EECS 442 – Computer vision

Multiple view geometry
Affine structure from Motion

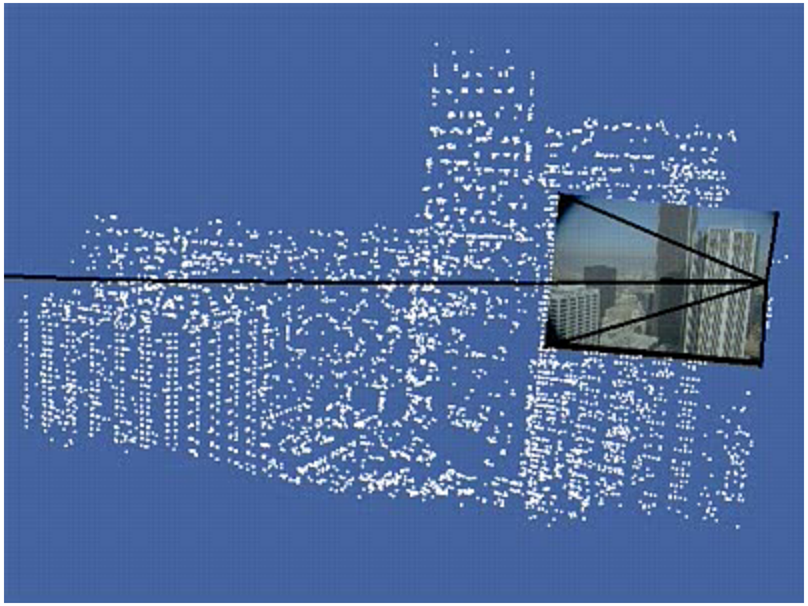
- Affine structure from motion problem
- Algebraic methods
- Factorization methods

Reading: [HZ] Chapters: 6,14,18
[FP] Chapter: 12

Some slides of this lectures are courtesy of prof. J. Ponce, prof FF Li, prof S. Lazebnik & prof. M. Hebert

Applications

Courtesy of Oxford **Visual Geometry Group**





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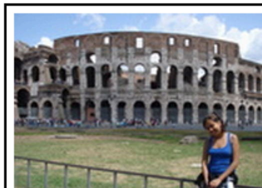

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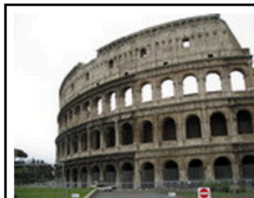

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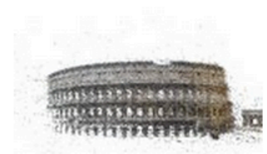
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Photo Tourism

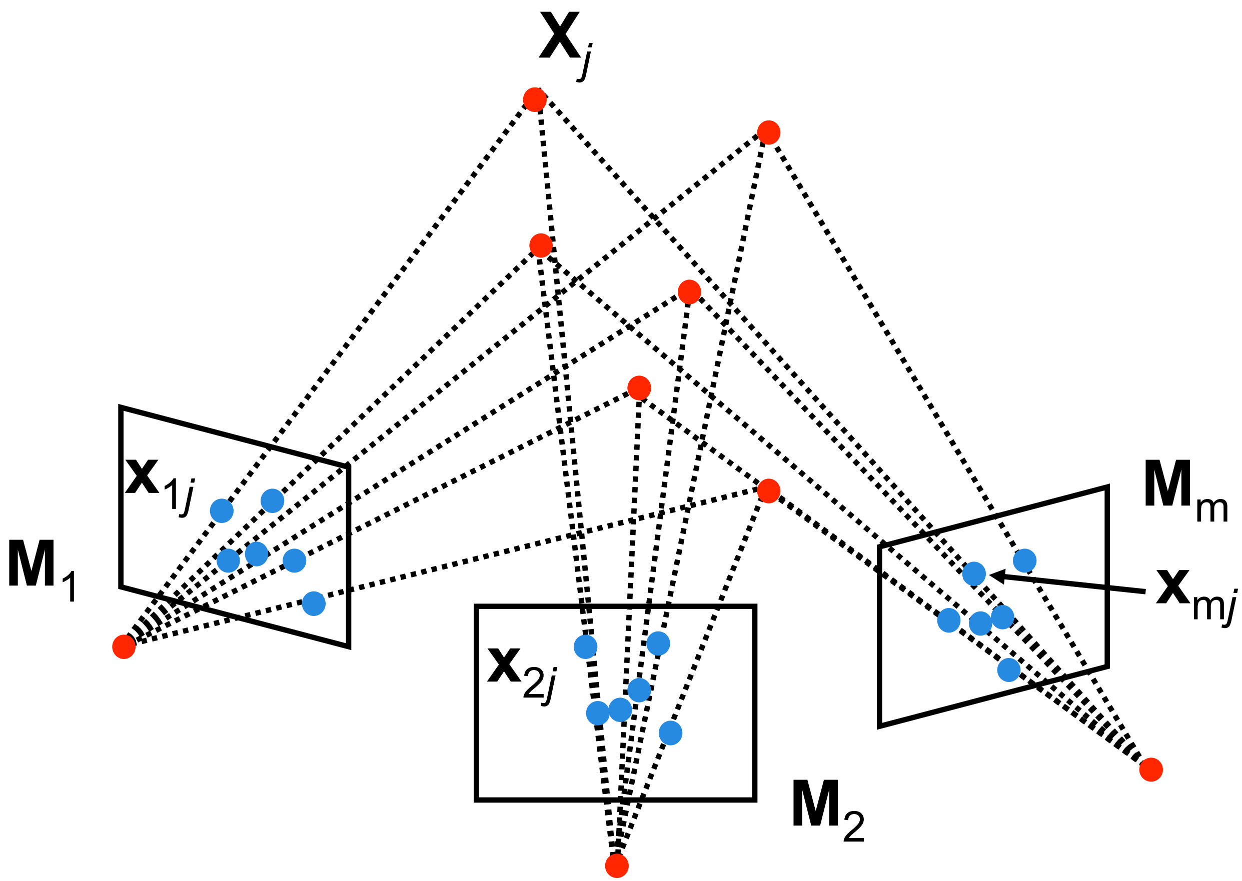
Exploring photo collections in 3D

Noah Snavely   Steven M. Seitz   Richard Szeliski

University of Washington   *Microsoft Research*

SIGGRAPH 2006

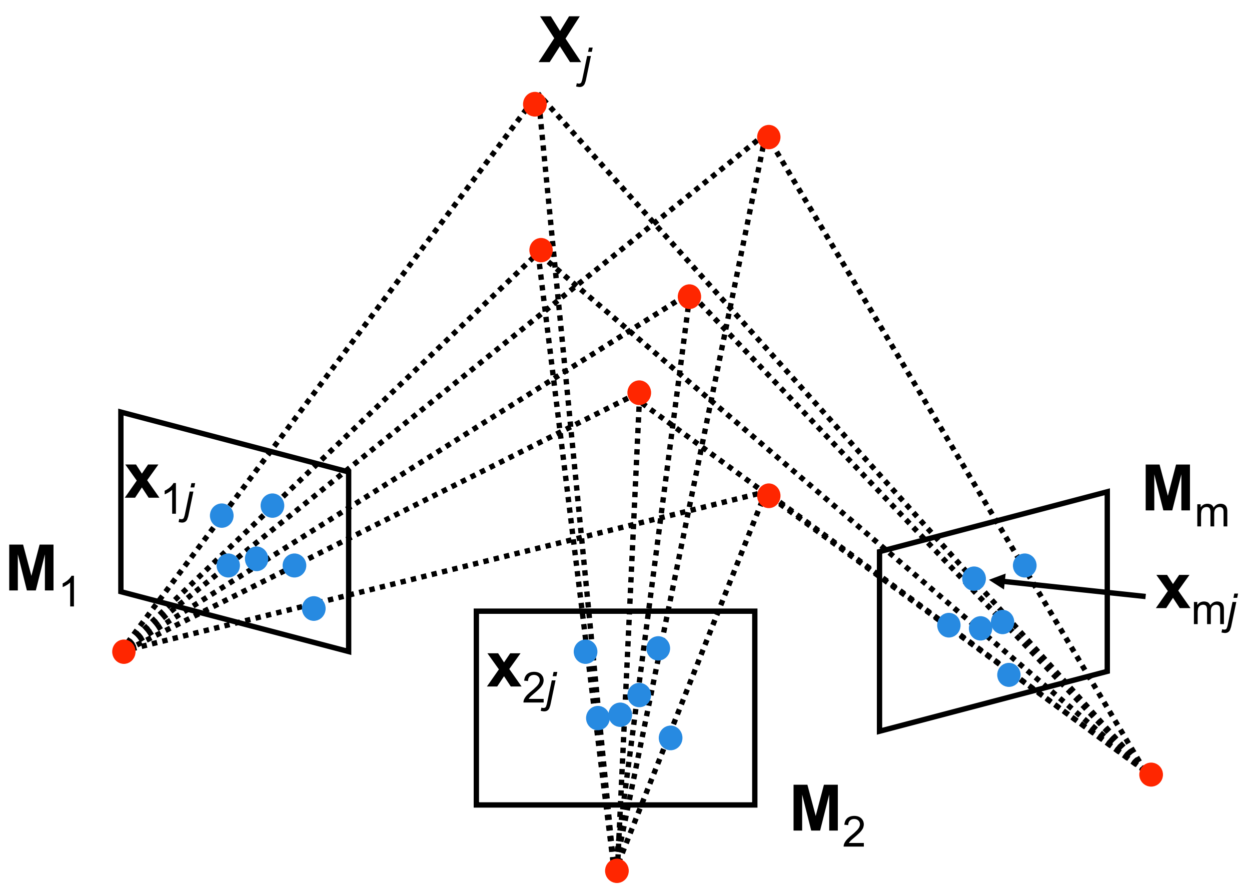
Structure from motion problem



Given *m* images of *n* fixed 3D points

- $\mathbf{x}_{ij} = \mathbf{M}_i \mathbf{X}_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$

Structure from motion problem



From the m×n correspondences $\mathbf{x}_{ij}$, estimate:

•m projection matrices $\mathbf{M}_i$	motion
•n 3D points $\mathbf{X}_j$	structure

Structure from motion ambiguity

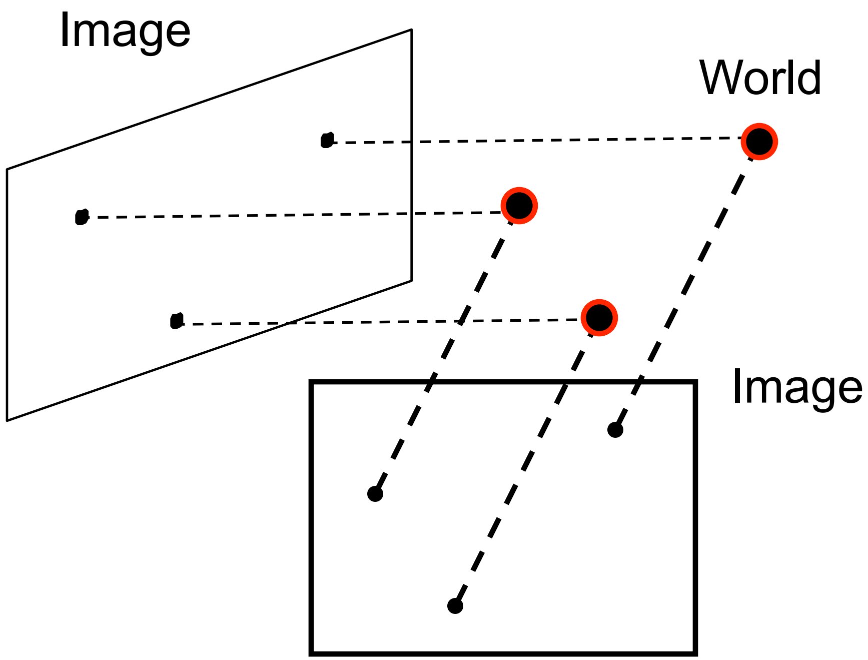
- If we scale the entire scene by some factor *k* and, at the same time, scale the camera matrices by the factor of 1/*k*, the projections of the scene points in the image remain exactly the same:

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\frac{1}{k}\mathbf{P}\right)(k\,\mathbf{X})$$

It is impossible to recover the absolute scale of the scene!

