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CISC2006
Algorithm Design and Analysis

Xiaowei Wu

SKL of IOTSC, Department of CIS

University of Macau

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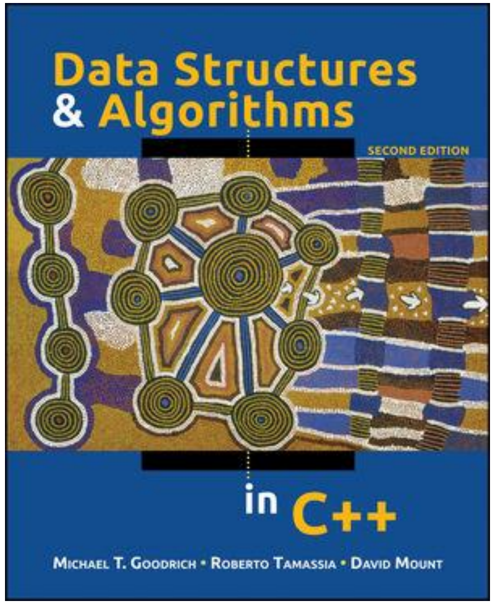
Materials

- **Textbook:**
  - Data Structures and Algorithms in C++ (2^{nd} Edition)
  - Data Structures and Algorithms in Java (6^{th} Edition)
  - by M.T. Goodrich, R. Tamassia, and M.H. Goldwasser
- Materials modified from
  - Algorithms and Data Structures 2
    by Monika Henzinger (University of Vienna)
  - Discrete Mathematics
    by T-H. Hubert Chan (University of Hong Kong)
  - Design and Analysis of Algorithms (Advanced Class)
    by Tak-Wah Lam (University of Hong Kong)



Topics (Part-1)

- Algorithm Basics (CISC2003)
  - Big-Oh, Array, Stacks, Queues, Linked Lists, Trees …
- Priority Queues and Binary Search Trees
  - Priority Queue Abstract Data Type and Implementation
  - Heap, Heapsort
  - Binary Search Trees: ADT, AVL Trees
- Sorting:
  - Merge-Sort, Quick-Sort
  - Complexity Analysis and Lower Bounds

Topics (Part-2)

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- Algorithm Design and Analysis Techniques
  - Recursion, Divide-and-Conquer, Greedy
  - Dynamic Programming

- Graphs:
  - Definition and Representation
  - Graph Traversals: BFS, DFS
  - Shortest Path (Dijkstra's Algorithm)
  - Minimum Spanning Tree (Kruskal's and Prim's Algorithm)

Algorithm Basics

Algorithm

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- **Problems**: input -> output
- **Algorithms**: a sequence of instructions that can produce the desired output from any given input.
- **Correctness**: always compute the desired output.
- **Time complexity/efficiency**:
  - how many steps: a function of the input size (usually denoted by $T(n)$, where n is the input size).
  - asymptotic behavior: the trend (i.e., when the input is large); big O notation. E.g., merge sort $O(n \log n)$ time; bubble sort $O(n^2)$ time.

Terminologies

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- **Problem vs. Instance**
  - E.g., sorting is a problem; array $(3,4,1,2)$ is an instance
  - E.g., MST is a problem; a graph $G(V,E)$ is an instance

- **Algorithm vs. Solution**
  - An algorithm is for solving a problem
  - A solution is to an instance

Big O notation

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- $T(n)$ be a function of n
- Big O notation: $T(n) = O(f(n))$ if

  there exist fixed positive constants c and N such that,

  for all $n \geq N$, $T(n) \leq c \cdot f(n)$.

Example:

- $T(n) = 2n^2 + 50n = O(n^2)$
- $T(n) = 5n \log n + 10\sqrt{n} = O(n \log n)$

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Big O notation

- $T(n)$ be a function of n
- Big O notation: $T(n) = O\big(f(n)\big)$ if

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Example:

- $T(n) = 2n^2 + 50n = O(n^3)$
- $T(n) = 5n\log n + 10\sqrt{n} = O(n^2)$

Big O notation

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- $T(n)$ be a function of n
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  there exist fixed positive constants c and N such that,

  for all $n \geq N$, $T(n) \leq c \cdot f(n)$.

- Big O notation reflects the asymptotic behavior:

  the complexity trend when the input is large.
- Big O only guarantees an **upper bound.**

Big Ω notation

- $T(n)$ be a function of n
- Big Ω notation: $T(n) = \Omega\big(f(n)\big)$ if

  there exist fixed positive constants c and N such that,

  for all $n \geq N$, $T(n) \geq c \cdot f(n)$.

Example:

- $T(n) = 2n^2 + 50n = \Omega(n^2)$
- $T(n) = 5n \log n + 10\sqrt{n} = \Omega(n)$

Big Ω notation

- $T(n)$ be a function of n
- Big Ω notation: $T(n) = \Omega(f(n))$ if

  there exist fixed positive constants c and N such that,

  for all $n \geq N$, $T(n) \geq c \cdot f(n)$.

- Big Ω only guarantees a **lower bound.**

Big Θ notation

- $T(n)$ be a function of n
- **Big Θ notation:** $T(n) = \Theta(f(n))$ if

  $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$

Example:

- $T(n) = 2n^2 + 50n = \Theta(n^2)$
- $T(n) = 5n\log n + 10\sqrt{n} = \Theta(n\log n)$
- $T(n) = n^3 + 10n\log^5 n = \Theta(n^3)$

Array

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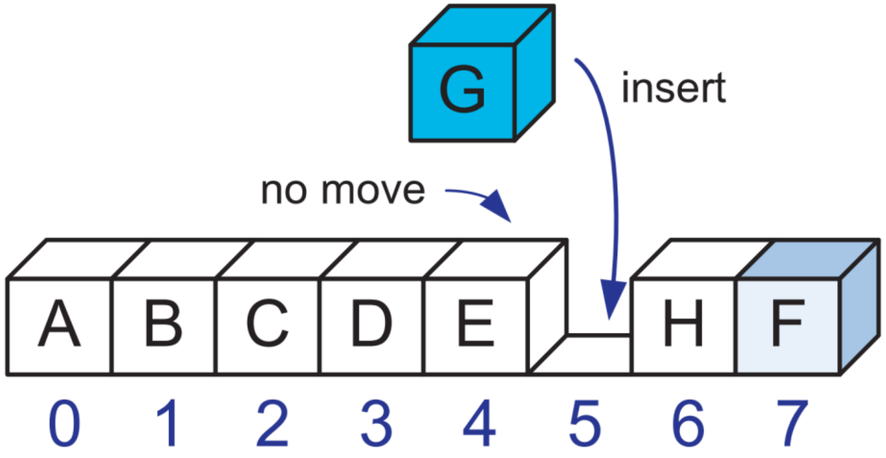
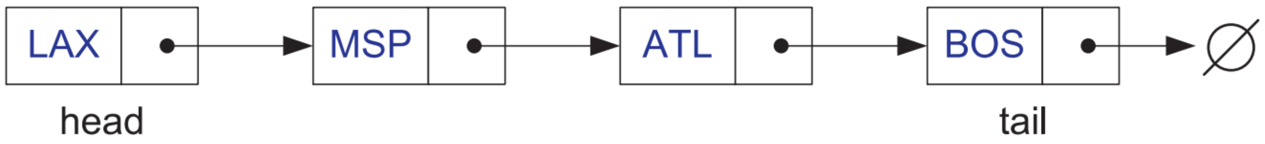
- An array data structure, or simply an array, is a data structure consisting of a collection of elements, each identified by at least one array index or key.
- Modify the "i-th element" in $O(1)$ time



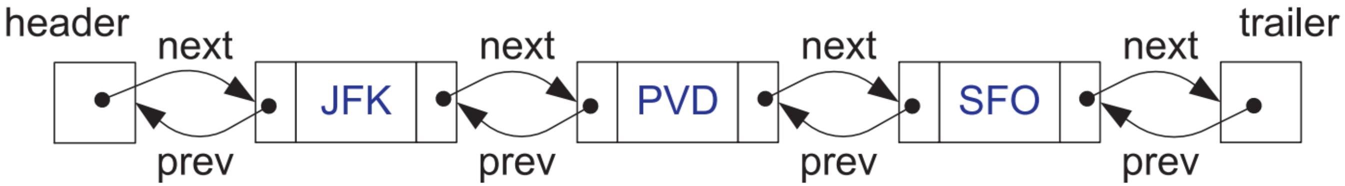
Linked List

- A linked list is a collection of nodes that together form a linear ordering.
- Modify the “i-th element” in $O(i)$ time
- Insert an element at a given position in $O(1)$ time
- Merging two linked lists in $O(1)$ time

- Singly linked list



- Doubly linked list



Stack

- A stack is a container of objects that are inserted and removed according to the last-in first-out (LIFO) principle.

- Stack Abstract Data Type (ADT):
  - push(e): insert element e
  - top(): return top element
  - pop(): return and remove top element
  - size(): return size of the stack
- Can be implemented by array/linked-list

Queue

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- A queue is a container of objects that are inserted and removed according to the first-in-first-out (FIFO) principle.

- Queue ADT:
    - enqueue(e): insert element e
    - front(): return front element
    - dequeue(): remove front element
    - size(): return size of the queue
- Can be implemented by array/linked-list

Trees

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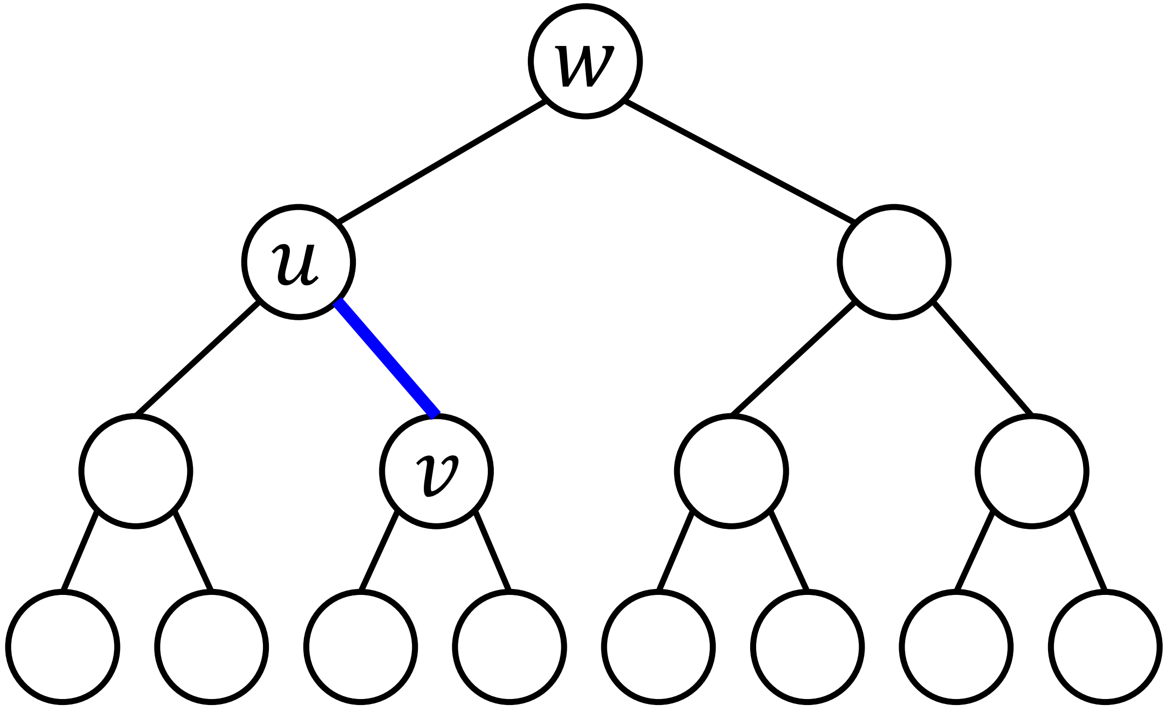
- Most important nonlinear data structures
- Nodes: objects
- Edges: relations

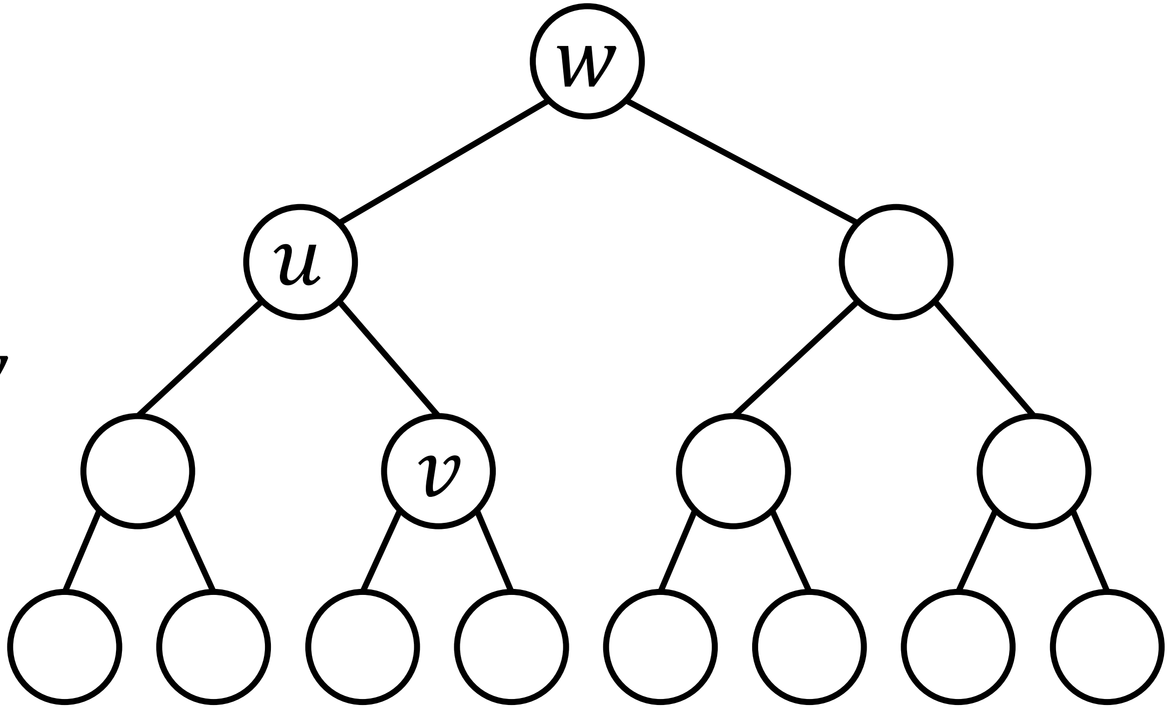
- Node u is parent of node v
- Node v is child of node u



Trees

- Most important nonlinear data structures
- Nodes: objects
- Edges: relations

- Node w is an ancestor of v
- Node v is a descendent of w



Trees

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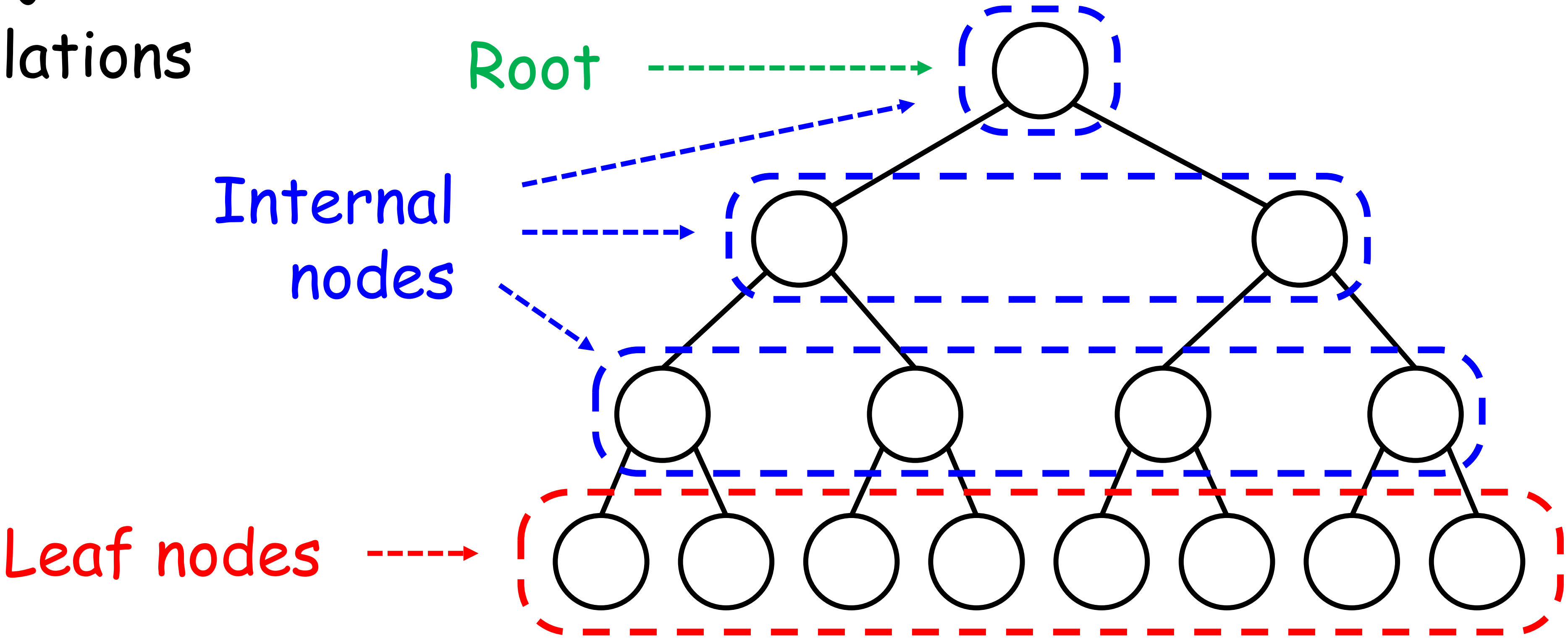

Trees

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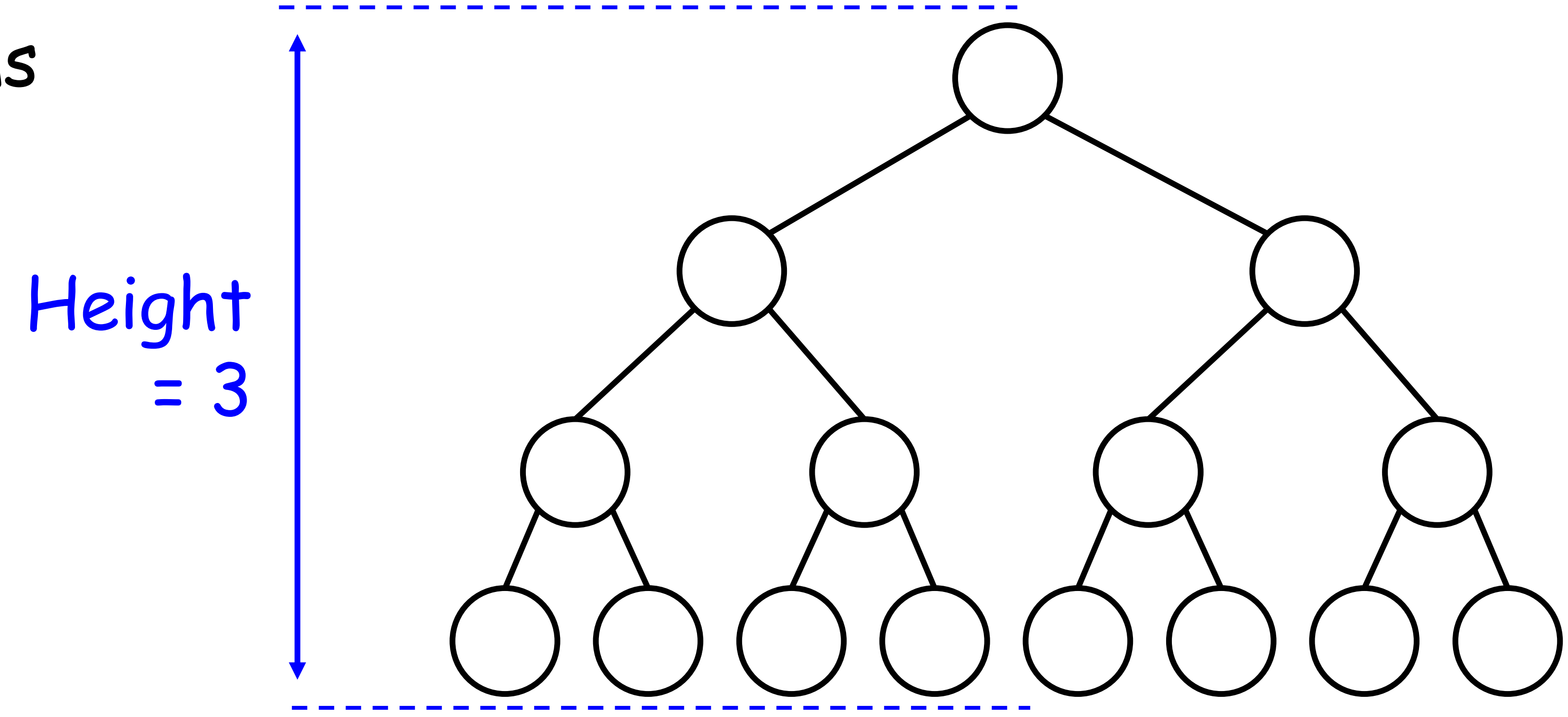

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Trees

- Most important nonlinear data structures
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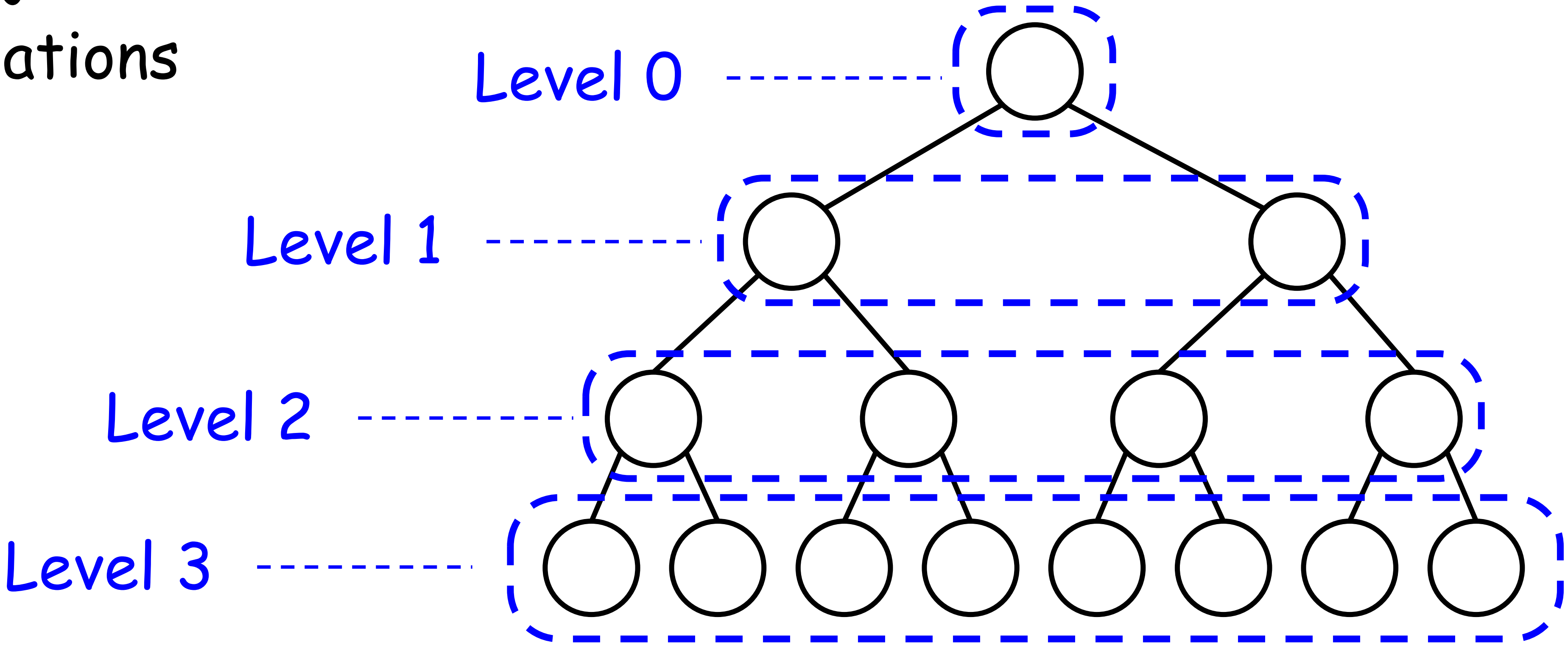
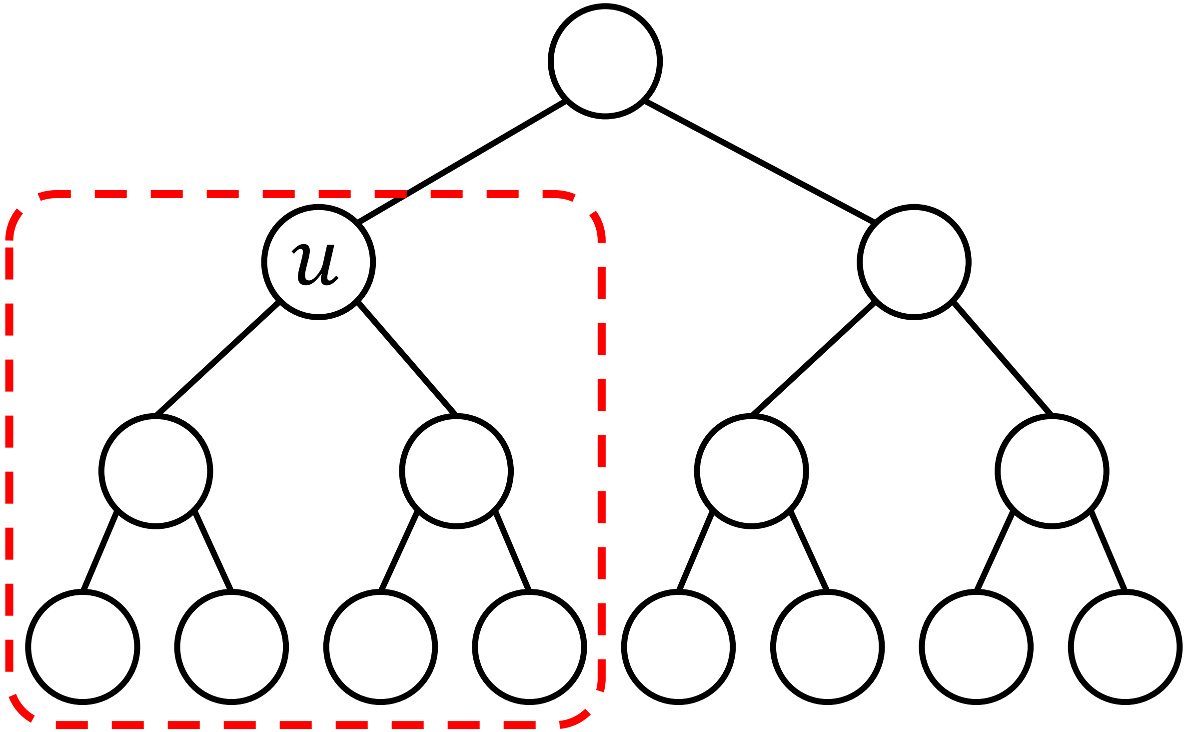


Trees

- Most important nonlinear data structures
- Nodes: objects
- Edges: relations

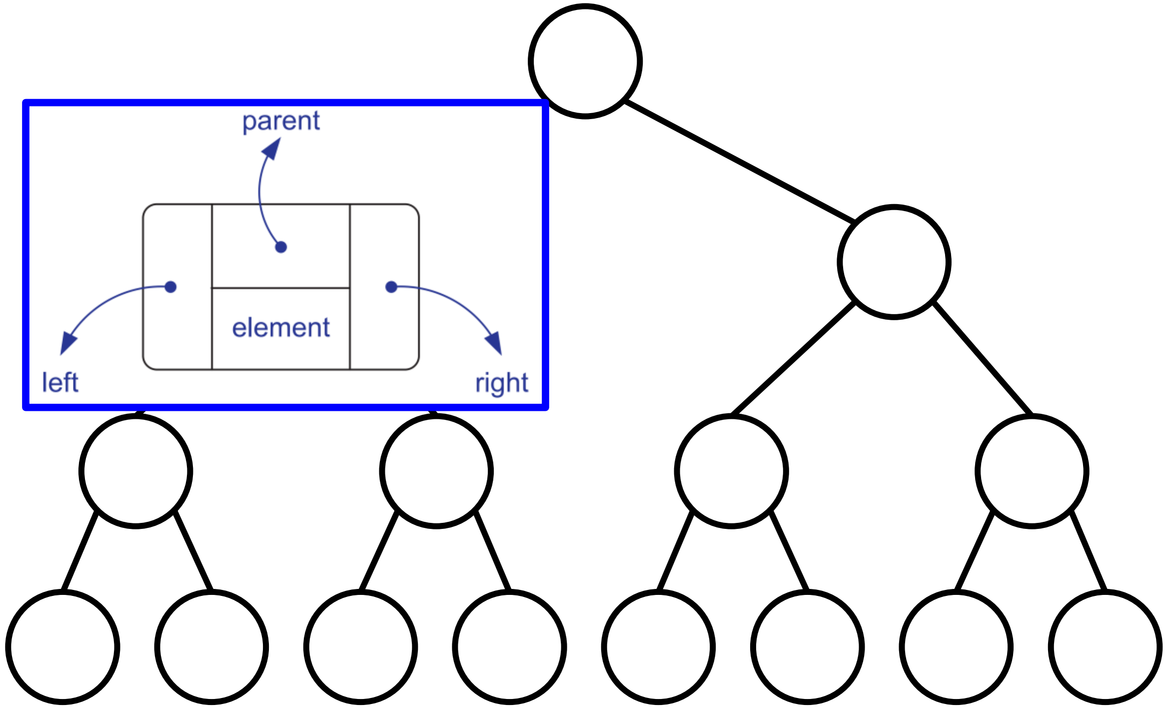
Subtree rooted at node u



Trees

- Most important nonlinear data structures

- Binary Tree ADT
    - size()
    - root()
    - p.parent()
    - p.left()
    - p.right()



Binary Search

- Given a sorted array,

1	3	6	7	9	11	12	16	19	23	24	26	28

> 9

- Task: search for a key value 9

Binary Search

- Given a sorted array,

1	3	6	7	9	11	12	16	19	23	24	26	28

< 20

- Task: search for a key value 20

Binary Search

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- Given a sorted array,

1	3	6	7	9	11	12	16	19	23	24	26	28

Output: Does not exist!

< 20

- Task: search for a key value 20
- Complexity: $O(\log n)$ for sorted array of length n

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Recursion

- Given a sorted array,

1	3	6	7	9	11	12	16	19	23	24	26	28

Search(L)

> 9

- Task: search for a key value 9

Recursion

- Given a sorted array,

1	3	6	7	9	11	12	16	19	23	24	26	28

< 20

Search(R)

- Task: search for a key value 20
- Complexity: $T(n) = T(n/2) + O(1) = O(\log n)$

Recursion

- During the execution of a function, a call to itself (with a smaller input) is made.

