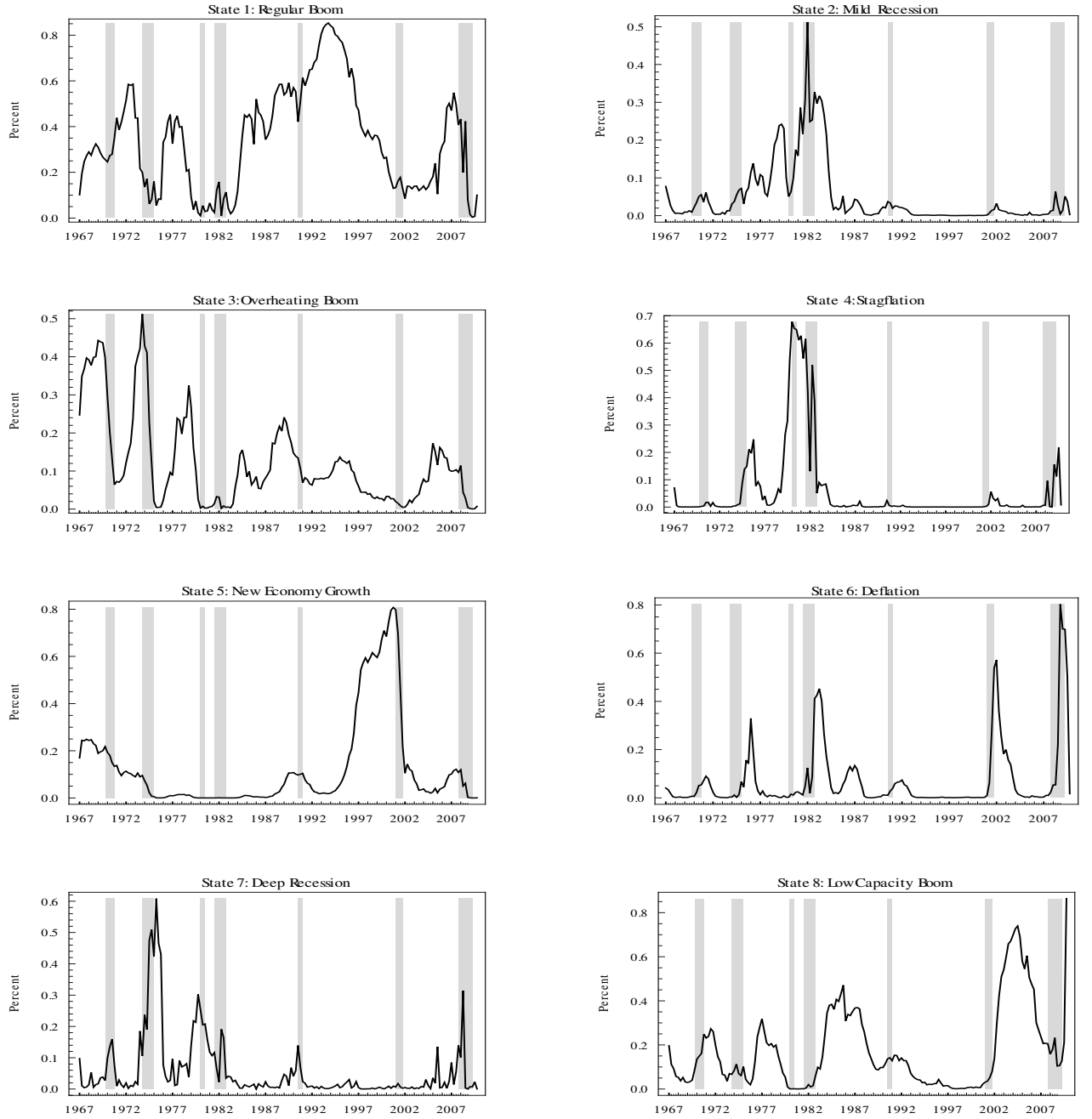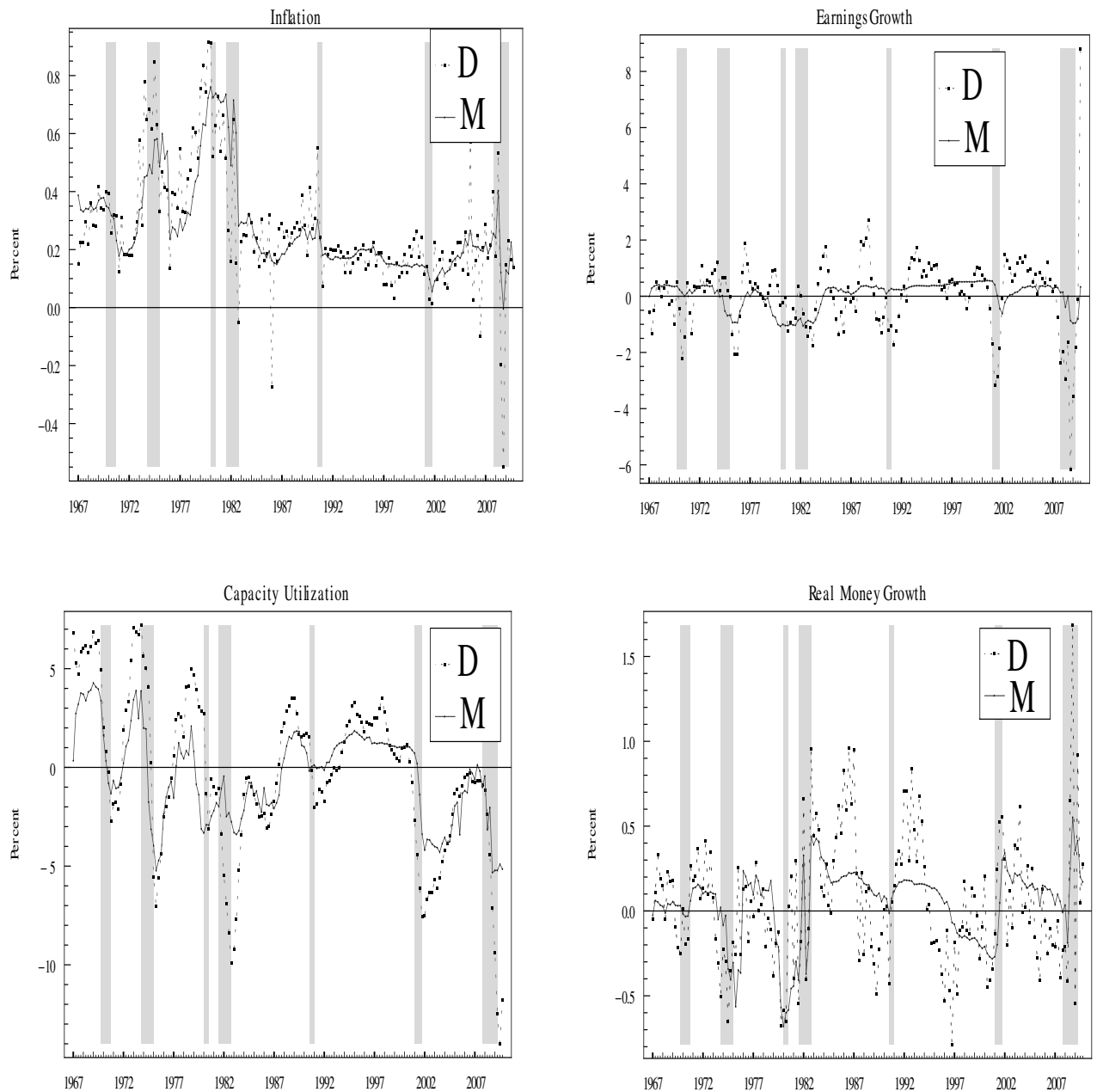

The regime definitions are in Table 2. β, θ, ρ, and ω are the drifts of inflation, earnings growth, de-meaned capacity utilization, and real money growth, respectively. The filtered beliefs are obtained from the SMM procedure in Appendix 2. The calibrated values of the parameters are shown in Table 1. Shaded areas represent NBER-dated recessions.