Affine structure from motion

(simpler problem)

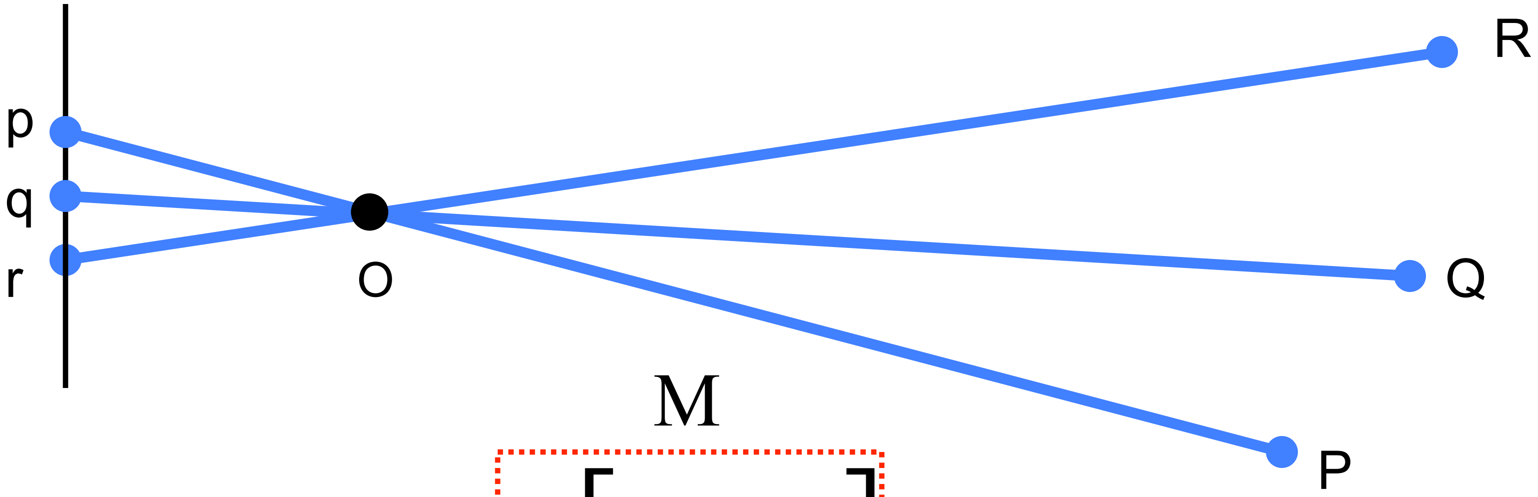


From the m×n correspondences $\mathbf{x}_{ij}$, estimate:

•*m* projection matrices $\mathbf{M}_i$ (affine cameras)

•*n* 3D points $\mathbf{X}_j$

Finite cameras

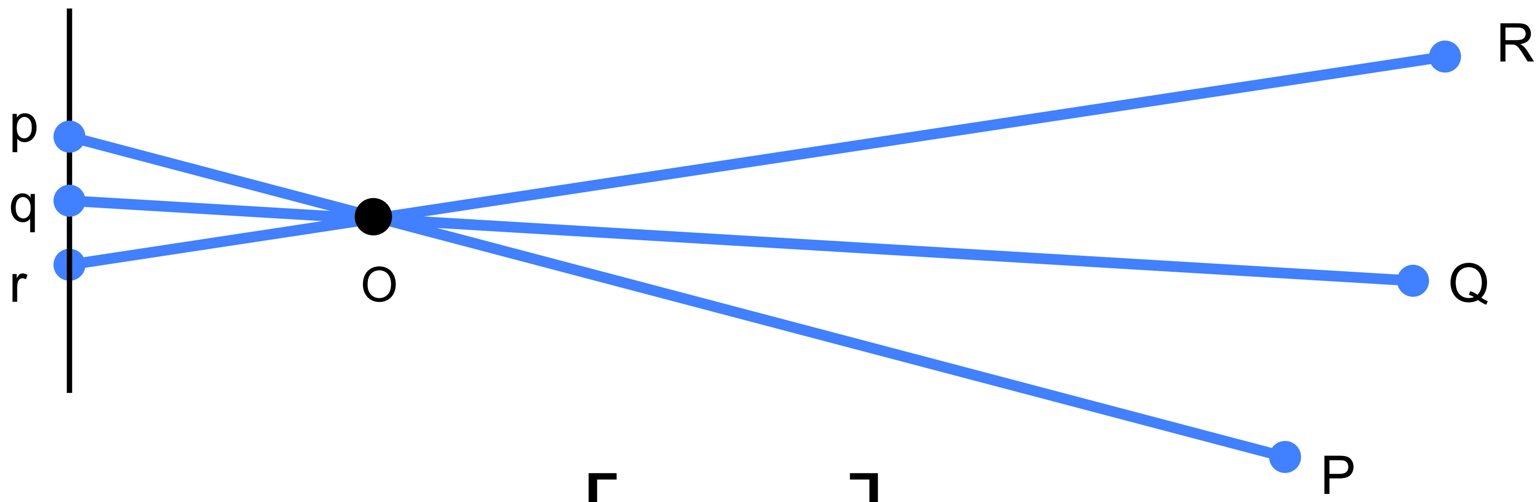


$$x = \underbrace{K\begin{bmatrix} R & T \end{bmatrix}}_{M} X$$

Question:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} = ??$$

Finite cameras



$$x = K\begin{bmatrix} R & T \end{bmatrix} X$$

Canonical perspective projection matrix

$$M = \boxed{K_{3x3}} \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \boxed{\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}}$$

Affine Homography (in 2D)

Affine homography (in 3D)

$$K = \begin{bmatrix} \alpha_x & s & x_o \\ 0 & \alpha_y & y_o \\ 0 & 0 & 1 \end{bmatrix}$$

Projective & Affine cameras

$$\mathrm{x} = \mathrm{K}\begin{bmatrix} \mathrm{R} & \mathrm{T} \end{bmatrix}\mathrm{X}$$

Projective case

$$\mathrm{K} = \begin{bmatrix} \boldsymbol{\alpha}_{\mathrm{x}} & \mathrm{s} & \mathrm{x}_{\mathrm{o}} \\ 0 & \boldsymbol{\alpha}_{\mathrm{y}} & \mathrm{y}_{\mathrm{o}} \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{M} = \mathrm{K}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{R} & \mathrm{T} \\ 0 & 1 \end{bmatrix}$$

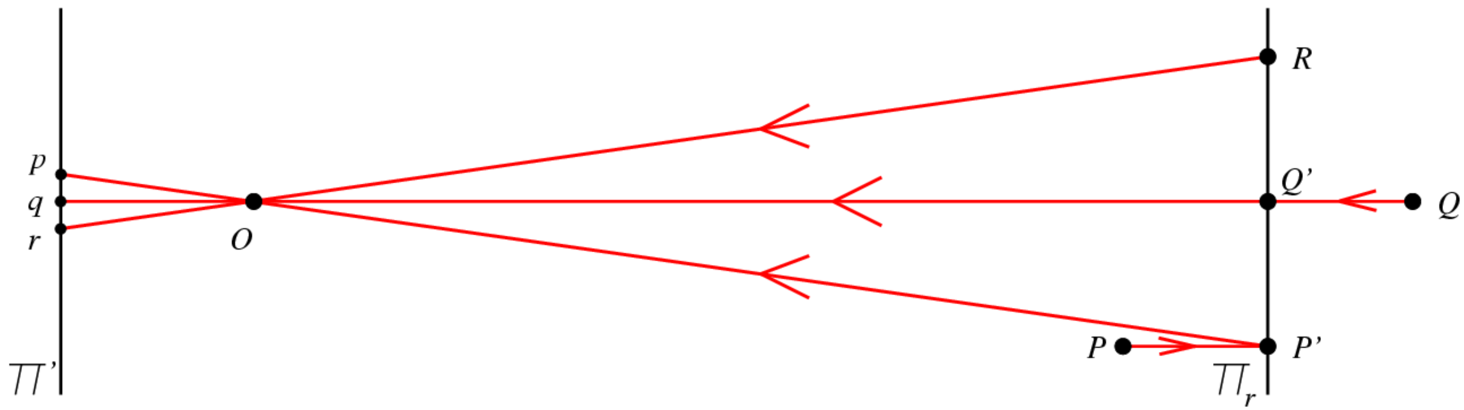
Affine case

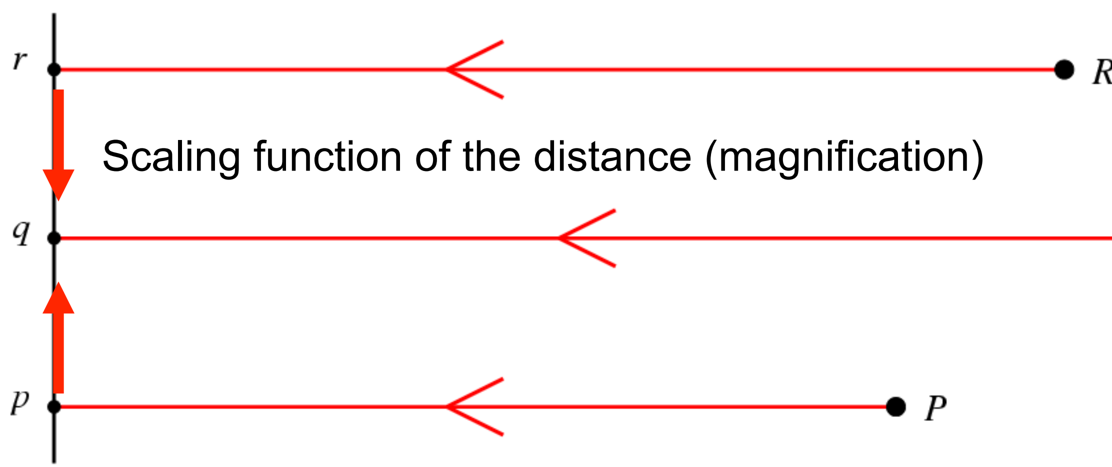
$$K = \begin{bmatrix} \alpha_x & s & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M = K\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

Weak perspective projection

When the relative scene depth is small compared to its distance from the camera

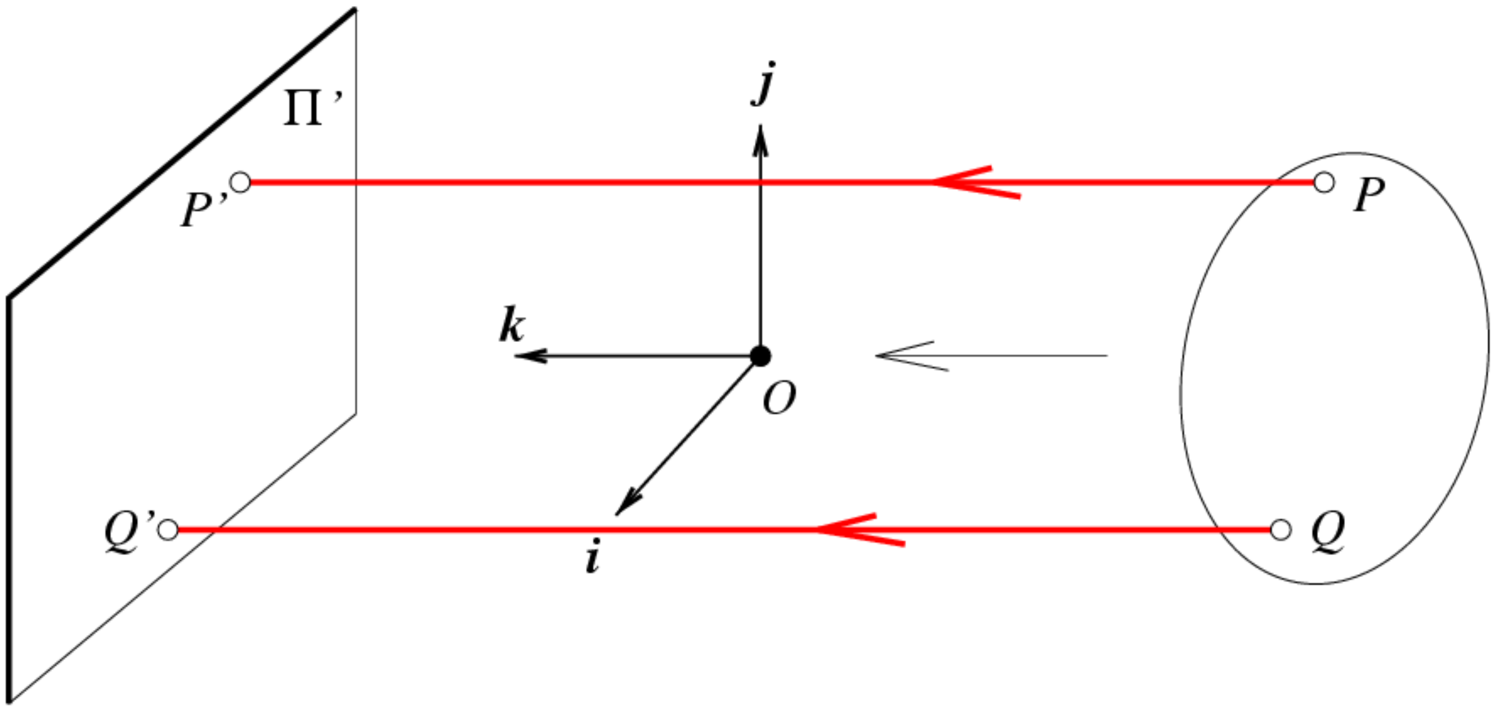
$$\begin{cases} x' = -mx \\ y' = -my \end{cases}$$