- Don't forget about…
  - Base case and boundary cases
  - Avoid infinite loops
  - Transforming solutions for the subproblems to the original problem

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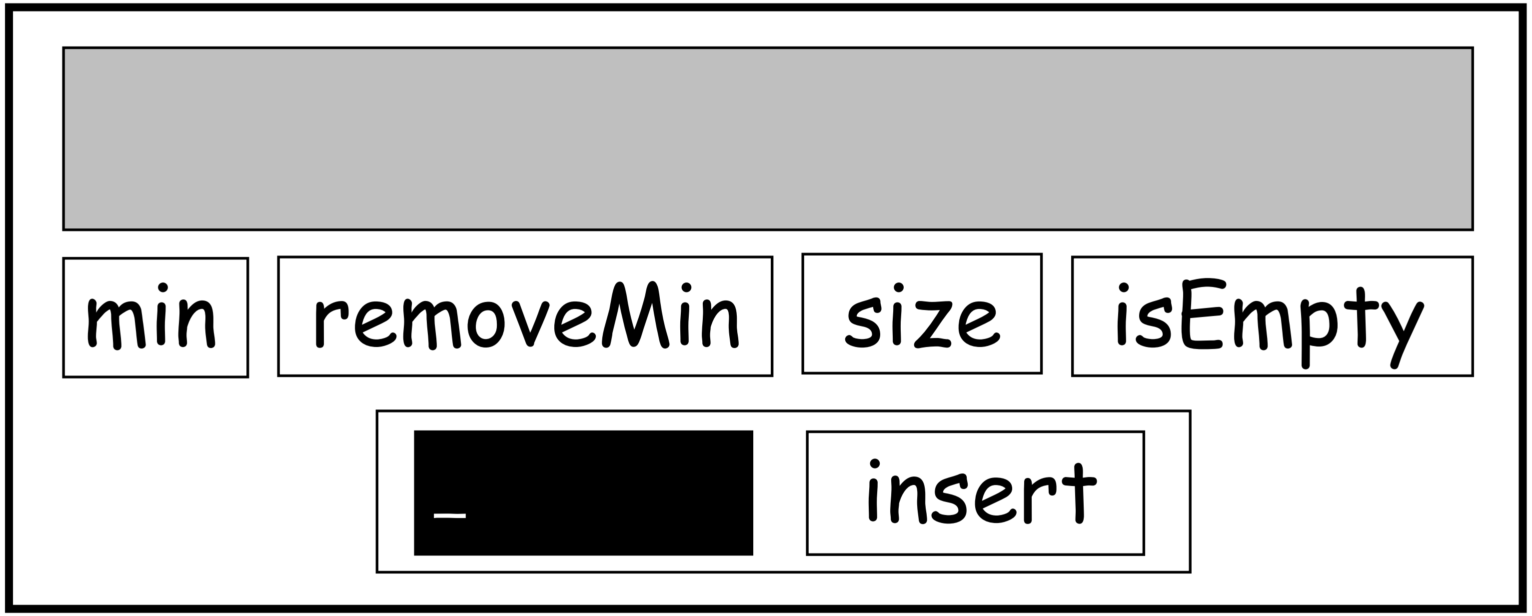
Priority Queue

Priority Queue

- An abstract data type storing a set of prioritized elements
- Supports arbitrary element insertion
- Supports removal of the elements of highest priority



Priority Queue

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- Priority Queue ADT (for priority queue P):
  - insert(e): insert element e to P
  - min(): return the minimum element of P
  - removeMin(): remove the minimum element of P
  - size(): return the number of elements in P
  - isEmpty(): return True if P is empty
- Extensions: general elements, e.g., e = (key, value, ...)
  - e.g., for storing multi-dimensional data
  - Need to define the comparator (to compare elements)

Sorting with Priority Queue

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- Given a sequence of numbers $A = (a_1, \dots, a_n)$, sort the numbers into $B = (b_1, \dots, b_n)$ such that $b_1 \le b_2 \le \cdots \le b_n$
- Sorting_with_Priority_Queue:
  - initialize an empty priority queue P
  - for $i = 1,2, \dots, n$:
    - $P.\mathrm{insert}(a_i)$
  - for $i = 1,2, \dots, n$:
    - $b_i \leftarrow P.\mathrm{min}()$
    - $P.\mathrm{removeMin}()$

Sorting with Priority Queue

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- Given a sequence of numbers $A = (a_1, \dots, a_n)$, sort the numbers into $B = (b_1, \dots, b_n)$ such that $b_1 \le b_2 \le \cdots \le b_n$
- Sorting_with_Priority_Queue:
  - suppose $P.\mathrm{insert}(e)$ takes T_i time
  - suppose $P.\mathrm{min}()$ takes T_m time
  - suppose $P.\mathrm{removeMin}()$ takes T_r time

  - Complexity: $n \cdot (T_i + T_m + T_r)$

Implementation

- How do we store the elements in P ?
  - Unsorted list
  - Sorted list
- How would it affect the performance?
- Example:
  - inserting elements (7,4,8,2,5,3,9) one-by-one
  - output and remove minimums one-by-one

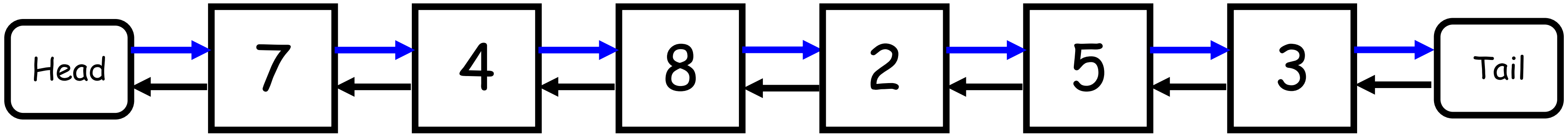
Implementation

- **Unsorted list:**
- insertion is easy
- output is difficult

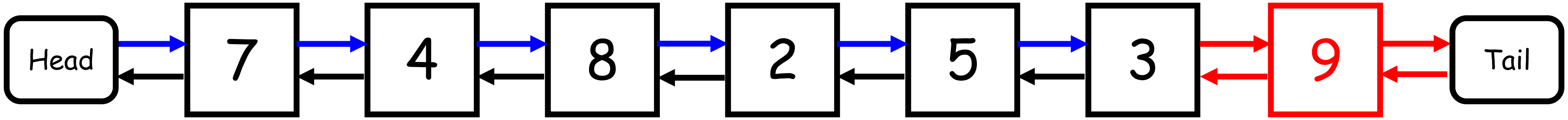
		List L	Priority Queue P
Input		(7,4,8,2,5,3,9)	()
Phase 1	(a)	(4,8,2,5,3,9)	(7)
	(b)	(8,2,5,3,9)	(7,4)
	⋮	⋮	⋮
	(g)	()	(7,4,8,2,5,3,9)
Phase 2	(a)	(2)	(7,4,8,5,3,9)
	(b)	(2,3)	(7,4,8,5,9)
	(c)	(2,3,4)	(7,8,5,9)
	(d)	(2,3,4,5)	(7,8,9)
	(e)	(2,3,4,5,7)	(8,9)
	(f)	(2,3,4,5,7,8)	(9)
	(g)	(2,3,4,5,7,8,9)	()

Implementation (linked list)

- **Unsorted list:**

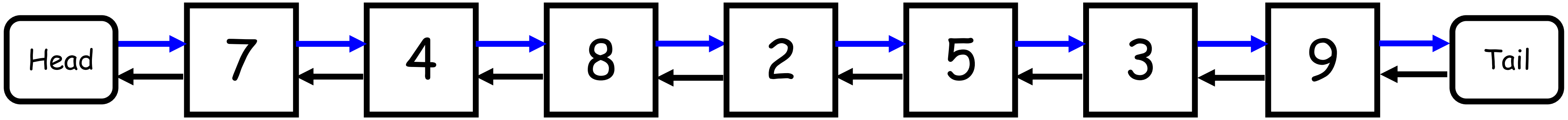


- insert(9) is easy ($O(1)$ time)

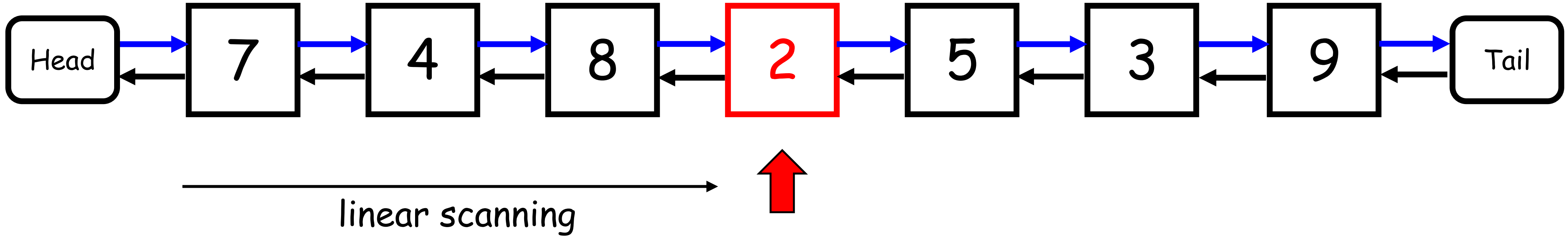


Implementation (linked list)

- **Unsorted list:**



- min() and removeMin() is difficult ($O(n)$ time)



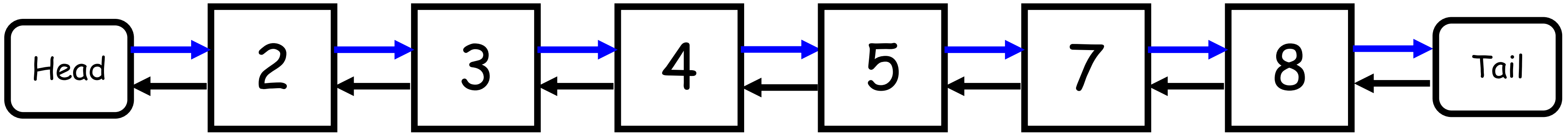
Implementation

- **Sorted list:**
- insertion is difficult
- output is easy

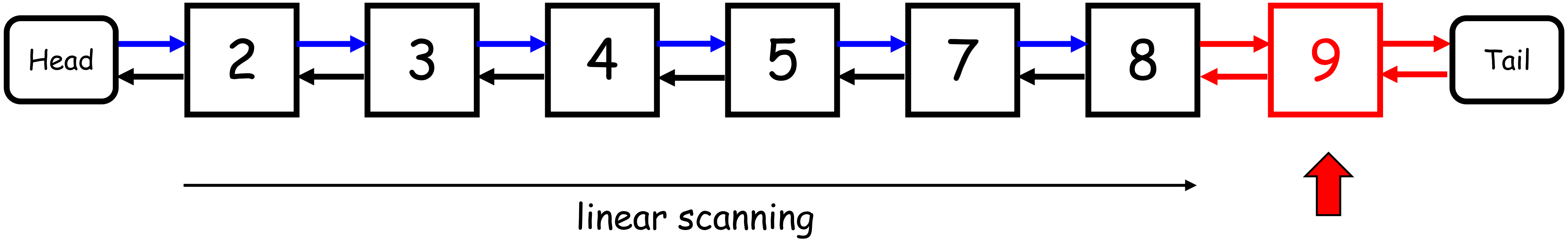
		List L	*Priority Queue P*
Input		(7,4,8,2,5,3,9)	()
Phase 1	(a)	(4,8,2,5,3,9)	(7)
	(b)	(8,2,5,3,9)	(4,7)
	(c)	(2,5,3,9)	(4,7,8)
	(d)	(5,3,9)	(2,4,7,8)
	(e)	(3,9)	(2,4,5,7,8)
	(f)	(9)	(2,3,4,5,7,8)
	(g)	()	(2,3,4,5,7,8,9)
Phase 2	(a)	(2)	(3,4,5,7,8,9)
	(b)	(2,3)	(4,5,7,8,9)
	⋮	⋮	⋮
	(g)	(2,3,4,5,7,8,9)	()

Implementation (linked list)

- **Sorted list:**

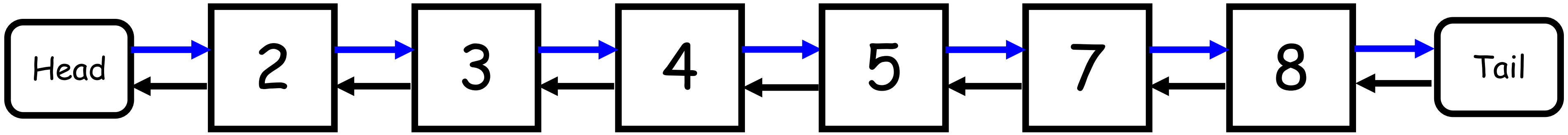


- insert(9) is difficult ($O(n)$ time)

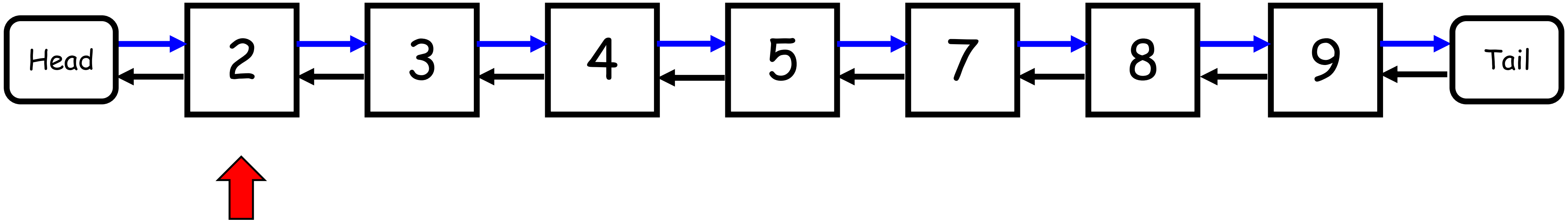


Implementation (linked list)

- **Sorted list:**



- min() and removeMin() is easy ($O(1)$ time)



Implementation (linked list)

- Complexity:

Operation	*Unsorted List*	*Sorted List*
size, empty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min, removeMin	$O(n)$	$O(1)$

- Sorting complexity: $O(n^2)$ for both implementation
  - Sorting requires inserting and removing n elements

Heaps

Heap

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- An efficient implementation of priority queue
- Store the n elements of priority queue in a tree structure
- Supports
  - size(), min() in $O(1)$ time
  - removeMin(), insert(e) in $O(\log n)$ time
- Heap-sort: $O(n \log n)$ time

Heap

- Data Structure: Complete binary tree T in which each node corresponds to one element of the priority queue.
  - Height of the tree: h
  - All levels $i \in \{0,1,\ldots,h-1\}$ are full (has 2^i nodes)
  - Nodes at level h fill this level from left to right.

- Heap-Order Property: for every node $v \neq$ root, the key associated with $v \geq$ key associated with v's parent.
  - Root = overall minimum (Min-Heap)

Heap

Lemma. Heap T storing n elements has height $h = \lfloor \log n \rfloor$.

Proof. Given a heap storing n elements and has height h:

- #elements at level i: 2^i (for all $0 \le i \le h-1$)
- #elements at level h: $n - \left(1 + 2 + \cdots 2^{h-1}\right) \in [1, 2^h]$

$$\Leftrightarrow \quad 1 \le n - \left(2^h - 1\right) \le 2^h$$

$$\Leftrightarrow \quad 2^h \le n \le 2^{h+1} - 1 < 2^{h+1}$$

$$\Leftrightarrow \quad h \le \log n < h + 1 \quad \Leftrightarrow \quad h = \lfloor \log n \rfloor \qquad \blacksquare$$

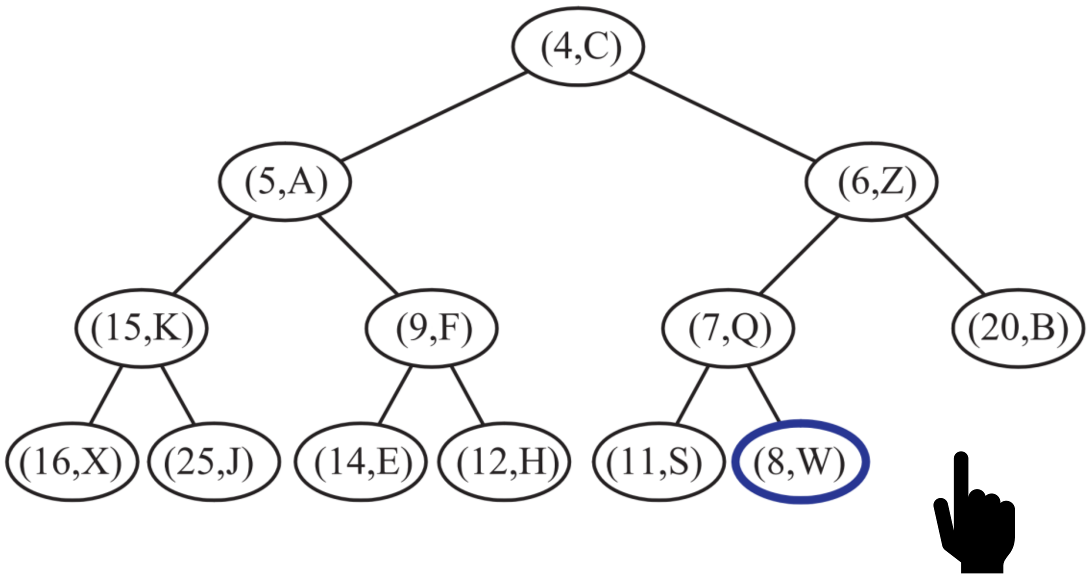
Heap

Operations of Heap (with height $= h$)

- size(): $O(1)$ time
- min(): $O(1)$ time
- insert(e): $O(h)$ time
    - ❖maintain complete binary tree and heap-order property
- removeMin(): $O(h)$ time
    - ❖maintain complete binary tree and heap-order property

Insert(e)

Maintenance of complete binary tree



Insert(e)

- Create a new node u to the right of the last node
  - If the last level is full, create a new level
- Set u.key $\leftarrow$ e.key and u.value $\leftarrow$ e.value
- **while**(u $\neq$ root and u.key $<$ u.parent.key)
  - swap keys and values of u and u.parent
  - u $\leftarrow$ u.parent

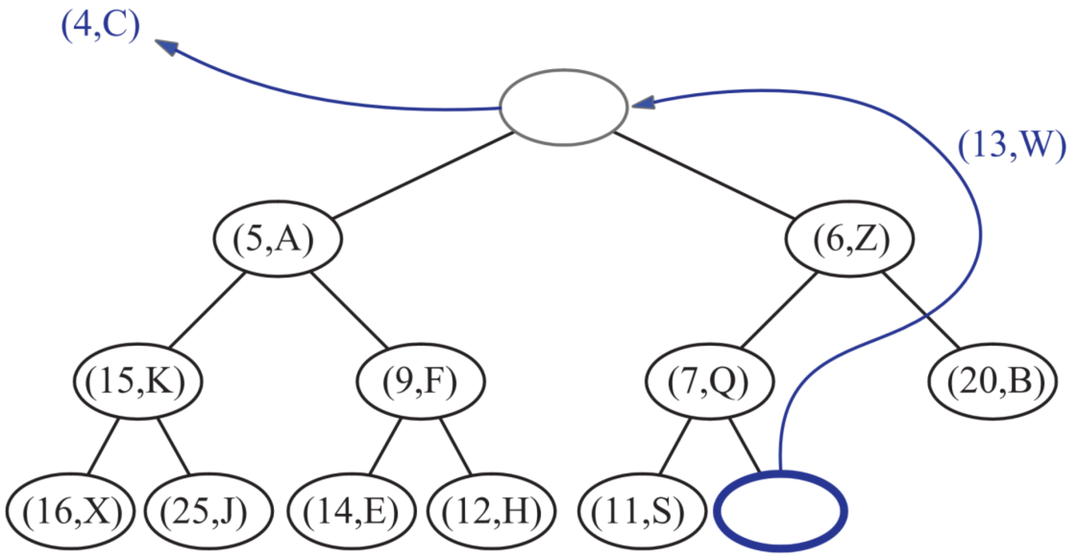
Up-heap Bubbling

Observation: at most h swaps are necessary

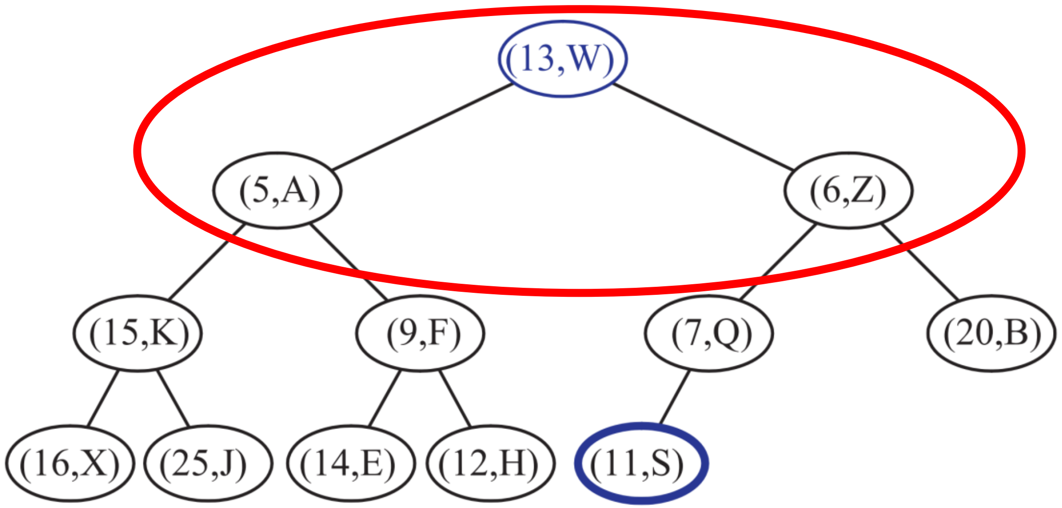
Lemma: insertion of elements can be implemented in $O(h)$ time

RemoveMin()

Maintenance of complete binary tree



move the last node to the root



need some fix to maintain the heap-order property !

RemoveMin()

- Let u point to the last node
- Set root.key ← u.key and root.value ← u.value
- Remove the last node (u points to) and let u point to the root
- **while**(u has child v such that v.key < u.key)
  - swap keys and values of u and child v of u with smaller value
  - u ← v

Down-heap Bubbling

Observation: at most h swaps are necessary

Lemma: removal of min can be implemented in $O(h)$ time

Heap-sort

Sort a given an array $A = (a_1, a_2, \ldots, a_n)$ using heap

- create an empty heap and array $B = (b_1, b_2, \ldots, b_n)$
- **for** $i = 1,2, \ldots, n$
  - insert(a_i)
- **for** $i = 1,2, \ldots, n$
  - $b_i \leftarrow$ min()
  - removeMin()
- **output** B

Complexity: $O(n \log n)$

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Binary Search Tree

Binary Search Tree

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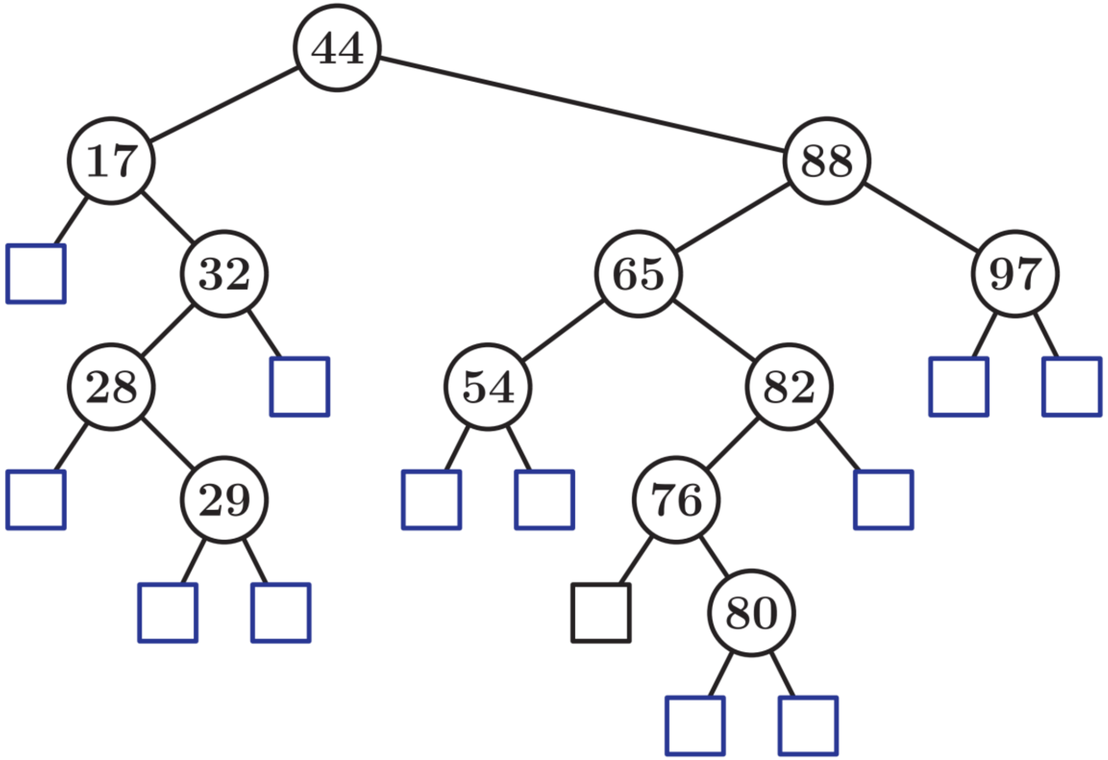
- Binary tree that supports efficient searching of elements
- Binary Search Tree (BST) supports
  - insert(e) insert an element, e.g., e = (key, value)
  - find(key) find element e with e.key = key
  - remove(key) remove the element e with e.key = key
  - remove(p) remove the element e point p points to

- **Application**: searching in a dictionary.

Binary Search Tree

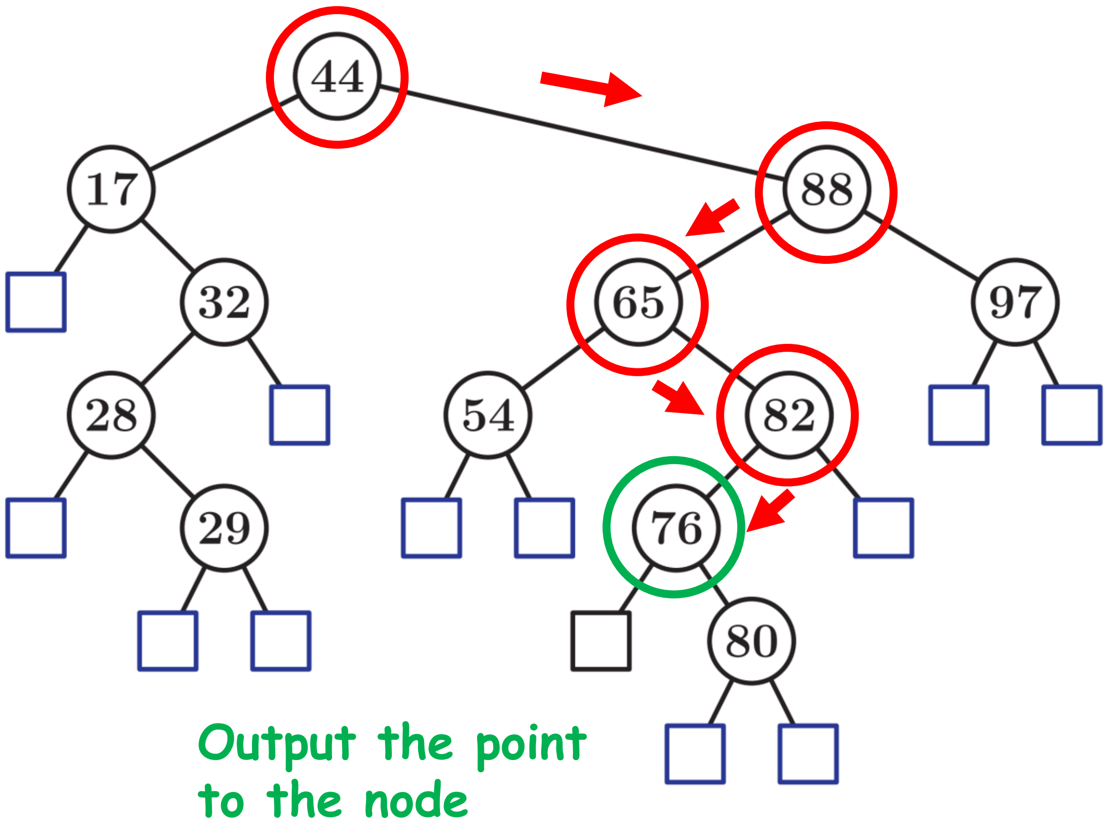
- A binary tree such that each internal node v stores an entry with key = k such that
  - Keys on left subtree are $\leq k$
  - Keys on right subtree are $\geq k$



Binary Search Tree

- Useful when searching target:
  - Go left if target < $u.key$
  - Go right if target > $u.key$
  - Fail if target $\neq u.key$ and u is a left node.

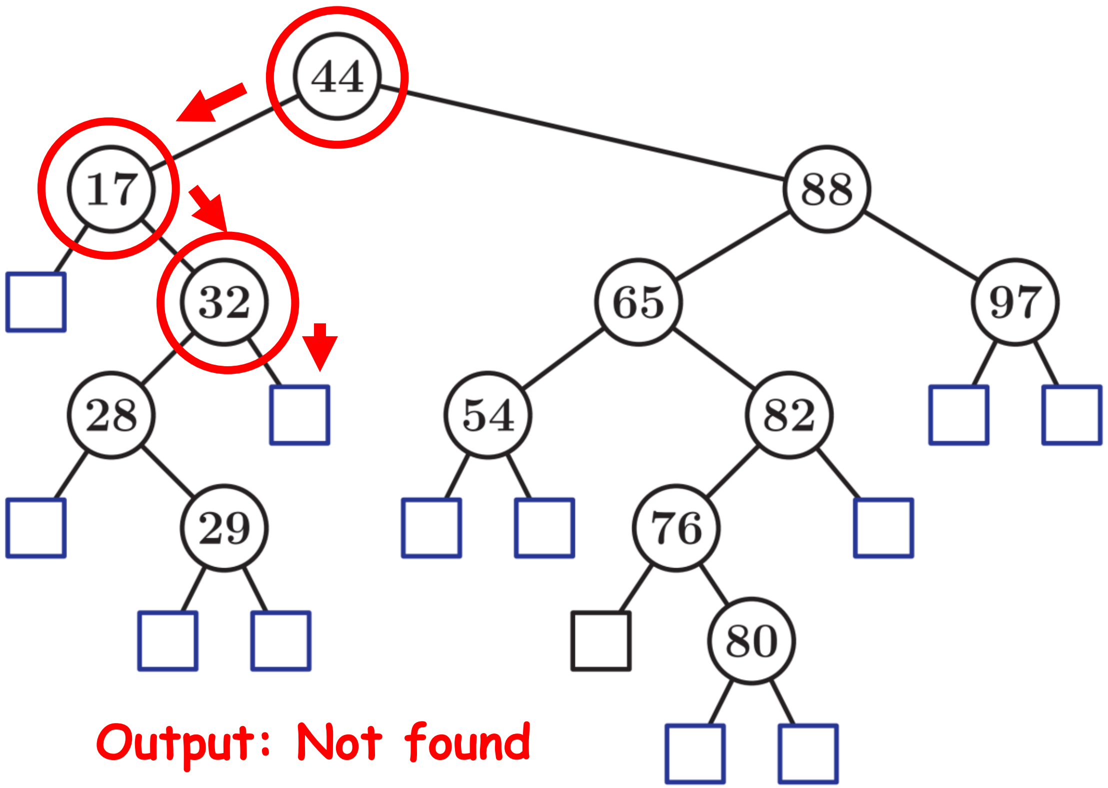
- Example: searching for 76



Binary Search Tree

- Useful when searching target:
  - Go left if target < $u.key$
  - Go right if target > $u.key$
  - Fail if target $\neq u.key$ and u is a left node.

