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Memorandum № 2

Cartesian State Vectors $\longrightarrow$ Keplerian Orbit Elements

Inputs

- cartesian state vectors
  - position vector $\mathbf{r}(t)$ [m]
  - velocity vector $\dot{\mathbf{r}}(t)$ $[\frac{\mathrm{m}}{\mathrm{s}}]$
- standard gravitational parameter $\mu = GM$ of the central body, if different from Sun (G...Newtonian constant of gravitation $[\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}]$, M...central body mass [kg])

Outputs

- a traditional set of Keplerian Orbit Elements
  - Semi-major axis a [m]
  - Eccentricity e [1]
  - Argument of periapsis ω [rad]
  - Longitude of ascending node (LAN) Ω [rad]
  - Inclination i [rad]
  - Mean anomaly M [rad]

1 Algorithm

1. Preparations:
   a) Calculate orbital momentum vector $\mathbf{h}$ $\left[\frac{\mathrm{m}^2}{\mathrm{s}}\right]$:
   $$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \tag{1}$$
   b) Obtain the eccentricity vector $\mathbf{e}$ [1] from
   $$\mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{\|\mathbf{r}\|} \tag{2}$$
   with standard gravitational parameter $\mu = \mu_{\odot} = 1.327\,124\,400\,18 \cdot 10^{20}\,(\pm 8 \cdot 10^{9})\,\frac{\mathrm{m}^3}{\mathrm{s}^2}$ for the Sun as central body.
   c) Determine the vector $\mathbf{n}$ $\left[\frac{\mathrm{m}^2}{\mathrm{s}}\right]$ pointing towards the ascending node and the true anomaly ν [rad] with
   $$\mathbf{n} = (0,0,1)^{\mathrm{T}} \times \mathbf{h} = (-h_y, h_x, 0)^{\mathrm{T}} \qquad \nu = \begin{cases} \arccos \frac{\langle \mathbf{e}, \mathbf{r} \rangle}{\|\mathbf{e}\|\|\mathbf{r}\|} & \text{for } \langle \mathbf{r}, \dot{\mathbf{r}} \rangle \geq 0 \\ 2\pi - \arccos \frac{\langle \mathbf{e}, \mathbf{r} \rangle}{\|\mathbf{e}\|\|\mathbf{r}\|} & \text{otherwise.} \end{cases} \tag{3}$$
2. Calculate the orbit inclination i by using the orbital momentum vector $\mathbf{h}$, where h_z is the third component of $\mathbf{h}$:
   $$i = \arccos \frac{h_z}{\|\mathbf{h}\|} \tag{4}$$
3. Determine the orbit eccentricity e [1], which is simply the magnitude of the eccentricity vector $\mathbf{e}$, and the eccentric anomaly E [1]:
   $$e = \|\mathbf{e}\| \qquad E = 2 \arctan \frac{\tan \frac{\nu}{2}}{\sqrt{\frac{1+e}{1-e}}} \tag{5}$$
4. Obtain the longitude of the ascending node Ω and the argument of periapsis ω:
   $$\Omega = \begin{cases} \arccos \frac{n_x}{\|\mathbf{n}\|} & \text{for } n_y \geq 0 \\ 2\pi - \arccos \frac{n_x}{\|\mathbf{n}\|} & \text{for } n_y < 0 \end{cases} \qquad \omega = \begin{cases} \arccos \frac{\langle \mathbf{n}, \mathbf{e} \rangle}{\|\mathbf{n}\|\|\mathbf{e}\|} & \text{for } e_z \geq 0 \\ 2\pi - \arccos \frac{\langle \mathbf{n}, \mathbf{e} \rangle}{\|\mathbf{n}\|\|\mathbf{e}\|} & \text{for } e_z < 0 \end{cases} \tag{6}$$
5. Compute the mean anomaly M with help of Kepler's Equation from the eccentric anomaly E and the eccentricity e:
   $$M = E - e \sin E \tag{7}$$
6. Finally, the semi-major axis a is found from the expression
   $$a = \frac{1}{\frac{2}{\|\mathbf{r}\|} - \frac{\|\dot{\mathbf{r}}\|^2}{\mu}}. \tag{8}$$

2 Constants and Conversion Factors

Universal Constants

Symbol	Description	Value	Source
G	Newtonian constant of gravitation[1]	$G = 6.67428(67) \cdot 10^{-11} \ \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$	[1, pp. 686–689]

Conversion Factors

Conversion		Source
Astronomical Units ⟶ Meters	$1\,\mathrm{AU} = 1.495\,978\,707\,00 \cdot 10^{11}\ (\pm 3)\,\mathrm{m}$	[4, p. 370 f.]
Julian Days ⟶ Seconds	$1\,\mathrm{d} = 86\,400\,\mathrm{s}$	[3]
Degrees ⟶ Radians	$1^\circ = 1^\circ \cdot \frac{\pi}{180^\circ}\,\mathrm{rad} \approx 0{,}017453293\,\mathrm{rad}$	

3 References

Equations 1 and 8: [2, p. 28]; Eq. 2: [8]; Eq. 3: [9, 12]; Eq. 5: [7, 10]; Eq. 4: [11]; Eq. 6: [6, 9]; Eq. 7: [5, p. 26]; Value for $\mu_\odot$: [3].

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[1] The numbers in parentheses in $6.67428(67) \cdot 10^{-11}$ are a common way to state the uncertainty; short notation for $(6.67428 \pm 0.0000067) \cdot 10^{-11}$.