Orthographic (affine) projection

When the camera is at a (roughly constant) distance from the scene



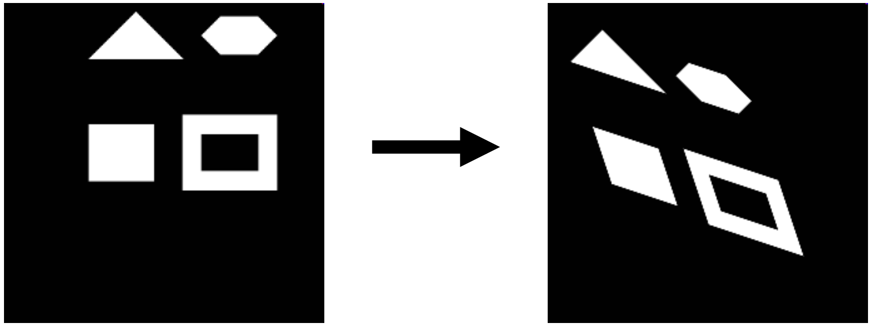
–Distance from center of projection
to image plane is infinite

$$\begin{cases} x' = x \\ y' = y \end{cases}$$

Transformation in 2D

Affinities:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Projective & Affine cameras

$$x = K\begin{bmatrix} R & T \end{bmatrix} X$$

Projective case

$$K = \begin{bmatrix} \alpha_x & s & x_o \\ 0 & \alpha_y & y_o \\ 0 & 0 & 1 \end{bmatrix} \qquad M = K\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

Affine case

$$K = \begin{bmatrix} \alpha_x & s & x_o \\ 0 & \alpha_y & y_o \\ 0 & 0 & 1 \end{bmatrix} \qquad M = K\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{0} & 1 \end{bmatrix}\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

Magnification (scaling term)

Parallel projection matrix
(points at infinity are mapped as points at infinity)

Affine cameras

$$\mathrm{x} = \mathrm{K}\begin{bmatrix}\mathrm{R} & \mathrm{T}\end{bmatrix}\mathrm{X}$$ [Homogeneous]

$$K = \begin{bmatrix} \alpha_x & 0 & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{M} = \mathrm{K}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathrm{R} & \mathrm{T} \\ 0 & 1 \end{bmatrix}$$

$$M = [3\times 3\,\text{affine}]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}[4\times 4\,\text{affine}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

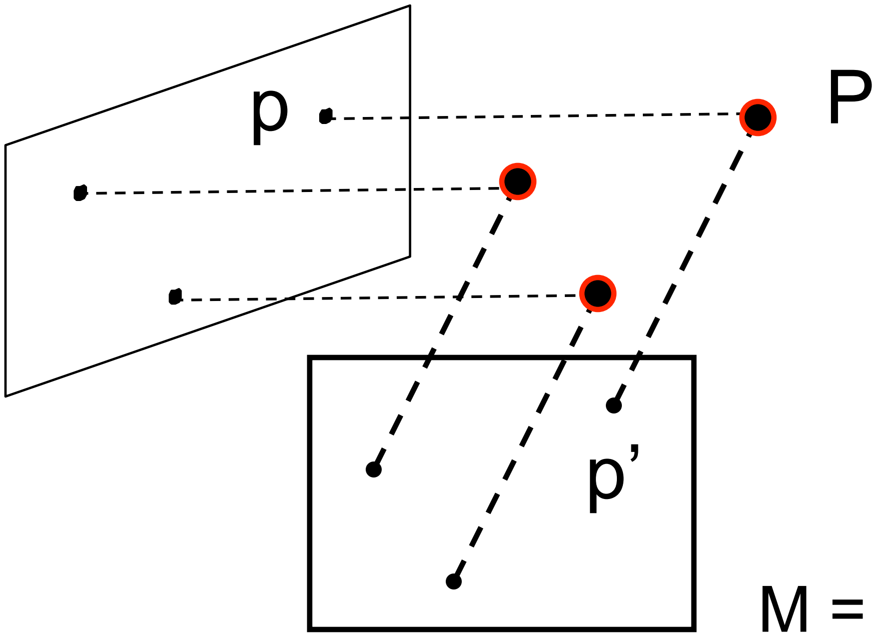
$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{AX} + \mathbf{b} = M_{Euc}\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = M_{Euc}\begin{bmatrix} \mathbf{P} \\ 1 \end{bmatrix};$$

$$\mathrm{M}_{\mathrm{Euc}} = \mathrm{M} = \begin{bmatrix}\mathrm{A} & \mathrm{b}\end{bmatrix}$$ [non-homogeneous image coordinates]

$$M = \begin{bmatrix} A & b \end{bmatrix}$$

- The point b is the image location of the world origin
- The center of the affine camera is infinity

Affine cameras



M = camera matrix

To recap:

from now on we define M as the camera matrix for the affine case

$$\mathbf{p} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{AP} + \mathbf{b} = M \begin{bmatrix} \mathbf{P} \\ 1 \end{bmatrix}; \qquad \mathrm{M} = \begin{bmatrix} \mathrm{A} & \mathrm{b} \end{bmatrix}$$

The Affine Structure-from-Motion Problem

Given m images of n fixed points P_j (=X_i) we can write

$$\boldsymbol{p}_{ij} = \mathcal{M}_i \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix} = \mathcal{A}_i \boldsymbol{P}_j + \boldsymbol{b}_i \quad \text{for} \quad i = 1, \ldots, \boxed{m} \quad \text{and} \quad j = 1, \ldots, \boxed{n}.$$

N of cameras N of points

Problem: estimate the m 2×4 matrices M_i and the n positions P_j from the m×n correspondences $\mathbf{p}_{ij}$.

How many equations and how many unknown?

2m × n equations in 8m+3n unknowns

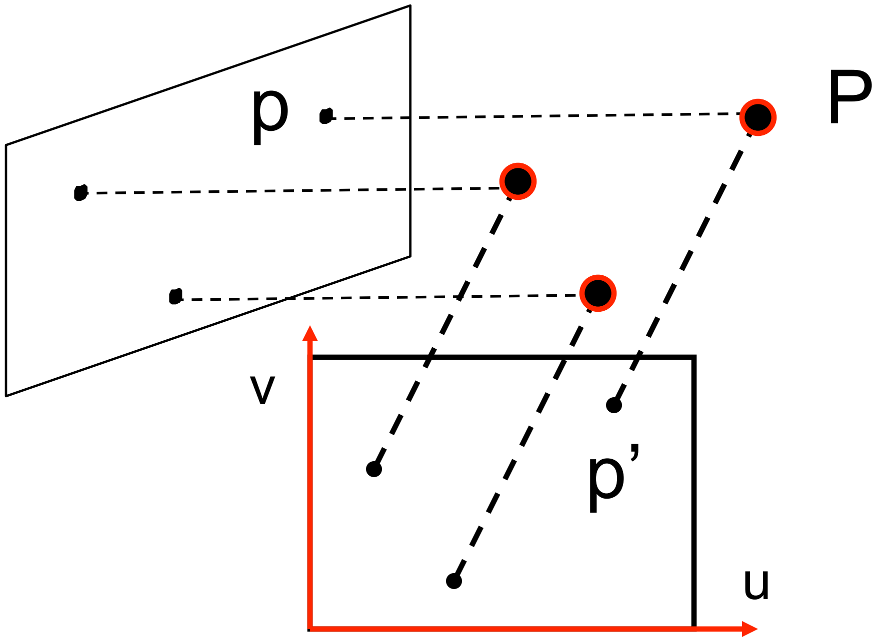
Two approaches:

- Algebraic approach (affine epipolar geometry; estimate F; cameras; points)
- Factorization method

Algebraic analysis (2-view case)

- Derive the fundamental matrix F_A for the affine case
- Compute F_A
- Use F_A to estimate projection matrices
- Use projection matrices to estimate 3D points

1. Deriving the fundamental matrix F_A



Homogeneous system

$$\begin{cases} \boldsymbol{p} = \mathcal{A}\boldsymbol{P} + \boldsymbol{b} \\ \boldsymbol{p}' = \mathcal{A}'\boldsymbol{P} + \boldsymbol{b}' \end{cases} \Rightarrow \begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} \begin{pmatrix} \boldsymbol{P} \\ -1 \end{pmatrix} = \mathbf{0}$$

Dim= ? 4x4

$$\Rightarrow \operatorname{Det} \begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} = 0 \Rightarrow \boxed{\alpha u + \beta v + \alpha' u' + \beta' v' + \delta = 0}$$

Deriving the fundamental matrix F_A

$$\alpha u + \beta v + \alpha' u' + \beta' v' + \delta = 0$$

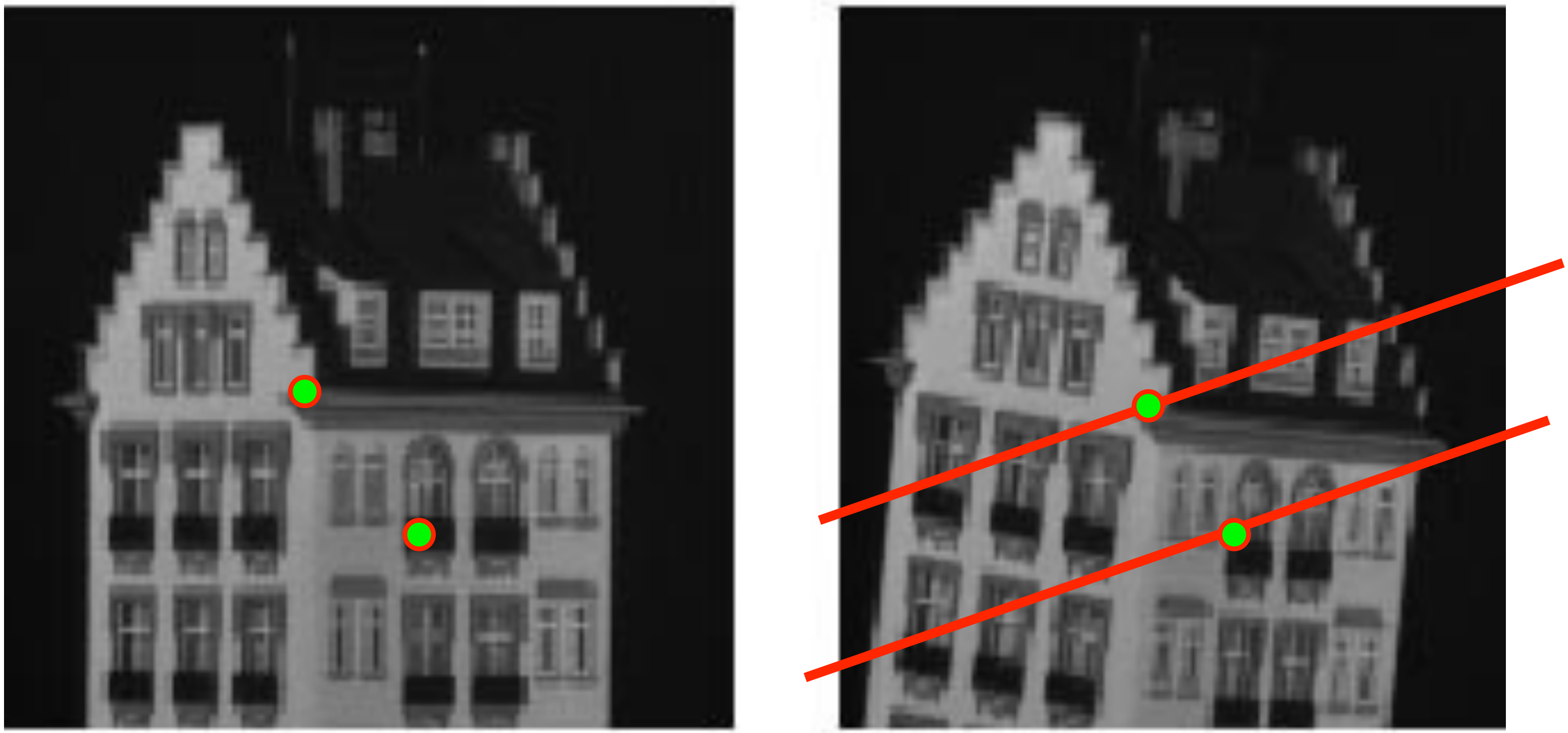
$$(u, v, 1)\mathcal{F}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