- Example: searching for 40



Binary Search Tree

Function to search for k in the subtree rooted at v

- TreeSearch($k,\ v$)
  - if $v = null$ then
    - Return: v
  - else if $v.key = k$ then
    - Return: v
  - else if $v.key > k$ then
    - Return TreeSearch($k,\ v.left$)
  - else
    - Return TreeSearch($k,\ v.right$)

Complexity: $O(h)$

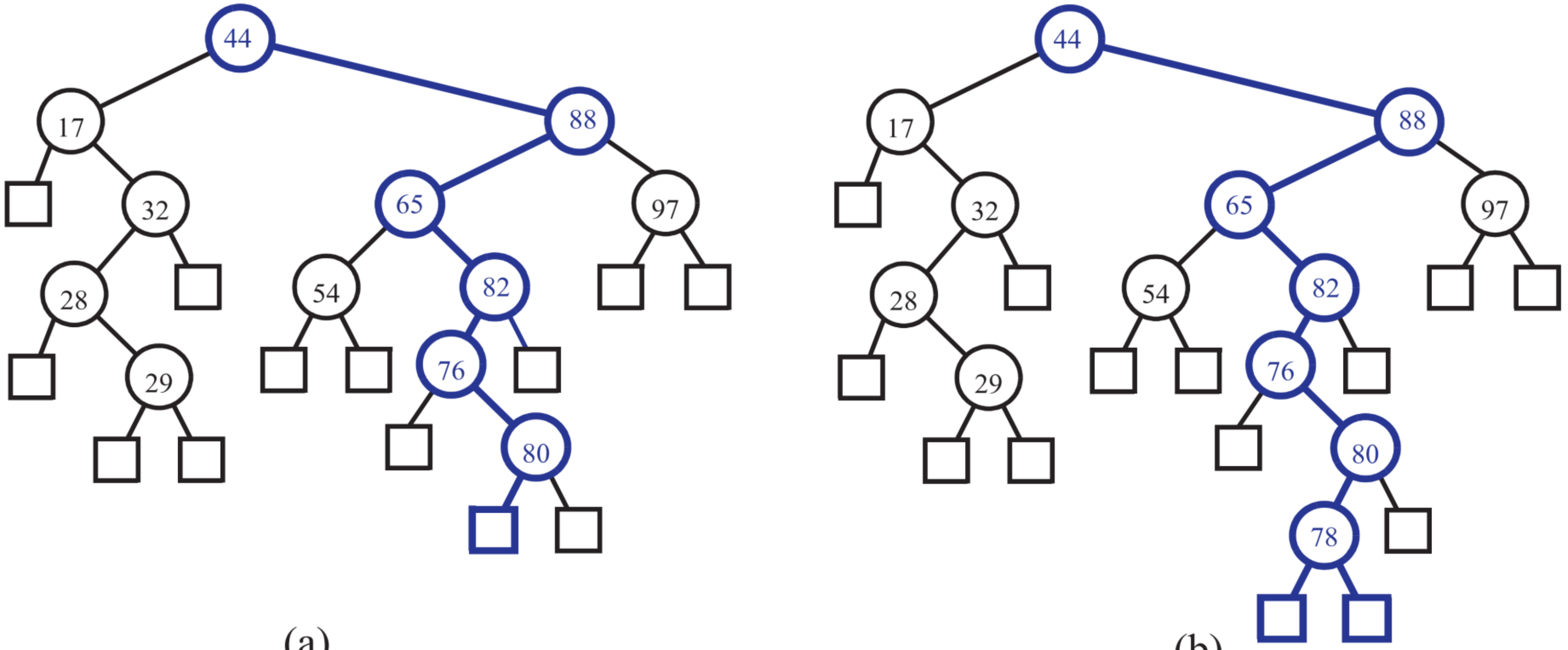
- Every recursion increases the level by one

Extension: Find-All(k, v)

- Output all matched element

Insertions

Insertion of element with key 78



(a) (b)

Insertions

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Function to insert $e = (key, value)$ to the subtree rooted at v

- TreeInsert(e, v)
  - if $v.key > e.key$ then
    - if $v.left = null$ then
      - Create a new node u for e and set $v.left \leftarrow u$
    - else
      - TreeInsert(e, $v.left$)
  - else
    - if $v.right = null$ then
      - Create a new node u for e and set $v.right \leftarrow u$
    - else
      - TreeInsert(e, $v.right$)

- new node u
- $u.key \leftarrow e.key$
- $u.value \leftarrow e.value$
- $u.left \leftarrow null$
- $u.right \leftarrow null$

Insertions

Function to insert $e = (key, value)$ to the subtree rooted at v

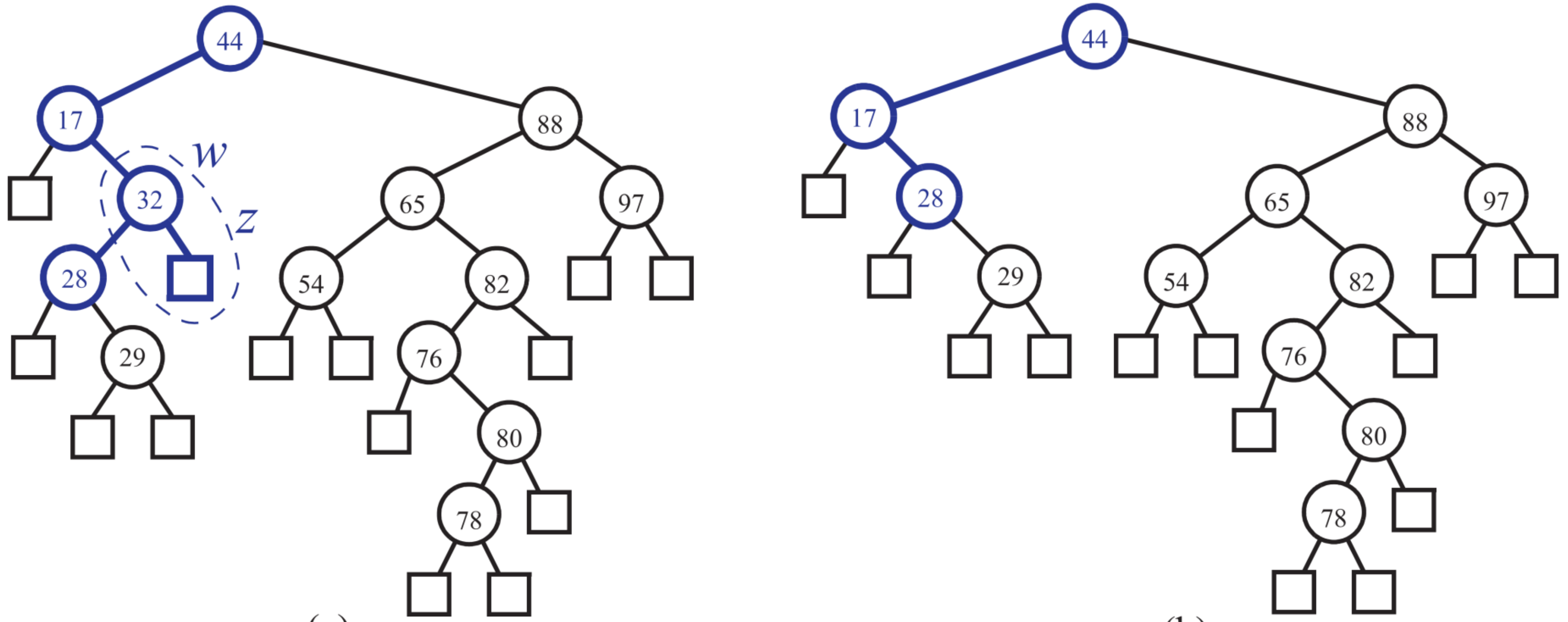
- TreeInsert(e, v)
  - if $v.key > e.key$ then
    - if $v.left = null$ then
      - Create a new node u for e and set $v.left \leftarrow u$
    - else
      - TreeInsert(e, $v.left$)
  - else
    - if $v.right = null$ then
      - Create a new node u for e and set $v.right \leftarrow u$
    - else
      - TreeInsert(e, $v.right$)

Complexity: $O(h)$

- Every recursion increases the level by one

Deletions

Removal of a node w: if w has ≤ 1 child



(a) (b)

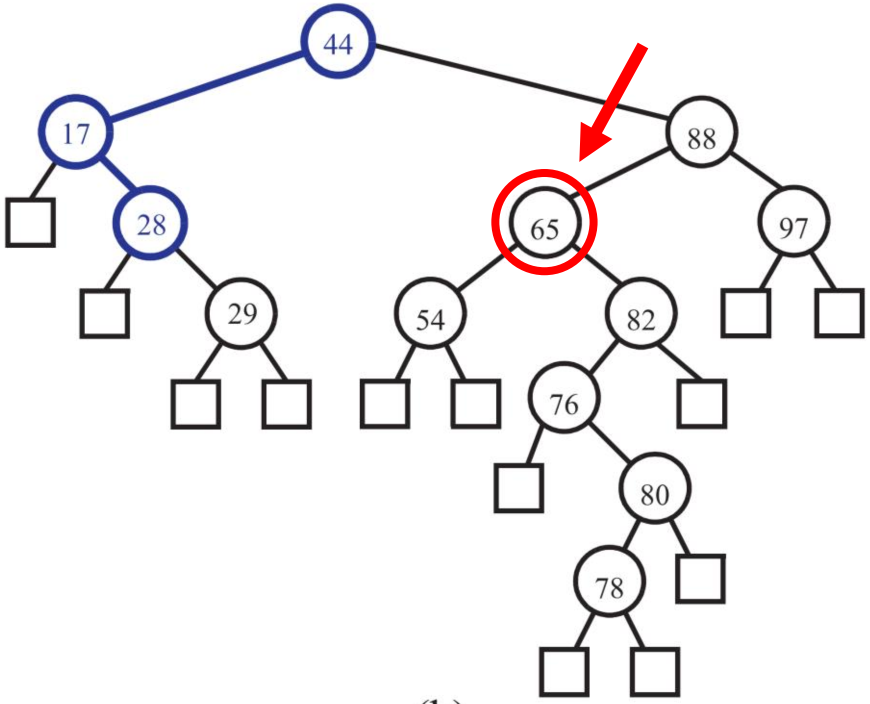
Deletions

Removal of a node w: if w has two children ?

Replace w with:

- maximum element on the left subtree of w, or
- minimum element on the right subtree of w

in this course



(b)

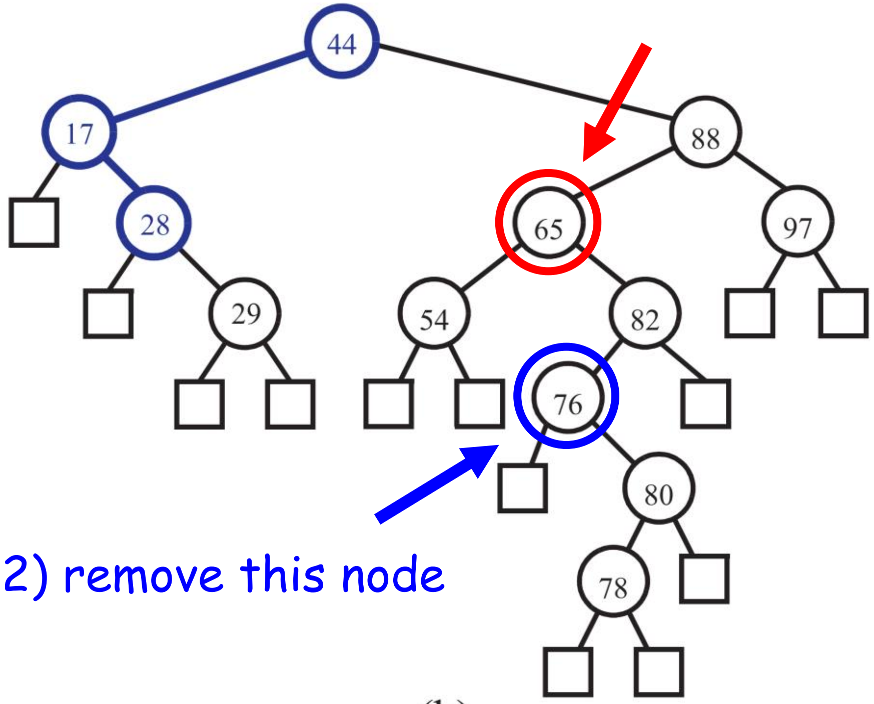
Deletions

Removal of a node w: if w has two children ?

Replace w with:

- maximum element on the left subtree of w, or
- left-most element on the right subtree of w

(1) replace this with node 76

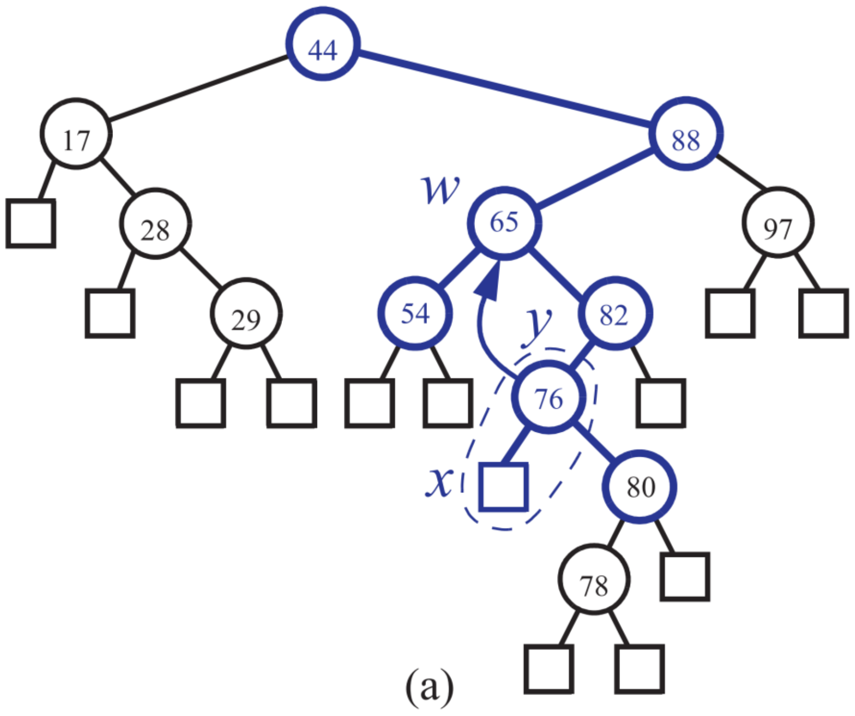


(2) remove this node

(b)

Deletions

Removal of a node w: if w has two children



Deletions

Removal of a node w:

- if w has no child:
  - Update corresponding pointer of parent of w to null
- if w has one child:
  - Replace w by its only child
- if w has two children
  - Replace w by the left-most node u on the right subtree
  - Remove node u (which has at most one child)

Complexity: $O(h)$

Binary Search Tree

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- Binary Search Tree (BST)
  - insert(e) $\Rightarrow O(h)$ time
  - find(key) $\Rightarrow O(h)$ time
  - remove(key) $\Rightarrow O(h)$ time
  - remove(p) $\Rightarrow O(h)$ time

- **Observation**: the tree is more efficient when h is small.
  - In worst case, h can be as large as n (example?)
- Question: how to maintain a small height?

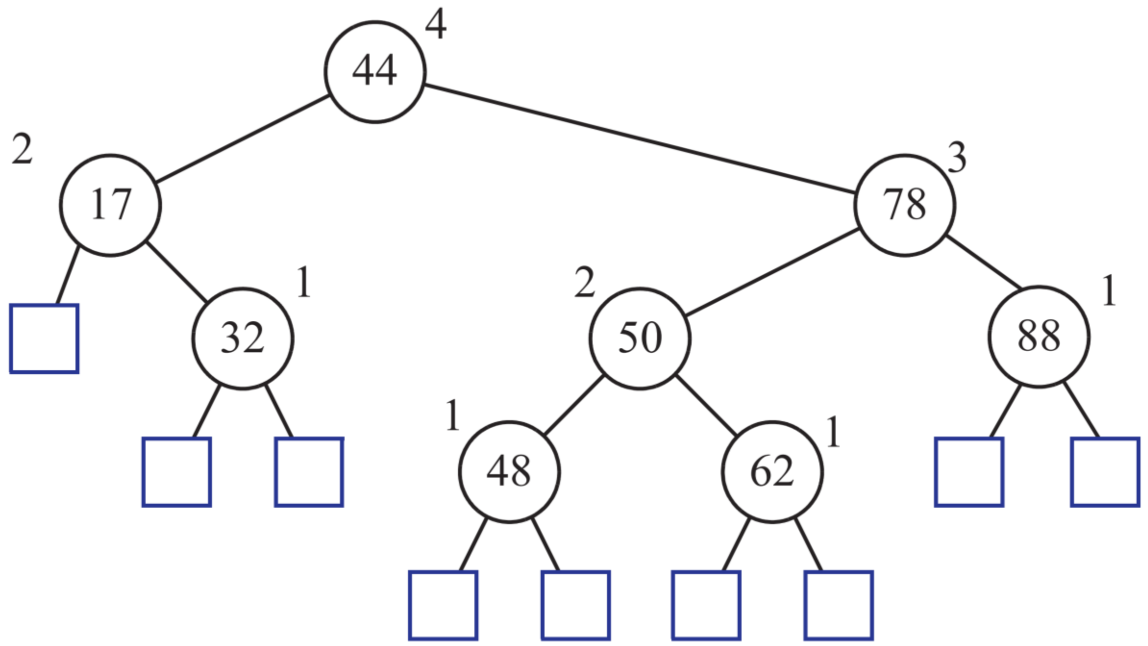
AVL Tree

AVL Tree

- A balanced binary tree
- Height-Balance Property:

  For every internal node v the heights of subtrees rooted at children of v differ by at most 1.

- Named by its inventors: Adel'son-Vel'skii and Landis.



AVL Tree

- **Lemma**. A binary tree with n internal nodes satisfying the height balance property has height $h = O(\log n)$.
- Proof 1.
  - Let $f(h)$ be the minimum number of nodes stored at a height-balanced binary tree of height h, e.g., $f(1) = 2$; $f(2) = 4$.
  - We have $f(h) = 1 + f(h-1) + f(h-2) > 2 \cdot f(h-2) > 2^i \cdot f(h - 2 \cdot i)$.
  - Let $i = \lceil h/2 \rceil - 1$, we have $f(h) > 2^i \cdot f(1) = 2^{i+1} \geq 2^{h/2}$.
  - Every height-balanced binary tree of height h has at least $2^{h/2}$ nodes.
  - Therefore, $n \geq 2^{h/2}$, which implies $h \leq 2 \cdot \log n = O(\log n)$.

AVL Tree

- **Lemma**. A binary tree with n internal nodes satisfying the height balance property has height $h = O(\log n)$.
- Proof 2.
  - Prove that $h \le 2\log n$ for a tree with n internal nodes
    - by Mathematical Induction on n
  - Base case: $n = 1$ and $h = 0$
  - Assume statement true for all $i < n$, and consider $n = 1 + n_l + n_r$
    - n_l (resp. n_r): internal nodes on the left (resp. right) subtree
  - By induction hypothesis: $h_l \le 2\log n_l$ and $h_r \le 2\log n_r$
    - h_l (resp. h_r): height of the left (resp. right) subtree
  - We have

$$h = 1 + \max\{h_l, h_r\} \le 2 + \min\{h_l, h_r\} \le 2 + \left(2\log\frac{n}{2} + 2\right) = 2\log n + 2$$

AVL Tree

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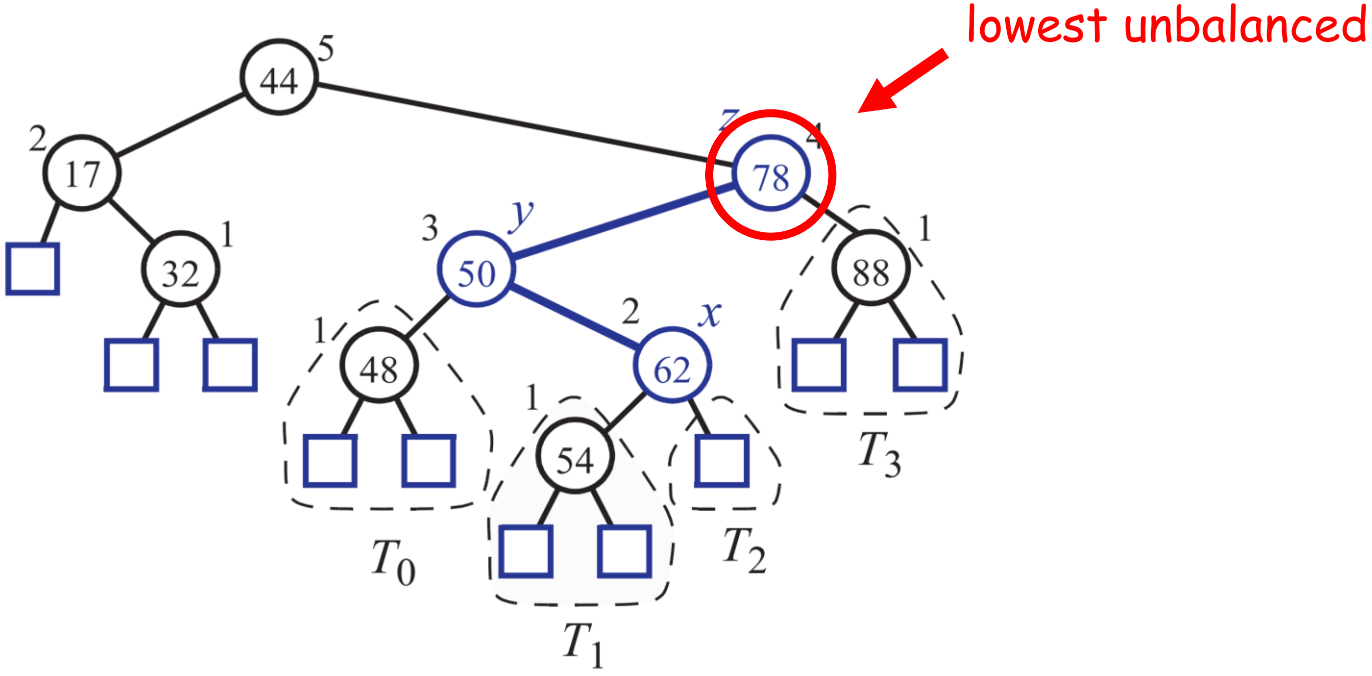
- **Lemma**. A binary tree with n internal nodes satisfying the height balance property has height $h = O(\log n)$.

- How to maintain the property ?
  - Height might get increased by Insert(e)
  - Height might get decreased by Remove(p)
  - Node v is balanced/unbalanced if the difference between subtrees rooted at its children differ by at-most/larger-than 1
  - Need to fix the unbalanced nodes after an update

Insertions

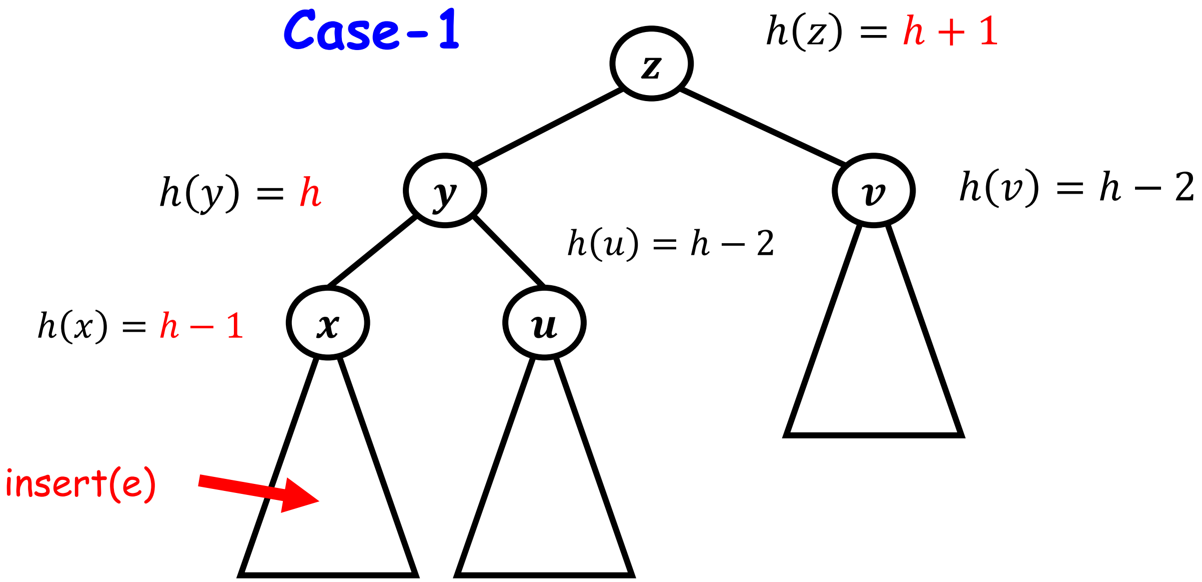
- Insertion of element with key 54



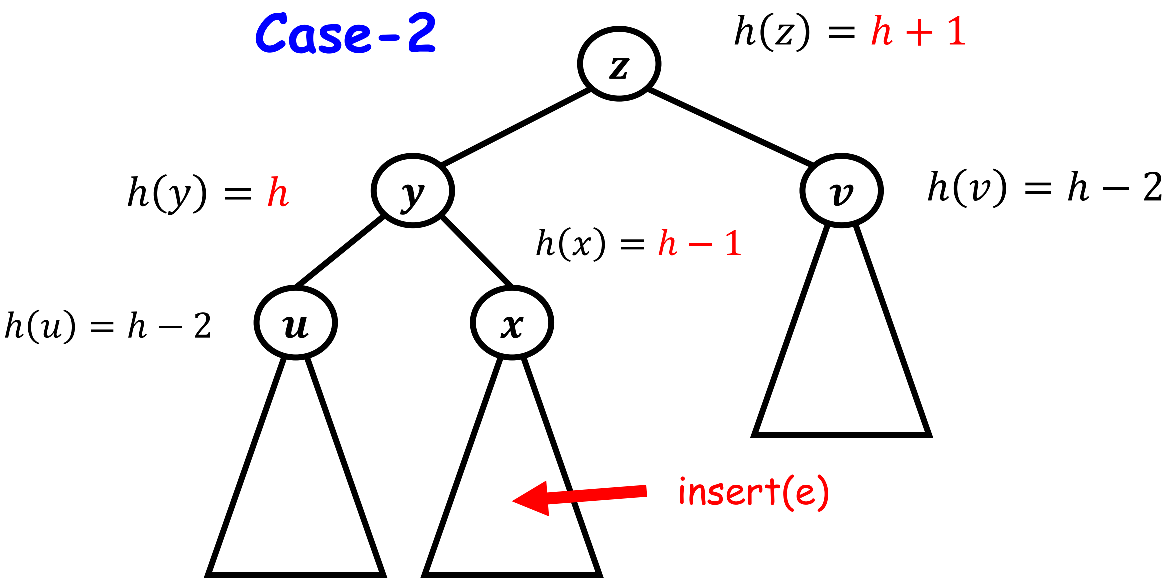
Insertions

- How to re-balance the tree?
- Find sub-trees of height $\leq h - 2$

Case-1

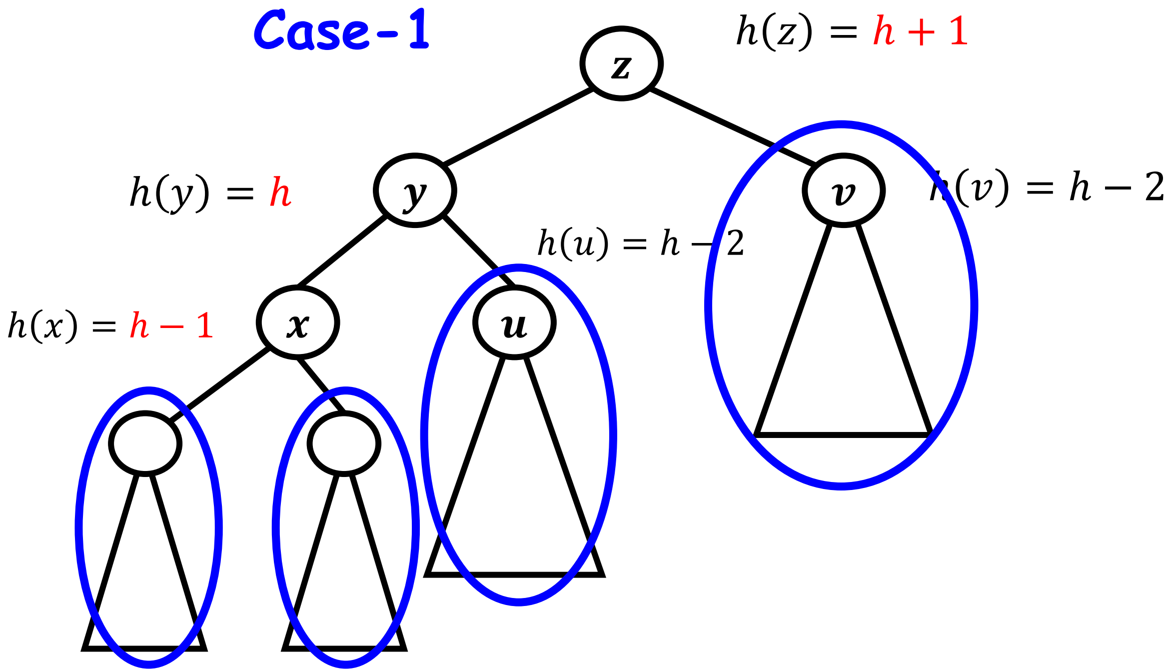


Case-2



Insertions

- How to re-balance the tree?
- Find sub-trees of height $\leq h-2$



Insertions

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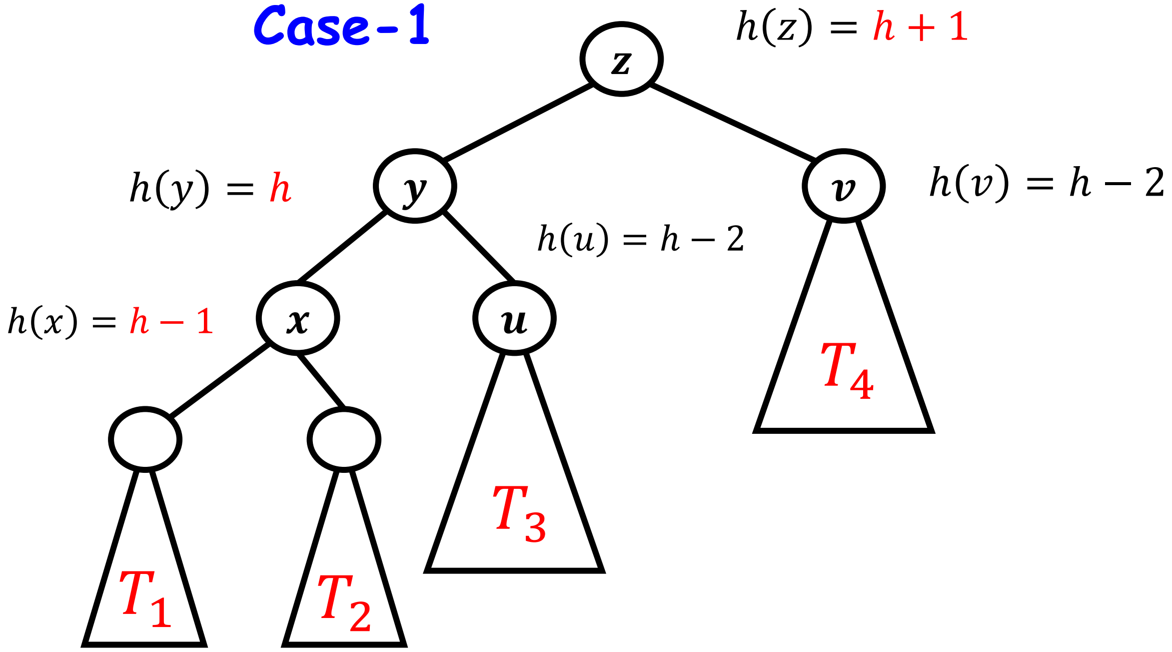
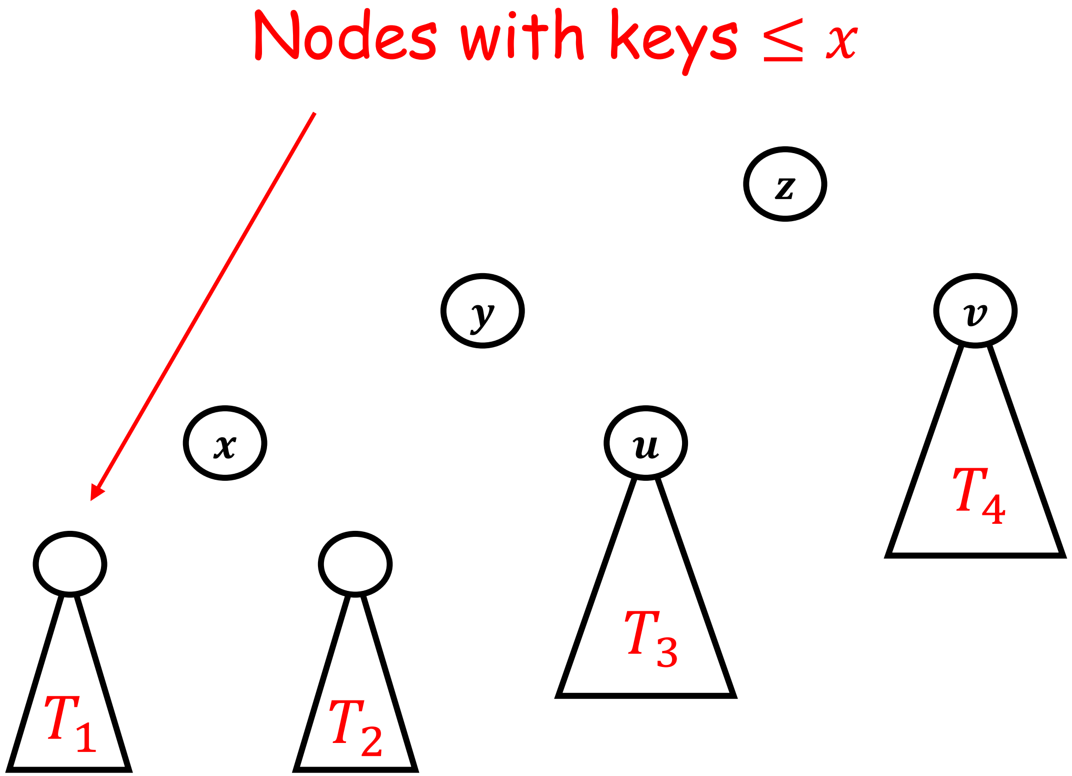
Case-1