Figure 16: Stock and Bond Prices: Empirical and Model Fitted (1960-2009) With Model Parameters Estimated on Data Until 1999


Historical values of financial and fundamental variables series (D) are in solid lines and their fitted values (M) from the SMM estimation procedure in the Appendix are in dashed lines. The estimated values of the parameters are shown in Table 1 and the implied asset price valuations are in Table 2 The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 17: Fundamentals: Empirical and Model Fitted (1960-2009) With Model Parameters Estimated on Data Until 1999

Historical values of financial and fundamental variables series (D) are in solid lines and their fitted values (M) from the SMM estimation procedure in Appendix B are in dashed lines. The calibrated values of the parameters are shown in Table 1. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 18: In Sample (1986:Q2-1999) and Out-of-Sample (2000-2009) Fit for ATMIV and P/C
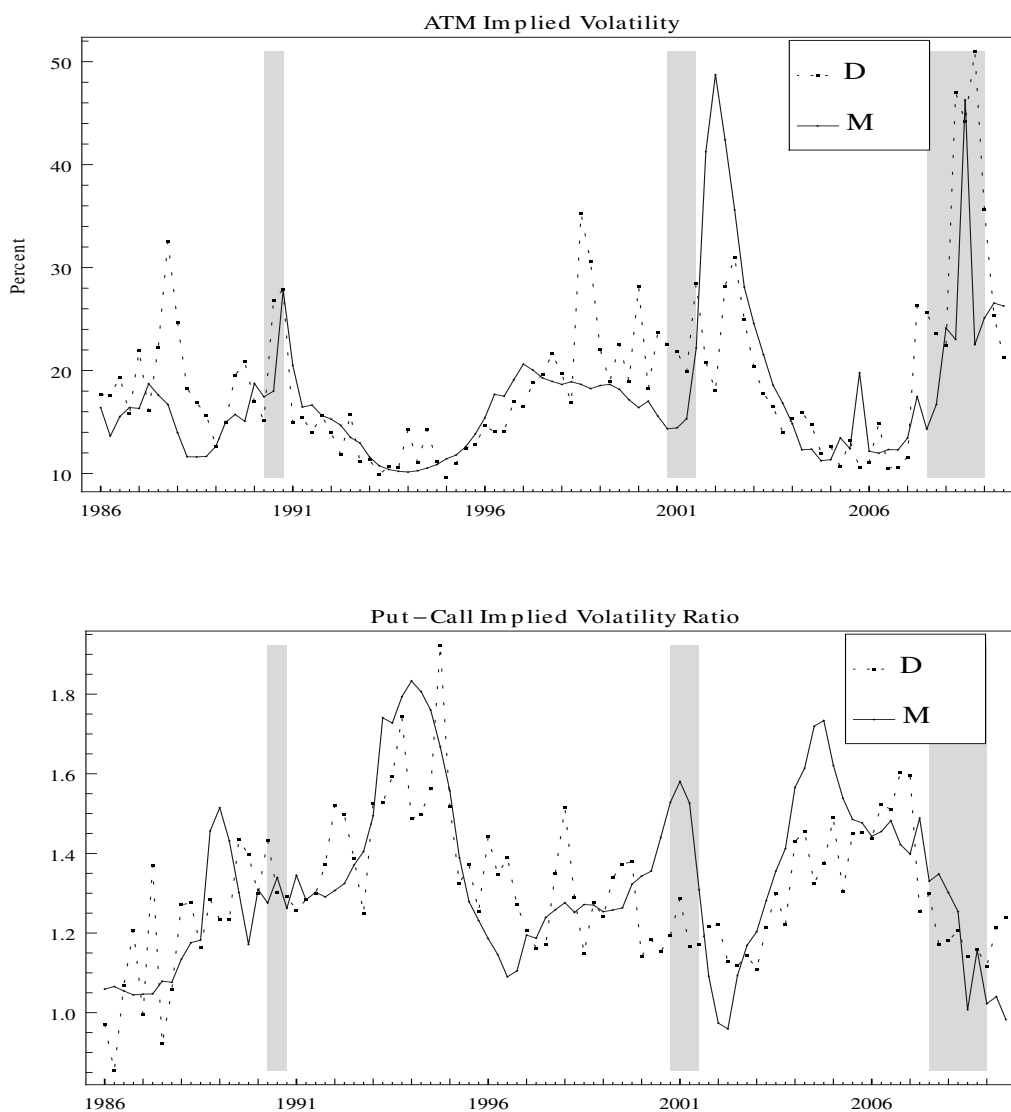
ATMIV and P/C are as in the footnote to Figure 1. The model parameters are estimated using fundamental and S&P 500 and Treasury data from 1967-1999, as well as option prices from 1986:Q2 – 1999. The fits of the option prices for 2000-2009 are obtained from using the fundamental data for this period to update investors' beliefs and using Monte Carlo simulations to calculate options prices as described in the above mentioned footnote. Shaded areas represent NBER-dated recessions.

Figure 19: Alternative Money Growth Measures

Nominal and Real Money Growth

Real Growth in M1 and M2