where

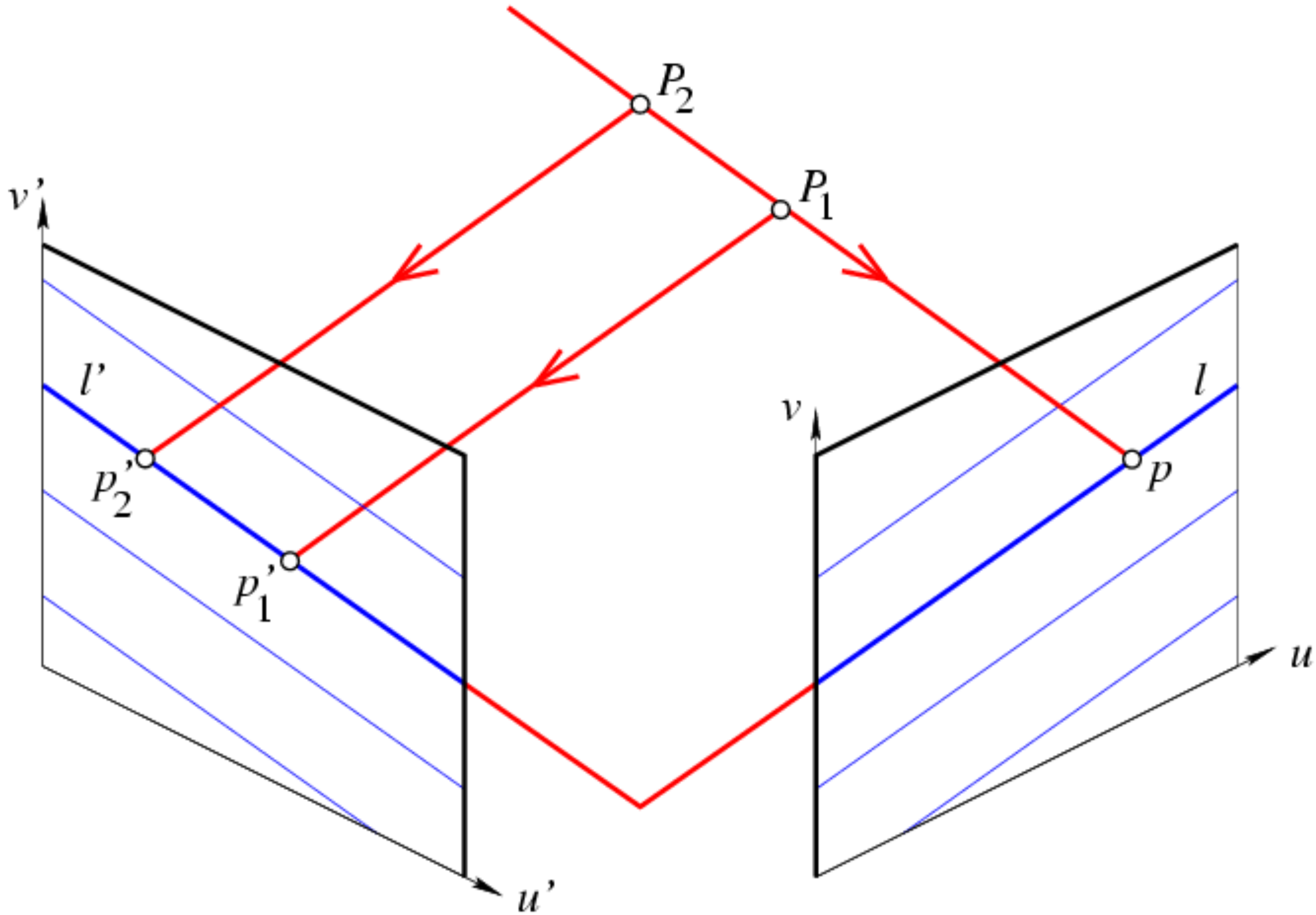
$$\mathcal{F} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ \alpha' & \beta' & \delta \end{pmatrix}$$

The Affine Fundamental Matrix!

Are the epipolar lines parallel or converging?



Affine Epipolar Geometry



Estimating F_A

$$\alpha u + \beta v + \alpha' u' + \beta' v' + \delta = 0$$

- Measurements: u, u’, v, v’
- From n correspondences, we obtain a linear system on the unknown alpha, beta, etc…

$$\begin{bmatrix} u_1' & v_1' & u_1 & v_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n' & v_n' & u_n & v_n & 1 \end{bmatrix} \mathbf{f} = 0$$

- Computed by least square and by enforcing |f|=1
- SVD

2. Estimating projection matrices from epipolar constraints

If M_i and P_i are solutions,
then M_i' and P_i' are also solutions,

where

$$\mathcal{M}_i' = \mathcal{M}_i \mathcal{Q} \quad \text{and} \quad \begin{pmatrix} \boldsymbol{P}_j' \\ 1 \end{pmatrix} = \mathcal{Q}^{-1} \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix}$$

and

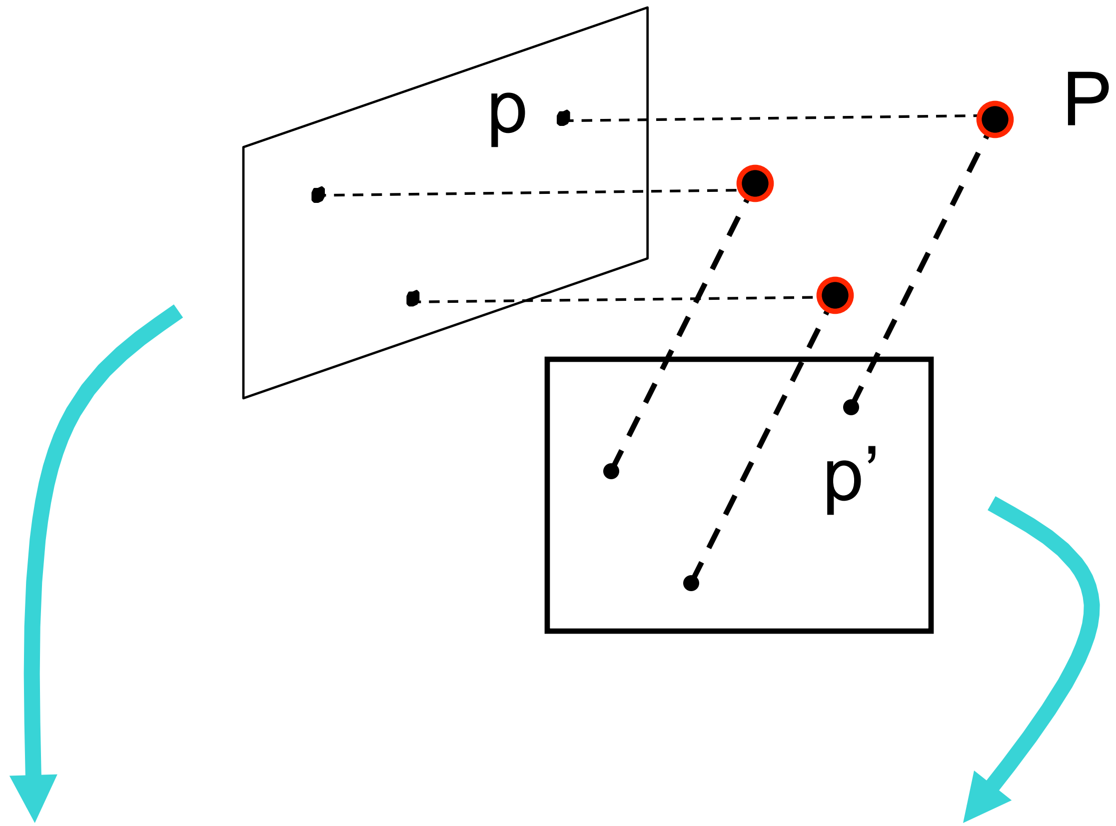
$$\mathcal{Q} = \begin{pmatrix} \mathcal{C} & \boldsymbol{d} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

Q is an affine transformation.

Proof:

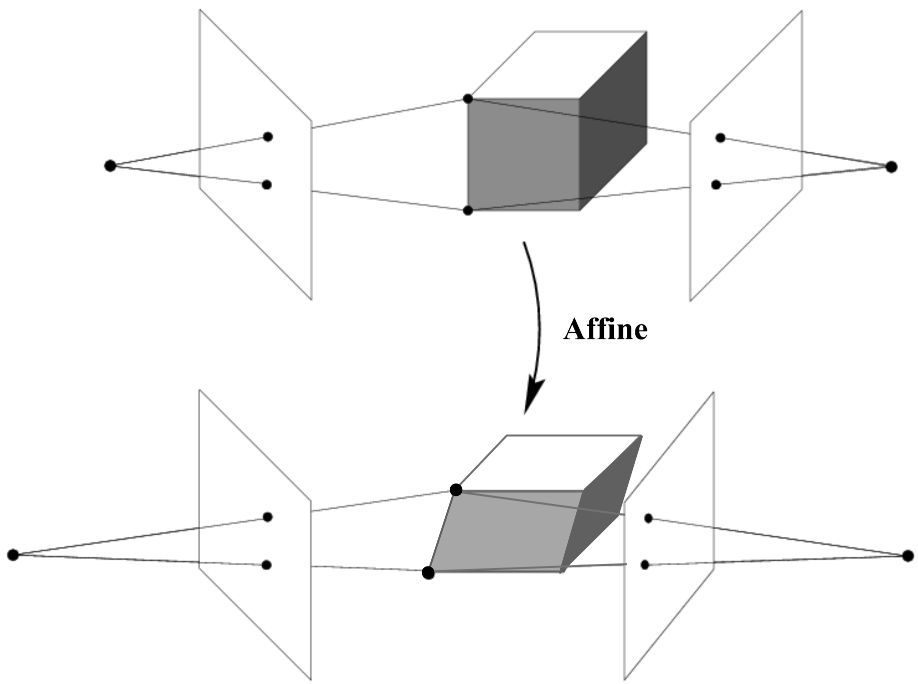
$$\boldsymbol{p}_{ij} = \mathcal{M}_i \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix} = (\mathcal{M}_i \mathcal{Q}) \left(\mathcal{Q}^{-1} \begin{pmatrix} \boldsymbol{P}_j \\ 1 \end{pmatrix} \right) = \mathcal{M}_i' \begin{pmatrix} \boldsymbol{P}_j' \\ 1 \end{pmatrix} \quad \blacksquare$$

Estimating projection matrices from F_A



$$\mathcal{M} = (\,\mathcal{A} \quad \boldsymbol{b}\,) \qquad \mathcal{M}' = (\,\mathcal{A}' \quad \boldsymbol{b}'\,)$$

Affine ambiguity



$$\mathbf{p} = \mathbf{M}\,\mathbf{P} = \left(\mathbf{M}\,\mathbf{Q}_A^{-1}\right)\left(\mathbf{Q}_A\,\mathbf{P}\right)$$

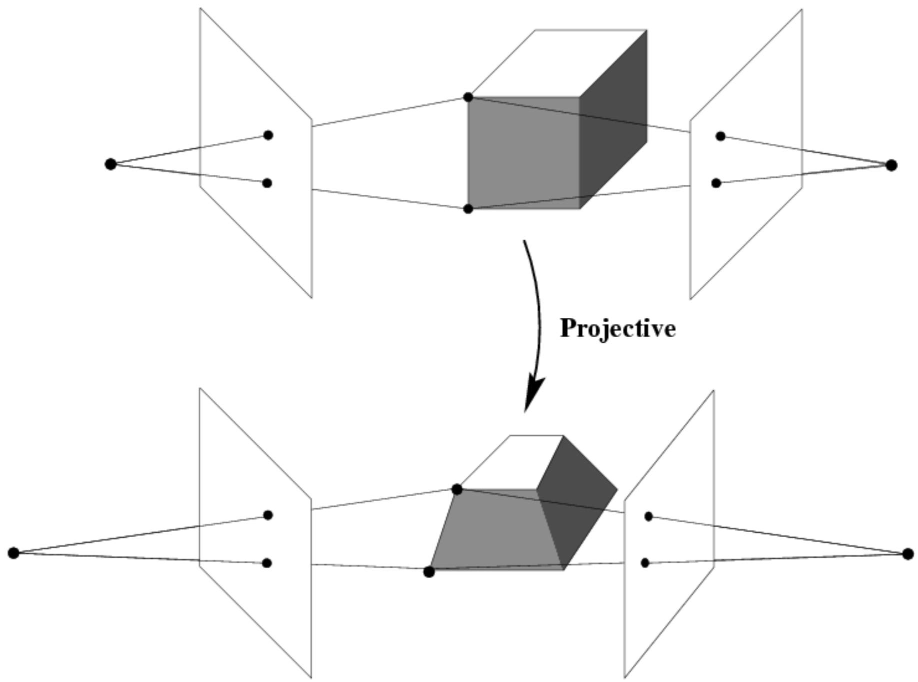
Structure from motion ambiguity

- If we scale the entire scene by some factor *k* and, at the same time, scale the camera matrices by the factor of 1/*k*, the projections of the scene points in the image remain exactly the same

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}^{-1}\right)\left(\mathbf{Q}\mathbf{X}\right)$$