Insertions

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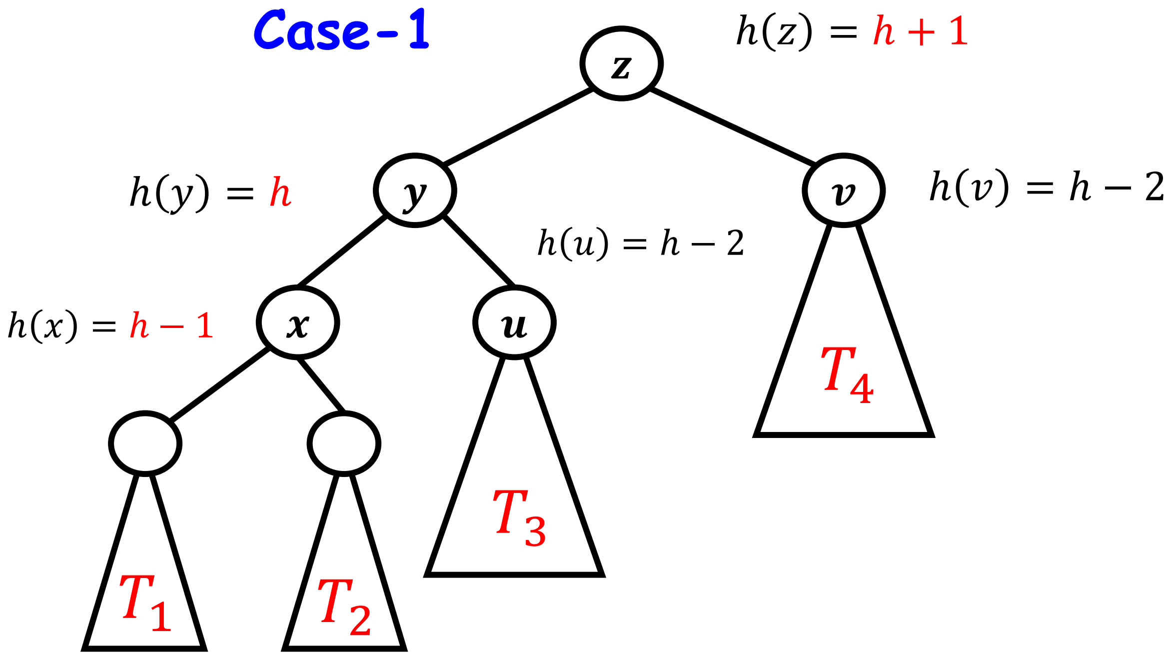
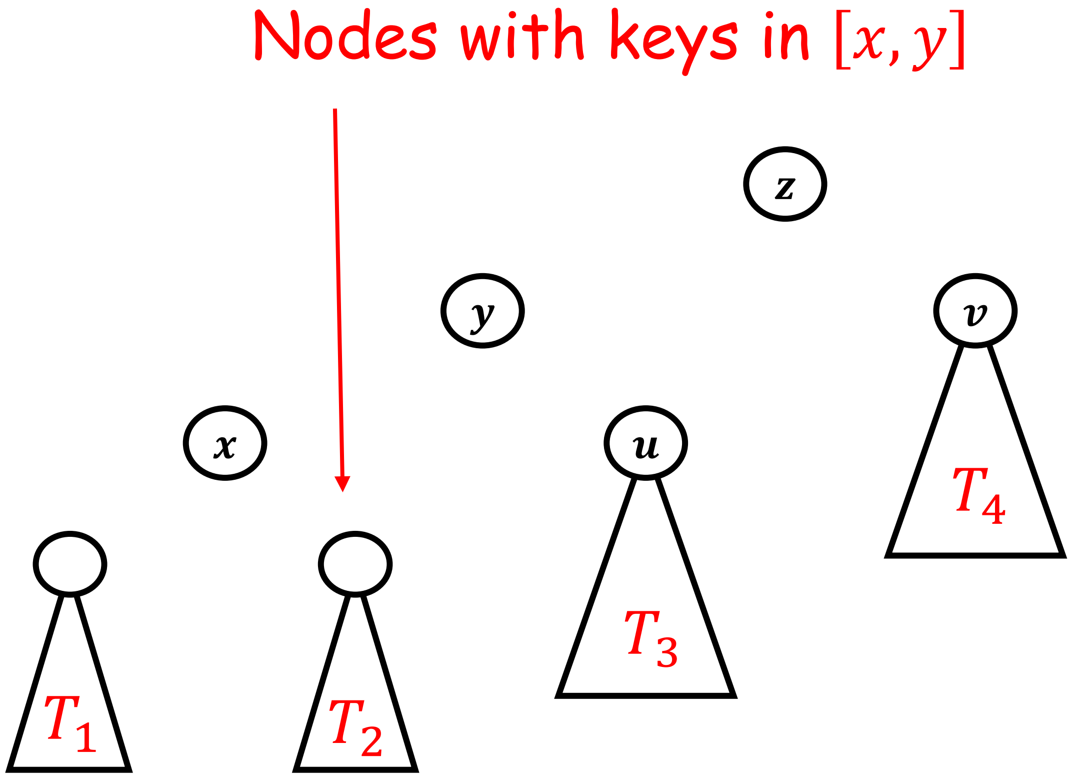
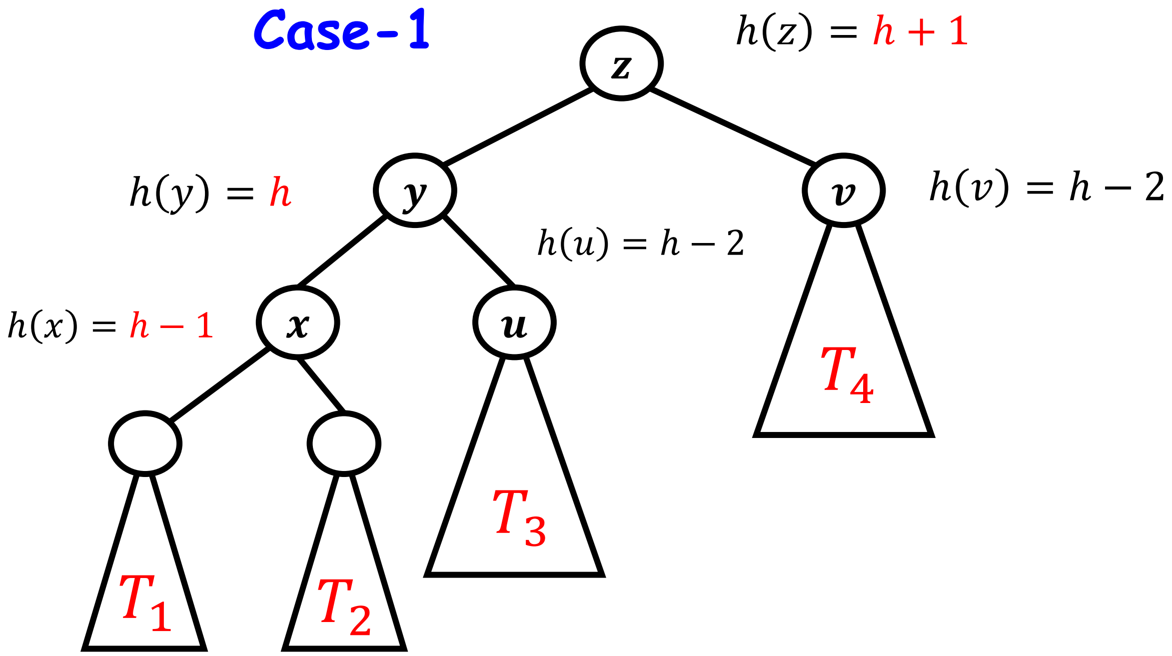
Case-1

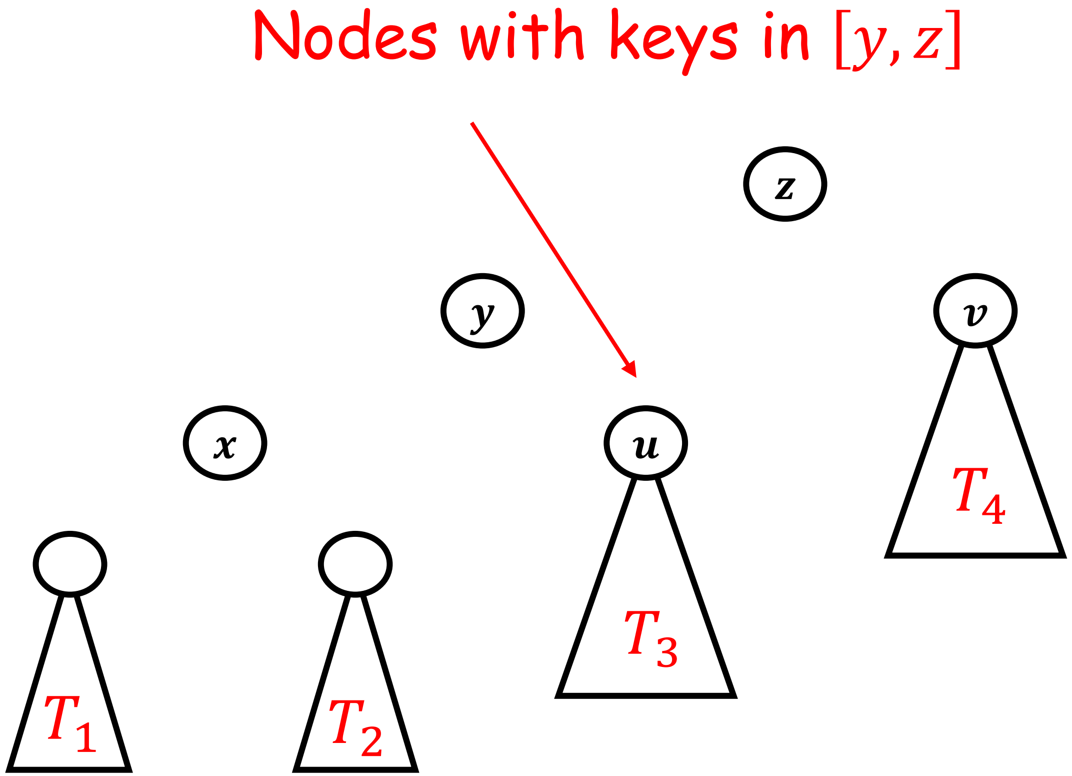




Insertions

- How to re-balance the tree?
- Find sub-trees of height $\leq h - 2$





Insertions

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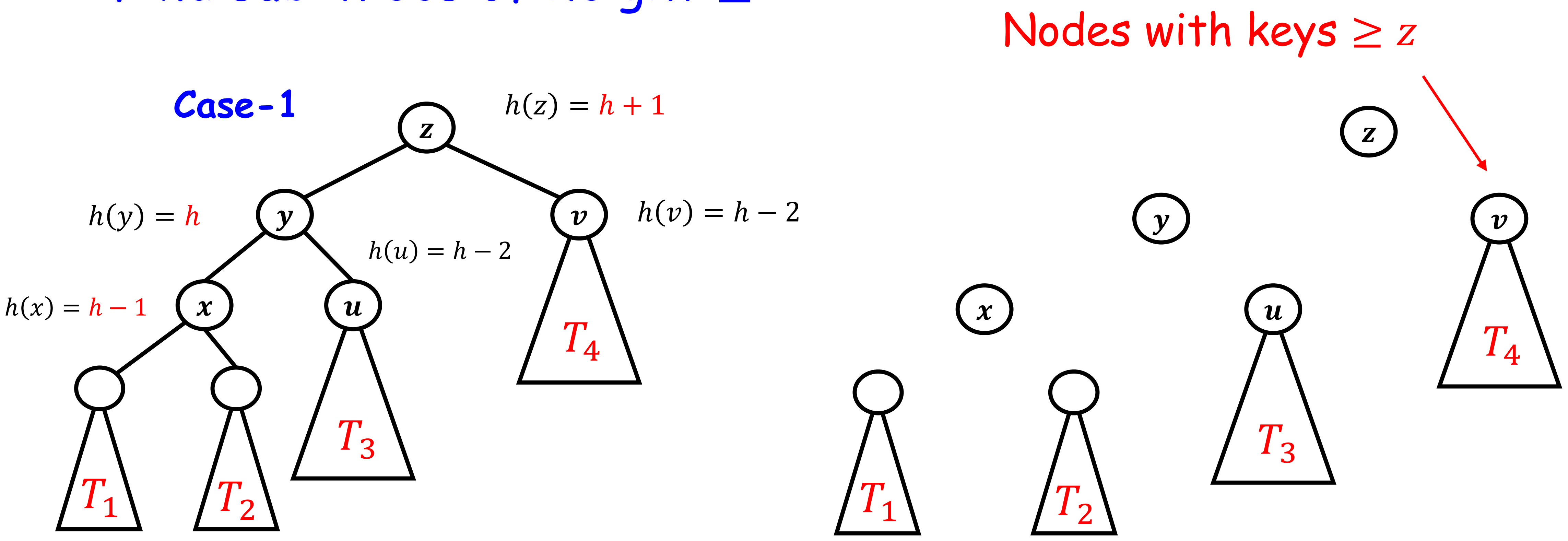
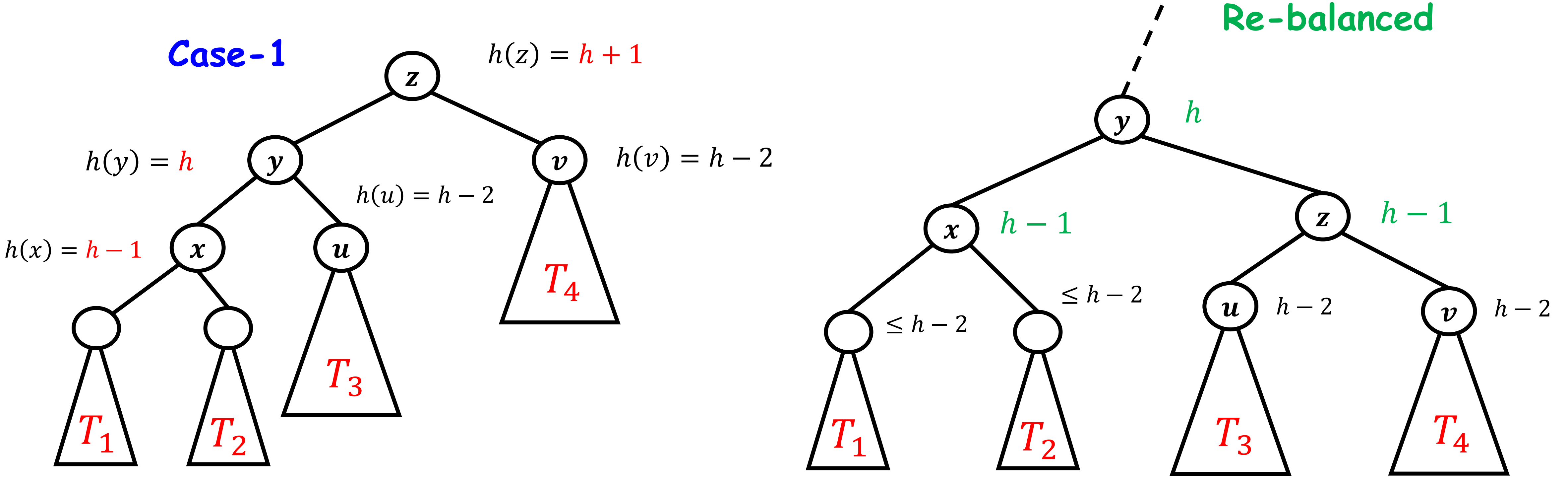
- How to re-balance the tree?
- Find sub-trees of height $\leq h - 2$



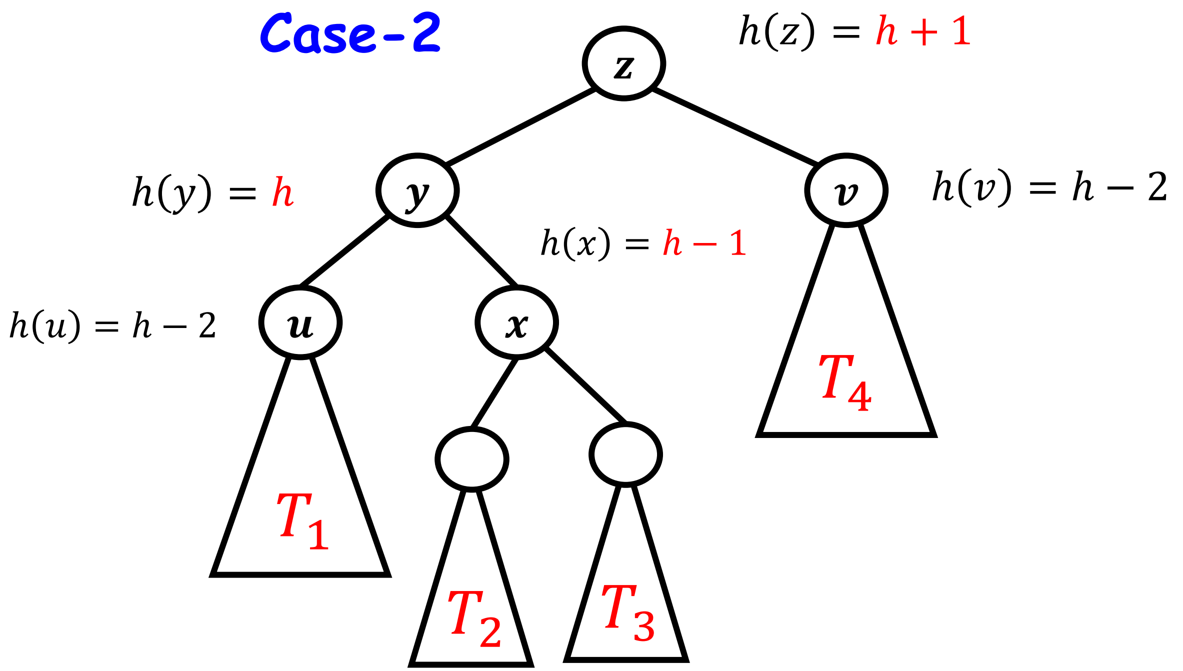
Insertions

- How to re-balance the tree?
- Find sub-trees of height $\leq h-2$



Insertions

- How to re-balance the tree?
- Find sub-trees of height $\leq h - 2$



Insertions

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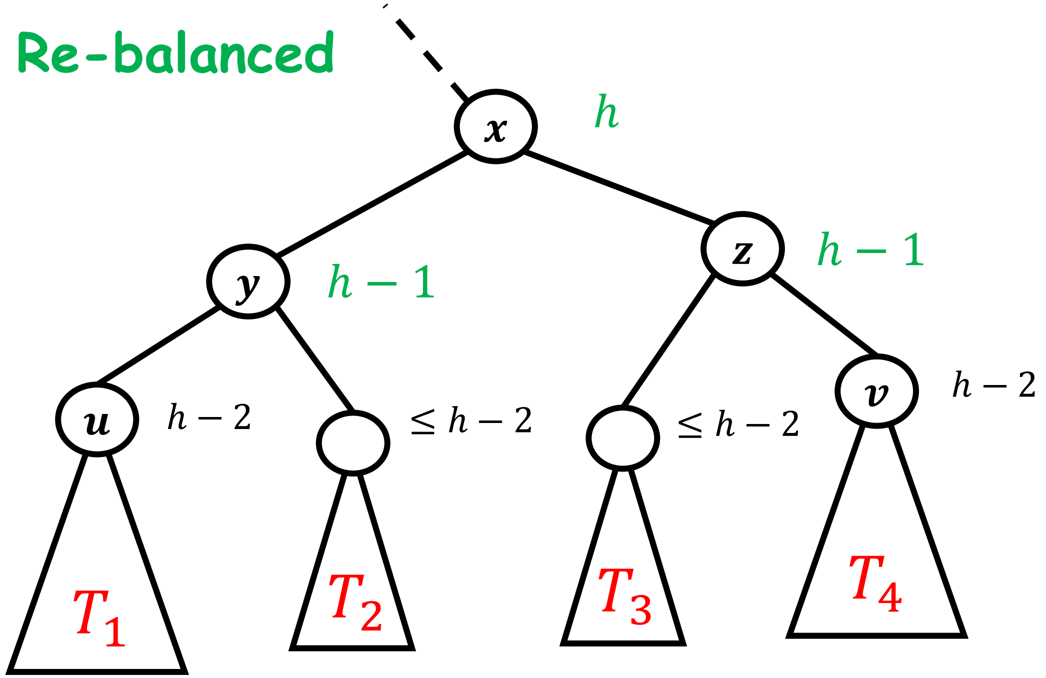
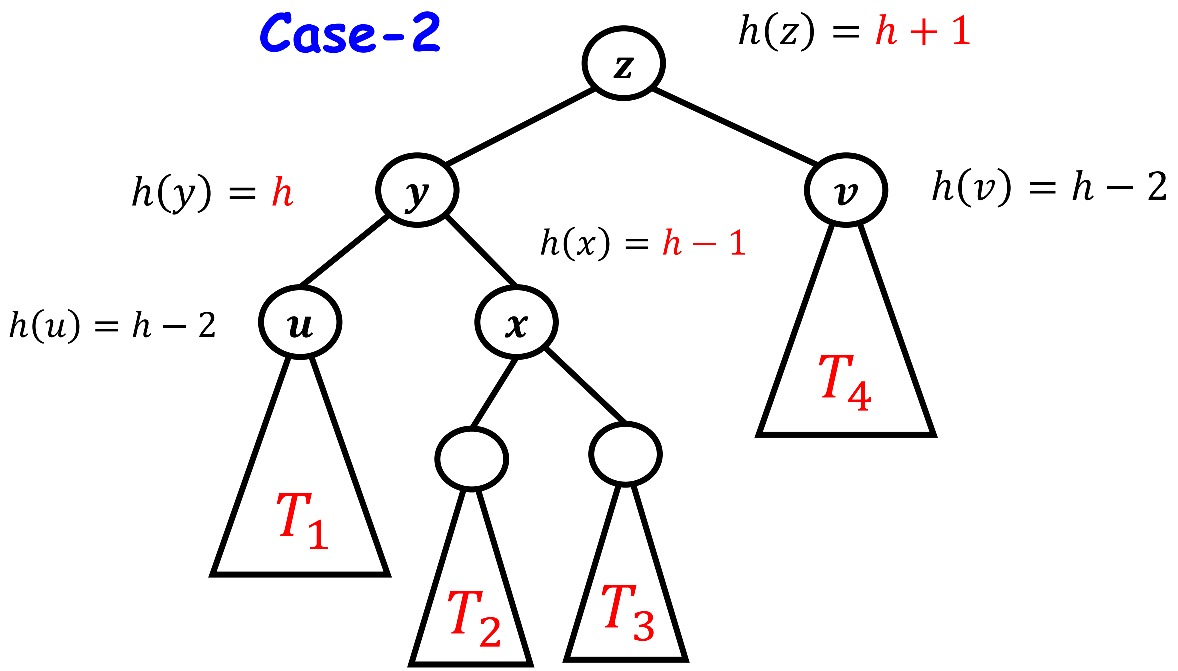
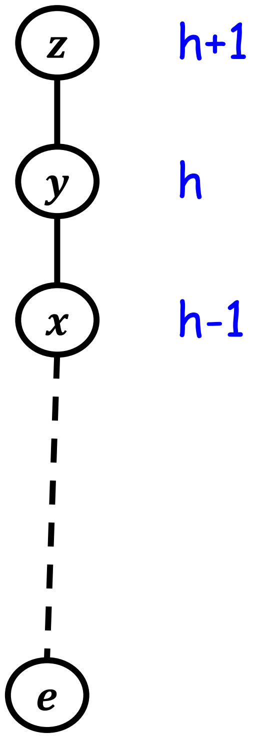
- How to re-balance the tree?
- Find sub-trees of height $\leq h-2$

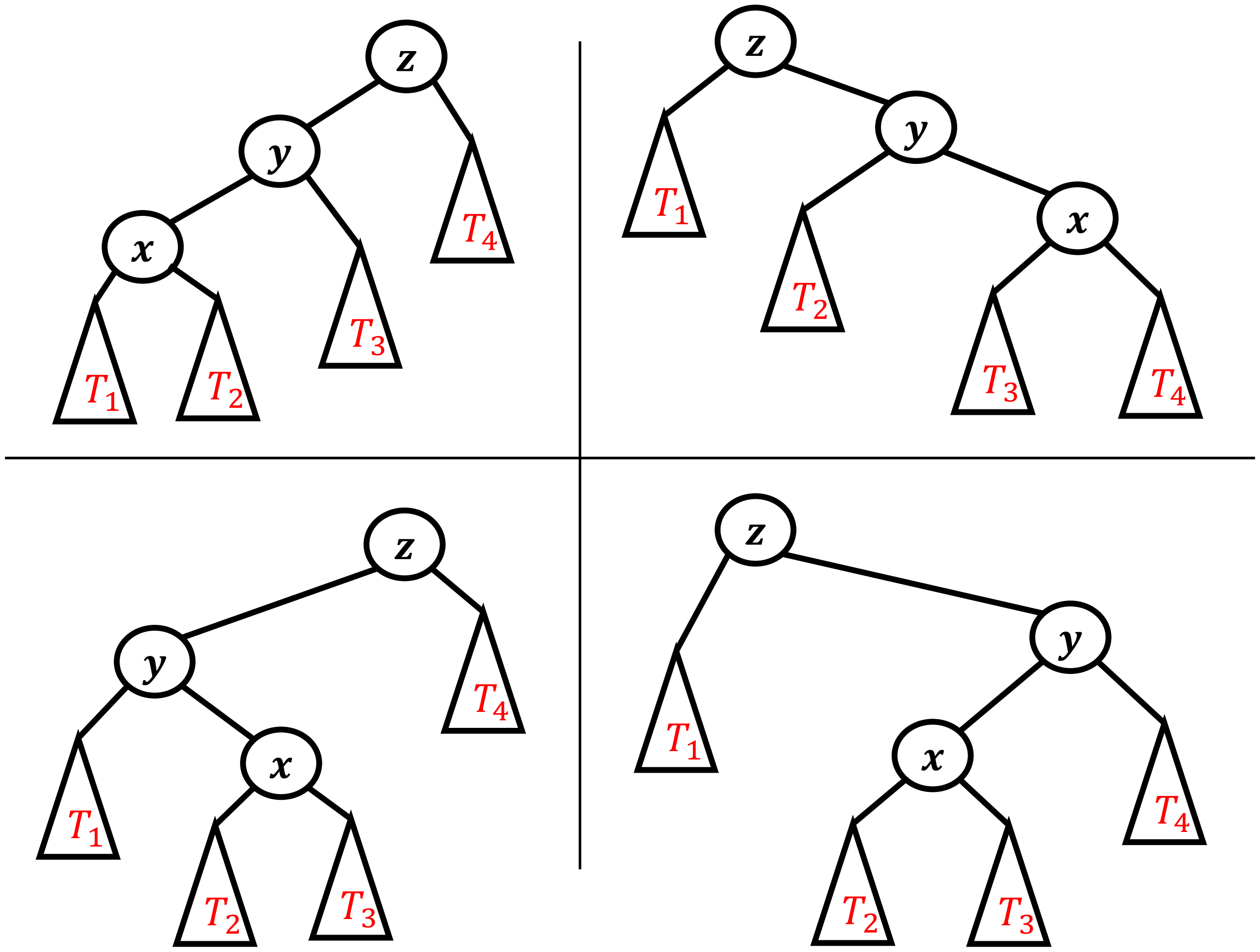




Insertions

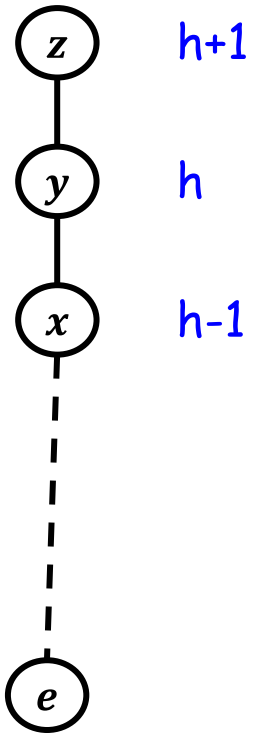
- After insert(e)

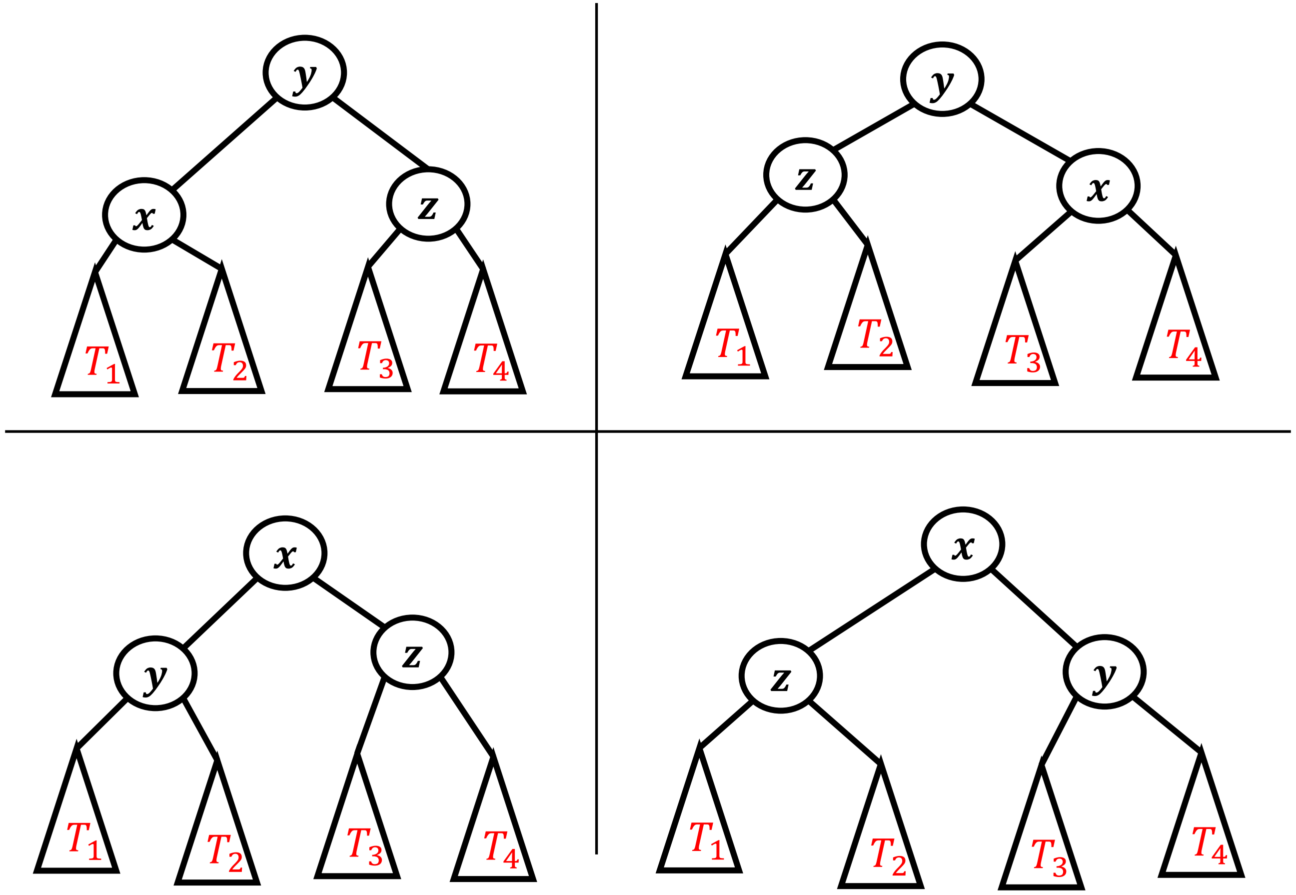




Insertions

- After insert(e)



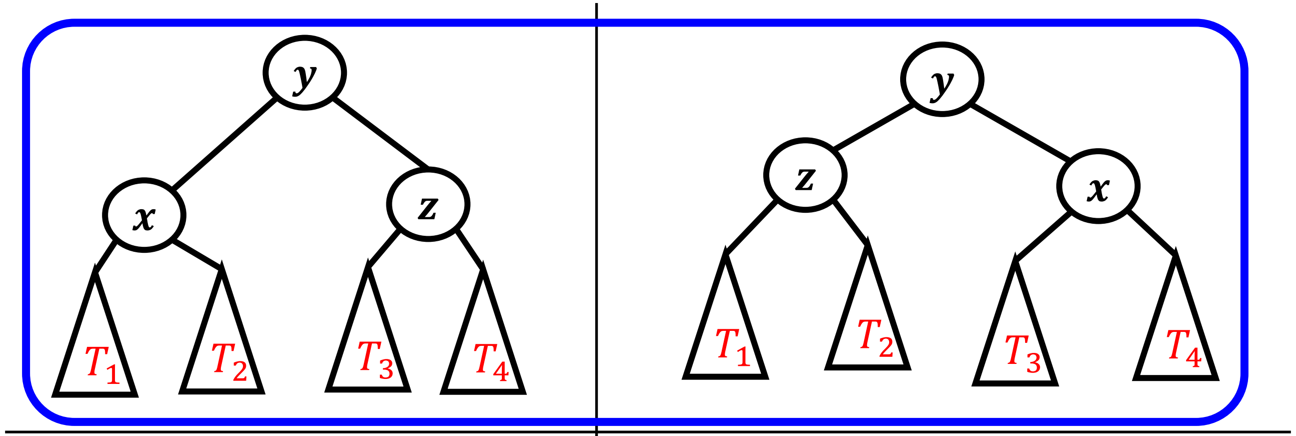


Insertions

- After insert(e)

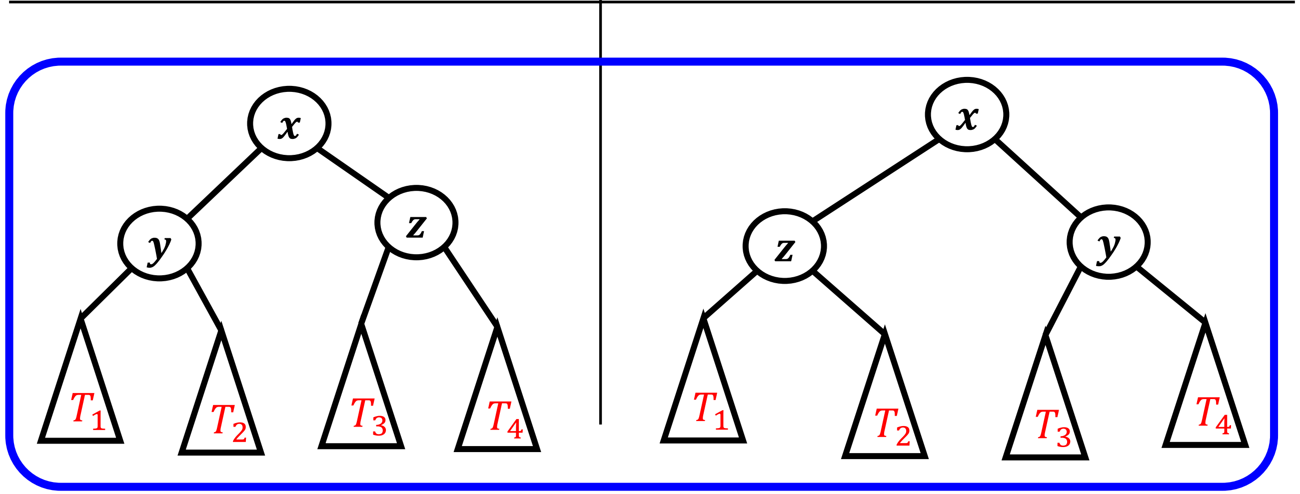
Single Rotation

- Rooted at y
- Subtree rooted at x does not change



Double Rotation

- Rooted at x
- Subtree rooted at x changed



Insertions

- After insert(e)
- Identify the lowest unbalanced node z
- Let x, y, z be the last three nodes on the path from e to z
- Perform single/double rotation to re-balance the subtree rooted at z.
    - Height of $T(z)$ changes from $h+1$ to h
- Observation. The whole tree is balanced again !
    - Height of $T(z)$: $h \rightarrow (h+1) \rightarrow h$
    - Complexity of re-balancing after insertion: $O(1)$

Deletions

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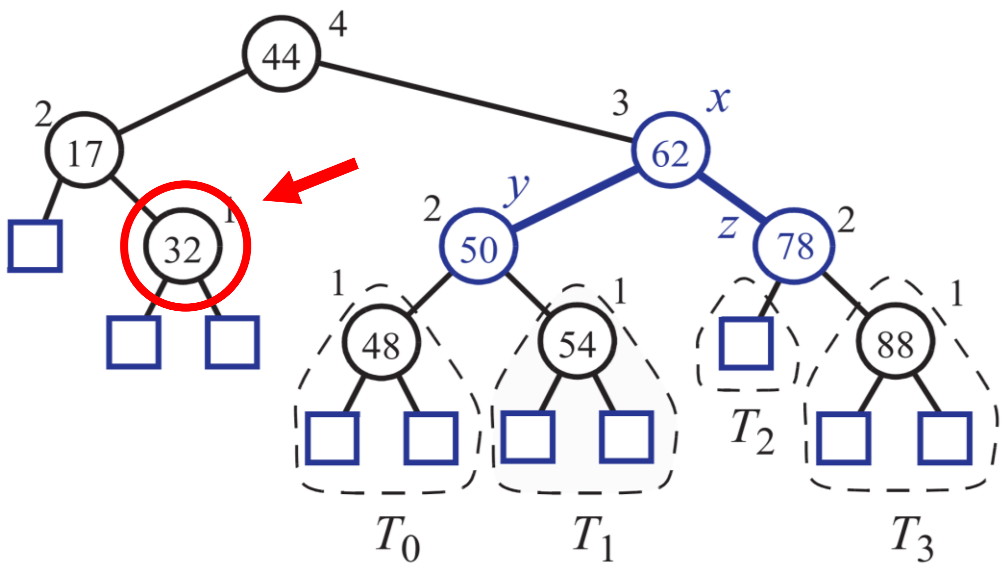
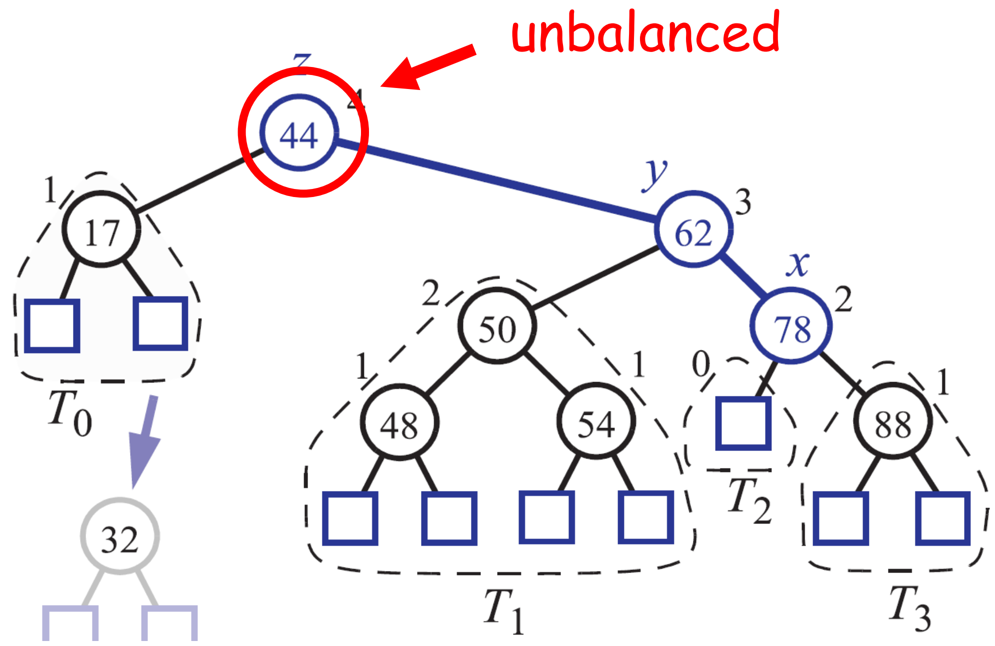
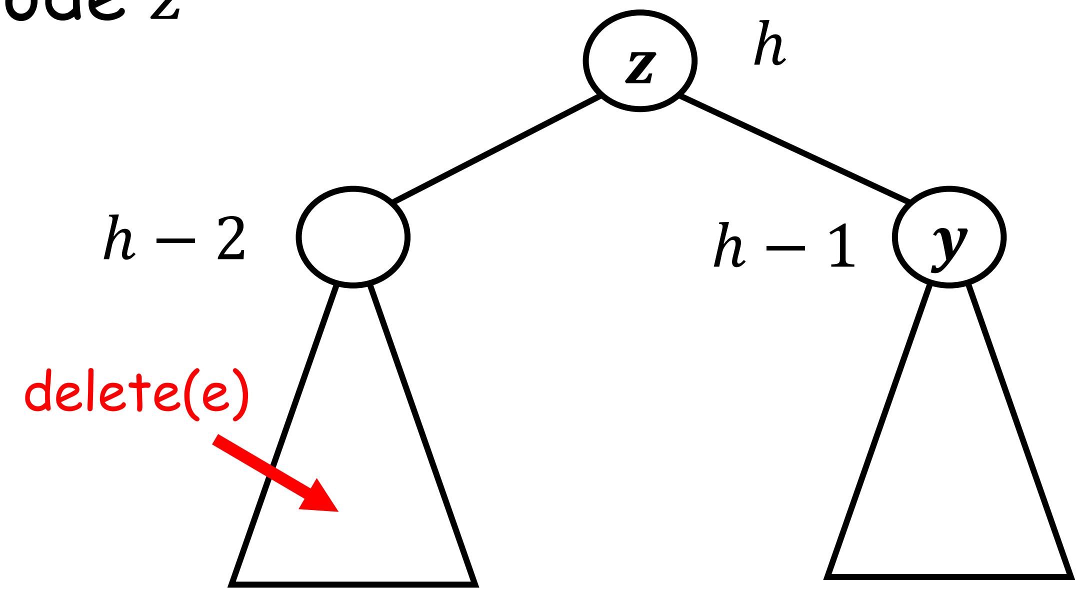
- Removal of element with key 32





Deletions

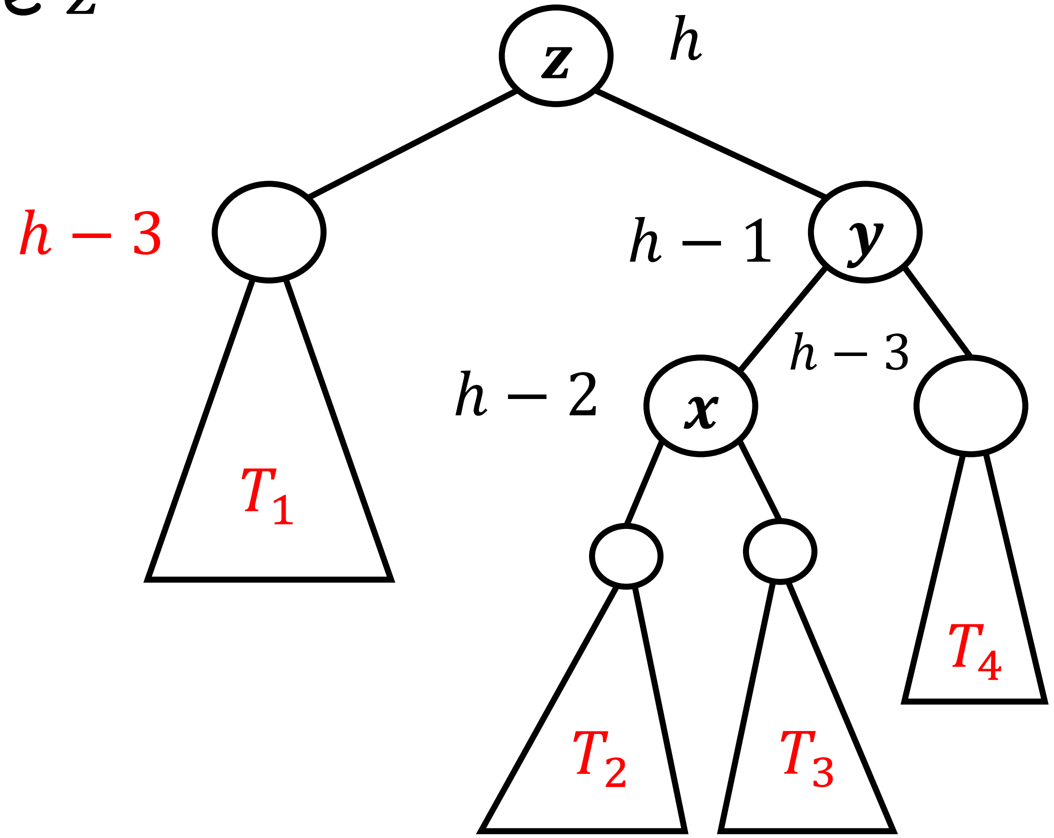
- Removal of element e
- Identify the lowest unbalanced node z
- Let y be the other child of z



Before deletion

Deletions

- Removal of element e
- Identify the lowest unbalanced node z
- Let y be the other child of z
- Let x be child of y such that
  - If subtrees of y have different heights: let x be the taller one



Deletions

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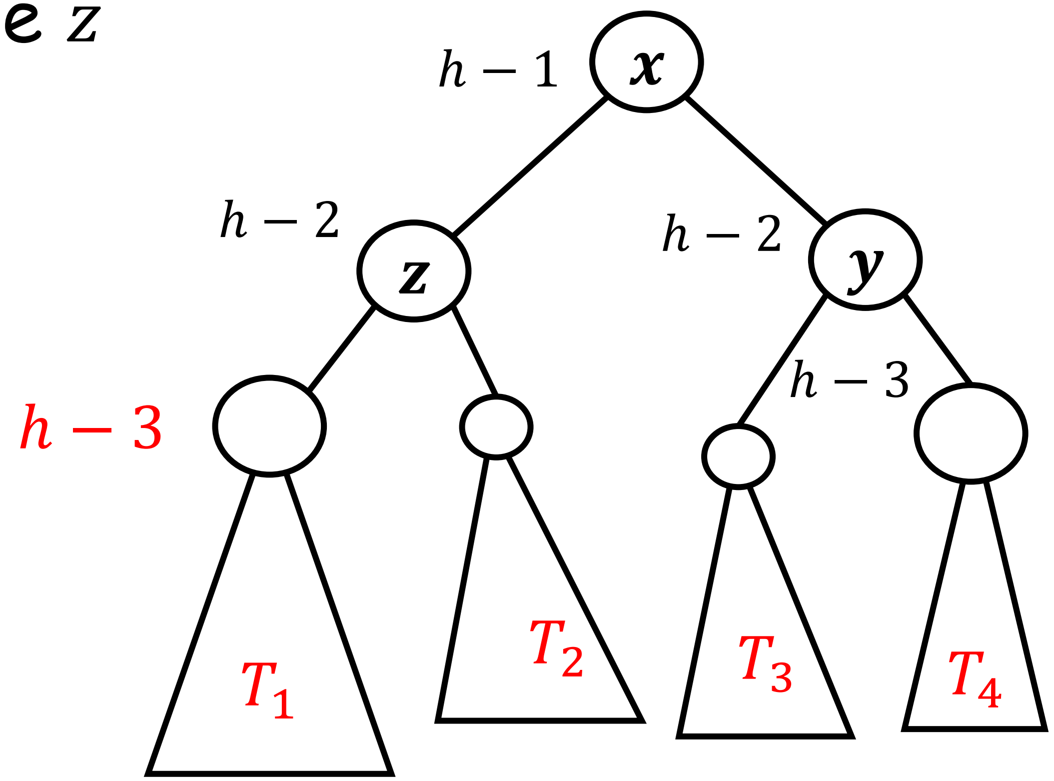
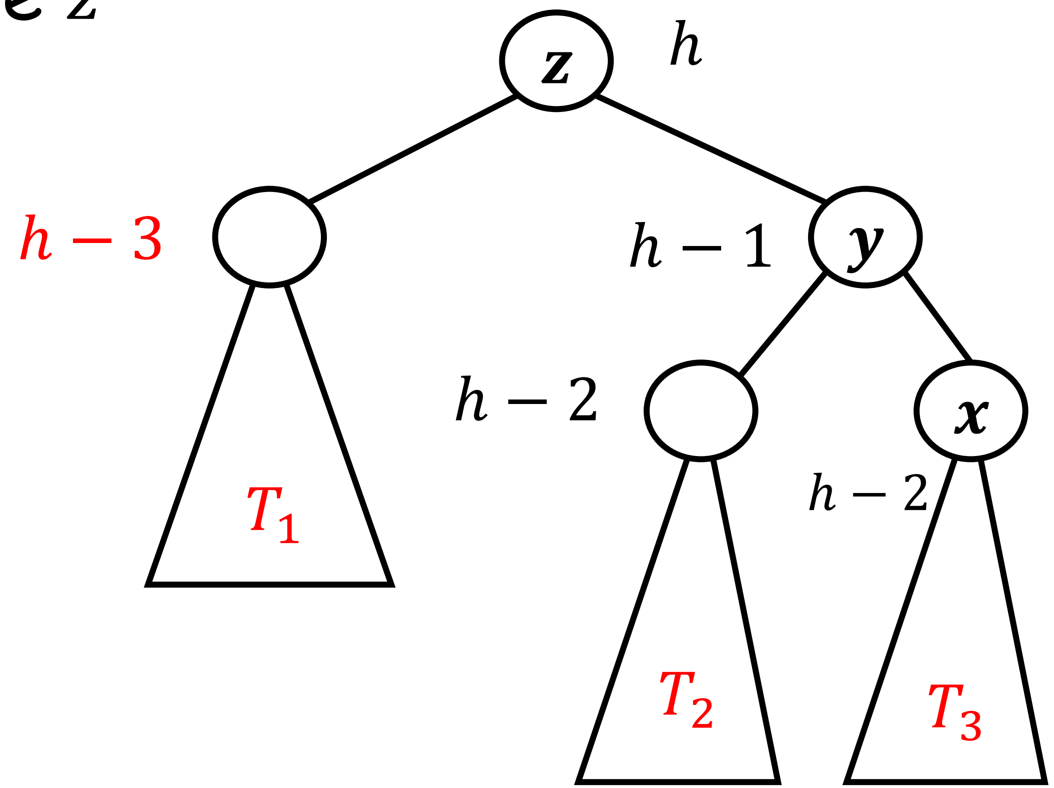
- Removal of element e
- Identify the lowest unbalanced node z
- Let y be the other child of z
- Let x be child of y such that
  - If subtrees of y have different heights: let x be the taller one

Re-balanced



Deletions

- Removal of element e
- Identify the lowest unbalanced node z
- Let y be the other child of z
- Let x be child of y such that
  - If subtrees of y have different heights: let x be the taller one
  - If subtrees of y have the same height: let x be on the same side as y being child of z.



Deletions

- Removal of element e
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- Let y be the other child of z
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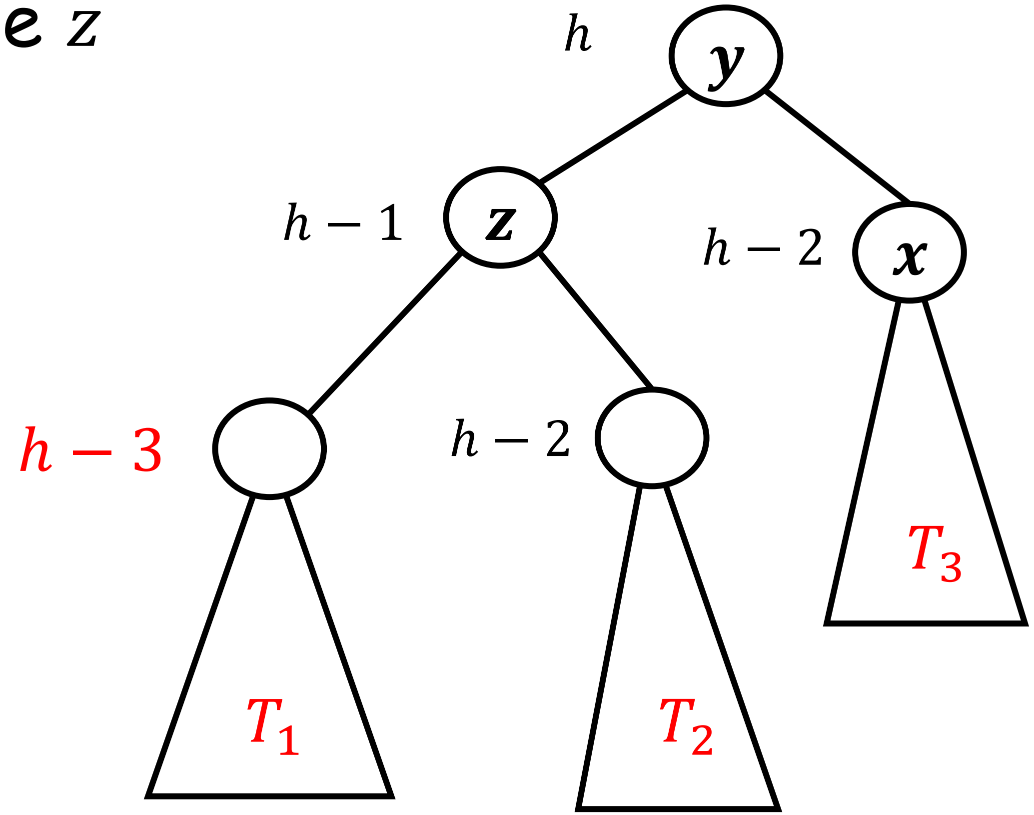
Re-balanced



Deletions

- Removal of element e
- Identify the lowest unbalanced node z
- Let y be the other child of z
- Let x be child of y such that
  - If subtrees of y have different heights: let x be the taller one
  - If subtrees of y have same height: let x be on the same side as y
- Re-balance $T(z)$ be single/double rotation

- ~~Observation: The whole tree is balanced again ?~~

Deletions

- Removal of element e
- While(there exists an unbalanced node)
  - Identify the lowest unbalanced node z
  - Re-balance $T(z)$ by single/double rotation

- Observation: at most h re-balancings are necessary
  - As re-balancing happens at higher and higher places
- Complexity: $O(h)$
  - Each re-balancing takes $O(1)$ time

Binary Search Tree

- Binary Search Tree (BST) supports
  - insert(e) insert an element, e.g., e = (key, value)
  - find(key) find element e with e.key = key
  - remove(key) remove the element e with e.key = key
  - remove(p) remove the element e point p points to

- Complexity of AVL tree:

Operation	**Time***Time*
size, empty	$O(1)$
find, insert, erase	$O(\log n)$

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Priority Queue vs. BST

- BST is more "powerful"
  - Supports searching of any key
  - Can be easily extended to support min() and removeMin()
- Priority Queue is more efficient in finding the minimum
  - $O(1)$ in priority queue; $O(\log n)$ for BST
- Tree structures with different properties
  - Heap (for priority queue) : heap order property
  - AVL tree (for BST) : left $\leq$ middle $\leq$ right

Sorting Algorithms

Sorting

- **Input**: a sequence of numbers $A = (a_1, a_2, \dots, a_n)$
- **Output**: a sorted sequence of numbers in A, which is denoted by $B = (b_1, b_2, \dots, b_n)$, where $b_1 \leq b_2 \leq \cdots \leq b_n$.

General version:

- Numbers -> Elements
- Need to define the comparator
  - elements $x \preccurlyeq y$ iff compare(x, y) = True

Sorting

- **Simple Sorting Algorithms:**
  - Insertion Sort
  - Bubble Sort
  - Selection Sort

- **Advanced Sorting Algorithms:**
  - Heap-Sort
  - Merge-Sort
  - Quick-Sort

Insertion Sort

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- **Main Idea**:
  - For each $i = 1,2,\dots,n$, sort prefix $(a_1, a_2, \dots, a_i)$
  - For each a_i, find the right position in $\{1,2,\dots,i\}$ by moving forward.

- **for** $(i = 2, \dots, n)$: // try to find a position for a_i
  - $e \leftarrow a_i, j \leftarrow i - 1$
  - **while** $(j \geq 1$ and $a_j > e)$ // a_j should appear after e
    - $a_{j+1} \leftarrow a_j, j \leftarrow j - 1$
  - $a_{j+1} \leftarrow e$

Insertion Sort

- **Main Idea:**
  - For each $i = 1,2,\dots,n$, sort prefix $(a_1, a_2, \dots, a_i)$
  - For each a_i, find the right position in $\{1,2,\dots,i\}$ by moving forward.

- **Example:** input = (5,7,2,6,9,3)
  - $i = 2$: (5,7,2,6,9,3) -> (5,7,2,6,9,3)
  - $i = 3$: (5,7,2,6,9,3) -> (2,5,7,6,9,3)
  - $i = 4$: (2,5,7,6,9,3) -> (2,5,6,7,9,3)
  - $i = 5$: (2,5,6,7,9,3) -> (2,5,6,7,9,3)
  - $i = 6$: (2,5,6,7,9,3) -> (2,3,5,6,7,9)

Insertion Sort

- **Main Idea:**
  - For each $i = 1,2,\dots,n$, sort prefix $(a_1, a_2, \dots, a_i)$
  - For each a_i, find the right position in $\{1,2,\dots,i\}$ by moving forward.

- **Complexity:** $O(n^2)$
  - There are $n-1$ for-loops
  - Each for-loop executes in $O(i) = O(n)$ time
    - It takes $O(i)$ time to locate the position to "insert" a_i

Bubble Sort

- **Main Idea:**
  - Do several linear scanning over the sequence
  - For each element, move it backward if it is larger than its successor

- **for** $(i = 1,2,\ldots,n-1)$: // round-i
  - **for** $(j = 1,2,\ldots,n-i)$ // linear scanning
    - **if** $(a_j > a_{j+1})$
      - swap the values of a_j and a_{j+1}

Bubble Sort

- **Main Idea:**
  - Do several linear scanning over the sequence
  - For each element, move it backward if it is larger than its successor
- **Example:** input = (5,7,2,6,9,3)
  - Round 1: (5,7,2,6,9,3) -> (5,2,6,7,3,9)
  - Round 2: (5,2,6,7,3,9) -> (2,5,6,3,7,9)
  - Round 3: (2,5,6,3,7,9) -> (2,5,3,6,7,9)
  - Round 4: (2,5,3,6,7,9) -> (2,3,5,6,7,9)
  - Round 5: (2,3,5,6,7,9) -> (2,3,5,6,7,9)

Bubble Sort

- **Correctness:**
  - After round-i, the top-i maximum elements are well-sorted
  - At most $n-1$ rounds are necessary
  - In round-i, we only need to scan elements up to a_{n-i}

- **Complexity:** $O(n^2)$
  - At most $n-1$ rounds
  - Each round terminates in $n-i=O(n)$ time

Swap-Based Sorting

- Both Insertion-Sort and Bubble-Sort are swap-based
  - Every operation swaps positions of two adjacent elements
- Complexity of the algorithms: number of inversions
  - Pair (a_i, a_j) is an inversion if $i < j$ but $a_i > a_j$
  - Each swap decreases the number of inversions by one (why?)
  - No inversion in the sorted list
- Expected number of inversions in a random sequence: $\Theta(n^2)$

Selection Sort

- **Main Idea:**
  - For $i = 1,2,\ldots,n$:
    (1) select the smallest element e from A;
    (2) set $b_i \leftarrow e$;
    (3) remove e from A.

- **Algorithms:**
  - Sorting with priority queues implemented by unsorted list
  - Complexity: $O(n^2)$

Heap Sort

- **Main Idea:**
  - For $i = 1,2, \ldots, n$:
    (1) select the smallest element e from A;
    (2) set $b_i \leftarrow e$;
    (3) remove e from A.

- **Algorithms:**
  - Sorting with heap
  - Complexity: $O(n \log n)$

Merge-Sort

Merge-sort is based on divide-and-conquer.

- Divide: If the input size is smaller than a certain threshold (say, one or two elements), solve the problem directly using a straightforward method and return the solution obtained. Otherwise, divide the input data into two or more disjoint subsets.
- Recurse: Recursively solve the subproblems associated with the subsets.
- Conquer: Take the solutions to the subproblems and "merge" them into a solution to the original problem.

Merge-Sort

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Merge-sort is based on divide-and-conquer.