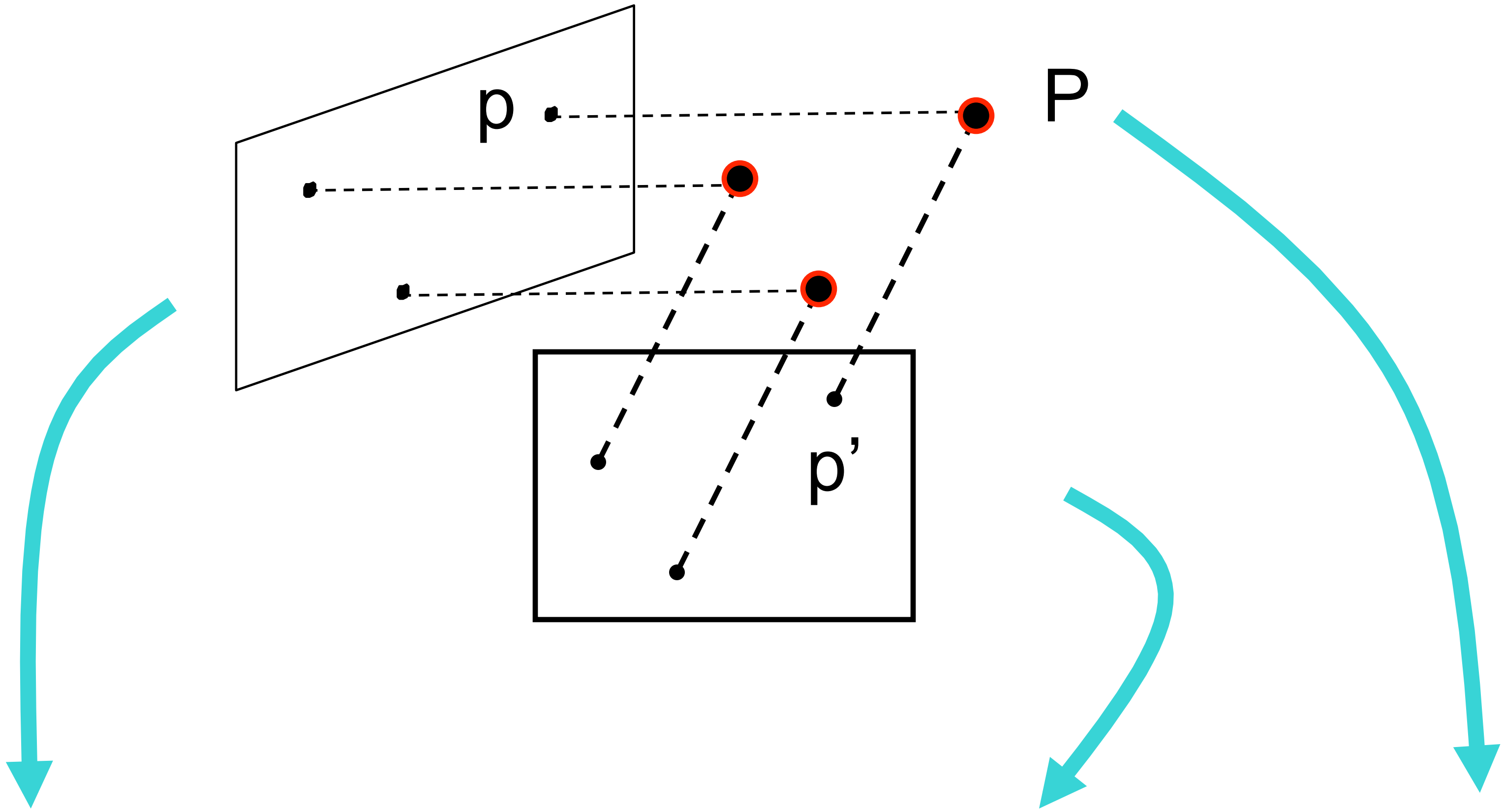
- More generally: if we transform the scene using a transformation **Q** and apply the inverse transformation to the camera matrices, then the images do not change

Projective ambiguity



$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}_{\mathbf{P}}^{-1}\right)\left(\mathbf{Q}_{\mathbf{P}}\mathbf{X}\right)$$

Estimating projection matrices from F_A



$$\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b}) \qquad \mathcal{M}' = (\mathcal{A}' \quad \boldsymbol{b}') \qquad \boldsymbol{P}$$

Estimating projection matrices from F_A

$$\mathcal{M} = \begin{pmatrix} \mathcal{A} & \boldsymbol{b} \end{pmatrix} \;\downarrow\; \tilde{\mathcal{M}} = \mathcal{M}\mathcal{Q} \;\downarrow\; \tilde{\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \;\downarrow\; \tilde{\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Choose Q such that…

$$\mathcal{M}' = \begin{pmatrix} \mathcal{A}' & \boldsymbol{b}' \end{pmatrix} \;\downarrow\; \tilde{\mathcal{M}}' = \mathcal{M}'\mathcal{Q} \;\downarrow\; \widetilde{M}' \;\downarrow\; \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix}$$

$$\boldsymbol{P} \;\downarrow\; \tilde{\boldsymbol{P}} = \mathcal{Q}^{-1}\boldsymbol{P} \;\downarrow\; \tilde{\boldsymbol{P}} \;\downarrow\; \tilde{\boldsymbol{P}} = \begin{pmatrix} u \\ v \\ u' \end{pmatrix}$$

Where a,b,c,d can be expressed as function of the parameters of F_A

See HZ page 348

4. Estimating the structure from F_A

$$\tilde{\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix}$$

A b A' b'

$$\begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} \begin{pmatrix} \boldsymbol{P} \\ -1 \end{pmatrix} = \boldsymbol{0} \quad \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u' \\ a & b & c & v' - d \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{P}} \\ -1 \end{pmatrix} = 0 \quad \Rightarrow \quad \boxed{\tilde{\boldsymbol{P}} = \begin{pmatrix} u \\ v \\ u' \end{pmatrix}}$$

Can be solved by least square again

Estimating projection matrices from epipolar constraints

$$\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b}) \qquad \mathcal{M}' = (\mathcal{A}' \quad \boldsymbol{b}') \qquad \boldsymbol{P}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\tilde{\mathcal{M}} = \mathcal{M}\mathcal{Q} \qquad \tilde{\mathcal{M}}' = \mathcal{M}'\mathcal{Q} \qquad \tilde{\boldsymbol{P}} = \mathcal{Q}^{-1}\boldsymbol{P}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

Choose Q such that…

$$\tilde{\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \qquad \tilde{\boldsymbol{P}}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \tilde{A}' = \begin{bmatrix} 0 & 0 & 1 \\ \boxed{a \quad b \quad c} \end{bmatrix}$$

$$\tilde{b} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \qquad \tilde{b}' = \begin{bmatrix} 0 & \boxed{d} \end{bmatrix}^T$$

Canonical affine cameras

Function of the parameters of F

Estimating projection matrices from epipolar constraints

$$\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b}) \qquad \mathcal{M}' = (\mathcal{A}' \quad \boldsymbol{b}') \qquad \boldsymbol{P}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\tilde{\mathcal{M}} = \mathcal{M}\mathcal{Q} \qquad \tilde{\mathcal{M}}' = \mathcal{M}'\mathcal{Q} \qquad \tilde{\boldsymbol{P}} = \mathcal{Q}^{-1}\boldsymbol{P}$$

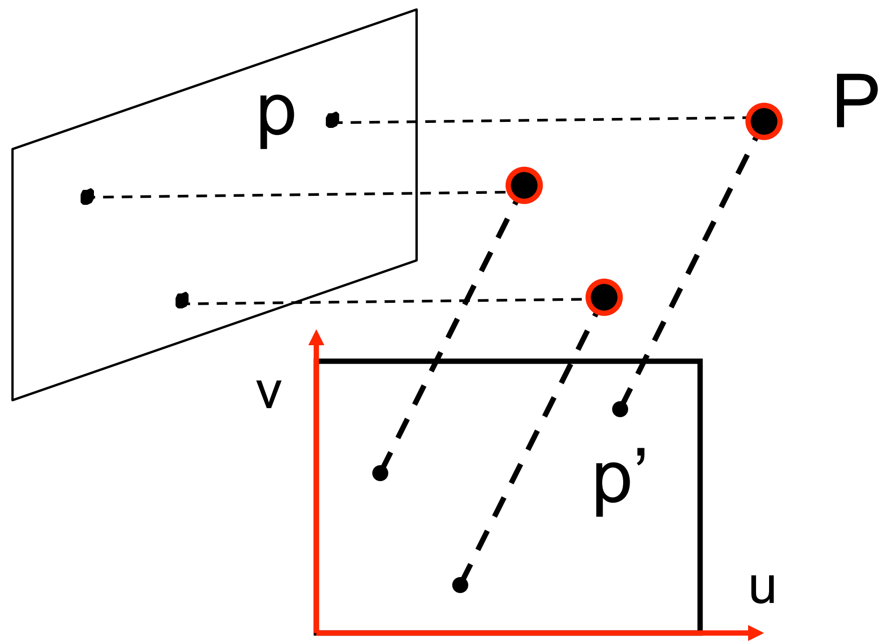
$$\downarrow \qquad \downarrow \qquad \downarrow$$

Choose Q such that…

$$\tilde{\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \quad \tilde{\boldsymbol{P}}$$

By re-enforcing the epipolar constraint, we can compute a, b, c, d directly from the measurements

Reminder: epipolar constraint



Homogeneous system

$$\begin{cases} \boldsymbol{p} = \mathcal{A}\boldsymbol{P} + \boldsymbol{b} \\ \boldsymbol{p}' = \mathcal{A}'\boldsymbol{P} + \boldsymbol{b}' \end{cases} \Rightarrow \begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} \begin{pmatrix} \boldsymbol{P} \\ -1 \end{pmatrix} = \boldsymbol{0}$$

$$\Rightarrow \boxed{\operatorname{Det} \begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} = 0} \Rightarrow \alpha u + \beta v + \alpha' u' + \beta' v' + \delta = 0$$

Estimating projection matrices from epipolar constraints

$$\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b}) \qquad \mathcal{M}' = (\mathcal{A}' \quad \boldsymbol{b}') \qquad \boldsymbol{P}$$

$$\tilde{\mathcal{M}} = \mathcal{M}\mathcal{Q} \qquad \tilde{\mathcal{M}}' = \mathcal{M}'\mathcal{Q} \qquad \tilde{\boldsymbol{P}} = \mathcal{Q}^{-1}\boldsymbol{P}$$

Choose Q such that…

$$\tilde{\mathcal{M}} = \left(\underbrace{\boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}}}_{\tilde{A}} \; \underbrace{\boxed{\begin{matrix} 0 \\ 0 \end{matrix}}}_{\tilde{b}}\right) \qquad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \qquad \tilde{\boldsymbol{P}}$$

Re-enforce the Epipolar constraint

$$\operatorname{Det}\begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} = 0 \quad \rightarrow \quad \operatorname{Det}\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u' \\ a & b & c & v' - d \end{pmatrix} = 0$$

Estimating projection matrices from epipolar constraints

$$\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b}) \qquad \mathcal{M}' = (\mathcal{A}' \quad \boldsymbol{b}') \qquad \boldsymbol{P}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\tilde{\mathcal{M}} = \mathcal{M}\mathcal{Q} \qquad \tilde{\mathcal{M}}' = \mathcal{M}'\mathcal{Q} \qquad \tilde{\boldsymbol{P}} = \mathcal{Q}^{-1}\boldsymbol{P}$$

Choose Q such that…

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\tilde{\mathcal{M}} = \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}}_{A} \; \underbrace{\begin{matrix} 0 \\ 0 \end{matrix}}_{b}\right) \qquad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \qquad \tilde{\boldsymbol{P}}$$

$$\boxed{\operatorname{Det}\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u' \\ a & b & c & v' - d \end{pmatrix} = au - bv + cu' + v' - d = 0}$$

Estimating projection matrices from epipolar constraints

$$\mathrm{Det}\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u' \\ a & b & c & v'-d \end{pmatrix} = au - bv + cu' + v' - d = 0$$

- Linear relationship between measurements and unknown

  Unknown: a, b, c, d
  Measurements: u, u', v, v'

- From at least 4 correspondences, we can solve this linear system and compute a, b, c, d (via least square)
- The cameras can be computed
- How about the structure?