- Divide:
  - (base case) if $n \leq 1$: the sequence is sorted.
  - otherwise let $k = \lfloor n/2 \rfloor$
  - partition $A = (a_1, a_2, \dots, a_n)$ into $L = (a_1, \dots, a_k)$ and $R = (a_{k+1}, \dots, a_n)$.
- Recurse:
  - Sort L into B_L; sort R into B_R
- Conquer:
  - Merge the two sequences B_L and B_R into a sorted sequence B
  - Since B_L and B_R are sorted, the merging can be done in $O(n)$ time

Merge-Sort(A)

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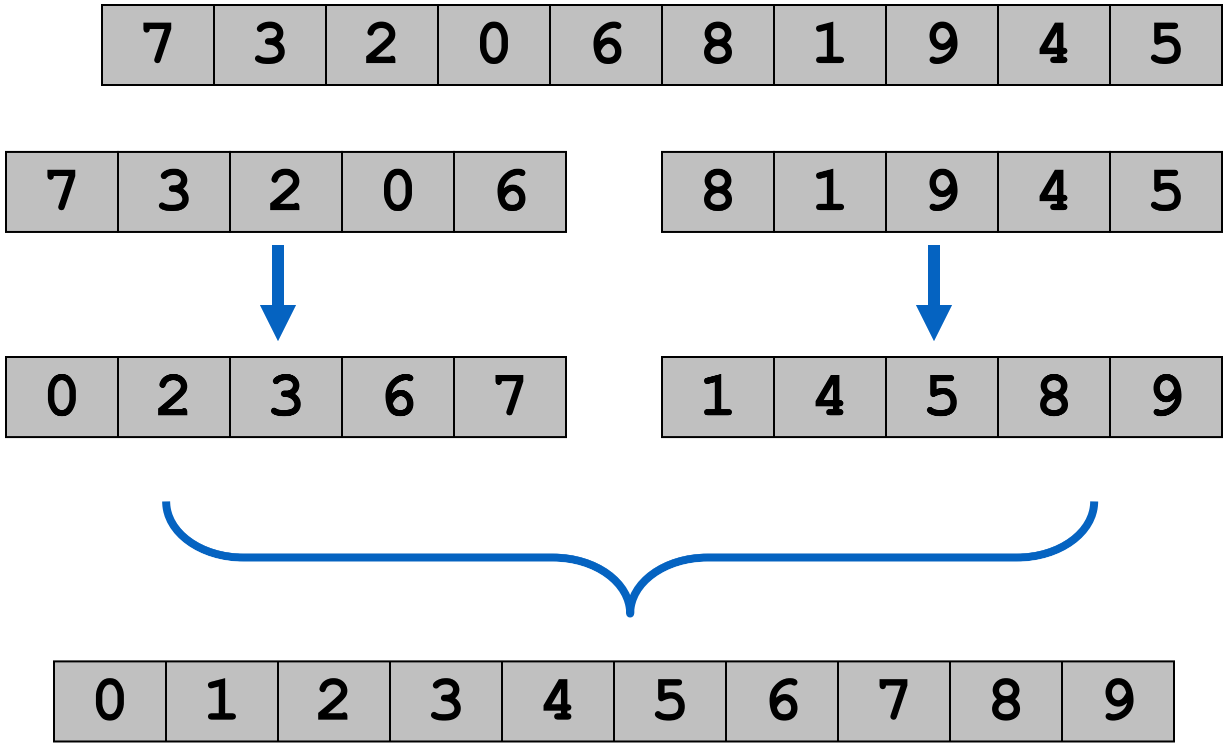
- **Input**: $A = (a_1, a_2, \dots, a_n)$
- **If** $(n \leq 1)$: **return** A
- **If** $(n \geq 2)$:
  - let $k \leftarrow \lfloor n/2 \rfloor$
  - partition $A = (a_1, \dots, a_n)$ into $L = (a_1, \dots, a_k)$ and $R = (a_{k+1}, \dots, a_n)$
  - $B_L \leftarrow$ Merge-Sort(L)
  - $B_R \leftarrow$ Merge-Sort(R)
  - $B \leftarrow$ Merge(B_L, B_R)
  - **return** B

Example

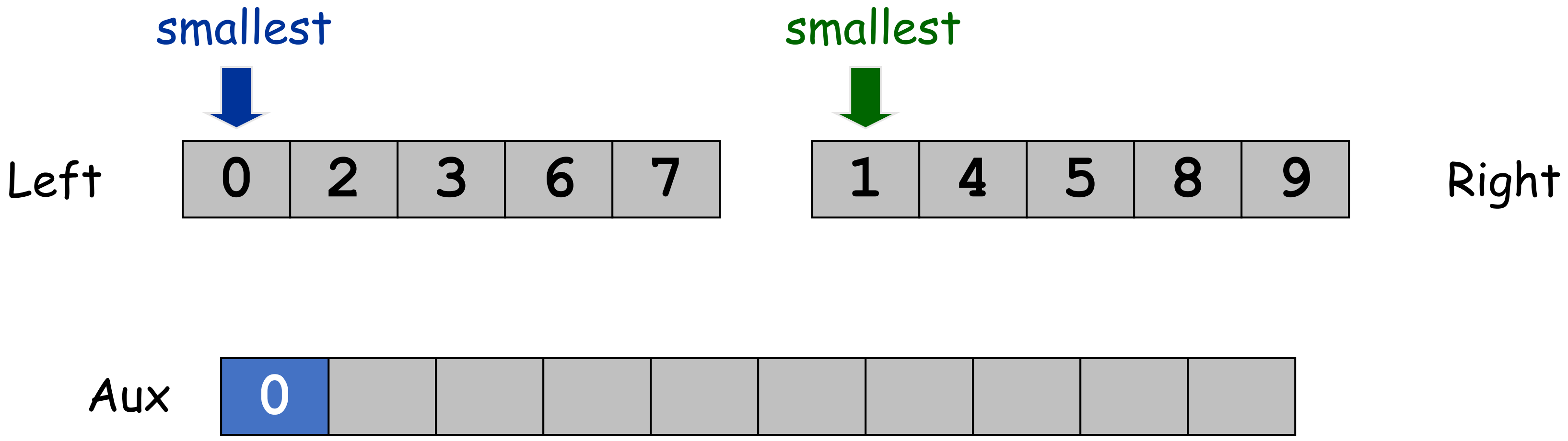


Divide: $O(1)$ time

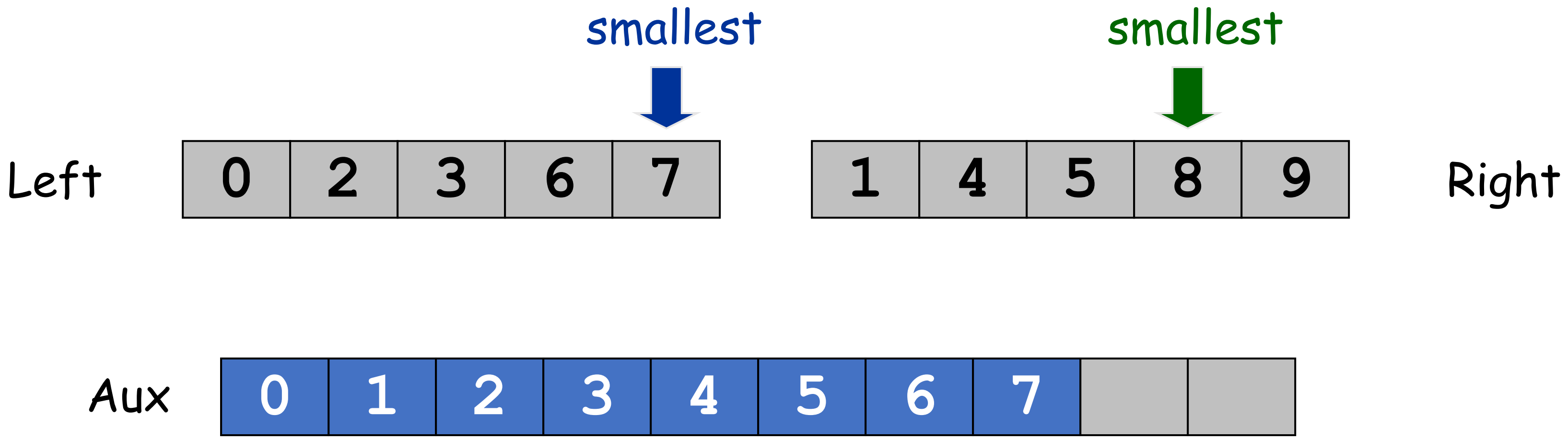
Sort recursively

Merge: $O(n)$ time

Merging



Merging



Merge(X, Y)

- **Input**: Sorted sequences $X = (x_1, \ldots, x_a)$ and $Y = (y_1, \ldots, y_b)$
- assume $x_{a+1} = y_{b+1} = \infty$
- let $i \leftarrow 1$ and $j \leftarrow 1$
- **for** $(k = 1,2, \ldots a + b)$:
  - if $(x_i \leq y_j)$: $z_k \leftarrow x_i, i \leftarrow i + 1$
  - if $(x_i > y_j)$: $z_k \leftarrow y_j, j \leftarrow j + 1$
- **return** $(z_1, \ldots, z_{a+b})$

Complexity of Merge

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- Lemma. Given any two sorted arrays X and Y, Merge(X, Y) finishes in $O(|X| + |Y|)$ time.
- Proof.
  - Consider the value of pointers $i + j$
  - initially $i + j = 2$ and after each for-loop $i + j$ increases by one
  - $i + j$ is at most $|X| + |Y| + 2$
  - each for-loop executes in $O(1)$ time
  - the whole algorithm finishes in $O(|X| + |Y|)$ time

Complexity of Merge-Sort

- Let $T(n)$ be the complexity with input size n.
  - $T(n)$ is the number of comparisons
- For all $n \geq 2$ we have (where $c \geq 1$ is a constant)
- $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + c \cdot n.$

- **Claim:** $T(n) \leq c \cdot n \cdot \log_2 n = O(n \log n)$

Complexity of Merge-Sort

- **Claim:** $T(n) \le c \cdot n \cdot \log_2 n.$

- **Proof 1.**

- $$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n = 2 \cdot \left(2 \cdot T\left(\frac{n}{4}\right) + \frac{c \cdot n}{2}\right) + c \cdot n$$
  $$= 4 \cdot T\left(\frac{n}{4}\right) + 2c \cdot n = 8 \cdot T\left(\frac{n}{8}\right) + 3c \cdot n = \cdots$$
  $$= \frac{n}{2} \cdot T(2) + \left(\log \frac{n}{2}\right) \cdot c \cdot n \le c \cdot n \cdot \log n.$$

Mathematical Induction

- Prove that: $n^3 - n$ is divisible by 3 for every $n \geq 2$.
- Let $p(n)$ be the above statement.

- Suppose $p(k)$ is true, we can prove $p(k+1)$ easily:
  - $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$
  - $= (k^3 - k) + 3(k^2 + k)$
  - $(k^3 - k)$ is divisible by $p(k)$,
  - $3(k^2 + k)$ is obviously divisible by 3
- Statement $p(k+1)$ is also true.

Mathematical Induction

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- Statement: $P(n)$ for all $n \in \boldsymbol{N}$.
- **Induction Principle:**
- $\left(P(1) \wedge \left(\forall k \in \boldsymbol{N}: P(k) \rightarrow P(k+1)\right)\right) \rightarrow \forall n \in \boldsymbol{N}: P(n)$.
- **Proof Steps:**
  - Base case: prove statement $P(1)$
  - Induction step:
    - **(Induction Hypothesis)** Assume $P(k)$ is true for some $k \in N$
    - Prove $P(k+1)$.
  - Conclusion: the statement $P(n)$ holds for all $n \in \boldsymbol{N}$.

Complexity of Merge-Sort

- **Claim:** $T(n) \leq c \cdot n \cdot \log_2 n.$

- **Proof 2.** (Mathematical Induction)
- Base Case: $n = 2$: $T(n) = 1 \leq c \cdot n \cdot \log_2 n$
- Induction hypothesis: $T(k) \leq c \cdot k \cdot \log k$ for all $k \leq n - 1$
- Induction step: $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n \leq c \cdot n \cdot \log\left(\frac{n}{2}\right) + c \cdot n = n \cdot \log_2 n$
- By M.I., we have $T(n) \leq c \cdot n \cdot \log_2 n$ for all $n \geq 2$.

Quick-Sort

Quick-sort is based on divide-and-conquer.

- Divide:
  - (base case) if $n \leq 1$: the sequence is sorted.
  - otherwise pick a **pivot** k and partition $A = (a_1, a_2, \ldots, a_n)$ into
    - $L = \{a_i \in A : a_i < k\}$ and
    - $M = \{a_i \in A : a_i = k\}$
    - $R = \{a_i \in A : a_i > k\}$.
  - $\Rightarrow O(n)$ time
- Recurse: Sort L into B_L; sort R into B_R $\quad\Rightarrow O(?)$ time
- Conquer: Return $B = B_L + M + B_R$ $\quad\Rightarrow O(1)$ time

Quick-Sort(A)

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- **Input**: $A = (a_1, a_2, \dots, a_n)$
- **If** $(n \leq 1)$: **return** A
- **If** $(n \geq 2)$:
  - $k \leftarrow$ PickPivot(A)
  - initialize $L \leftarrow \emptyset, M \leftarrow \emptyset, R \leftarrow \emptyset$
  - **For** $(i = 1,2, \dots, n)$
    - put a_i in (L / M / R) if ($a_i < k$ / $a_i = k$ / $a_i > k$)
  - $B_L \leftarrow$ Quick-Sort(L), $B_R \leftarrow$ Quick-Sort(R)
  - $B \leftarrow B_L + M + B_R$
  - **return** B

PickPivot(A)

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How to pick the pivot?

- Median
  + produces perfectly balanced recursions
  - time consuming to find median (best algorithm: $\Omega(n)$)
- First element
  + very convenient
  - may produce highly unbalanced recursions when A is structured
- Random element
  + efficient and produce balanced recursions on average

Quick-Sort(A)

- PickPivot(A): pick a random element

- **Theorem**: The expected running time of randomized Quick-Sort on a sequence of n elements is $O(n \log n)$.
- **Justification:**
  - $T(n) = T(|L|) + T(|R|) + O(n)$
  - The probability of having $|L| < n/4$ or $|R| < n/4$ is low
  - It is fine to have some unbalanced recursions

Quick-Sort(A)

- PickPivot(A): pick a random element
- **Theorem**: The expected running time of randomized Quick-Sort on a sequence of n elements is $O(n \log n)$.
- Worst Case complexity is still $O(n^2)$
- In practice: median of three random elements

Linear-time Sorting

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- **Input**: a sequence of numbers $A = (a_1, a_2, \dots, a_n)$, where all numbers are integers in $\{1,2,\dots,k\}$, for some $k < n$.
- E.g., sorting a group of people by ages.

- Linear-time Sorting: $\Rightarrow O(k+n) = O(n)$ time
  - Initialize array $L[1,\dots,k]$, each entry corresponds to an initially empty linked list. $\Rightarrow O(k)$ time
  - Insert each element $a_i = t$ into linked list $L[t]$. $\Rightarrow O(n)$ time
  - Output L as a sorted array. $\Rightarrow O(n)$ time

Linear-time Sorting

- Is it possible to have an $O(n)$ time sorting algorithm for general input instances? **NO !**

- Theorem: The running time of any comparison-based algorithm for sorting an n-element sequence is $\Omega(n \log n)$ in the worst case
- **Justification**: given $A = (a_1, \ldots, a_n)$
  - Total number of possible outputs: $n! = n \times (n-1) \times \cdots \times 2 \times 1$.
  - Each comparison eliminates at most half of the possible outputs

Linear-time Sorting

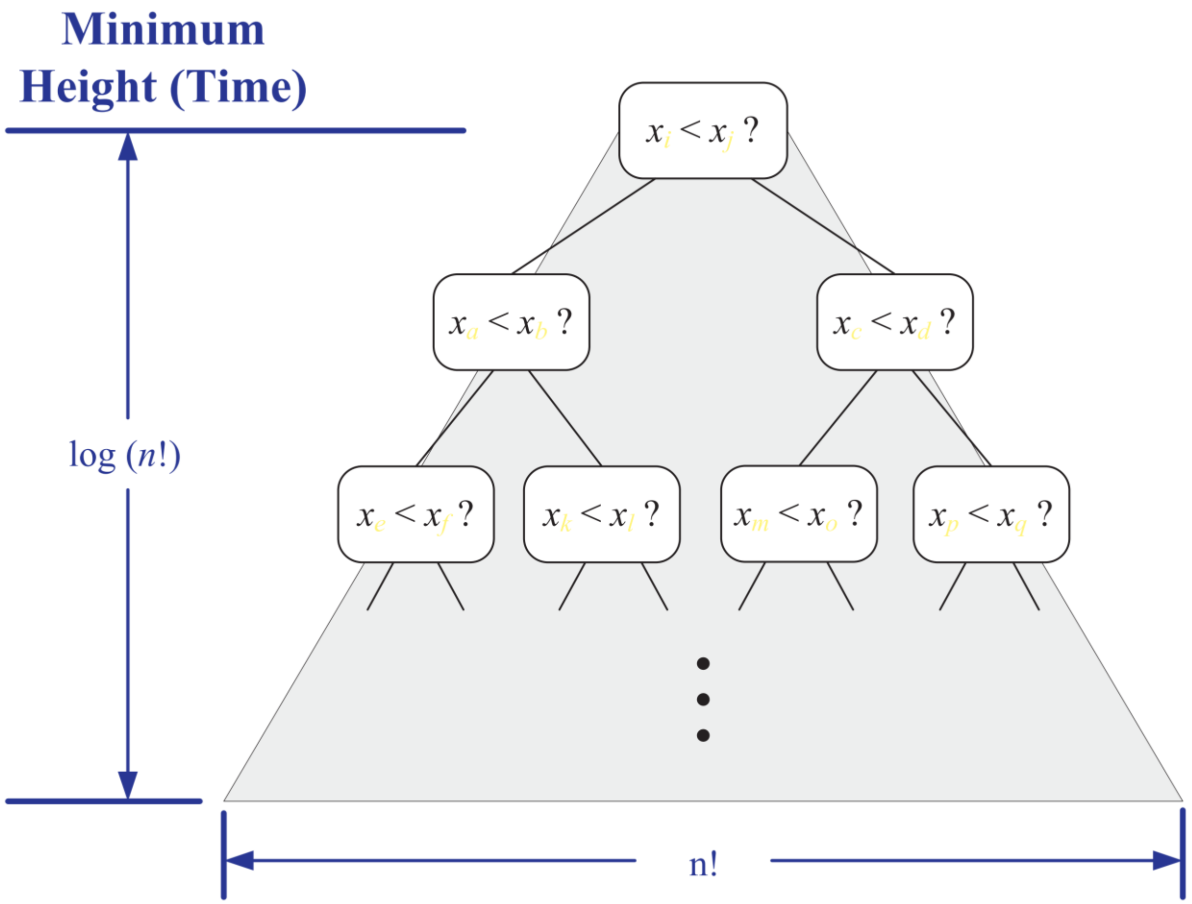
- Decision tree

- Comparisons needed:

$$\log(n!) = \Omega(n \log n)$$

  - Stirling's approximation

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$



Algorithm Design Techniques

Problems

- **Decision Problem**
  - Given an instance of the problem, decides YES/NO
  - **Example**: "Given x, decide if it is a prime number."

- **Optimization Problem**
  - **Constraints**: any feasible solution must satisfy
  - **Objective**: maximize/minimize some function of a solution
  - **Example**: Knapsack problem

Optimization Problem

- **Optimization Problem**
  - **Constraint**: feasibility of solution
  - **Objective**: maximize/minimize ….
- Feasible Solution: a solution that meets all constraints
- Optimal Solution: feasible solution with best objective
- Optimal Objective: objective of the optimal solution
  - For a maximization problem: if there is no feasible solution, then $\mathrm{OPT} = -\infty$
  - For a minimization problem: if there is no feasible solution, then $\mathrm{OPT} = +\infty$

Divide and Conquer

- Divide and Conquer
  - Break up a problem into several parts.
  - Solve each part recursively.
  - Combine solutions to sub-problems into overall solution.
- Most common usage
  - Break up a problem of size n into two equal parts of size $\frac{1}{2}n$.
  - Combine two solutions into overall solution in linear time.
  - Typical time complexity: $O(n \log n)$.
- Example: Merge-Sort

Greedy Algorithms

Problem: among a collection of elements, select a subset of them to maximize/minimize some objective

Greedy Algorithm

- Fix some ordering of the elements.
- Consider the element in this order one-by-one.
- Select an element if
  (1) it does not cause infeasibility;
  (2) selecting the element gives a better objective

A Schedule Problem

- **Problem**: n professors want to use the same classroom. Each of them has a time interval during which he wants to use the place. Find the largest non-conflicting subset of professors.

- **Input**: A set N of n professors such that
  - Each professor $i \in N$ has interval (s_i, f_i) for the use of classroom.
- **Output**: Subset $A \subseteq N$ such that
  - (constraint) for any two different $i, j \in A$, $(s_i, f_i) \cap (s_j, f_j) = \emptyset$.
  - (objective) the size $|A|$ is as large as possible

Greedy Algorithms

- Fix some ordering of the intervals.
- Consider the intervals in this order one-by-one.
- Include an interval if it does not cause infeasibility.
  - Accept those compatible with the ones already accepted.

Greedy Algorithms

- [Earliest start time first]
  - Consider intervals in ascending order of start time s_j.
- **[Earliest finish time first]**
  - Consider intervals in ascending order of finish time f_j.
- [Shortest interval first]
  - Consider intervals in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts first]
  - For each interval, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

Greedy Algorithm

- Sort the intervals in increasing order of finish time.
  - "earliest finish time first"
- Consider the intervals one by one; accept an interval if it is compatible with the ones already accepted.

- **Theorem**: The above Greedy algorithm always computes an optimal solution, for any given input.

Greedy Algorithm

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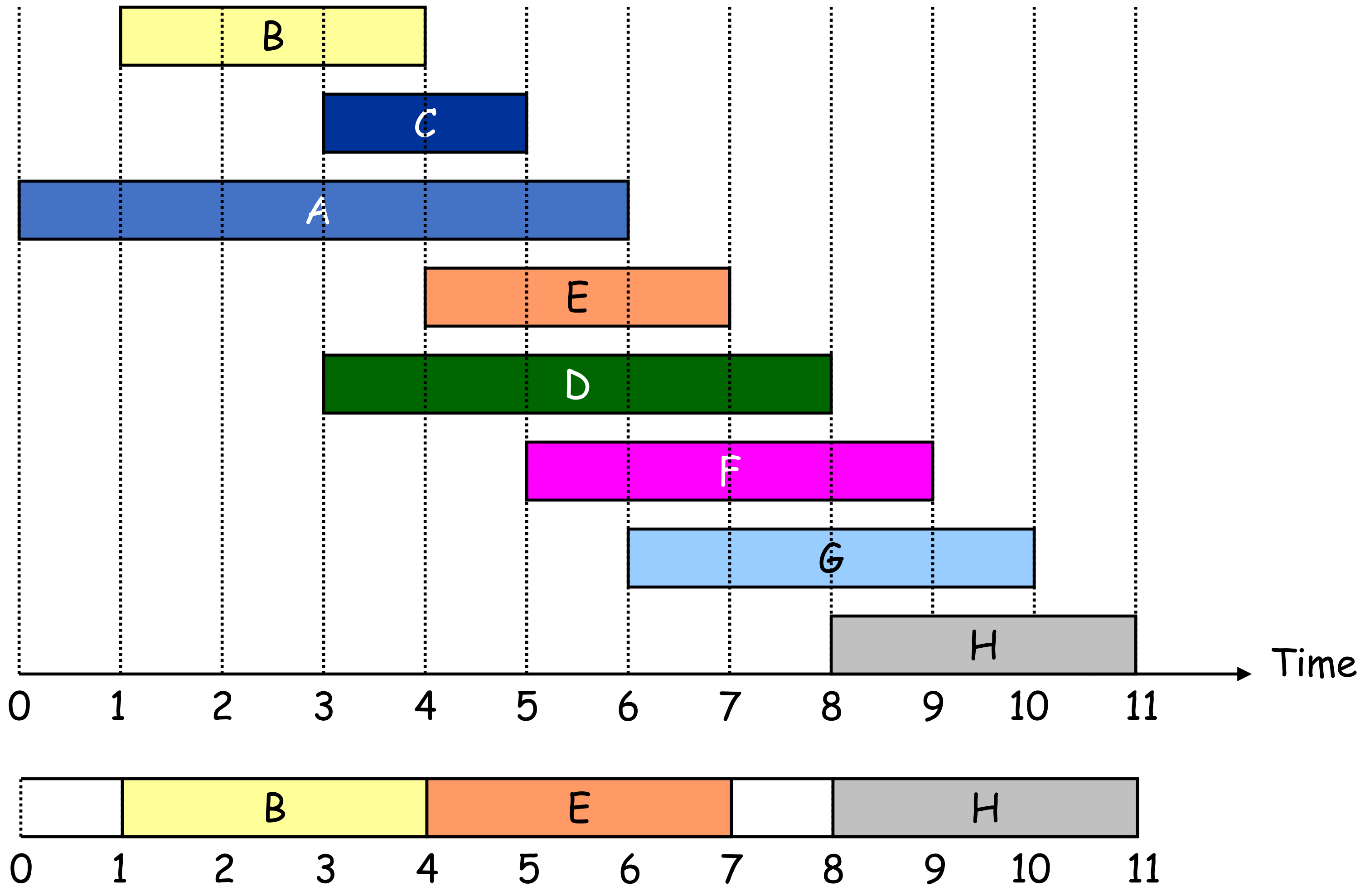

Observation

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- **Claim**: there must exists an optimal solution that includes the interval with minimum finish time.

- **Proof Sketch**.
  - Fix any optimal solution and suppose it does not contain interval 1 (the interval with minimum finish time).
  - Remove the interval in the solution with minimum finish time.
  - Include interval 1.
  - The resulting solution must be feasible and optimal.

Observation

- **Theorem**: Greedy algorithm computes an **optimal solution.**

- **Proof.**
  - There exists an optimal solution containing interval 1.
  - Intervals conflicting with interval 1 do not appear in this optimal solution. We can remove these intervals.
  - //It remains to compute an optimal solution for the resulting instance
  - For the resulting instance, there exists an optimal solution containing the interval with minimum finish time.

Greedy Algorithms

- **Most common usage**
  - Consider the elements of the problem one by one.
  - Take/remove each element if it helps improve the solution.

- Greedy algorithms are usually efficient and easy to implement.
- Greedy algorithms are not always optimal.

Classic Greedy Algorithms

- **Optimal**:
  - Scheduling algorithms
  - Kruskal's algorithm, Prim's algorithm
  - Caching algorithms
- **Good Approximations**:
  - Greedy matching
  - Set cover problem
  - Online load balancing

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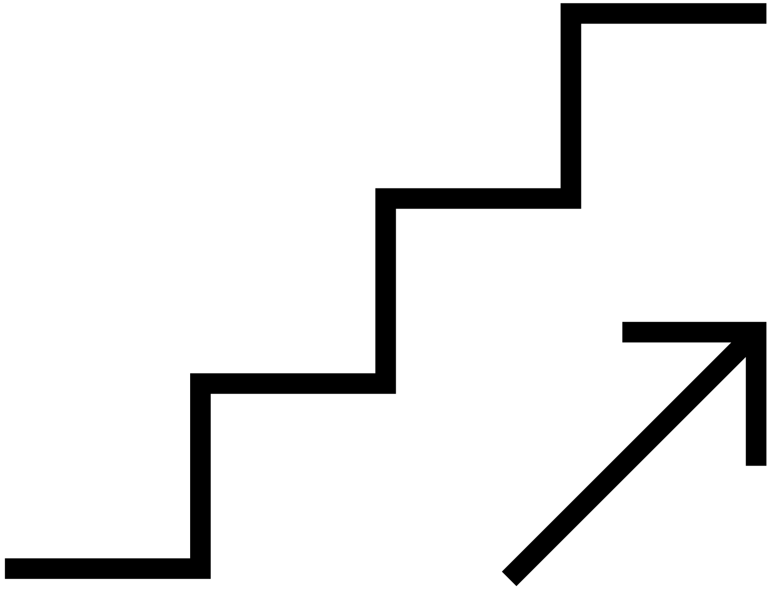
Dynamic Programming

Number of Walks

- Walk from stair 0 to stair n
- Step size: 1 and 2

- **Question:** how many ways?

- **Example:** n=5, f(n)=8



Number of Walks

- Let $f(n)$ be the number of ways to reach stair n.
  - Number of sequences of $\{1,2\}$ that sum to n.

- We have: $f(n) = f(n-1) + f(n-2)$

number of ways to reach stair n - 2

number of ways to reach stair n - 1

Number of Walks

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- Let f(n) be the number of ways to reach stair n.
  - Number of sequences of {1,2} that sum to n.

- To compute f(n): ~~recursion?~~ Record f(1), f(2), …, f(n-1).

Number of Walks

- **General Problem:** compute the number f(n) of sequences of $\{s_1, s_2, \ldots, s_k\}$ that sum to n.
  - Target number: n
  - Step sizes: $s_1, s_2, \ldots, s_k$

- **Observation:** $f(n) = \sum_{i \le k : s_i \le n} f(n - s_i)$

Number of Walks

- **General Problem:** compute the number f(n) of sequences of $\{s_1, s_2, \ldots, s_k\}$ that sum to n.
    - Target number: n
    - Step sizes: $s_1, s_2, \ldots, s_k$

- **Observation:** $f(n) = \sum_{i=1}^{k} f(n - s_i)$
- Base case: $f(0) = 1$; $f(x) = 0$ if $x < 0$.

- **Solution:** compute f(1), then f(2), then f(3), ...

Number of Walks

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- **General Problem**: compute the number f(n) of sequences of $\{s_1, s_2, \dots, s_k\}$ that sum to n.
  - Target number: n
  - Step sizes: $s_1, s_2, \dots, s_k$

- **Observation**: $f(n) = \sum_{i \le k : s_i \le n} f(n - s_i)$
- Base case: $f(0) = 1$

- **Solution**: compute $f(i)$ for $i = 1,2, \dots, n$

Best Walk

- **General Problem**: compute the length of shortest sequence of $\{s_1, s_2, \ldots, s_k\}$ that sums to n.
  - Target number: n
  - Step sizes: $s_1, s_2, \ldots, s_k$

- **Example**: $n = 33$, $s_1 = 3$ and $s_2 = 7$:
- **Answers**: 7 (steps): (7,7,7,3,3,3,3)

Change Making

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- **Given**: coins of integer values $s_1, s_2, \dots, s_k$
  - Each coin has infinitely many of them
- **Target value**: n
- **Problem**: compute the minimum number $f(n)$ of coins that sum to exactly n.

- **Recursion**: $f(n) = 1 + \min\{f(n - s_i) : s_i \leq n\}$
- **Base Case**: $f(0) = 0$

Change Making

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- **Given**: coins of integer values $s_1, s_2, \ldots, s_k$
  - Each coin has infinitely many of them
- **Target value**: n
- **Problem**: compute the minimum number $f(n)$ of coins that sum to exactly n.

- **Recursion**: $f(n) = 1 + \min\{f(n - s_i) : i \leq k\}$
- **Base Case**: $f(0) = 0$, $\forall x < 0$: $f(x) = +\infty$

Pseudo-code

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- let n be the target value and $s_1, s_2, \dots, s_k$ be the coin values
- initialize $f(0) \leftarrow 0$
- **for** ($i = 1,2, \dots, n$):
  - initialize $f(i) \leftarrow \infty$
  - **for** ($j = 1, \dots, k$):
    - **if** ($s_j \le i$): $f(i) \leftarrow \min\{f(i), 1 + f(i - s_j)\}$
- **return** $f(n)$

Knapsack Problem

- **Knapsack** of capacity C
- **Items** N: each item $i \in N$ has a value v_i and a size s_i
- **Problem**: find a subset $A \subseteq N$ of items with total size at most C, such that the total value is maximized.
    - Size constraint: $s(A) = \sum_{i \in A} s_i \leq C$
    - Objective: maximize $v(A) = \sum_{i \in A} v_i$

- **Assumption**: all sizes and the capacity are integers.

Example

- **Knapsack** of capacity $C = 10$
- **Items**: each item i is represented by (v_i, s_i)
  - $N = \{(3,2), (2,3), (4,5), (1,3), (6,4), (7,6)\}$

    1 2 3 4 5 6

- Question: should we take item 6 ?
  - What is the best solution if we take it?
  - What is the best solution if we don't?
  - Choose the better solution between the two.