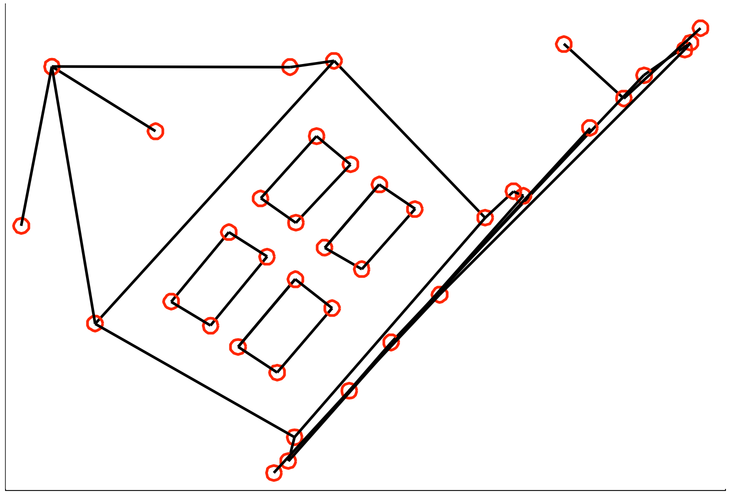
4. Estimating the structure from F_A

$$\tilde{\mathcal{M}} = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}}_{A} & \underbrace{\begin{matrix} 0 \\ 0 \end{matrix}}_{b} \end{pmatrix} \quad \tilde{\mathcal{M}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \quad \tilde{\boldsymbol{P}}$$

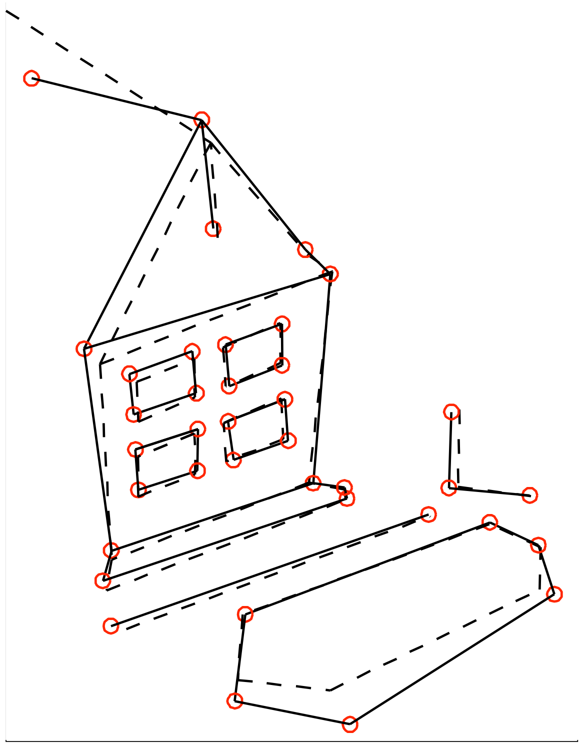
$$\begin{pmatrix} \mathcal{A} & \boldsymbol{p} - \boldsymbol{b} \\ \mathcal{A}' & \boldsymbol{p}' - \boldsymbol{b}' \end{pmatrix} \begin{pmatrix} \boldsymbol{P} \\ -1 \end{pmatrix} = \boldsymbol{0} \quad \Rightarrow$$

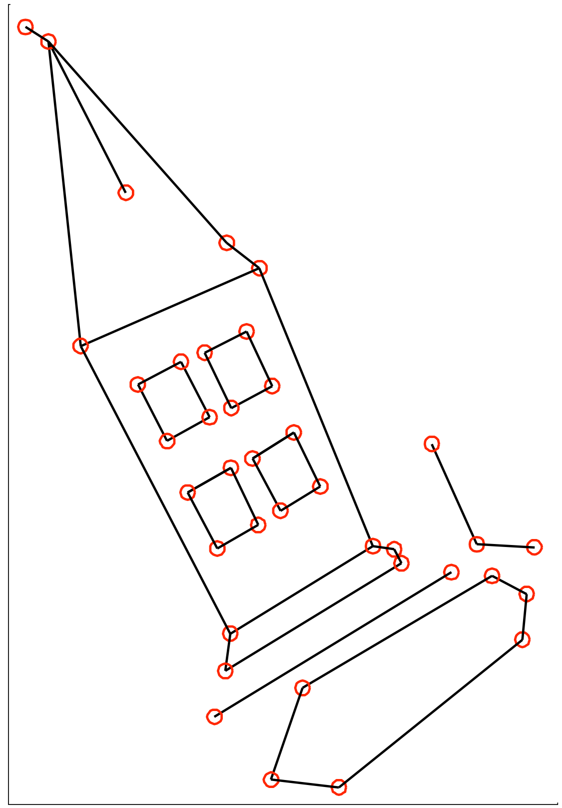
$$\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u' \\ a & b & c & v' - d \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{P}} \\ -1 \end{pmatrix} = 0 \quad \Rightarrow \quad \boxed{\tilde{\boldsymbol{P}} = \begin{pmatrix} u \\ v \\ u' \end{pmatrix}}$$

Can be solved by least square again



First reconstruction. Mean re-projection error: 1.6pixel





Second reconstruction. Mean re-projection error: 7.8pixel

A factorization method – Tomasi & Kanade algorithm

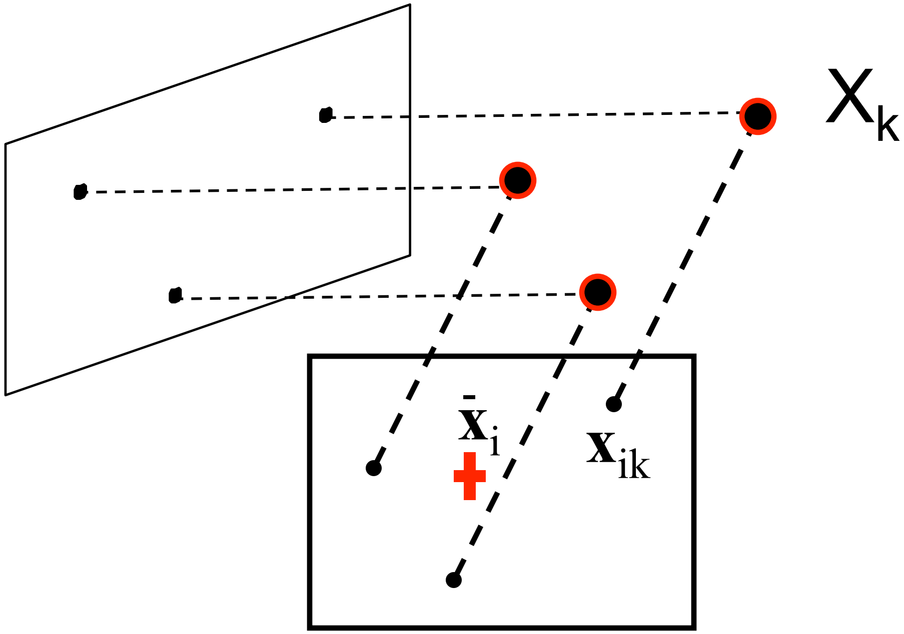
C. Tomasi and T. Kanade.
Shape and motion from image streams under orthography: A factorization method. *IJCV*, 9(2):137-154, November 1992.

- Centering the data
- Factorization

A factorization method - Centering the data

Centering: subtract the centroid of the image points

$$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \boxed{\frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik}}\ \bar{\mathbf{x}}_i$$



A factorization method - Centering the data

Centering: subtract the centroid of the image points

$$\begin{cases} \hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_{ik} = \mathbf{A}_i \mathbf{X}_j + \mathbf{b}_i - \frac{1}{n}\sum_{k=1}^{n} \left(\mathbf{A}_i \mathbf{X}_k + \mathbf{b}_i \right) \\ \mathbf{x}_{ij} = \mathbf{A}_i \mathbf{X}_j + \mathbf{b}_i \end{cases}$$

A factorization method - Centering the data

Centering: subtract the centroid of the image points

$$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_j + \mathbf{b}_i - \frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{A}_i\mathbf{X}_k + \mathbf{b}_i\right)$$

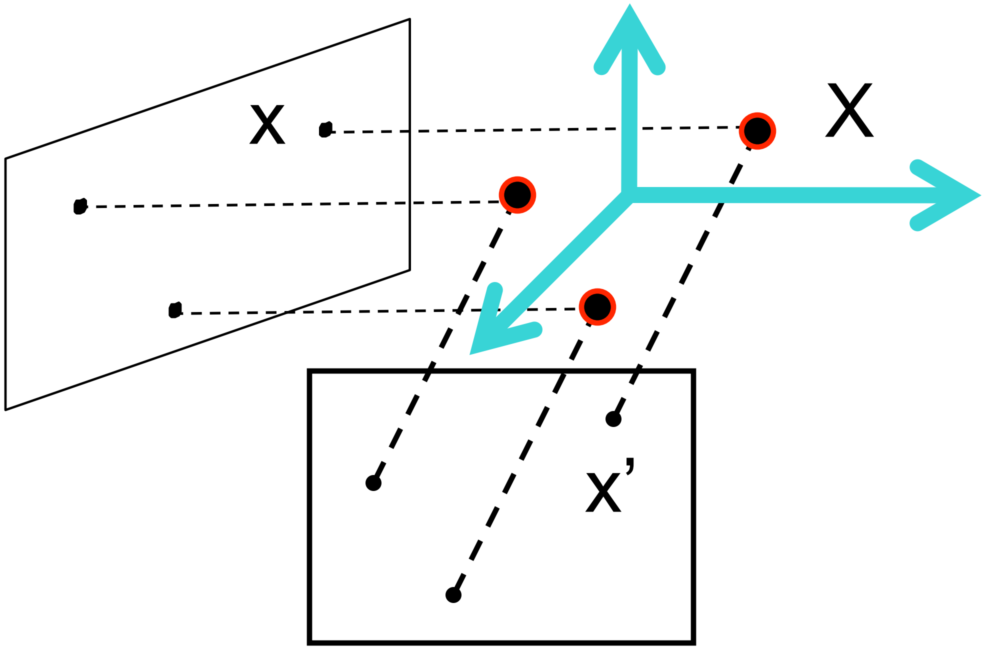
$$= \mathbf{A}_i\left(\mathbf{X}_j - \boxed{\frac{1}{n}\sum_{k=1}^{n}\mathbf{X}_k}\right)$$

Assume that the origin of the world coordinate system is at the centroid of the 3D points

After centering, each normalized point $\mathbf{x}_{ij}$ is related to the 3D point $\mathbf{X}_i$ by

$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i\mathbf{X}_j$$

A factorization method - Centering the data



$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i \mathbf{X}_j$$

A factorization method - factorization

Let's create a $2m \times n$ data (measurement) matrix:

$$\begin{array}{cc} \mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix} & \text{cameras } (2m) \\ \text{points } (n) & \end{array}$$

A factorization method - factorization

Let's create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix}}_{\mathrm{M}} \overbrace{\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n \end{bmatrix}}^{\mathrm{S}}$$

$(2m \times n)$

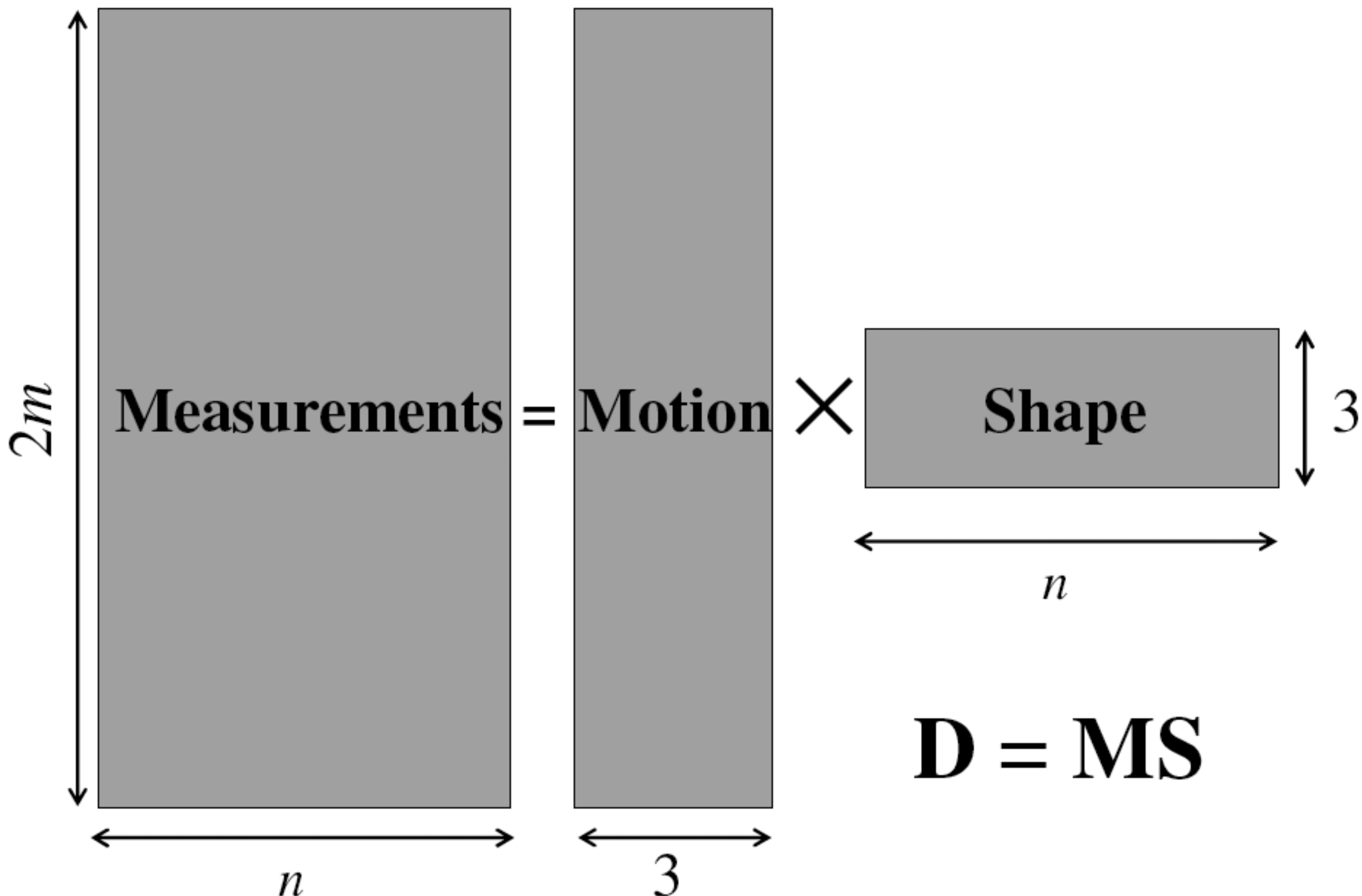
cameras $(2m \times 3)$

points $(3 \times n)$

The measurement matrix **D = M S** has rank 3

(it's a product of a 2mx3 matrix and 3xn matrix)