Example

- What is the **optimal solution** if we take 6 ?
  - Remaining items:

    1 2 3 4 5

  - Remaining capacity: $C - s_6 = 10 - 6 = 4$

- Compute the **optimal solution** $B \subseteq \{1,2,3,4,5\}$ for the sub-problem with capacity 4.
- Our solution: $A = B \cup \{6\}$ (with value $v(A) = v(B) + v_6$)

Example

- What is the **optimal solution** if we take 6 ?
  - Remaining items:

    $\{(3,2),(2,3),(4,5),(1,3),\boxed{(6,4)}\}$

  - Remaining capacity: $C - s_6 = 10 - 6 = 4$

- Compute the **optimal solution** $B = \{5\}$ for the sub-problem with capacity 4.
- Our solution: $A = B \cup \{6\}$ (with value $v(A) = v(B) + v_6$)

Example

- What is the **optimal solution** if we don't take 6 ?
  - Remaining items:

    1 2 3 4 5

  - Remaining capacity: $C = 10$

- Compute the **optimal solution** $B \subseteq \{1,2,3,4,5\}$ for the sub-problem with capacity 10.
- Our solution: $A = B$ (with value $v(A) = v(B)$)

Example

- What is the **optimal solution** if we don't take 6 ?
  - Remaining items:

$$\{\boxed{(3,2),(2,3)},(4,5),(1,3),\boxed{(6,4)}\}$$

  - Remaining capacity: $C = 10$

- Compute the **optimal solution** $B = \{1,2,5\}$ for the sub-problem with capacity 10.
- Our solution: $A = B$ (with value $v(A) = v(B)$)

Example

- **Optimal solution** if we take 6 :
- $A = \{5, 6\}$ (with value $v(A) = v_5 + v_6 = 13$)

- **Optimal solution** if we don't take 6 :
- $A = \{1,2,5\}$ (with value $v(A) = 11$)

Knapsack Problem

- **Knapsack** of capacity C
- **Items** $N = \{1,2,\ldots,n\}$: each $i \in N$ has a value v_i and a size s_i

- Let $f(i,b)$ be the value of optimal solution
  - on items $\{1,2,\ldots,i\}$ $(i \leq n)$
  - with capacity b $(b \leq C)$

Knapsack Problem

- Let $f(i, b)$ be the value of optimal solution
  - on items $\{1,2, \dots, i\}$ $(i \leq n)$
  - with capacity b $(b \leq C)$
- If $b \geq s_i$:

$$\boxed{f(i, b) = \max\{f(i-1, b-s_i) + v_i, f(i-1, b)\}}$$

- If $b < s_i$: $\boxed{f(i, b) = f(i-1, b)}$

- **Base Case**: $f(i, b) = 0$ if $i = 0$

Pseudo-code

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- let C be the capacity and (v_i, s_i) be size and value of item i
- initialize $f(0,b) \leftarrow 0$ for all $b = 0,1,\ldots,C$
- **for** $(i = 1,2,\ldots,n)$:
    - **for** $(b = 0,1,\ldots,C)$:
        - **if** $(b \geq s_i)$: $f(i,b) = \max\{f(i-1,b-s_i)+v_i, f(i-1,b)\}$
        - **else**: $f(i,b) = f(i-1,b)$
- **return** $f(n,C)$

Example

	Capacity									
	1	2	3	4	5	6	7	8	9	10
1 (3,2)	0	3	3	3	3	3	3	3	3	3
2 (2,3)	0	3	3	3	5	5	5	5	5	5
3 (4,5)	0	3	3	3	5	5	7	7	7	9
4 (1,3)	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9
5 (6,4)	0	3	3	6	6	9	9	9	11	11
6 (7,6)	0	3	3	6	6	9	9	10	11	$\boxed{13}$

items

"obtained by taking item 6"

Example

Capacity

items	1	2	3	4	5	6	7	8	9	10
5 (6,4)	0	3	3	6	6	9	9	9	11	11
6 (7,6)	0	3	3	6	6	9	9	10	11	13

Maintain a table $d(i, b)$ such that

- $d(i, b) =$ Y if $f(i, b)$ is achieved by taking item i
- $d(i, b) =$ N if $f(i, b)$ is achieved by not taking item i

Example

items		Capacity									
		1	2	3	4	5	6	7	8	9	10
	1 (3,2)	N	y	y	y	y	y	y	y	y	y
	2 (2,3)	N	N	N	N	y	y	y	y	y	y
	3 (4,5)	N	N	N	N	N	N	y	y	y	y
	4 (1,3)	N	N	N	N	N	N	N	N	N	N
	5 (6,4)	N	N	N	y	y	y	y	y	y	y
	6 (7,6)	N	N	N	N	N	N	N	y	N	y

Example

items	Capacity 1	2	3	4	5	6	7	8	9	10
1 (3,2)	N	y	y	y	y	y	y	y	y	y
2 (2,3)	N	N	N	N	y	y	y	y	y	y
3 (4,5)	N	N	N	N	N	N	y	y	y	y
4 (1,3)	N	N	N	N	N	N	N	N	N	N
5 (6,4)	N	N	N	y	y	y	y	y	y	y
6 (7,6)	N	N	N	N	N	N	N	y	N	y

Pseudo-code

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- let C be the capacity and (v_i, s_i) be size and value of item i
- initialize $f(0, b) \leftarrow 0$ for all $b = 0,1, \ldots, C$
- **for** $(i = 1,2, \ldots, n)$:
  - **for** $(b = 0,1, \ldots, C)$:
    - **if** $(b \geq s_i)$: $f(i, b) = \max\{f(i-1, b-s_i) + v_i, f(i-1, b)\}$
      - **if** $f(i-1, b-s_i) + v_i > f(i-1, b)$: $d(i, b) \leftarrow Y$
      - **else**: $d(i, b) \leftarrow N$
    - **else**: $f(i, b) = f(i-1, b)$ and $d(i, b) \leftarrow N$
- print_solution(d, n, C) and **return** $f(n, C)$

Pseudo-code

print_solution(d, i, b)

- **if** ($i = 0$ or $b = 0$)
  - **return**
- **Elseif** $d(i, b) = Y$
  - print i
  - print_solution($d, i-1, b-s_i$)
- **Elseif** $d(i, b) = N$
  - print_solution($d, i-1, b$)

Longest Common Subsequence

- **Problem:** Given two strings X and Y, compute the longest string Z that is a common subsequence of X and Y.
  - A subsequence Z of a string X is a string that can be obtained by removing some letters from X.

- **Example:** X=CTGACA and Y=ACGCTAC
  - Z=CGA is a common subsequence
  - Z=CGAC is the longest common subsequence (How to find it?)

Observations

- $X = X_n = (x_1, x_2, \dots, x_n)$ and $Y = Y_m = (y_1, y_2, \dots, y_m)$
- **Prefix**: $X_i = (x_1, x_2, \dots, x_i)$ and $Y_j = (y_1, y_2, \dots, y_j)$
- Length of LCS of X_i and Y_j: $c(i, j)$

- **Observation 1.** If $x_i = y_j$ then

$$c(i, j) = c(i-1, j-1) + 1. \quad \text{(proof?)}$$

Observations

- $X = X_n = (x_1, x_2, \dots, x_n)$ and $Y = Y_m = (y_1, y_2, \dots, y_m)$
- **Prefix**: $X_i = (x_1, x_2, \dots, x_i)$ and $Y_j = (y_1, y_2, \dots, y_j)$
- Length of LCS of X_i and Y_j: $c(i,j)$

- **Observation 2.** If $x_i \neq y_j$ then

$$c(i,j) = \max\{c(i,j-1), c(i-1,j)\}.$$ (proof?)

Observations

- If $x_i = y_j$ then

$$c(i,j) = c(i-1,j-1) + 1.$$

- If $x_i \neq y_j$ then

$$c(i,j) = \max\{c(i,j-1), c(i-1,j)\}.$$

- Base case: $c(i,0) = c(0,j) = 0$, for all i,j

Example

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	•	A	C	G	C	T	A	C
•	0	0	0	0	0	0	0	0
C	0	0	1	1	1	1	1	1
T	0	0	1	1	1	2	2	2
G	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
C	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Observations

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- If $x_n = y_m$ then

$$c(n,m) = c(n-1, m-1) + 1.$$

$$b(n,m) = \nwarrow$$

- If $x_n \neq y_m$ then

$$c(n,m) = \max\{c(n, m-1), c(n-1, m)\}.$$

$$b(n,m) = \leftarrow \qquad\qquad b(n,m) = \uparrow$$

Example

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	•	A	C	G	C	T	A	C
•	0	0	0	0	0	0	0	0
C	0	↑	↖	←	↖	←	←	↖
T	0	↑	↑	↑	↑	↖	←	←
G	0	↑	↑	↖	←	↑	↑	↑
A	0	↖	↑	↑	↑	↑	↖	←
C	0	↑	↖	↑	↖	←	↑	↖
A	0	↖	↑	↑	↑	↑	↖	↑

Example

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	•	A	C	G	C	T	A	C
•	0	0	0	0	0	0	0	0
C	0	↑	↖	←	↖	←	←	↖
T	0	↑	↑	↑	↑	↖	←	←
G	0	↑	↑	↖	←	↑	↑	↑
A	0	↖	↑	↑	↑	↑	↖	←
C	0	↑	↖	↑	↖	←	↑	↖
A	0	↖	↑	↑	↑	↑	↖	↑

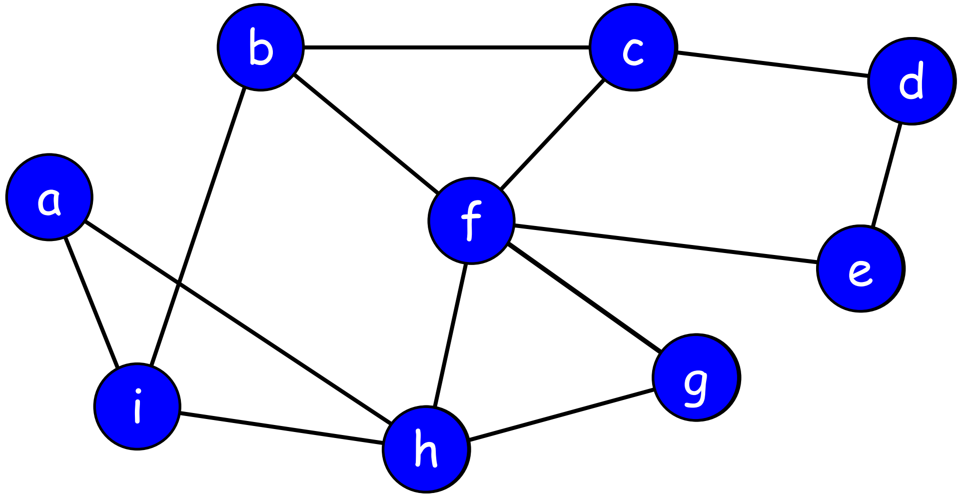
General Framework for DP

- Define the function to capture the problem
  - e.g., $f(i, b)$ for the Knapsack problem
- Formulate the recursion between the function values on different inputs
  - e.g., $f(i, b) = \max\{f(i-1, b-s_i) + v_i, f(i-1, b)\}$
- Compute the value table

Graphs

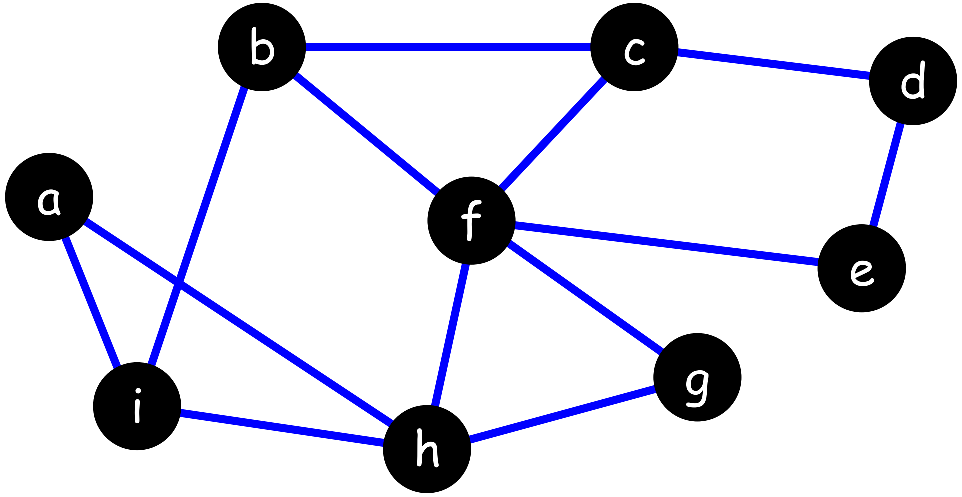
Graphs

- **Nodes:** $V = \{a, b, c, d, e, f, g, h, i\}$
- **Edges**: relations between pairs of objects



Graphs

- **Nodes:** $V = \{a, b, c, d, e, f, g, h, i\}$
- **Edges:** $E = \{(a,i), (a,h), (b,i), (b,f), (b,c), (c,d), (c,f), (d,e), (e,f), (f,g), (f,h), (g,h), (h,i)\}$



Graphs

- **Graph** $G(V, E)$ on nodes V and edges $E \subseteq V \times V$.

- **Examples:**
- Undirected graph: Facebook network.
  - $(u, v) \in E$ if users u and v are friends of each other.
- Directed graph: Twitter network
  - $(u, v) \in E$ if u follows v.

Graphs

- **Graph** $G(V, E)$ on nodes V and edges $E \subseteq V \times V$.

- **More Examples of Undirected Graphs:**
- Co-authorship network:
  - Nodes: authors;
  - Edges: $(u, v) \in E$ if u and v are co-authors.
- Road network:
  - Nodes: cities;
  - Edges: $(u, v) \in E$ if there is a rail connecting cities u and v.

Basic Concepts of Graphs

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- If $(u, v) \in E$:
  - u and v are **adjacent**
  - u and v are **neighbors** of each other

- Set of neighbors $N(u) = \{v \in V: (u, v) \in E\}$ of node u.
- Degree of node u: $d(u) = |N(u)|$.

- **Claim**: $\sum_{u \in V} d(u) = 2 \cdot |E|$.

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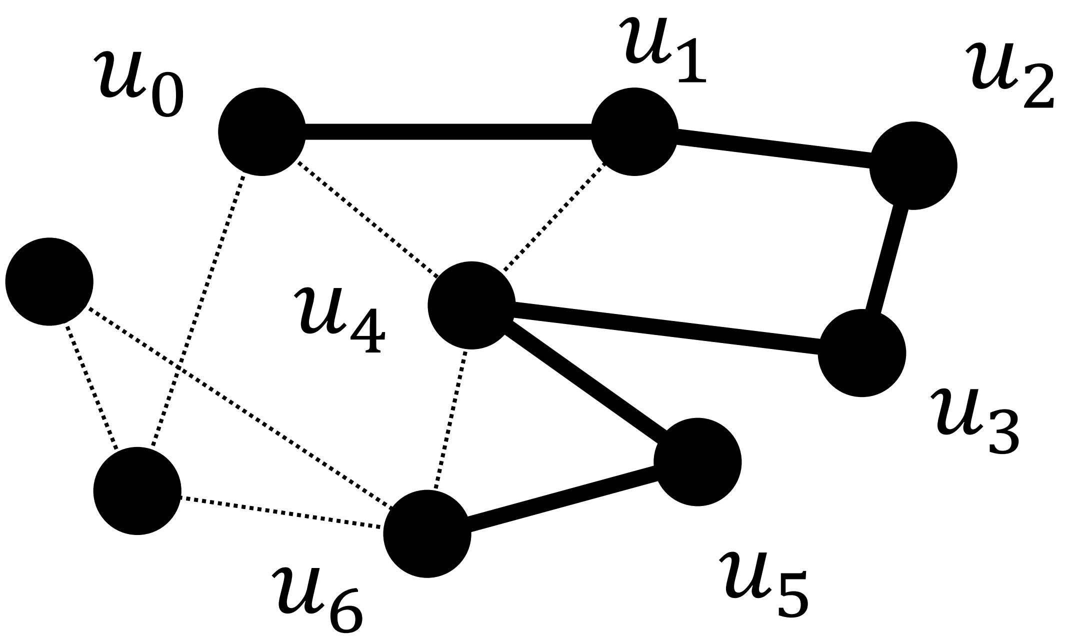
Basic Concepts of Graphs

- Path $(u_0, u_1, \dots, u_l)$ connecting node u_0 and node u_l:
  - for all $i = 1,2, \dots, l$, we have $(u_{l-1}, u_l) \in E$
  - l is the length of the path (number of edges)



Basic Concepts of Graphs

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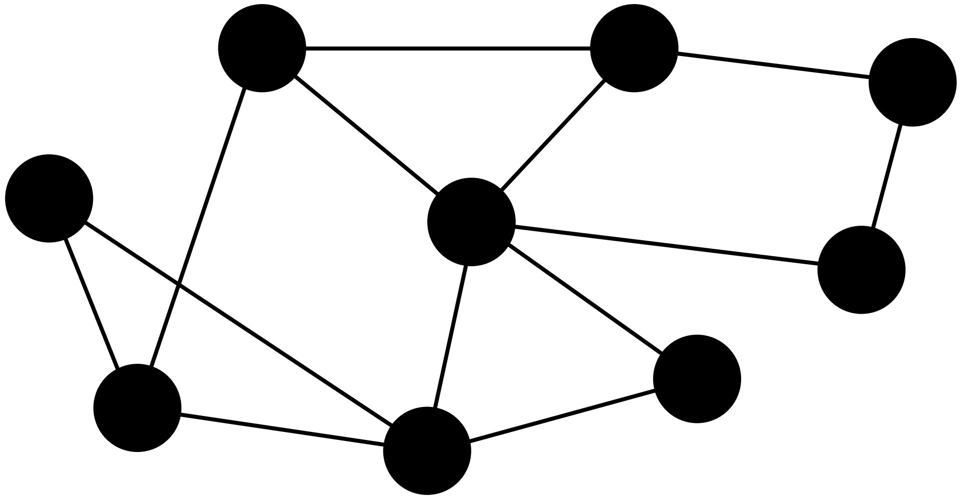
- Nodes u and v are connected if there exists a path connecting u and v.
- A graph $G(V, E)$ is connected if any two nodes $u, v \in V$ are connected.



Connected

Basic Concepts of Graphs

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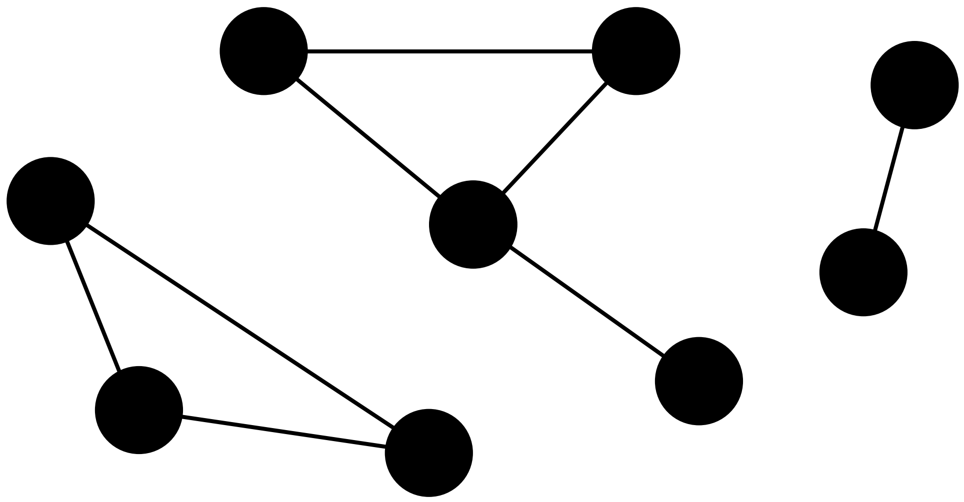
- Nodes u and v are connected if there exists a path connecting u and v.
- A graph $G(V,E)$ is connected if any two nodes $u, v \in V$ are connected.



Disconnected

Basic Concepts of Graphs

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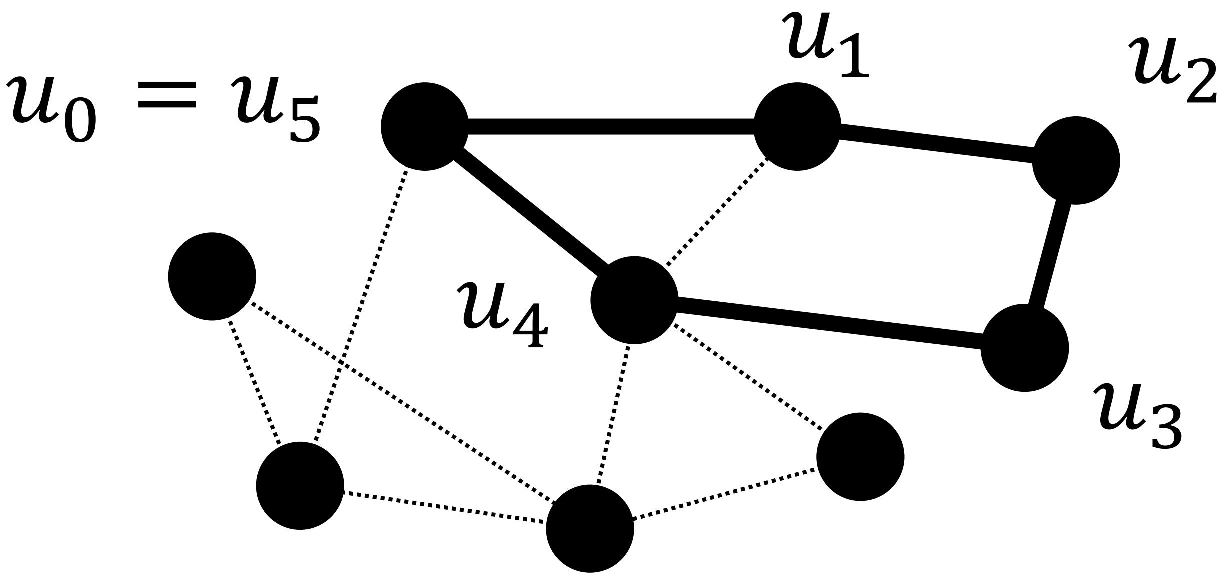
- Cycle $(u_0, u_1, \dots, u_l)$ is a path with $u_0 = u_l$:
  - for all $i = 1,2, \dots, l$, we have $(u_{i-1}, u_i) \in E$
  - l is the length of the cycle (number of edges)



Graph ADT

- Two basic types: **nodes** and **edges**
  - nodes(): return the set of nodes V
  - edges(): return the set of edges E
  - each node u supports
    - incidentEdges(): return the set of edges incident to u
    - neighbors(): return the set of neighbors of u
    - isNeighbor(v): check whether v is a neighbor of u
  - each edge $e = (u, v)$ supports
    - endNodes(): return the set $\{u, v\}$
    - opposite(u): return node v (the endpoint other than u)
    - isAdjacentTo(f): check whether e and f have common endpoints

Graph ADT

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- Two basic types: **nodes** and **edges**
- Update operations:
  - insertNode(u): insert a node u to the graph (without incident edge)
  - insertEdge(u, v): insert an edge between existing nodes u and v
  - removeNode(u): remove node u and all its incident edges
  - removeEdge(u, v): remove the edge between node u and v

Implementation (trivial)

Every graph $G(V, E)$ maintains

- A set of n node objects, e.g., as an array/linked list
- A set of m edge objects, e.g., as an array/linked list

Implementation (trivial)

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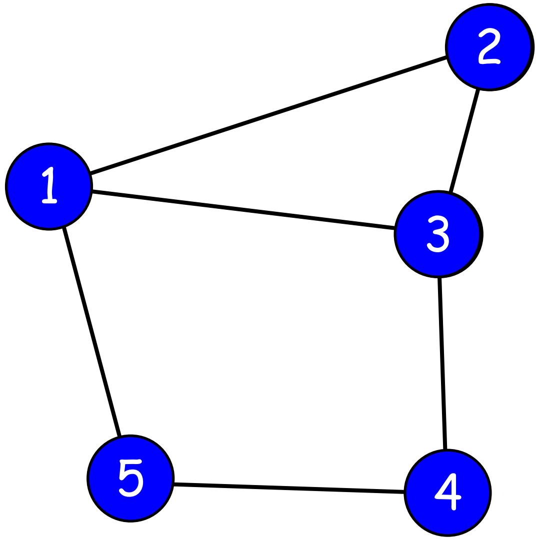
- Two basic types: **nodes** and **edges**
  - nodes(): return the set of nodes V — $O(n)$ time
  - edges(): return the set of edges E — $O(m)$ time
  - each node u supports
    - incidentEdges(): — $O(m)$ time
    - neighbors(): — $O(m)$ time
    - isNeighbor(v): — $O(m)$ time
  - each edge $e = (u, v)$ supports
    - endNodes(): — $O(1)$ time
    - opposite(u): — $O(1)$ time
    - isAdjacentTo(f): — $O(1)$ time

Adjacency Matrix

- Adjacency Matrix $A = \{a_{ij}\}_{V\times V}$: binary $a_{uv} = 1 \Leftrightarrow (u, v) \in E$

	1	2	3	4	5
1	0	1	1	0	1
2	1	0	1	0	0
3	1	1	0	1	0
4	0	0	1	0	1
5	1	0	0	1	0



Adjacency Matrix

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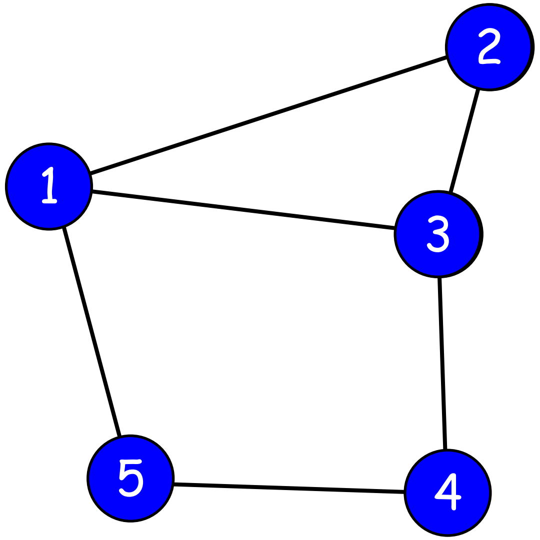
- Two basic types: **nodes** and **edges**
  - nodes(): return the set of nodes V — $O(n)$ time
  - edges(): return the set of edges E — $O(m)$ time
  - each node u supports
    - incidentEdges(): — $O(n)$ time
    - neighbors(): — $O(n)$ time
    - isNeighbor(v): — $O(1)$ time
  - each edge $e = (u, v)$ supports
    - endNodes(): — $O(1)$ time
    - opposite(u): — $O(1)$ time
    - isAdjacentTo(f): — $O(1)$ time

Adjacency Matrix

- Adjacency Matrix $A = \{a_{ij}\}_{V \times V}$: binary $a_{uv} = 1 \Leftrightarrow (u, v) \in E$

	1	2	3	4	5
1	0	1	1	0	1
2	1	0	1	0	0
3	1	1	0	1	0
4	0	0	1	0	1
5	1	0	0	1	0

Space complexity : $O(n^2)$



Adjacency List

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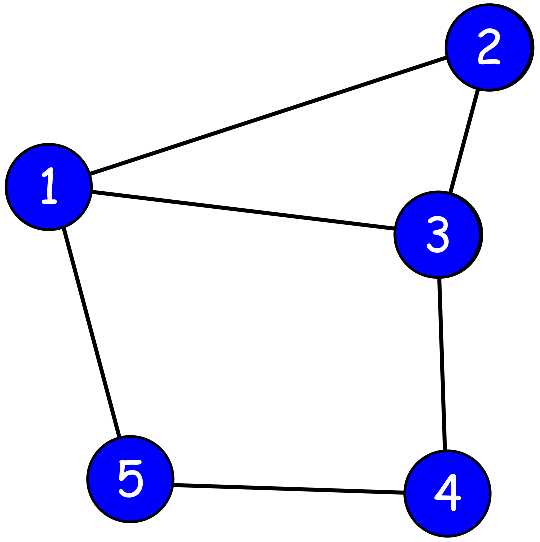
- Each node u maintains the set of incident-edges/neighbors
- Example:
  - $u_1.neighbors() = \{u_2, u_3, u_5\}$
  - $u_2.neighbors() = \{u_1, u_3\}$
  - $u_3.neighbors() = \{u_1, u_2, u_4\}$
  - $u_4.neighbors() = \{u_3, u_5\}$
  - $u_5.neighbors() = \{u_1, u_4\}$



Space complexity : $O(m)$

Adjacency List

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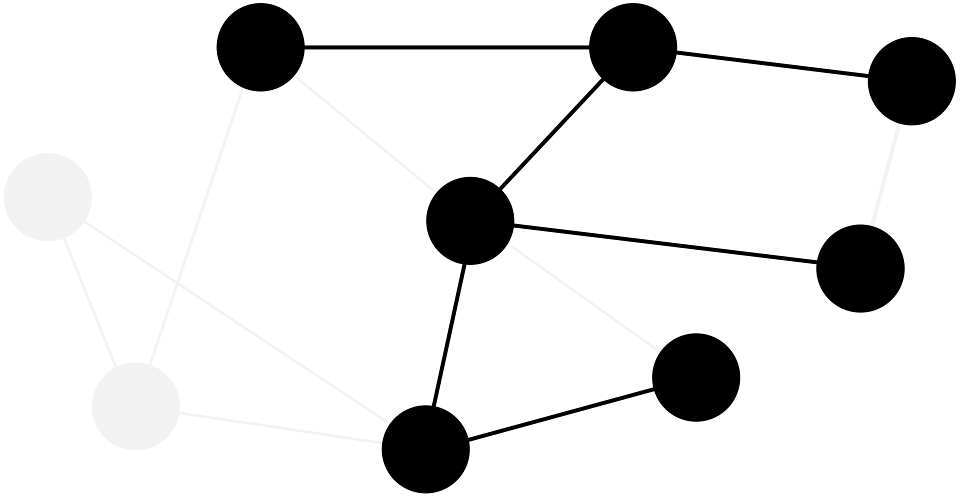
- Two basic types: **nodes** and **edges**
  - nodes(): return the set of nodes V — $O(n)$ time
  - edges(): return the set of edges E — $O(m)$ time
  - each node u supports
    - incidentEdges(): — $O(d(u))$ time
    - neighbors(): — $O(d(u))$ time
    - isNeighbor(v): — $O(d(u))$ time
  - each edge $e = (u, v)$ supports
    - endNodes(): — $O(1)$ time
    - opposite(u): — $O(1)$ time
    - isAdjacentTo(f): — $O(1)$ time

Subgraphs

- Graph $H(V_H, E_H)$ is a subgraph of $G(V, E)$ if
  - $V_H \subseteq V$ and $E_H \subseteq E$

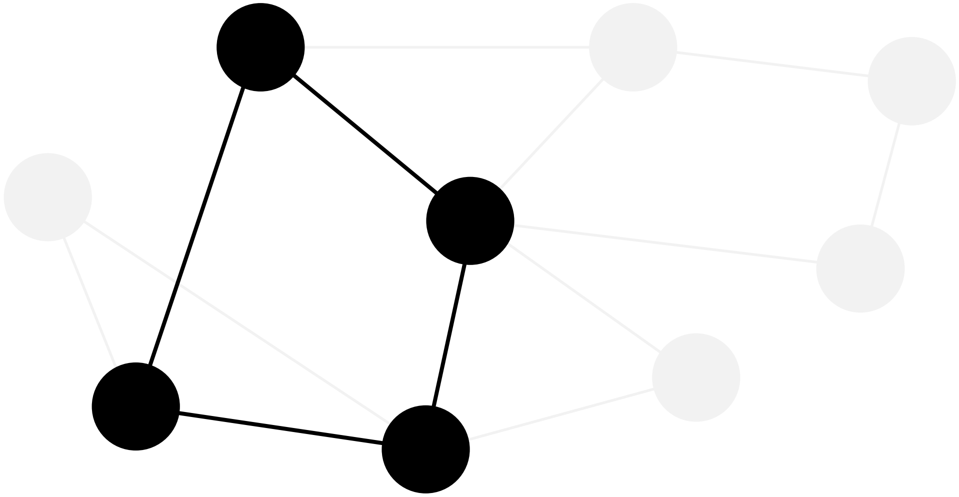
$H(V_H, E_H)$



Induced Subgraphs

- Graph $H(V_H, E_H)$ is an induced subgraph of $G(V, E)$ if
  - $V_H \subseteq V$ and $E_H = \{(u, v) \in E : u \in V_H \text{ and } v \in V_H\}$
  - Notation: $H(V_H, E_H) = G[V_H]$

$H(V_H, E_H)$



Connected Components

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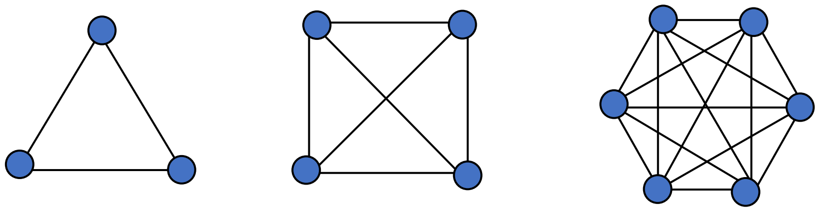
- The connected components of $G(V, E)$ is the collection of all maximal connected induced subgraphs of $G(V, E)$.