Factorizing the measurement matrix

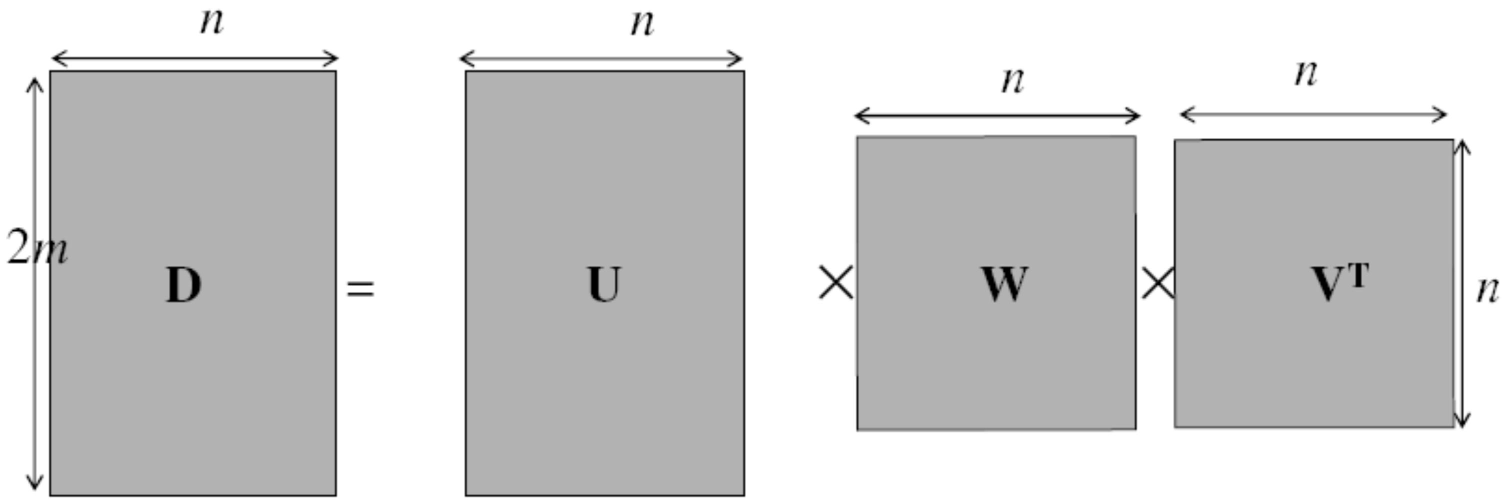


$$\mathbf{D} = \mathbf{M}\mathbf{S}$$

Source: M. Hebert

Factorizing the measurement matrix

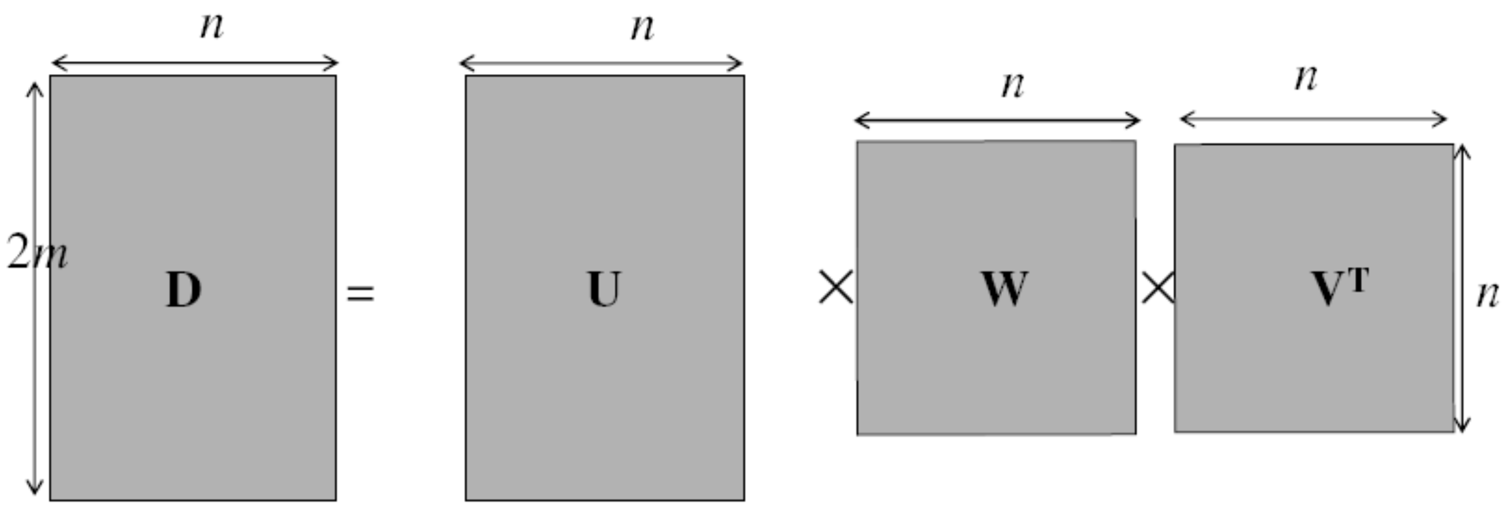
Singular value decomposition of D:



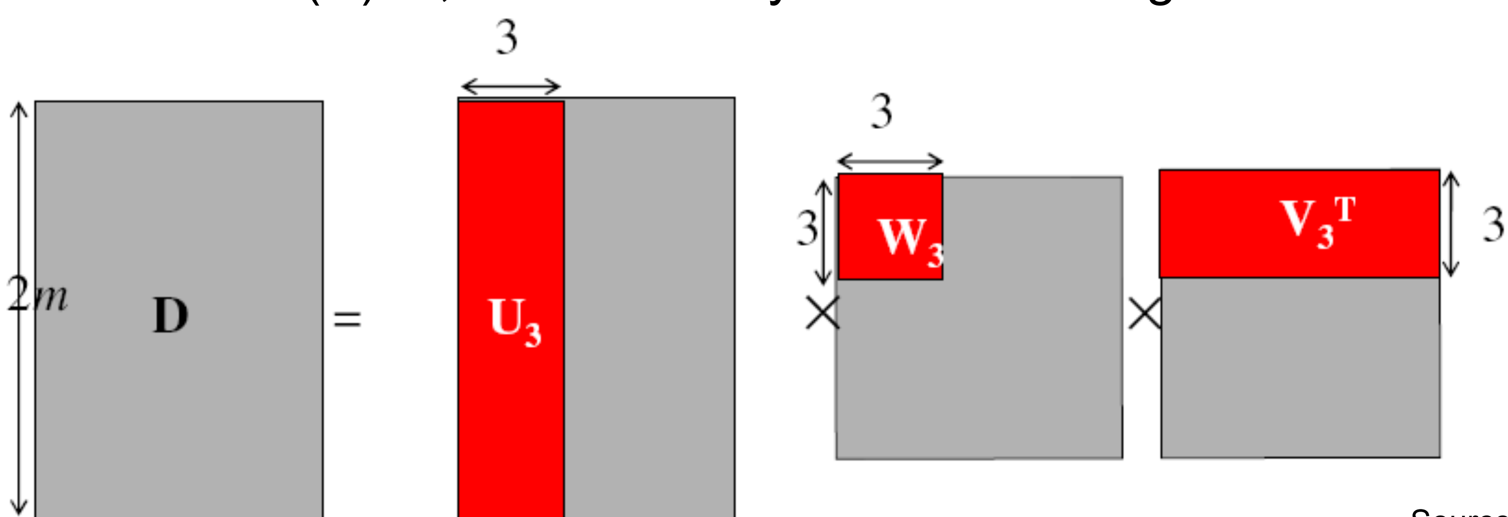
Source: M. Hebert

Factorizing the measurement matrix

Singular value decomposition of D:



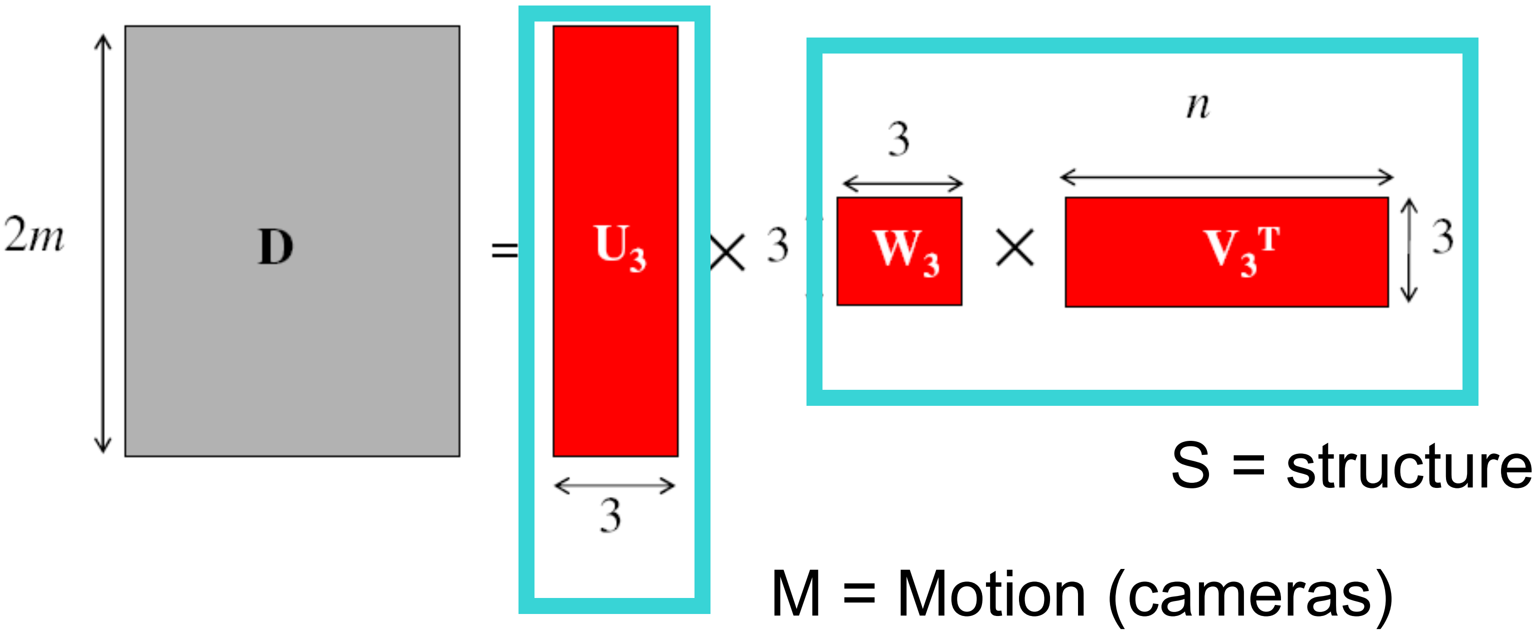
Since rank (D)=3, there are only 3 non-zero singular values



Source: M. Hebert

Factorizing the measurement matrix

Obtaining a factorization from SVD:

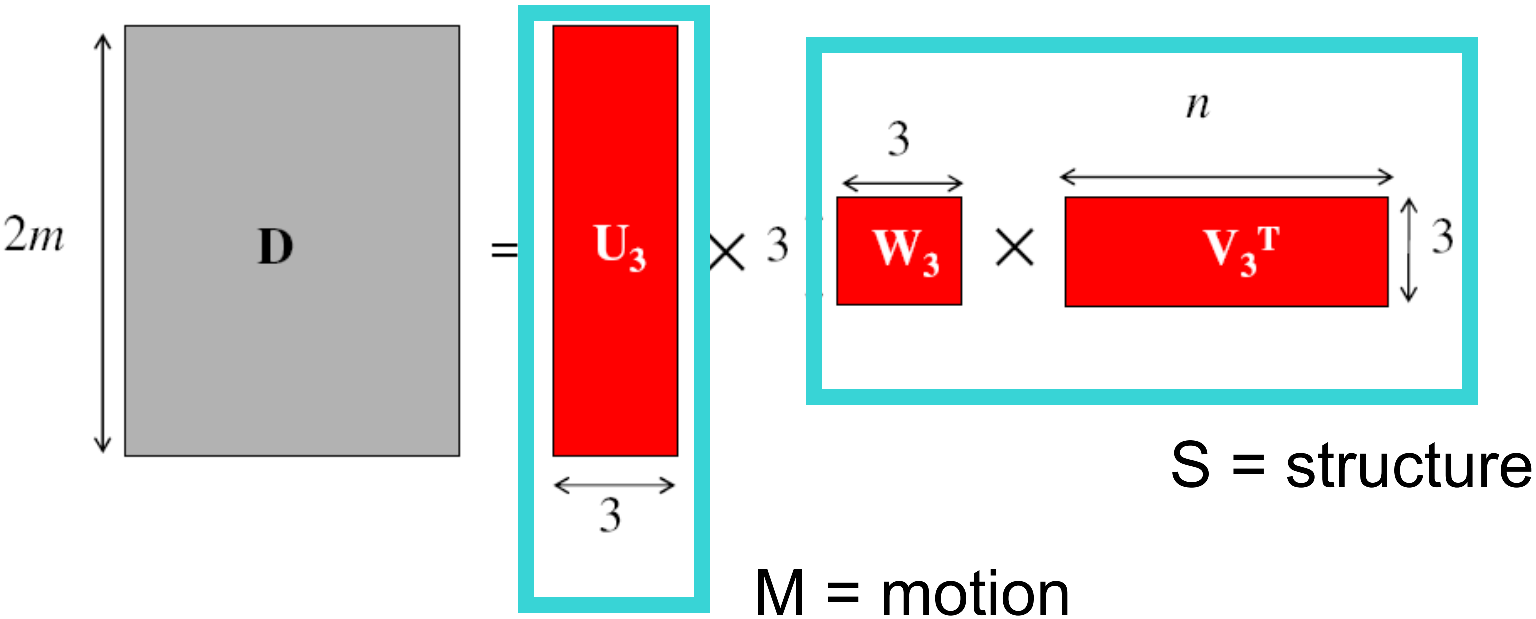


What is the issue here?

D has rank>3 because of - measurement noise
- affine approximation

Factorizing the measurement matrix

Obtaining a factorization from SVD:

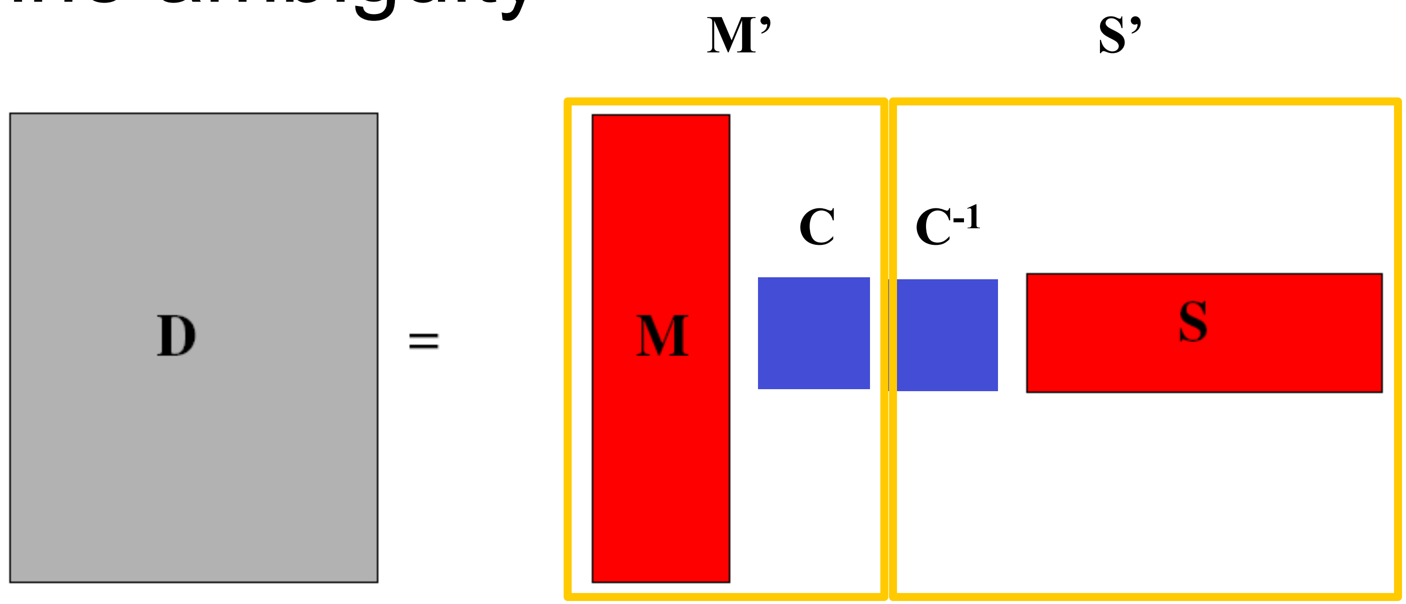


Theorem: When D has a rank greater than p, $\mathcal{U}_p\mathcal{W}_p\mathcal{V}_p^T$ is the best possible rank-p approximation of D in the sense of the Frobenius norm.

$$\mathcal{D} = \mathcal{U}_3\mathcal{W}_3\mathcal{V}_3^T$$

$$\boxed{\begin{cases} \mathcal{A}_0 = \mathcal{U}_3 \\ \mathcal{P}_0 = \mathcal{W}_3\mathcal{V}_3^T \end{cases}}$$

Affine ambiguity



The decomposition is not unique. We get the same **D** by using any 3×3 matrix **C** and applying the transformations **M** → **MC**, **S** → $\mathbf{C}^{-1}\mathbf{S}$

We can enforce some Euclidean constraints to resolve this ambiguity (more on next lecture!)

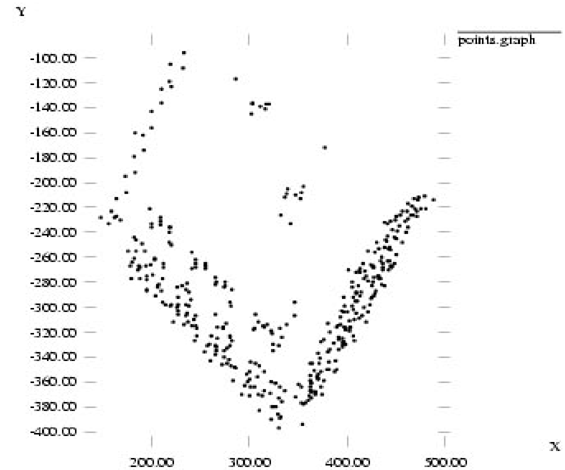
Algorithm summary

1. Given: *m* images and *n* features $\mathbf{x}_{ij}$
2. For each image *i, c*enter the feature coordinates
3. Construct a 2*m* × *n* measurement matrix **D**:
   - Column *j* contains the projection of point *j* in all views
   - Row *i* contains one coordinate of the projections of all the *n* points in image *i*
4. Factorize **D**:
   - Compute SVD: $\mathbf{D} = \mathbf{U}\ \mathbf{W}\ \mathbf{V}^{\mathbf{T}}$
   - Create $\mathbf{U}_3$ by taking the first 3 columns of **U**
   - Create $\mathbf{V}_3$ by taking the first 3 columns of **V**
   - Create $\mathbf{W}_3$ by taking the upper left 3 × 3 block of **W**
5. Create the motion and shape matrices:
   - $\mathbf{M} = \mathbf{M} = \mathbf{U}_3$ and $\mathbf{S} = \mathbf{W}_3\mathbf{V}_3^{T}$ (or $\mathbf{U}_3\mathbf{W}_3^{½}$ and $\mathbf{S} = \mathbf{W}_3^{½}\ \mathbf{V}_3^{T}$)
6. Eliminate affine ambiguity

Reconstruction results

1

60



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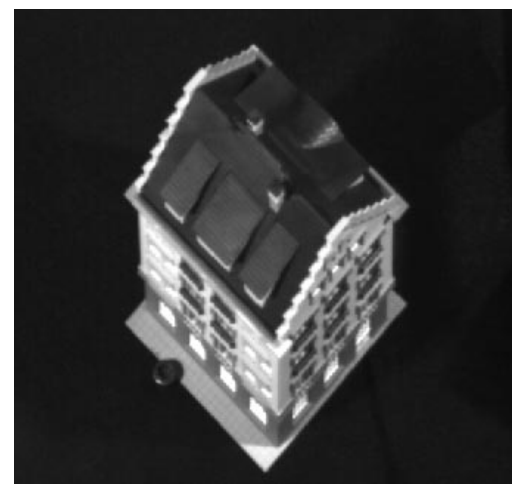
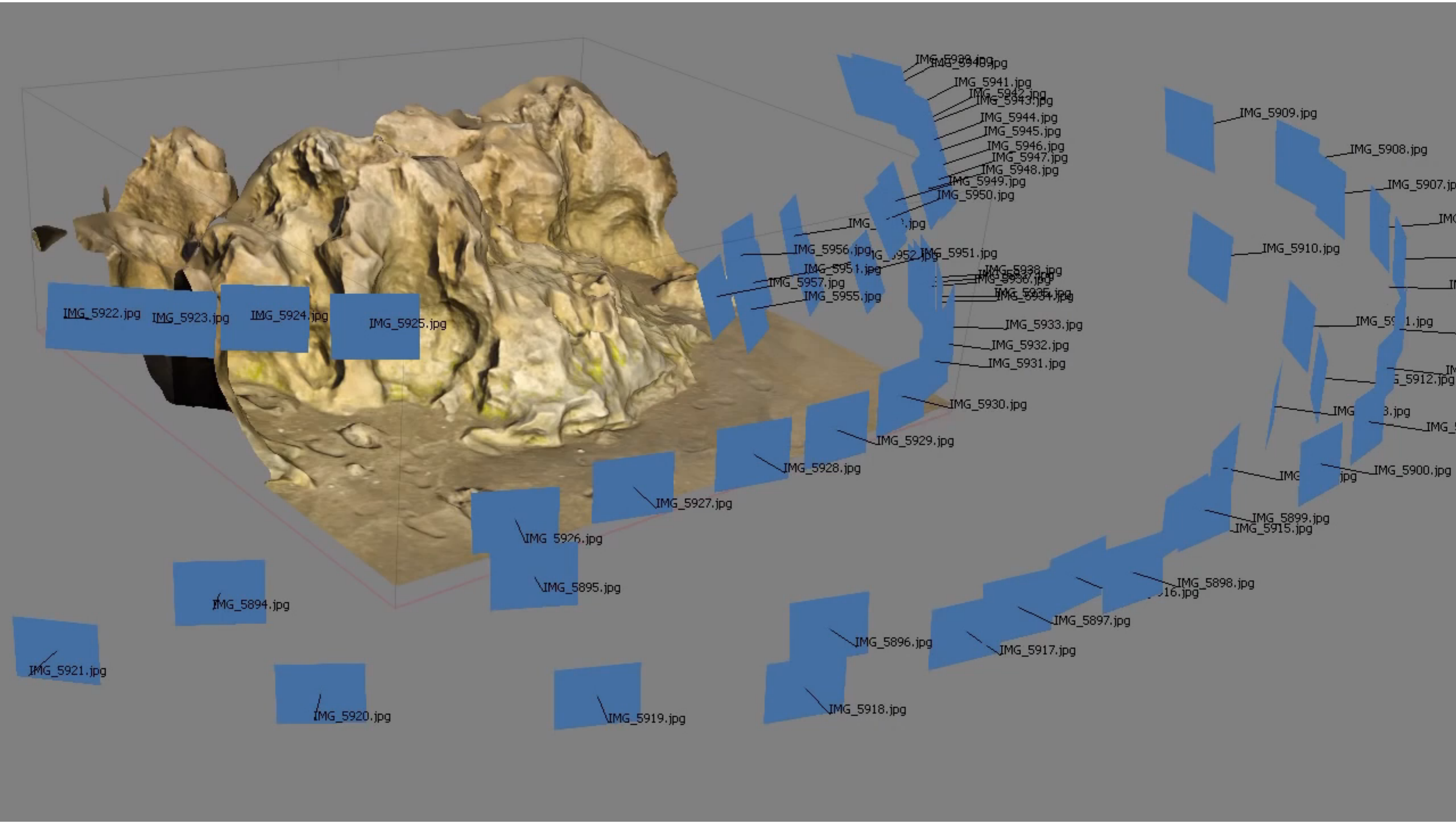


C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.](#) *IJCV*, 9(2):137-154, November 1992.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Next lecture

Multiple view geometry

Perspective structure from Motion