- Maximal Connected Induced Subgraph: $H(V_H, E_H)$:
  - Induced $H(V_H, E_H) = G[V_H]$
  - $H(V_H, E_H)$ is Connected
  - Maximal: for any node $x \in V \setminus V_H$: $G[V_H \cup \{x\}]$ is **not** connected

Some Special Graphs

- Complete graphs
  - $G(V, E)$ with $E = V \times V$
- Trees
  - unrooted
- Bipartite graphs

Complete graphs K_n



K_3 K_4 K_6

$|V| = n$, $|E| = n(n-1)/2$

Degree = n-1

Trees

- Connected graph $G(V, E)$ with $|E| = |V| - 1$.

- **Claim**: any connected graph on nodes V must have at least $|V| - 1$ edges.
- Proof:
  - Any connected graph $G(V, E)$ can be created by adding edges to an initially empty graph on nodes V (no edge).
  - Inserting an edge to a graph reduces the number of connected components by at most one.
  - Connected components: initially $n \rightarrow$ eventually 1

Trees

Property: a graph is a tree <u>if and only if</u>

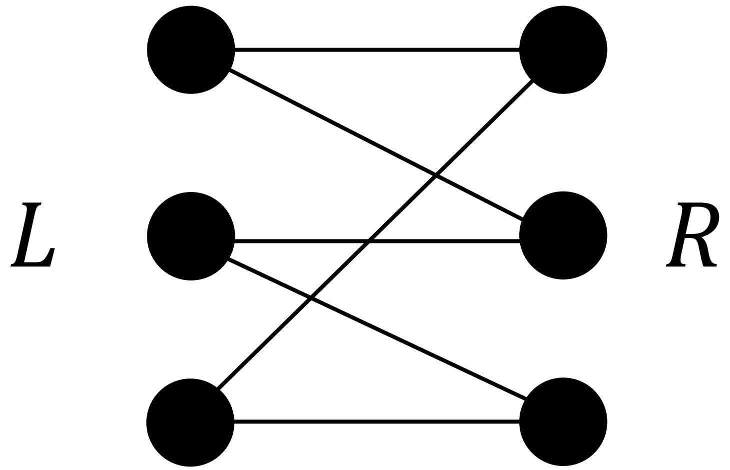
it is connected and contains no cycle.

Proof:

- **tree -> no cycle**: if there exists a cycle, then after removing any edge in the cycle, the graph remains connected. Hence (before the deletion) the graph contains at least |V| edges, which is not a tree.
- **no cycle -> tree**: removing any edge (u,v) disconnects u and v (otherwise there is a cycle). Hence there are at most |V|-1 edges in the graph. Since the graph is connected, it is a tree.

Bipartite Graphs

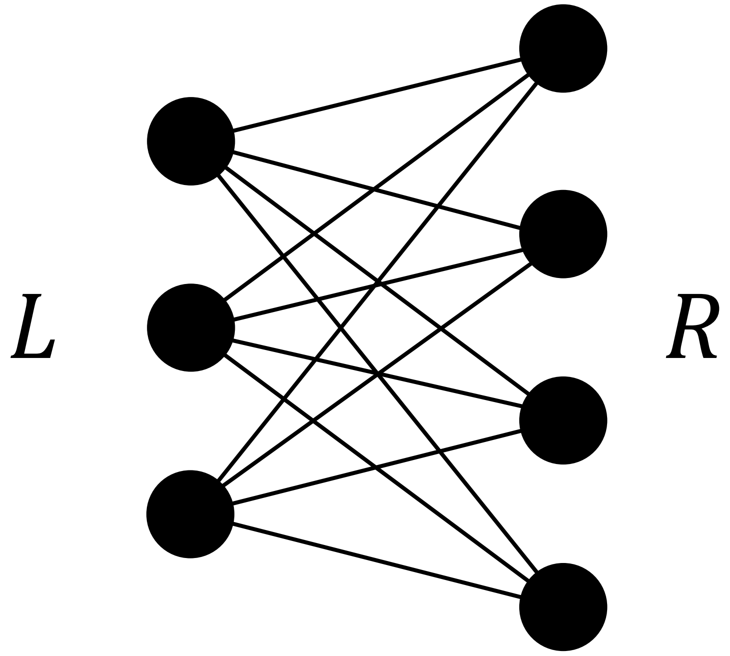
- Graph $G(V, E)$ in which the nodes V can be partitioned into two sets L and R such that every edge $e \in E$ connects a node from L and a node from R.
- Notation: $G(L \cup R, E)$, where $E \subseteq L \times R$.



Complete Bipartite Graphs

- Bipartite graph $G(L \cup R, E)$ with $E = L \times R$.

$K_{3,4}$



Bipartite Graphs

Property: $G(V, E)$ is bipartite <u>if and only if</u> it contains no odd cycle.

- bipartite → no odd cycle:
  - In any cycle $(u_0, u_1, \ldots, u_l)$, if $u_i \in L$ then $u_{i+1} \in R$
  - $u_0 = u_l \rightarrow l$ (length of cycle) is even
- no odd cycle → bipartite:
  - Color an arbitrary node black.
  - Repeat: color neighbors of black nodes white; color neighbors of white nodes black.
  - no odd cycle → no conflict
  - L: black nodes; R: white nodes

Graphs Traversal

Graph Traversal

- A traversal is a systematic procedure for exploring a graph by examining all its nodes and edges.
- Depth-first-search (DFS) from a node $u \in V$
  - recursively visit an unvisited neighbor
  - easy to implement
- Breadth-first-search (BFS) from a node $u \in V$
  - visit all neighbors of u, then neighbors of neighbors u, etc
  - closest nodes are visited first

Depth-first-search (DFS)

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DFS(u)

- label u as "visited" and print u
- **for** (each neighbor v of u):
  - **if** (v is not visited):
    - label edge (u, v) as a discovery-edge
    - DFS(v)
  - **elseif** (edge (u, v) has not been labelled)
    - label edge (u, v) as a back-edge

Depth-first-search (DFS)

- Produce a traversal order of the nodes
- Classify all edges into discovery-edges and back-edges
- **Complexity**: $O(n+m)$
  - Excluding recursive calls, DFS(u) runs in $O(d(u)+1)$ time.
  - For each node v, DFS(v) will be called exactly once
  - Total time = $\sum_{v \in V} O(d(v)+1) = O(n+m)$

Breadth-first-search (BFS)

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BFS(u)

- initialize an empty queue Q
- Q.enqueue(u) and label u as visited
- **while** (Q is not empty)
    - $v \leftarrow Q$.dequeue() and print v
    - **for** (each neighbor w of v)
        - **if** (w is not visited)
            - Q.enqueue(w) and label w as visited
            - label (v, w) as a discovery-edge

Breadth-first-search (BFS)

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- Produce a traversal order of the nodes
- Classify all edges into discovery-edges and cross-edges
- **Complexity**: $O(n+m)$
  - There is no recursive calls
  - Every node is enqueued and dequeued at most once
  - After node v is dequeued, $d(v)$ for-loops are executed
  - Total time = $\sum_{v \in V} O\big(1+d(v)\big) = O(n+m)$

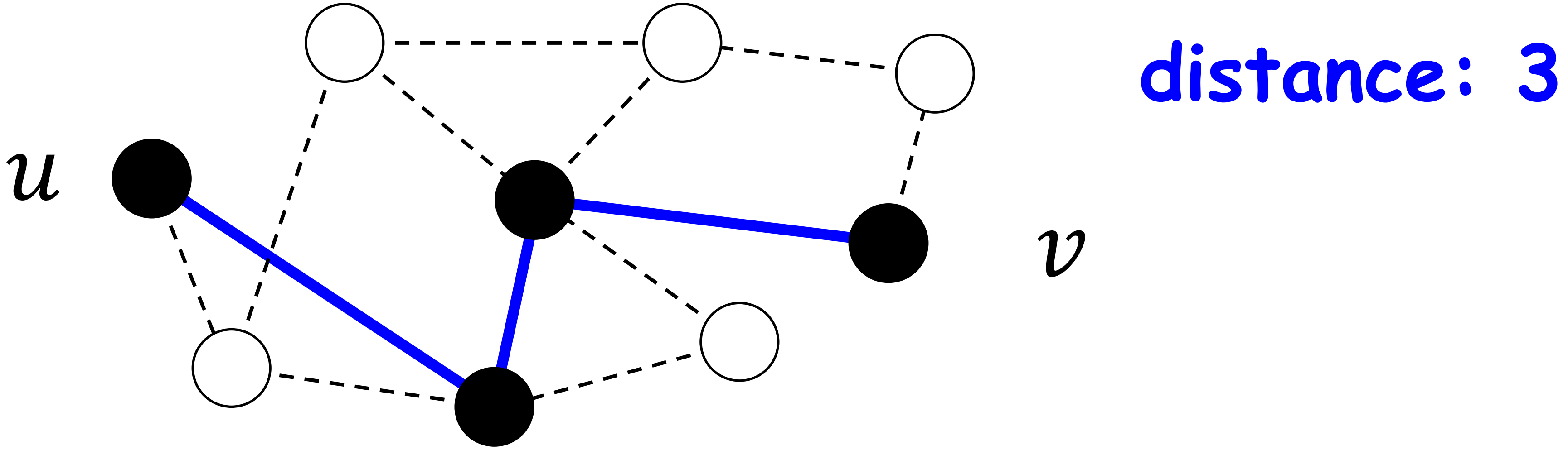
Graph Traversal

- A traversal is a systematic procedure for exploring a graph by examining all its nodes and edges.
- Depth-first-search (DFS) from a node $u \in V$
- Breadth-first-search (BFS) from a node $u \in V$
- Both runs in $O(n + m)$ time
- Outputs a tree containing all nodes connected to u
- Can be used to test whether G is connected

Shortest Path

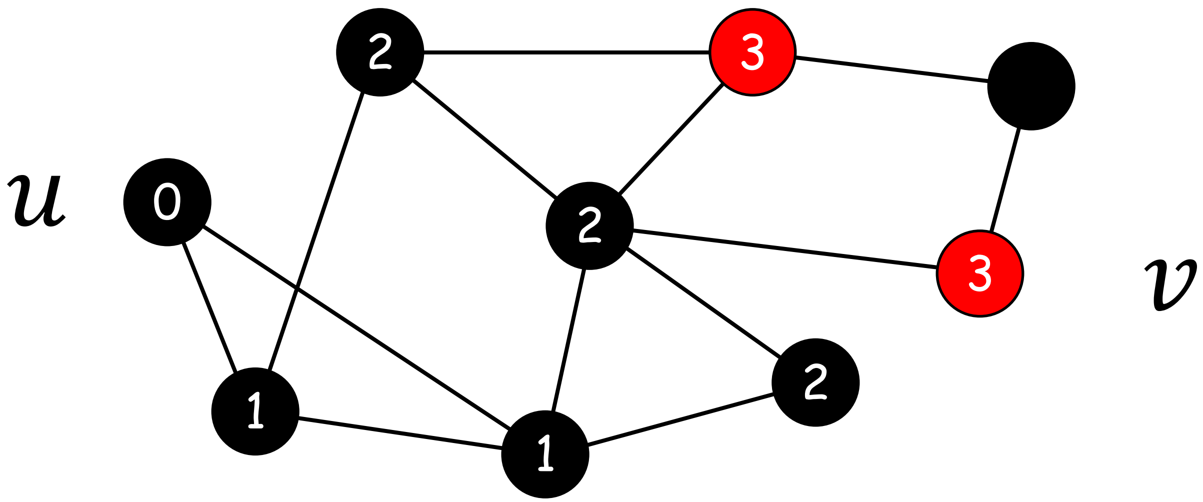
Shortest Path

- **Shortest path** between two nodes u and v : the path connecting u and v with the smallest length.
- **Distance** between u and v : length of the shortest path.



Compute Distances

- Let $d(u, v)$ be the distance between u and v.
- How to compute $d(u, v)$?



Unweighted Distance by BFS

procedure UnweightedDistance(G, u)

- let Q be a queue
- Q.enqueue(u), set $d(u,u) \leftarrow 0$ and label u as "visited"
- **while** Q is not empty **do**
  - $x \leftarrow$ Q.dequeue()
  - **for** each edge $(x,y) \in E$ **do**
    - **if** y is not "visited" **then**
      - Q.enqueue(y), set $d(u,y) \leftarrow d(u,x) + 1$
      - label y as "visited"

Edge-weighted Graphs

- Graph $G(V, E)$ on nodes V and edges $E \subseteq V \times V$.
- Every edge $e = (u, v) \in E$ has a weight $w_e = w_{uv} > 0$.
  - Unweighted graphs: $w_e = 1$ for all edge $e \in E$

- Path $(u_0, u_1, \dots, u_l)$ has **length** $w_{u_0 u_1} + w_{u_1 u_2} + \cdots + w_{u_{l-1} u_l}$
- **Distance** between u and v: length of the shortest path connecting u and v.

Single Source Shortest Path

- Given edge-weighted undirected graph $G(V, E)$.

- Let $d(s, u)$ be the (weighted) distance between s and u.

- **Problem**: for a fixed node $s \in V$ (the source node), compute the distance $d(s, u)$ for all $u \in V$.
  - Single Source Shortest Path (SSSP) problem

Dijkstra's Algorithm

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- S = { all nodes u for which $d(s,u)$ is **computed** }.
- Initially $S = \{s\}$, $d(s,s) = 0$.

- For each other node u, compute an estimate $d'(s,u)$ of the distance from s to u.
  - $d'(s,u)$ is an upper bound of the real distance
- Initially, let $d'(s,u) = w_{su}$ if edge $(s,u) \in E$ exists;
- otherwise let $d'(s,u) = \infty$.

Dijkstra's Algorithm

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- In each step: how can we be sure that the estimate is correct, i.e., $d'(s,u)$ is the actual distance $d(s,u)$?
- Answer: the node u with the **smallest** $d'(s,u)$

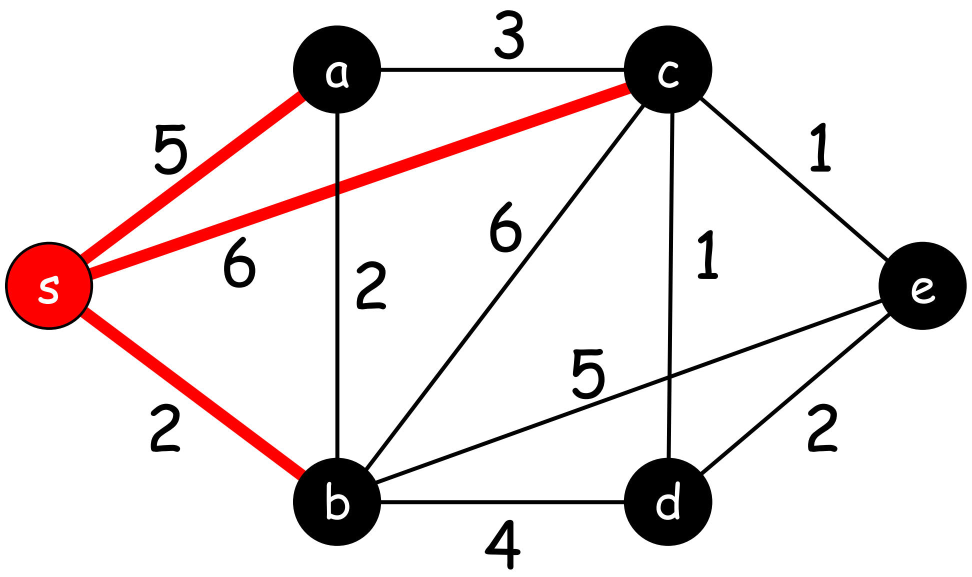
- In this case, we know $d(s,u) = d'(s,u)$.
- Insert u into S & record its distance:
  - $S \leftarrow S \cup \{u\}$.
  - $d(s,u) \leftarrow d'(s,u)$.

Update the Estimate

Initialize $S \leftarrow \{s\}$, $d(s, s) \leftarrow 0$.
$d'(s, u) \leftarrow w_{su}$ if edge $(s, u) \in E$ exists;
otherwise let $d'(s, u) \leftarrow \infty$.

1. **Repeat**
2. let $u \in V \setminus S$ have the **smallest** $d'(s, u)$;
3. $d(s, u) \leftarrow d'(s, u)$; insert u into S: $S \leftarrow S \cup \{u\}$.
4. For each edge $e = (u, v) \in E$ and $v \in V \setminus S$,
5. if $d'(s, v) > d(s, u) + w_{uv}$ then update $d'(s, v) \leftarrow d(s, u) + w_{uv}$
6. **until** $S = V$.

Example



$d(s, s) = 0$

$d'(s, a) = 5$

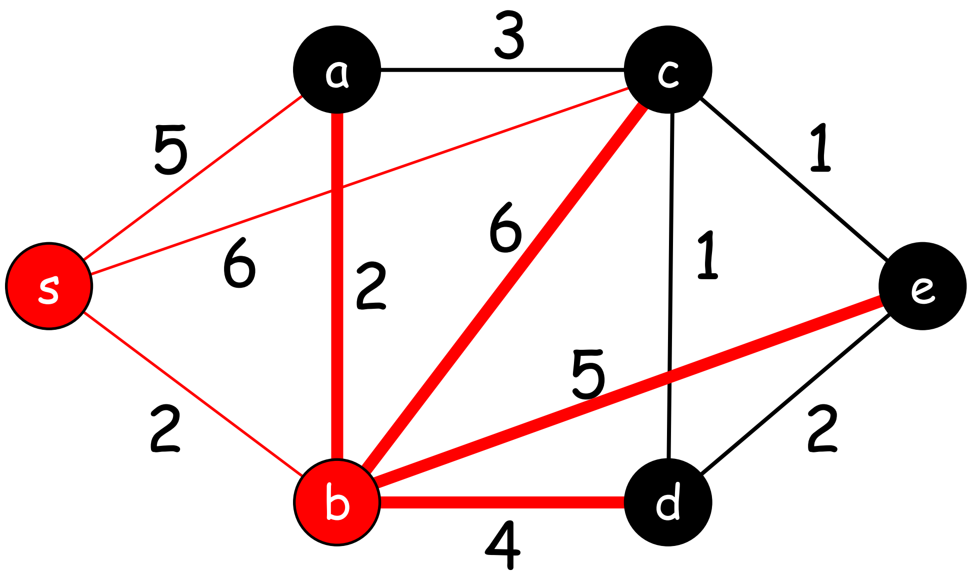
$d'(s, b) = 2$

$d'(s, c) = 6$

$d'(s, d) = \infty$

$d'(s, e) = \infty$

Example



$d(s,s) = 0$

$d'(s,a) = 4$

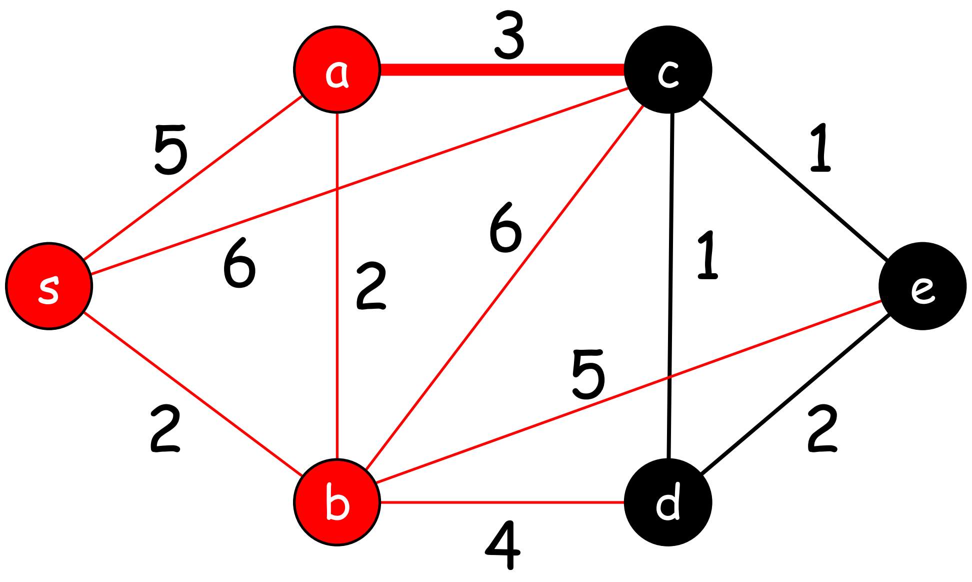
$d(s,b) = 2$

$d'(s,c) = 6$

$d'(s,d) = 6$

$d'(s,e) = 7$

Example



$d(s,s) = 0$

$d(s,a) = 4$

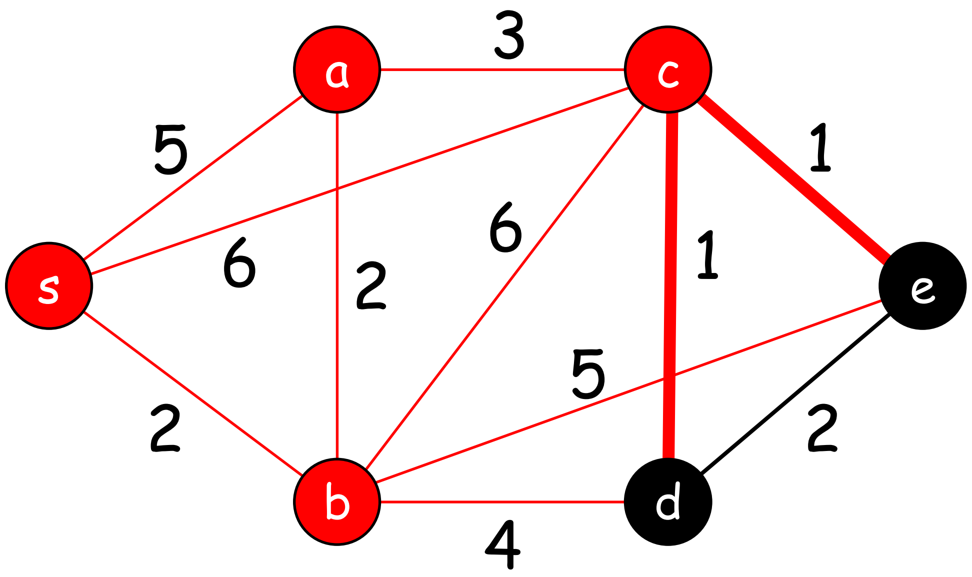
$d(s,b) = 2$

$d'(s,c) = 6$

$d'(s,d) = 6$

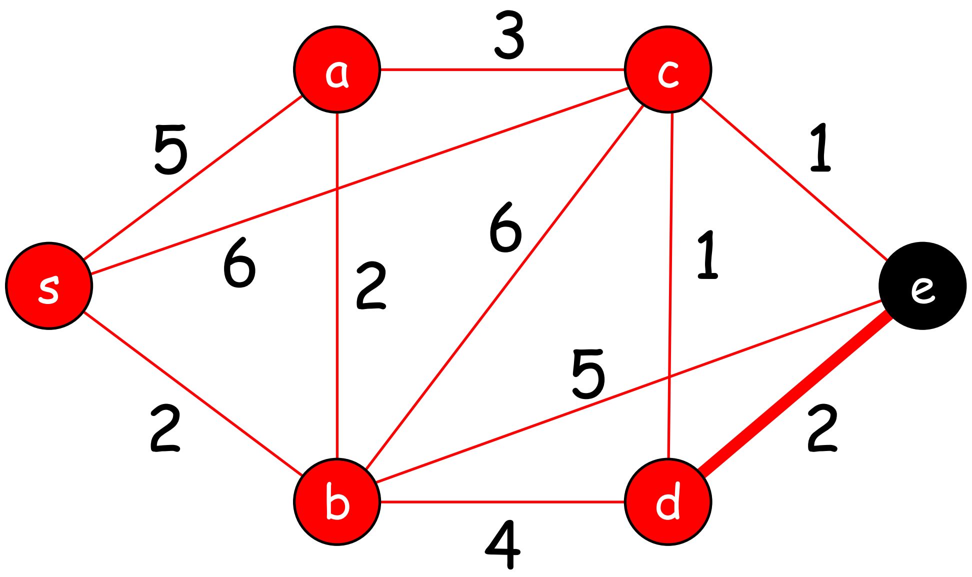
$d'(s,e) = 7$

Example



$d(s,s) = 0$
$d(s,a) = 4$
$d(s,b) = 2$
$d(s,c) = 6$
$d'(s,d) = 6$
$d'(s,e) = 7$

Example



$d(s,s) = 0$
$d(s,a) = 4$
$d(s,b) = 2$
$d(s,c) = 6$
$d(s,d) = 6$
$d'(s,e) = 7$

Implementation

- **Trivial implementation:** $O(n^2)$ time
  - $n-1$ rounds in total
  - each round takes $O(n)$ time (find-min and update estimates)
- Better implementation: Use an efficient data structure to store the estimates for all nodes in $V \setminus S$.
  - Query the minimum estimate.
  - Deletion of the minimum estimate.
  - Update of estimate $d'(s,u)$.

Implementation

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- **Binary Search Tree (AVL-Tree):**
  - Query the minimum estimate O(log n) (need to implement)
  - Deletion of the minimum estimate O(log n)
  - Update of estimate $d'(s, u)$ O(log n)
- **Priority Queue (Min-Heap):**
  - Query the minimum estimate O(1)
  - Deletion of the minimum estimate O(log n)
  - Update of estimate $d'(s, u)$ O(log n) (need to implement)
    - In practice: insert a new value

Running Time

- At most n Delete min operations.

- Each edge causes at most one Update operation.
- There are at most m Update key operations.

- Time complexity: $O((n + m) \log n)$.

Dynamic Programming Solution

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Problem: Given edge-weighted graph $G(V, E)$ and source node s, compute $d(s, u)$ for all $u \in V$.

Dynamic Programming Solution:

- Definition: $d(s, u, k)$ = smallest length of all paths between s and u using at most k edges.
- Observation: $d(s, u) = d(s, u, n - 1)$.
- Recursion: $d(s, u, k) = \min\{d(s, v, k - 1) + w_{uv}: v \in N(u)\}$
- Base case: $d(s, u, 1) = w_{su}$ if $u \in N(s)$; $d(s, u, 1) = \infty$ otherwise

Minimum Spanning Tree

Minimum Spanning Tree

- Given an edge-weighted undirected connected graph $G(V, E)$:
- A spanning tree of $G(V, E)$ is a subgraph of G that is a tree and connects all nodes in V.

Minimum Spanning Tree (MST) Problem:

- Compute a spanning tree $T(V, E_T)$ on V such that
  - $E_T \subseteq E$ and $\sum_{e \in E_T} w_e$ is minimized.
- For unweighted graphs: any spanning tree is an MST.

Applications

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- **Network design**: telephone, electrical, hydraulic, TV cable, computer, road networks.
- Approximation algorithms for NP-hard problems
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein

Algorithms

- Two Greedy algorithms:
  - Simple to understand and implement
  - Non-trivial to prove its correctness
  - Require special data structures to allow efficient implementation

- **Kruskal's Algorithm.**
- **Prim's Algorithm.**

Kruskal's Algorithm

- Start with an empty graph $T(V, F)$ with $F = \emptyset$.
- Repeatedly add the next **lightest edge e** that **does not create a cycle**.
  - Consider edges one-by-one in ascending order of edge weight.
- Terminate when all nodes are connected in T.

Implementation

- Start with an empty graph $T(V,F)$ with $F = \emptyset$.
- Sort the edges in ascending order of weights
- Repeatedly add the next lightest edge e that does not create a cycle.

- How to determine whether the new edge $e = (u,v)$ creates a cycle?
- **Trivial Algorithm:** Run DFS/BFS on T starting from u, and check whether v can be visited.
  - Running Time: $O(|V| + |F|) = O(n)$

Implementation

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- Start with an empty graph $T(V,F)$ with $F = \emptyset$.
- Sort the edges in ascending order of weights
- Repeatedly add the next lightest edge e that does not create a cycle.

- How to determine whether the new edge $e = (u,v)$ creates a cycle?
- **Union-Find** data structure.
  - Two nodes are connected in T if and only if they are in the same set of the Union-Find data structure

Union-Find Data Structure

- Maintains a collection of sets of elements and supports the following operations:

- Make-Set(x): Create a set containing only element x;
- Find-Set(x): Return the pointer to the set containing x;
- Union(p1, p2): Given pointers p1 and p2 to two different sets, merge the two sets.

Union-Find Data Structure

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- Maintains a collection of sets of elements and supports the following operations:

- Make-Set(x): O(1) time;
- Find-Set(x): O(log n) time;
- Union(p1, p2): O(1) time.

- **Question**: how to implement such a data structure?
  - Hint: organize elements in one set as a tree

Pseudocode

procedure Kruskal(G)

Sort edges in ascending order of weights, and initialize $F \leftarrow \emptyset$
For each node $u \in V$ **do**
    Make-Set(u)
For each edge $e = (u, v) \in E$ in ascending order of w_e **do**
    If (Find-Set(u) $\neq$ Find-Set(v)) **then**
        $F \leftarrow F \cup \{e\}$.
        Union(Find-Set(u), Find-Set(v)).
Return F

Complexity

- Sorting all edges: O(m log m) = O(m log n) time
- Make-Set for each node: O(n) total time
- Check cycle for an edge: O(log n) time
  - m edges -> O(m log n) total time
- Insert an edge to F and Union: O(1) time
  - n-1 edges in F -> O(n) total time

- **Overall: O(m log n) time.**

Prim's Algorithm

- Start with a given node s
- Greedily grow a tree T from s outward.
- At each step, add the **lightest edge e** to T that connects to a node **outside T**.

Prim's Algorithm

- Start with a given node s. Let $V_T = \{s\}$ and $F = \emptyset$.
- Greedily grow a tree $T(V_T, F)$ from s outward.

- Boundary Edges: $\delta(V_T) = \{(u, v) \in E : u \in V_T, v \notin V_T\}$
  - Find edge $e \in \delta(V_T)$ with smallest edge weight w_e
  - Update the solution: Add e to F, and v to V_T
  - Update the data structure $\delta(V_T)$:
    - Remove $(x, v) \in \delta(V_T)$ for all $x \in V_T$
    - Include (v, y) to $\delta(V_T)$ for all $y \notin V_T$

Implementation

- **Binary Search Tree (AVL-Tree):**
  - Query the lightest edge O(log n) (need to implement)
  - Insertion of an edge O(log n)
  - Removal of an edge O(log n)
- **Priority Queue (Min-Heap):**
  - Query the lightest edge O(1)
  - Insertion of an edge O(log n)
  - Removal of an edge O(log n) (need to implement)

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Pseudocode

procedure Prim($G(V, E)$, s)

Initialize $F \leftarrow \emptyset$ and Visited$[u] \leftarrow \text{False}$ for all $u \in V$

Create a Heap containing all edges incident to s and set Visited$[s] \leftarrow \text{True}$

While (Heap is not empty) **do**

    let (u, v) be the lightest edge in the Heap, with Visited$[v] = \text{False}$

    $F \leftarrow F \cup \{(u, v)\}$ and set Visited$[v] \leftarrow \text{True}$

    For each $x \in N(v)$ **do**

        If (Visited$[x] = \text{True}$) **then** Remove (x, v) from the Heap

        If (Visited$[x] = \text{False}$) **then** Insert (v, x) into the Heap

Return F

Complexity

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- Initialization and update of Visited[*]: O(n) time
- Insert/remove an edge in the data structure: O(log n) time
  - Each edge is inserted and removed at most once
  - m edges -> O(m log n) total time
- Get min and insert the edge to the tree: O(log n) time
  - n-1 edges in F -> O(n log n) total time

- **Overall: O(m log n) time.**